Appendix

Here we add some important results and calculations which were used in the main part of the work and which are important by themselves.

A.1. Integral symmetric bilinear forms. Elements of the discriminant forms technique

Here, for readers' convenience, we review results about integral symmetric bilinear forms (lattices) which we used. We follow [Nik80b].

A.1.1. Lattices

Everywhere in the sequel, by a **lattice** we mean a free $\mathbb{Z}$-module of finite rank, with a nondegenerate symmetric bilinear form with values in the ring $\mathbb{Z}$ of rational integers (thus, "lattice" replaces the phrase "nondegenerate integral symmetric bilinear form").

A lattice M is called **even** if $x^2 = x \cdot x$ is even for each $x \in M$, and **odd** otherwise (here we denote by $x \cdot y$ the value of the bilinear form of M at the pair (x, y)). By $M_1 \oplus M_2$ we denote the orthogonal direct sum of lattices M_1 and M_2. If M is a lattice, we denote by $M(a)$ the lattice obtained from M by multiplying the form of M by the rational number $a \neq 0$, assuming that $M(a)$ is also integral.

A.1.2. Finite symmetric bilinear and quadratic forms

By a **finite symmetric bilinear form** we mean a symmetric bilinear form $b : \mathfrak{A} \times \mathfrak{A} \to \mathbb{Q}/\mathbb{Z}$ defined on a finite Abelian group $\mathfrak{A}$.

By a **finite quadratic form** we mean a map $q : \mathfrak{A} \to \mathbb{Q}/2\mathbb{Z}$ satisfying the following conditions:

1) $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$ and $a \in \mathfrak{A}$.

2) $q(a+a') - q(a) - q(a') \equiv 2b(a,a') \pmod 2$, where $b : \mathfrak{A}\times\mathfrak{A} \to \mathbb{Q}/\mathbb{Z}$ is a finite symmetric bilinear form, which we call the **bilinear form** of q.

A finite quadratic form q is nondegenerate when b is nondegenerate. In the usual way, we introduce the notion of orthogonality ($\perp$) and of orthogonal sum ($\oplus$) of finite symmetric bilinear and quadratic forms.

A.1.3. The discriminant form of a lattice

The bilinear form of a lattice M determines a canonical embedding $M \subset M^* = \mathrm{Hom}(M,\mathbb{Z})$. The **discriminant group** of the lattice M is the factor group $\mathfrak{A}_M = M^*/M$. It is finite and Abelian, and its order is equal to $|\det(M)|$. We remind that the **determinant** $\det(M)$ of M equals $\det(e_i \cdot e_j)$ for some basis e_i of the lattice M. A lattice L is called **unimodular** if $\det(L) = \pm 1$.

We extend the bilinear form of M to one on M^*, taking values in $\mathbb{Q}$. We put

$$b_M(t_1 + M, t_2 + M) = t_1 \cdot t_2 + \mathbb{Z}$$

where $t_1, t_2 \in M^*$, and

$$q_M(t+M) = t^2 + 2\mathbb{Z},$$

if M is even, where $t \in M^*$.

We obtain the **discriminant bilinear form** $b_M : \mathfrak{A}_M \times \mathfrak{A}_M \to \mathbb{Q}/\mathbb{Z}$ and the **discriminant quadratic form** $q_M : \mathfrak{A}_M \to \mathbb{Q}/2\mathbb{Z}$ (if M is even) of the lattice M. They are nondegenerate.

Similarly, one can define discriminant forms of p-adic lattices over the ring $\mathbb{Z}_p$ of p-adic integers for a prime p. The decomposition of $\mathfrak{A}_M$ as a sum of its p-components $(\mathfrak{A}_M)_p$ defines the decomposition of b_M and q_M as the orthogonal sum of its p-components $(b_M)_p$ and $(q_M)_p$. They are equal to the discriminant forms of the corresponding p-adic lattices $M \otimes \mathbb{Z}_p$.

We denote by:

$K_\theta^{(p)}(p^k)$ the 1-dimensional p-adic lattice determined by the matrix $\langle \theta p^k \rangle$, where $k \geq 1$ and $\theta \in \mathbb{Z}_p^*$ (taken $\mod (\mathbb{Z}_p^*)^2$);

$U^{(2)}(2^k)$ and $V^{(2)}(2^k)$ the 2-dimensional 2-adic lattices determined by the matrices

$$\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}, \quad \begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}$$

respectively;

$q_\theta^{(p)}(p^k)$, $u_+^{(2)}(2^k)$ and $v_+^{(2)}(2^k)$, the discriminant quadratic forms of $K_\theta^{(p)}(p^k)$, $U^{(2)}(2^k)$ and $V^{(2)}(2^k)$ respectively;

$b_\theta^{(p)}(p^k)$, $u_-^{(2)}(2^k)$ and $v_-^{(2)}(2^k)$, the bilinear forms of $q_\theta^{(p)}(p^k)$, $u_+^{(2)}(2^k)$ and $v_+^{(2)}(2^k)$ respectively.

These p-adic lattices and finite quadratic and bilinear forms are called **elementary**. Any p-adic lattice (respectively finite non-degenerate quadratic, bilinear form) is an orthogonal sum of elementary ones *(Jordan decomposition).*

A.1.4. Existence of an even lattice with a given discriminant quadratic form

The **signature** of a lattice M is equal to $\operatorname{sign} M = t_{(+)} - t_{(-)}$ where $t_{(+)}$ and $t_{(-)}$ are numbers of positive and negative squares of the corresponding real form $M \otimes \mathbb{R}$. The formula

$$\operatorname{sign} q_M \mod 8 = \operatorname{sign} M \mod 8 = t_{(+)} - t_{(-)} \mod 8$$

where M is an even lattice, correctly defines the signature $\mod 8$ for non-degenerate finite quadratic forms. For elementary finite quadratic forms we obtain respectively

$$\operatorname{sign} q_\theta^{(p)}(p^k) \equiv k^2(1-p) + 4k\eta \mod 8$$

where p is odd and $\left(\frac{\theta}{p}\right) = (-1)^\eta$, where we use Legendre symbol;

$$\operatorname{sign} q_\theta^{(2)}(2^k) \equiv \theta + 4\omega(\theta)k \mod 8$$

where $\omega(\theta) \equiv (\theta^2-1)/8 \mod 2$;

$$\operatorname{sign} v_+^{(2)}(2^k) \equiv 4k \mod 8;$$

$$\operatorname{sign} u_+^{(2)}(2^k) \equiv 0 \mod 8.$$

In particular, $\operatorname{sign} L \equiv 0 \mod 8$ if L is an even unimodular lattice.

We denote by $l(\mathfrak{A})$ the minimal number of generators of a finite Abelian group $\mathfrak{A}$. We consider an even lattice M with the invariants $(t_{(+)}, t_{(-)}, q)$ where $t_{(+)}$, $t_{(-)}$ are its numbers of positive and negative squares, and $q \cong q_M$; we denote by $\mathfrak{A}_q$ the group where q is defined. *The invariants* $(t_{(+)}, t_{(-)}$, $q)$ *are equivalent to the genus of* M (see Corollary 1.9.4 in [Nik80b]) . Thus, they define the isomorphism classes of the p-adic lattices $M \otimes \mathbb{Z}_p$ for all prime p, and of $M \otimes \mathbb{R}$.

We have (see Theorem 1.10.1 in [Nik80b]):

Theorem A.1. *An even lattice with invariants* $(t_{(+)}, t_{(-)}, q)$ *exists if and only if the following conditions are simultaneously satisfied:*

1) $t_{(+)} - t_{(-)} \equiv \operatorname{sign} q \mod 8$.

2) $t_{(+)} \geq 0$, $t_{(-)} \geq 0$, $t_{(+)} + t_{(-)} \geq l(\mathfrak{A}_q)$.

3) $(-1)^{t_{(-)}}|\mathfrak{A}_q| \equiv \det(K(q_p)) \mod (\mathbb{Z}_p^*)^2$ *for all odd primes* p *for which* $t_{(+)} + t_{(-)} = l(\mathfrak{A}_{q_p})$ *(here* $K(q_p)$ *is the unique p-adic lattice with the discriminant quadratic form* q_p *and the rank* $l(\mathfrak{A}_{q_p})$*).*

4) $|\mathfrak{A}_q| \equiv \pm\det(K(q_2)) \mod (\mathbb{Z}_2^*)^2$ *if* $t_{(+)} + t_{(-)} = l(\mathfrak{A}_{q_2})$ *and* $q_2 \neq q_\theta^{(2)}(2) \oplus q_2'$ *(here* $K(q_2)$ *is the unique 2-adic lattice with the discriminant quadratic form* q_2 *and the rank* $l(\mathfrak{A}_{q_2})$*).*

From $l(\mathfrak{A}_q) = \max_p l(\mathfrak{A}_{q_p})$, we obtain the important corollary.

Corollary A.2. *An even lattice with invariants* $(t_{(+)}, t_{(-)}, q)$ *exists if the following conditions are simultaneously satisfied:*

1) $t_{(+)} - t_{(-)} \equiv \operatorname{sign} q \mod 8$.

2) $t_{(+)} \geq 0$, $t_{(-)} \geq 0$, $t_{(+)} + t_{(-)} > l(\mathfrak{A}_q)$.

Theorem 1.16.5 and Corollary 1.16.6 in [Nik80b] give similar results for odd lattices.

A.1.5. Primitive embeddings into even unimodular lattices

We have a simple statement (see Proposition 1.4.1 in [Nik80b]).

Proposition A.3. *Let* M *be an even lattice. Its overlattice* $M \subset N$ *of finite index is equivalent to the isotropic subgroup* $H = N/M \subset \mathfrak{A}_M$ *with respect to* q_M*. Moreover, we have* $q_N = q_M|(H^\perp)/H$.

An embedding of lattices $M \subset L$ is primitive if L/M is a free $\mathbb{Z}$-module.

Let L be an even unimodular lattice, $M \subset L$ its primitive sublattice, and $T = M_L^\perp$. Then $M \oplus T \subset L$ is an overlattice of a finite index. Applying Proposition A.3, we obtain that $H = L/(M \oplus T)$ is the graph of an isomorphism $\gamma : q_M \cong -q_T$, and this is equivalent to a primitive embedding $M \subset L$ into an even unimodular lattice with $T = M_L^\perp$. Thus we have (see Proposition 1.6.1 in [Nik80b])

Proposition A.4. *A primitive embedding of an even lattice* M *into an even unimodular lattice, in which the orthogonal complement is isomorphic to* T*, is determined by an isomorphism* $\gamma : q_M \cong -q_T$.

Two such isomorphisms γ *and* γ' *determine isomorphic primitive embeddings if and only if they are conjugate via an automorphism of* T.

From Theorem A.1 and Corollary A.2 we then obtain (Theorem 1.12.2 and Corollary 1.12.3 in [Nik80b])

Theorem A.5. *The following properties are equivalent:*

a) There exists a primitive embedding of an even lattice M *with invariants* $(t_{(+)}, t_{(-)}, q)$ *into some even unimodular lattice of signature* $(l_{(+)}, l_{(-)})$.

b) There exists an even lattice with invariants $(l_{(+)}-t_{(+)}, l_{(-)}-t_{(-)}, -q)$

c) There exists an even lattice with invariants $(l_{(-)}-t_{(-)}, l_{(+)}-t_{(+)}, q)$.

d) The following conditions are simultaneously satisfied:

1) $l_{(+)} - l_{(-)} \equiv 0 \mod 8$.

2) $l_{(+)} - t_{(+)} \geq 0$, $l_{(-)} - t_{(-)} \geq 0$, $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} \geq l(\mathfrak{A}_q)$.

3) $(-1)^{l_{(+)}-t_{(+)}}|\mathfrak{A}_q| \equiv \det(K(q_p)) \mod (\mathbb{Z}_p^*)^2$ *for all odd primes p for which* $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} = l(\mathfrak{A}_{q_p})$ *(here* $K(q_p)$ *is the unique p-adic lattice with the discriminant quadratic form* q_p *and the rank* $l(\mathfrak{A}_{q_p})$*).*

4) $|\mathfrak{A}_q| \equiv \pm\det(K(q_2)) \mod (\mathbb{Z}_2^*)^2$ *if* $l_{(+)}+l_{(-)}-t_{(+)}-t_{(-)} = l(\mathfrak{A}_{q_2})$ *and* $q_2 \neq q_\theta^{(2)}(2) \oplus q_2'$ *(here* $K(q_2)$ *is the unique 2-adic lattice with the discriminant quadratic form* q_2 *and the rank* $l(\mathfrak{A}_{q_2})$*).*

Corollary A.6. *There exists a primitive embedding of an even lattice* M *with invariants* $(t_{(+)}, t_{(-)}, q)$ *into some even unimodular lattice of signature* $(l_{(+)}, l_{(-)})$ *if the following conditions are simultaneously satisfied:*

1) $l_{(+)} - l_{(-)} \equiv 0 \mod 8$.

2) $l_{(-)} - t_{(-)} \geq 0$, $l_{(+)} - t_{(+)} \geq 0$, $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} > l(\mathfrak{A}_q)$.

It is well-known that an even unimodular lattice of signature $(l_{(+)}, l_{(-)})$ is unique if it is indefinite (e. g. see [Ser70]). The same is valid if $l_{(+)} + l_{(-)} \leq 8$. Thus, Theorem A.5 and Corollary A.6 give existence of a primitive embedding of M into these unimodular lattices.

A.1.6. Uniqueness

We restrict ourselves to the following uniqueness result (see Theorem 1.14.2 in [Nik80b] and Theorem 1.2′ in [Nik80a]). We note that this is based on fundamental results about spinor genus of indefinite lattices of the rank ≥ 3 due to M. Eichler and M. Kneser.

Theorem A.7. *Let* T *be an even indefinite lattice with the invariants* $(t_{(+)}, t_{(-)}, q)$ *satisfying the following conditions:*

a) $\operatorname{rk} T \geq l(\mathfrak{A}_{q_p}) + 2$ *for all* $p \neq 2$.

b) If $\operatorname{rk} T = l(\mathfrak{A}_{q_2})$*, then* $q_2 \cong u_+^{(2)}(2) \oplus q'$ *or* $q_2 \cong v_+^{(2)}(2) \oplus q'$.

Then the lattice T *is unique (up to isomorphisms), and the canonical homomorphism* $O(T) \to O(q_T)$ *is surjective.*

Applying additionally Proposition A.4, we obtain the following Analogue of Witt's Theorem for primitive embeddings into even unimodular lattices (see Theorem 1.14.4. in [Nik80b]):

Theorem A.8. *Let* M *be an even lattice of signature* $(t_{(+)}, t_{(-)})$*, and let* L *be an even unimodular lattice of signature* $(l_{(+)}, l_{(-)})$*. Then a primitive embedding of* M *into* L *is unique (up to isomorphisms), provided the following conditions hold:*

1) $l_{(+)} - t_{(+)} > 0$ *and* $l_{(-)} - t_{(-)} > 0$.

2) $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} \geq 2 + l(\mathfrak{A}_{M_p})$ *for all* $p \neq 2$.

3) If $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} = l(\mathfrak{A}_{M_2})$, *then* $q_M \cong u_+^{(2)}(2) \oplus q'$ *or* $q_M \cong v_+^{(2)}(2) \oplus q'$.

A.2. Classification of main invariants and their geometric interpretation

Here we apply results of Section A.1 to classify main invariants S and (r, a, δ) of non-symplectic involutions of K3 surfaces and equivalent right DPN surfaces; moreover, we give their geometric interpretation (types: elliptic, parabolic, hyperbolic, and invariants (k, g, δ)) (see Section 2.3). We also give proofs of results of Section 2.3 which were only cited there. All these results had been obtained in [Nik80a, Nik80b, Nik79, Nik83, Nik87] and are well-known. We follow these papers.

We follow notations and considerations of Section 2.3.

According to Section 2.3, the set of main invariants S of K3 surfaces with non-symplectic involution is exactly *the set of isomorphism classes of 2-elementary even hyperbolic lattices* S *having a primitive embedding* $S \subset L$ *where* $L \cong L_{K3}$ *is an even unimodular lattice of signature* $(3, 19)$. Further we denote $L = L_{K3}$. We remind that a lattice M is called **2-elementary** if its discriminant group $\mathfrak{A}_M = M^*/M \cong (\mathbb{Z}/2\mathbb{Z})^a$ is 2-elementary.

Since the lattice $T = S_L^{\perp}$ is also 2-elementary, let us more generally consider all even 2-elementary lattices M. We denote by $(t_{(+)}, t_{(-)})$ the numbers of positive and negative squares of M. Since M is 2-elementary, the discriminant group $\mathfrak{A}_M \cong (\mathbb{Z}/2\mathbb{Z})^a$ is a 2-elementary group where 2^a is its order. We get the important **invariant** a of the discriminant group $\mathfrak{A}_M$ (and M itself). We have $a \in \mathbb{Z}$ and $a \geq 0$.

The discriminant form q_M of M is a non-degenerate finite quadratic form on the 2-elementary group $\mathfrak{A}_M$ (we call such a form 2-elementary.) By Jordan decomposition (see Section A.1.3), the form q_M is orthogonal sum of elementary finite quadratic forms $u_+^{(2)}(2)$, $v_+^{(2)}(2)$ and $q_{\pm 1}^{(2)}(2)$ with the signature $0 \mod 8$, $4 \mod 8$ and $\pm 1 \mod 8$ respectively. If q_M is sum of only elementary forms $u_+^{(2)}(2)$ and $v_+^{(2)}(2)$, then q_M is **even:** it takes values only in $\mathbb{Z}/2\mathbb{Z}$. Otherwise it is **odd:** at least one of its values belongs to $\{-1/2, 1/2\} \mod 2$. Therefore, we introduce an important **invariant** $\delta \in \{0, 1\}$ of q_M (and of M itself). The $\delta = 0$ if q_M is even, and $\delta = 1$ if q_M is odd.

We have the important relations between elementary forms: $2u_+^{(2)}(2) \cong 2v_+^{(2)}(2)$, $3q_{\pm 1}^{(2)} \cong q_{\mp 1}^{(2)}(2) \oplus v_+^{(2)}(2)$, $q_1^{(2)}(2) \oplus q_{-1}^{(2)}(2) \oplus q_{\pm 1}^{(2)}(2) \cong u_+^{(2)}(2) \oplus q_{\pm 1}^{(2)}(2)$. It follows that q_M can be written in the **canonical form** depending on its invariants a, δ and $\sigma \mod 8 = \operatorname{sign} q_M \mod 8 \equiv t_{(+)} - t_{(-)} \mod 8$. We have several cases:

$\delta = 0$: then $a \equiv 0 \mod 2$, $\sigma \equiv 0 \mod 4$, and $\sigma \equiv 0 \mod 8$ if $a = 0$. We have

$$q_M \cong s v_+^{(2)}(2) \oplus (a/2 - s) u_+^{(2)}(2)$$

where $s = 0$ or 1 and $\sigma \equiv 4s \mod 8$.

$\delta = 1$: then $a \geq 1$, $\sigma \equiv a \mod 2$, $\sigma \equiv \pm 1 \mod 8$ if $a = 1$, and $\sigma \not\equiv 4 \mod 8$ if $a = 2$. We have

$$q_M \cong q_{\pm 1}^{(2)}(2) \oplus ((a-1)/2)\, u_+^{(2)}(2)\ if\ \sigma \equiv \pm 1 \mod 8;$$

$$q_M \cong 2q_{\pm 1}^{(2)} \oplus (a/2 - 1) u_+^{(2)}(2)\ if\ \sigma \equiv \pm 2 \mod 8;$$

$$q_M \cong q_1^{(2)}(2) \oplus q_{-1}^{(2)}(2) \oplus (a/2 - 1) u_+^{(2)}(2)\ if\ \sigma \equiv 0 \mod 8;$$

$$q_M \cong q_1^{(2)}(2) \oplus q_{-1}^{(2)}(2) \oplus v_+^{(2)}(2) \oplus (a/2 - 2) u_+^{(2)}(2)\ if\ \sigma \equiv 4 \mod 8.$$

Thus, the discriminant form q_M is determined by its invariants $(\sigma \equiv t_{(+)} - t_{(-)} \mod 8, a, \delta)$. Moreover, we have listed above all conditions of existence of q_M for the given invariants $\sigma = t_{(+)} - t_{(-)} \mod 8$, $a \geq 0$ and $\delta \in \{0, 1\}$.

Assume that these conditions are satisfied. By Corollary A.2, a 2-elementary lattice M with invariants $(t_{(+)}, t_{(-)}, q_M)$ exists if $t_{(+)} \geq 0$, $t_{(-)} \geq 0$ and $t_{(+)} + t_{(-)} > a = l(\mathfrak{A}_M)$. The condition $t_{(+)} + t_{(-)} \geq a$ is necessary for the existence. Assume that $t_{(+)} + t_{(-)} = a$. If $\delta = 1$, then $q_M \cong q_{\pm 1}^{(2)}(2) \oplus q'$, and the lattice M also does exist by Theorem A.1. If $\delta = 0$, then $M(1/2)$ will be an even unimodular lattice. It follows that the condition $t_{(+)} - t_{(-)} \equiv 0 \mod 8$ must be satisfied, and it is sufficient for the existence of M since an even unimodular lattice $M(1/2)$ with the invariants $(t_{(+)}, t_{(-)})$ does exist under this condition. Thus, we finally listed all conditions of existence of an even 2-elementary lattice M with the invariants $(t_{(+)}, t_{(-)}, a, \delta)$.

Moreover, we had proved that the invariants $(t_{(+)}, t_{(-)}, a, \delta)$ define the discriminant quadratic form q_M of M. We have $l(\mathfrak{A}_{M_p}) = 0$ if a prime p is odd. If $a \geq 3$, then $q_M \cong u_+^{(2)}(2) \oplus q'$ or $q_M \cong v_+^{(2)}(2) \oplus q'$. By Theorem A.7, then the lattice M is unique if it is indefinite and $\operatorname{rk} M \geq 3$. Moreover, then the canonical homomorphism $O(M) \to O(q_M)$ is epimorphic. If $\operatorname{rk} M \leq 2$, then M is one of lattices: $\langle \pm 2 \rangle$, $\langle \pm 2 \rangle \oplus \langle \pm 2 \rangle$, U or $U(2)$. One can easily check for them the same statements directly.

Thus, finally we get the following classification result about 2-elementary even indefinite (except few exceptions) lattices. It is Theorems 3.6.2 and 3.6.3 from [Nik80b].

Theorem A.9. *The genus of an even 2-elementary lattice M is determined by the invariants $(t_{(+)}, t_{(-)}, a, \delta)$; and if either M is indefinite or* $\mathrm{rk}\, M = 2$, *these invariants determine the isomorphism class of M, and the canonical homomorphism $O(M) \to O(q_M)$ is epimorphic.*

An even 2-elementary lattice M with invariants $(t_{(+)}, t_{(-)}, a, \delta)$ exists if and only if all the following conditions are satisfied (it being assumed that $\delta = 0$ or 1, and that $a, t_{(+)}, t_{(-)} \geq 0$):

1) $a \leq t_{(+)} + t_{(-)}$;

2) $t_{(+)} + t_{(-)} + a \equiv 0 \mod 2$;

3) $t_{(+)} - t_{(-)} \equiv 0 \mod 4$ *if* $\delta = 0$;

4) $(\delta = 0,\ t_{(+)} - t_{(-)} \equiv 0 \mod 8)$ *if* $a = 0$;

5) $t_{(+)} - t_{(-)} \equiv \pm 1 \mod 8$ *if* $a = 1$;

6) $\delta = 0$ *if* $(a = 2,\ t_{(+)} - t_{(-)} \equiv 4 \mod 8)$;

7) $t_{(+)} - t_{(-)} \equiv 0 \mod 8$ *if* $(\delta = 0,\ a = t_{(+)} + t_{(-)})$.

Let S be a main invariant and $r = \mathrm{rk}\, S$. Since S is 2-elementary even hyperbolic, by Theorem A.9 it is then determined by its invariants $(t_{(+)} = 1, t_{(-)} = r - 1, a, \delta)$.

By Theorem A.5, existence of a primitive embedding $S \subset L_{K3}$ is equivalent to existence of a 2-elementary even lattice $T = S^{\perp}$ with invariants $(t_{(+)} = 2, t_{(-)} = 20 - r, a, \delta)$ (indeed, $q_T \cong -q_S$ has the same invariants a and δ).

Thus, *the set of main invariants S is equal to the set of (r, a, δ) such that both $(1, r - 1, a, \delta)$ and $(2, 20 - r, a, \delta)$ satisfy conditions 1) — 7) of Theorem A.9. It consists of exactly (r, a, δ) which are presented in Figure 1.*

By Theorem A.9, the orthogonal complement $T = S_L^{\perp}$ is uniquely determined by (r, a, δ), and the canonical homomorphism $O(T) \to O(q_T)$ is epimorphic. By Proposition A.4 the primitive embedding $S \subset L_{K3}$ is unique up to automorphisms of L_{K3}.

Let us show that $O(S \subset L_{K3})$ contains an automorphism of spinor norm -1 (i. e. it changes two connected components of the quadric $\Omega_{S \subset L_{K3}}$, see (30)). Using Theorem A.9, it is easy to see that either $T = \langle 2 \rangle \oplus \langle 2 \rangle$, or $T = U \oplus T'$, or $T = U(2) \oplus T'$, or $T = \langle 2 \rangle \oplus \langle -2 \rangle \oplus T'$. If $T \cong \langle 2 \rangle \oplus \langle 2 \rangle$, we consider the automorphism α of T which is $+1$ on the first $\langle 2 \rangle$ and -1 on the second $\langle 2 \rangle$. In remaining cases we consider an automorphism α of T which is -1 on the first 2-dimensional hyperbolic summand of T, and which is $+1$ on T'. It is easy to see that α changes connected components of the $\Omega_{S \subset L_{K3}}$. On the other hand, α is identical on the T^*/T and can be continued identically on S. This extension gives an automorphism of L_{K3} which is identical on S and has spinor norm -1.

It follows (see Section 2.3) that for the fixed main invariants (r, a, δ) the moduli space $\mathrm{Mod}_{(r,a,\delta)}$ of K3 surfaces with non-symplectic involution is irreducible.

Now let us consider the geometric interpretation of main invariants (r, a, δ) in terms of the set $C = X^\theta$ of the fixed points. The set C is non-singular. Indeed, if $x \in C$ a singular point of C, then θ is the identity in the tangent space T_x. Then $\theta^*(\omega_X) = \omega_X$ for any $\omega_X \in H^{2,0}(X)$ and θ is symplectic. We get a contradiction.

For a non-singular irreducible curve C on a K3 surface X we have $g(C) = (C^2 + C \cdot K_X)/2 + 1 = C^2/2 + 1 \geq 0$. It follows that $C^2 > 0$, if $g(C) > 1$. Since the Picard lattice S_X is hyperbolic, it follows that any two curves on X of genus ≥ 2 must intersect. It then follows that X^θ has one of types A, B, C listed below:

Case A: $X^\theta = C_g + E_1 + \cdots + E_k$ where C_g is a non-singular irreducible curve of genus $g \geq 0$ and $C_g \neq \emptyset$, the curves $E_1, \ldots, E_k$ are non-singular irreducible rational (i. e. $E_i^2 = -2$). All curves C_g, E_i are disjoint to each other.

Case B: $X^\theta = C_1^{(1)} + \cdots + C_1^{(m)} + E_1 + \cdots + E_k$ where $C_1^{(1)} + \cdots + C_1^{(m)}$ is disjoint union of $m > 1$ elliptic curves (we shall prove in a moment that actually $m = 2$ and $k = 0$).

Case C: $X^\theta = \emptyset$.

By Lefschetz formula, the Euler characteristics $\chi(X^\theta) = 2 + r - (22 - r) = 2r - 20$.

By Smith Theory (see [Kha76]), the total Betti number over $\mathbb{Z}/2\mathbb{Z}$ satisfies

$$\dim H^*(X^\theta, \mathbb{Z}/2\mathbb{Z}) = \dim H^*(X, \mathbb{Z}/2\mathbb{Z}) - 2a = 24 - 2a \text{ if } X^\theta \neq \emptyset.$$

For any 2-dimensional cycle $Z \subset X$ one evidently has $Z \cdot \theta(Z) \equiv Z \cdot X^\theta \mod 2$. Thus, $X^\theta \sim 0 \mod 2$ in $H^2(X, \mathbb{Z})$ if and only if $(x, \theta^* x) \equiv 0 \mod 2$ for any $x \in H^2(X, \mathbb{Z})$. Let us write $x \in L = H^2(X, \mathbb{Z})$ as $x = x_+ + x_-$ where $x_+ \in S^*$ and $x_- \in (S^\perp)^*$. Then $x \cdot \theta^*(x) = x_+^2 - x_-^2$. Moreover, $x^2 = x_+^2 + x_-^2 \equiv 0 \mod 2$ because L is even. Taking the sum, we get $x \cdot \theta^*(x) \equiv 2x_+^2 \mod 2$. Since $H^2(X, \mathbb{Z})$ is unimodular, any $x_+ \in S^*$ appears in this identity. It follows that $X^\theta \sim 0 \mod 2$ in $H^2(X, \mathbb{Z})$ if and only if $x_+^2 \in \mathbb{Z}$ for any $x_+ \in S^*$. Equivalently, the invariant $\delta = 0$. Therefore

$$\delta = 0 \text{ if and only if } X^\theta \sim 0 \mod 2 \text{ in } H^2(X, \mathbb{Z}).$$

In case B, elliptic curves $C_1^{(i)}$ belong to one elliptic pencil $|C|$ of elliptic curves where it is known (see [PS-Sh71]) that C is primitive in Picard lattice

S_X. Assume that either $m > 2$ or $k > 0$. Then θ is trivial on the base $\mathbb{P}^1$ of the elliptic pencil. Since it is also trivial on a fibre $C_1^{(i)}$ which is not multiple, θ is symplectic, and we get a contradiction. Thus, in Case B we have $k = 0$, $m = 2$ and $\delta = 0$.

In case C, the quotient $Y = X/\{1, \theta\}$ is an Enriques surface. It follows that $r = a = 10$ and $\delta = 0$ in this case.

Combining all these arguments, we obtain the geometric interpretation of the invariants (r, a, δ) cited in Section 2.3

A.3. The analogue of Witt's theorem for 2-elementary finite forms

Here we follow Section 1.9 in [Nik84b] to prove an important Lemma 2.7.

We consider a 2-elementary finite bilinear forms $b : B \times B \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ and 2-elementary finite quadratic forms $q : Q \to \frac{1}{2}\mathbb{Z}/2\mathbb{Z}$ on finite 2-elementary groups B, Q.

In the previous section we gave classification of non-degenerate 2-elementary finite quadratic forms. Similarly one can classify non-degenerate 2-elementary finite bilinear forms. They are orthogonal sums of elementary forms $b_1^{(2)}(2)$ and $u_-^{(2)}(2)$. The form $b_1^{(2)}(2)$ is the bilinear form of quadratic forms $q_\pm^{(2)}(2)$, and the form $u_-^{(2)}(2)$ is the bilinear form of quadratic forms $u_+^{(2)}(2)$ and $v_+^{(2)}(2)$. We denote by s_b the characteristic element of b, i. e. $b(x,x) = b(s_b, x)$ for all $x \in B$. It is easy to see that any non-degenerate 2-elementary finite bilinear form b is

$$b \cong m u_-^{(2)}(2)$$

if $b(x,x) = 0$ for all $x \in B$ (equivalently the characteristic element $s_b = 0$, these bilinear forms are the same as skew-symmetric ones);

$$b \cong b_1^{(2)}(2) \oplus m u_-^{(2)}(2),$$

if $b(s_b, s_b) = \frac{1}{2} \mod 1$;

$$b \cong 2b_1^{(2)} \oplus m u_-^{(2)}(2)$$

if $s_b \neq 0$ but $b(s_b, s_b) = 0$.

We prove (see Section 1.9 in [Nik84b])

Proposition A.10. *Let b be a non-degenerate bilinear form on a finite 2-elementary group B and $\theta : H_1 \to H_2$ be an isomorphism of subgroups of B which preserves the restrictions $b|H_1$ and $b|H_2$ and that maps the characteristic element of the form b to itself (if, of course, it belongs to H_1). Then θ extends to an automorphism of b.*

Proposition A.11. *Let q be a quadratic form on a finite 2-elementary group Q whose kernel is zero; that is*

$$\{x \in Q \mid x \perp Q \text{ and } q(x) = 0\} = 0.$$

Let $\theta : H_1 \to H_2$ be an isomorphism of two subgroups of Q that preserves the restrictions $q|H_1$ and $q|H_2$ and that maps the elements of the kernel and the characteristic elements of the bilinear form q into the same sort of elements (of course, if they belong to H_1). Then ϕ extends to an automorphism of q (an element $s \in Q$ is called **characteristic** *if $q(s,x) = q(x,x)$ for all $x \in Q$).*

We shall prove the propositions by induction on the number of generators of B and Q. Let us begin with Proposition A.10. Suppose there exist $x_1 \in H_1$ and $x_2 = \theta(x_1) \in H_2$ such that $b(x_1, x_1) = b(x_2, x_2) = \frac{1}{2} \mod 1$. Write $B_1 = (x_1)^{\perp}_{B}$, $B_2 = (x_2)^{\perp}_{B}$, $b_1 = b|B_1$, $b_2 = b|B_2$, $H_1' = (x_1)^{\perp}_{H_1}$, $H_2' = (x_2)^{\perp}_{H_2}$ and $\theta' = \theta|H_1$. Then the same conditions hold for the nondegenerate forms b_1 and b_2 defined on the subgroups B_1 and B_2, their subgroups $H_1' \subset B_1$ and $H_2' \subset B_2$, and an isomorphism $\theta' : H_1' \cong H_2'$. Everything reduces to extending θ'. Since x_1 and x_2 are characteristic simultaneously, b_1 and b_2 are isomorphic (this follows from classification of nondegenerate bilinear forms). Therefore, the existence of an extension of θ' follows from the induction hypothesis. To complete the proof it remains to consider the case when the function $b(x,x)$ on H_1 and H_2 is zero. Denote by s the characteristic element of B. It is easy to check (using the classification again) that the natural homomorphism

$$O(b) \to O(s^{\perp})$$

is epimorphic (we always consider a subgroup with the restriction of the form b on the subgroup). In our case H_1 and H_2 lie in $s^{\perp}$; therefore, it suffices to extend θ to an automorphism of $s^{\perp}$. If $s \notin s^{\perp}$, this is obvious since in this case $s^{\perp}$ is a nondegenerate skew-symmetric form; for them the proposition is well-known and obvious. If $s \in s^{\perp}$, then $[s]$ is the kernel of $s^{\perp}$, and, by the hypothesis,

$$[s] \cap H_1 = \theta([s] \cap H_1) = [s] \cap H_2.$$

Let

$$\overline{\theta} : H_1/([s] \cap H_1) \cong H_2/([s] \cap H_2)$$

be the isomorphism $\theta \mod [s] \cap H_1$. Then, because $s^{\perp}/[s]$ is nondegenerate and skew-symmetric, $\overline{\theta}$ extends to an automorphism $\overline{\psi} \in O(s^{\perp}/[s])$. Let ψ be a lifting of $\overline{\psi}$ to an automorphism of $(s)^{\perp}$. Then $\psi(x) - \theta(x) = \overline{g}(x)s$, if $x \in H_1$, where $\overline{g} : H_1 \to \mathbb{Z}/2\mathbb{Z}$ is a linear function. Extending $\overline{g}$ to a linear function $g : s^{\perp} \to \mathbb{Z}/2\mathbb{Z}$, we put

$$\widetilde{\psi}(x) = \psi(x) + \widetilde{f}(x)s \; if \; x \in s^{\perp}.$$

Evidently $\widetilde{\psi} \in O(s^{\perp})$ is the desired extension of θ.

Let us prove Proposition A.11. Assume that the bilinear form of q has a nonzero kernel. Then it is generated by an element r, and $q(r) = 1 \mod 2$. Using Proposition A.10, we can extend θ to an automorphism ψ of the bilinear form of q. The function $f(x) = q(x) - q(\psi(x)) \in \mathbb{Z}/2\mathbb{Z}$, where $x \in Q$, is linear and vanishes on H_1. Evidently, $\widetilde{\psi}(x) = \psi(x) + f(x)r$, where $x \in Q$, is the desired extension of θ.

It remains to examine the case when the bilinear form q is nondegenerate. The case where there exist elements $x_1 \in H_1$ and $x_2 = \theta(x_1) \in H_2$ for which $q(x_1) = q(x_2) = \frac{1}{2}\theta$ where $\theta \notin 2\mathbb{Z}$, can be examined similarly to the corresponding case of Proposition A.10. Therefore, we assume that $q(x) \equiv 0 \mod 1$ if $x \in H_1$. For a characteristic element $s \in Q$ the natural homomorphism

$$O(q) \to O(s^{\perp})$$

is epimorphic (this easily follows from the classification of nondegenerate quadratic forms on 2-elementary groups given in Section A.2). In our case H_1 and H_2 lie in $s^{\perp}$, and it suffices to extend θ to an automorphism of $s^{\perp}$. If the bilinear form on $s^{\perp}$ is non-degenerate, this follows from the classical Witt theorem over the field with two elements. Suppose it is degenerate; then the kernel of $s^{\perp}$ is generated by s. In the case $q(s) = 0$, one can pass to a form on $s^{\perp}/[s]$ and we argue in the same way as in the proof of Proposition A.10. But if $q(s) \equiv 1 \mod 2$, then we pass to the first case, already treated. Proposition A.11 is proved.

A.4. Calculations of fundamental chambers

Here we outline calculations of fundamental chambers for hyperbolic reflection groups which had been used in the main part of the work.

A.4.1. Fundamental chambers $\mathcal{M}^{(2,4)}$ of 2-elementary even hyperbolic lattices of elliptic type (Table 1).

We consider all 50 types of 2-elementary even hyperbolic lattices S of elliptic type given by their full invariants (r, a, δ). We outline the calculation of a fundamental chamber $\mathcal{M}^{(2,4)}$ (equivalently, the corresponding Dynkin diagram $\Gamma(P(\mathcal{M}^{(2,4)}))$) for the full reflection group $W^{(2,4)}(S) = W(S)$. This is the group generated by reflections s_f in all roots f of S. They are elements $f \in S$ either with $f^2 = -2$ or with $f^2 = -4$ and $(f, S) \equiv 0 \mod 2$. The reflection $s_f \in O(S)$ is then given by

$$x \mapsto x - \frac{2(x, f)f}{f^2}, \quad \forall x \in S.$$

We use Vinberg's algorithm [Vin72] which we describe below. It can be applied to any hyperbolic lattice S and any of its reflection subgroup $W \subset W(S)$ which is generated by reflections in some precisely described subset $\Delta \subset S$ of primitive roots of S which is W-invariant.

First, we should choose a non-zero $H \in S$ with $H^2 \geq 0$. Then H defines the half cone $V^+(S)$ such that $H \in \overline{V^+(S)}$. We want to find a fundamental chamber $\mathcal{M} \subset \mathcal{L}(S) = V^+(S)/\mathbb{R}^+$ of W containing $\mathbb{R}^+h$.

Step 0. We consider the subset Δ_0 of all roots from Δ which are orthogonal to H. This set is either a finite root system or affine root system. One should choose a bases P_0 in Δ_0. For example, one can take another element $H_1 \in S$ such that $H_1^2 \geq 0$, $(H, H_1) > 0$ and (H_1, Δ_0) does not contain zero. Then

$$\Delta_0^+ = \{f \in \Delta_0 \mid (f, H_1) > 0\},$$

and $P_0 \subset \Delta_0^+$ consists of roots from Δ_0^+ which are not non-trivial sums of others. For $f \in \Delta$ with $(f, H) \geq 0$ we introduce the height

$$h(f) = \frac{2(f, H)^2}{-f^2}.$$

The height is equivalent to the hyperbolic distance between the point $\mathbb{R}^+H$ and the hyperplane $\mathcal{H}_f$ which is orthogonal to f. The set of all possible heights is a discrete ordered subset

$$h_0 = 0,\ h_1,\ h_2,\ \dots\ h_i,\ \dots, \tag{90}$$

of $\mathbb{R}^+$. It is always a subset of non-negative integers, and one can always take $\mathbb{Z}^+$ as the set of possible heights.

The fundamental chamber $\mathcal{M} \subset \mathcal{L}(S)$ is defined by the set $P(\mathcal{M}) \subset \Delta$ of orthogonal roots to $\mathcal{M}$ which is

$$P(\mathcal{M}) = \bigcup_{0 \leq j} P_j \tag{91}$$

where P_0 is defined above and P_j for $j > 0$ consists of all $f \in \Delta$ such that $(f, H) > 0$, the height $h(f) = h_j$, and

$$\left(f, \bigcup_{0 \leq i \leq j-1} P_i\right) \geq 0. \tag{92}$$

Then

$$\mathcal{M} = \{\mathbb{R}^+x \in \mathcal{L}(S) \mid (x, P(\mathcal{M})) \geq 0\}. \tag{93}$$

If $\mathcal{M}$ has finite volume, the algorithm terminates after a finite number m of steps (i. e. all P_j are empty for $j > m$, and

$$P(\mathcal{M}) = \bigcup_{0\le j\le m} P_j \tag{94}$$

whenever (93) defines a polyhedron $\mathcal{M}$ of finite volume in $\mathcal{L}(S)$ for $P(\mathcal{M})$ given by (94).

Below we apply this algorithm to 2-elementary hyperbolic lattices S of elliptic type and $W = W(S) = W^{(2,4)}(S)$. The set $\Delta \subset S$ consists of all $f \in S$ such that either $f^2 = -2$ or $f^2 = -4$ and $(f, S) \equiv 0 \mod 2$.

Cases $S = \langle 2\rangle \oplus lA_1$ where $0 \le l \le 8$. Then $(r, a, \delta) = (1+l, 1+l, 1)$, $0 \le l \le 8$. We use the standard orthogonal basis h for $\langle 2\rangle$ where $h^2 = 2$, and the standard orthogonal basis $v_1, \dots, v_l$ for lA_1 where $v_1^2 = \cdots = v_l^2 = -2$.

We take $H = h$, $H_1 = th + v_1 + 2v_2 + \cdots + lv_l$ where $t >> 0$. Then $P(\mathcal{M}^{(2,4)})$ consists of roots: $\beta_0 = h - v_1 - v_2$ if $l = 2$; $\beta_0 = h - v_1 - v_2 - v_3$ if $l \ge 3$; $\beta_1 = v_1 - v_2, \dots, \beta_{l-1} = v_{l-1} - v_l$ if $l \ge 2$; $\beta_l = v_l$ if $l \ge 1$.

For $2 \le l \le 8$ the polyhedron $\mathcal{M}^{(2,4)}$ is obviously a simplex in $\mathcal{L}(S)$ of finite volume. The Gram matrix (β_i, β_j) gives the Dynkin diagrams of cases $N = 1$, $N = 3 - 10$ of Table 1. Here we repeated calculations by Vinberg in [Vin72].

Cases $S = U \oplus lA_1$, $0 \le l \le 8$. Then $(r, a, \delta) = (2, 0, 0)$ if $l = 0$, and $(r, a, \delta) = (2 + l, l, 1)$ if $1 \le l \le 8$. We use the standard basis c_1, c_2 for U where $c_1^2 = c_2^2 = 0$ and $(c_1, c_2) = 1$, and the standard orthogonal basis $v_1, \dots, v_l$ for lA_1 as above.

We take $H = c_1$. We can take P_0 which consists of $\beta_0 = c_1 - v_1$ if $l \ge 1$; $\beta_1 = v_1 - v_2, \dots, \beta_{l-1} = v_{l-1} - v_l$ if $l \ge 2$; $\beta_l = v_l$ if $l \ge 1$. Then $P(\mathcal{M}^{(2,4)})$ consists of P_0, $e = -c_1 + c_2$ and additional elements $\gamma_1 = 2c_1 + 2c_2 - v_1 - v_2 - v_3 - v_4 - v_5$ if $l = 5$, and $\gamma_1 = 2c_1 + 2c_2 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6$ if $l \ge 6$.

We shall discuss the finiteness of volume of $\mathcal{M}^{(2,4)}$ later. We obtain the diagrams of cases $N = 11 - 20$ excluding $N = 15$ of Table 1.

Cases $S = U \oplus D_4 \oplus lA_1$, $0 \le l \le 5$. Then $(r, a, \delta) = (6, 2, 0)$ if $l = 0$, and $(r, a, \delta) = (6+l, 2+l, 1)$ if $1 \le l \le 5$. We use the standard bases c_1, c_2 for U and $v_1, \dots, v_l$ for lA_1 as above. We use the standard orthogonal basis $\epsilon_1, \dots, \epsilon_m$ for $D_m \otimes \mathbb{Q}$ where $\epsilon_1^2 = \cdots = \epsilon_m^2 = -1$; the lattice D_m consists of all $x_1\epsilon_1 + \cdots + x_m\epsilon_m$ where $x_i \in \mathbb{Z}$ and $x_1 + \cdots + x_m \equiv 0 \mod 2$.

We take $H = c_1$, and we can take P_0 which consists of $\alpha_0 = c_1 - \epsilon_1 - \epsilon_2$, $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, $\alpha_3 = -\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4$, $\alpha_4 = 2\epsilon_4$, $\beta_0 = c_1 - v_1$ if $l \ge 1$; $\beta_1 = v_1 - v_2, \dots, \beta_{l-1} = v_{l-1} - v_l$ if $l \ge 2$; $\beta_l = v_l$ if $l \ge 1$.

Then $P(\mathcal{M}^{(2,4)})$ consists of P_0, $e = -c_1 + c_2$, $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - v_1 - v_2 - v_3$ if $l = 3$, and $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - v_1 - v_2 - v_3 - v_4$ if $l \geq 4$; $\gamma_2 = 2c_1 + 2c_2 - v_1 - v_2 - v_3 - v_4 - v_5$ if $l = 5$.

We obtain the diagrams of cases $N = 21 - 27$ excluding $N = 25$ of Table 1.

Cases $S = U \oplus D_m \oplus lA_1$ **where** $m \equiv 0 \mod 2$, $m \geq 6$, $l \geq 0$, $m + 2l \leq 14$. Then $r = 2 + m + l$, $a = l + 2$. Moreover $\delta = 0$ if $l = 0$ and $m \equiv 0 \mod 4$, otherwise $\delta = 1$.

We use the standard bases c_1, c_2 for U, and $\epsilon_1, \ldots, \epsilon_m$ for $D_m \otimes \mathbb{Q}$, and $v_1, \ldots, v_l$ for lA_1 as above.

We take $H = c_1$, and we can take P_0 which consists of $\alpha_0 = c_1 - \epsilon_1 - \epsilon_2$, $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{m-1} = \epsilon_{m-1} - \epsilon_m$, $\alpha_m = 2\epsilon_m$; $\beta_0 = c_1 - v_1$ if $l \geq 1$; $\beta_1 = v_1 - v_2, \ldots, \beta_{l-1} = v_{l-1} - v_l$ if $l \geq 2$; $\beta_l = v_l$ if $l \geq 1$.

Then $P(\mathcal{M}^{(2,4)})$ consists of P_0, $e = -c_1 + c_2$, and some additional elements γ_i depending on $m \geq 6$ and $l \geq 0$ where we always assume that $m + 2l \leq 14$ and $m \equiv 0 \mod 2$.

If $m = 6$ and $l = 2$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_6 - v_1 - v_2$.

If $m = 6$ and $l = 3$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_6 - v_1 - v_2 - v_3$, $\gamma_2 = 2c_1 + 2c_2 - 2\epsilon_1 - v_1 - v_2 - v_3$.

If $m = 6$ and $l = 4$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_6 - v_1 - v_2 - v_3$, $\gamma_2 = 2c_1 + 2c_2 - 2\epsilon_1 - v_1 - v_2 - v_3 - v_4$.

If $m = 8$ and $l = 1$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_8 - v_1$.

If $m = 8$ and $l = 2$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_8 - v_1 - v_2$.

If $m = 8$ and $l = 3$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_8 - v_1 - v_2$, $\gamma_2 = 2c_1 + 2c_2 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - v_1 - v_2 - v_3$, $\gamma_3 = 2c_1 + 2c_2 - 2\epsilon_1 - v_1 - v_2 - v_3$,

If $m = 10$ and $l = 0$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_{10}$.

If $m = 10$ and $l = 1$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_{10} - v_1$.

If $m = 10$ and $l = 2$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_{10} - v_1$, $\gamma_2 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_6 - v_1 - v_2$, $\gamma_3 = 4c_1 + 4c_2 - 3\epsilon_1 - \epsilon_2 - \cdots - \epsilon_{10} - 2v_1 - 2v_2$.

If $m = 12$ and $l = 0$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_{12}$.

If $m = 12$ and $l = 1$, one must add $\gamma_1 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_{12}$, $\gamma_2 = 2c_1 + 2c_2 - \epsilon_1 - \cdots - \epsilon_8 - v_1$, $\gamma_3 = 6c_1 + 6c_2 - 4\epsilon_1 - 2\epsilon_2 - 2\epsilon_3 - \cdots - 2\epsilon_{11} - 3v_1$.

We obtain the diagrams for $N = 28 - 50$ of Table 1 except $N = 30, 34, 40, 41, 44, 49, 50$ when either $\delta = 0$ and $a > 2$, or $a \leq 1$.

Cases $S = U(2) \oplus D_m$ **where** $m \equiv 0 \mod 4$ **and** $0 \leq m \leq 12$. Then $(r, a, \delta) = (2 + m, 4, 0)$. We use the standard bases c_1, c_2 for $U(2)$ where $c_1^2 = c_2^2 = 0$ and $(c_1, c_2) = 2$, and the standard basis $\epsilon_1, \ldots, \epsilon_m$ for $D_m \otimes \mathbb{Q}$ as above.

We use $H = c_1$ and denote $e = -c_1 + c_2$ with $e^2 = -4$.

If $m = 0$, then $P_0 = \emptyset$ and $P(\mathcal{M}^{(2,4)})$ consists of e.

If $m = 4$, then P_0 consists of $\alpha_0 = c_1 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4$, $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, $\alpha_3 = -\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4$, $\alpha_4 = 2\epsilon_4$. Then $P(\mathcal{M}^{(2,4)})$ consists of P_0 and e.

If $m \geq 8$, then P_0 consists of $\alpha_0 = c_1 - 2\epsilon_1$, $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{m-1} = \epsilon_{m-1} - \epsilon_m$, $\alpha_m = 2\epsilon_m$.

If $m = 8$, then $P(\mathcal{M}^{(2,4)})$ consists of P_0, e and $\gamma_1 = c_1 + c_2 - \epsilon_1 - \cdots - \epsilon_8$.

If $m = 12$, then $P(\mathcal{M}^{(2,4)})$ consists of P_0, e and $\gamma_1 = 2c_1 + c_2 - \epsilon_1 - \cdots - \epsilon_{12}$, $\gamma_2 = c_1 + c_2 - \epsilon_1 - \cdots - \epsilon_6$.

We obtain the diagrams for $N = 2, 15, 30, 44$ of Table 1.

Case $U(2) \oplus D_4 \oplus D_4$. Then $(r, a, \delta) = (10, 6, 0)$. We use standard bases c_1, c_2 for $U(2)$ and $\epsilon_1^{(1)}, \ldots, \epsilon_4^{(1)}$ for the first D_4, and $\epsilon_1^{(2)}, \ldots, \epsilon_4^{(2)}$ for the second D_4.

We take $H = c_1$ and P_0 which consists of $\alpha_0^{(1)} = c_1 - \epsilon_1^{(1)} - \epsilon_2^{(1)} - \epsilon_3^{(1)} - \epsilon_4^{(1)}$, $\alpha_1^{(1)} = \epsilon_1^{(1)} - \epsilon_2^{(1)}$, $\alpha_2^{(1)} = \epsilon_2^{(1)} - \epsilon_3^{(1)}$, $\alpha_3^{(1)} = -\epsilon_1^{(1)} + \epsilon_2^{(1)} + \epsilon_3^{(1)} - \epsilon_4^{(1)}$, $\alpha_4^{(1)} = 2\epsilon_4^{(1)}$ and $\alpha_0^{(2)} = c_1 - \epsilon_1^{(2)} - \epsilon_2^{(2)} - \epsilon_3^{(2)} - \epsilon_4^{(2)}$, $\alpha_1^{(2)} = \epsilon_1^{(2)} - \epsilon_2^{(2)}$, $\alpha_2^{(2)} = \epsilon_2^{(2)} - \epsilon_3^{(2)}$, $\alpha_3^{(2)} = -\epsilon_1^{(2)} + \epsilon_2^{(2)} + \epsilon_3^{(2)} - \epsilon_4^{(2)}$, $\alpha_4^{(2)} = 2\epsilon_4^{(2)}$.

Then $P(\mathcal{M}^{(2,4)})$ consists of P_0 and $e = -c_1 + c_2$.

We obtain the diagram for $N = 25$ of Table 1.

Cases $S = U \oplus E_7, U \oplus E_8, U \oplus E_8 \oplus A_1, U \oplus E_8 \oplus E_7, U \oplus E_8 \oplus E_8$. Respectively $(r, a, \delta) = (9, 1, 1), (10, 0, 0), (11, 1, 1), (17, 1, 1), (18, 0, 0)$. We use the standard basis c_1, c_2 for U. For each irreducible root lattice $R_i = A_1, E_7, E_8$ of the rank t_i we use its standard basis $r_1^{(i)}, \ldots, r_{t_i}^{(i)}$ of roots with the corresponding Dynkin diagram. We denote by $r_{max}^{(i)}$ the maximal root of R_i corresponding to this basis.

For $S = U \oplus R$ where R is the sum of irreducible root lattices R_i above, we take $H = c_1$ and P_0 which consists of standard bases $r_1^{(i)}, \ldots, r_{t_i}^{(i)}$ of R_i and $r_0^{(i)} = c_1 - r_{max}^{(i)}$.

Then $P(\mathcal{M}^{(2,4)})$ consists of P_0 and $e = -c_1 + c_2$, and one additional element γ_1 if $S = U \oplus E_8 \oplus E_7$. The element $\gamma_1 \in S$ is shown on the diagram $N = 49$ of Table 1 as the right-most vertex. It can be easily computed using pairings (γ_1, ξ_i) prescribed by this diagram for basis elements ξ_i of S given above.

We obtained the remaining diagrams of cases $N = 34, 40, 41, 49, 50$ of Table 1.

Finiteness of volume of polyhedra $\mathcal{M}^{(2,4)}$ above. To prove finiteness of volume of the polyhedra $\mathcal{M}^{(2,4)}$ defined by the subsets $P = P(\mathcal{M}^{(2,4)}) \subset S$ calculated above with the corresponding diagrams $\Gamma = \Gamma(P)$ of Table 1, one can use methods developed by Vinberg in [Vin72].

We remind that a subset $T \subset P$ is called **elliptic, parabolic, hyperbolic,** if its Gram matrix is respectively negative definite, semi-negative definite, hyperbolic. A hyperbolic subset T is called **Lannér** if each its proper subset is elliptic. Dynkin diagrams of all Lannér subsets are classified by Lannér, e.g. see Table 3 in [Vin72]. They have at most 5 elements.

We exclude trivial cases $N = 1, 2, 3, 11$ when $\operatorname{rk} S \leq 2$. In all other cases, from our calculations, it easily follows that P generates $S \otimes \mathbb{Q}$, and $\Gamma(P)$ is connected. Moreover, by the classification of affine Dynkin diagrams, one can check that all connected components (for its Dynkin diagram) of any maximal parabolic subset $T \subset P$ are also parabolic, and sum of their ranks is $\operatorname{rk} S - 2$. We remind that the rank of a connected parabolic subset $T \subset P$ is equal to $\#T - 1$.

From the classification of Lannér subsets, it easily follows that the graph Γ has no Lannér subgraphs if $N \neq 45, 47$. By Proposition 1 in [Vin72]), then $\mathcal{M}^{(2,4)}$ has finite volume.

Assume that $N = 45$ or $N = 47$. Then the only Lannér subset $L \subset P$ consists of two elements defining the broken edge (it is the only one) of Γ. Finiteness of volume of $\mathcal{M}^{(2,4)}$ is then equivalent to $L^\perp \cap \widetilde{\mathcal{M}^{(2,4)}} = \emptyset$. Here

$$\widetilde{\mathcal{M}^{(2,4)}} = \{x \in S \otimes \mathbb{R} \mid (x, P(\mathcal{M})) \geq 0\}/\mathbb{R}^+ \tag{95}$$

is the natural extension of $\mathcal{M}^{(2,4)}$. Let $K \subset P$ consists of all elements which are orthogonal to L. Looking at the diagrams Γ in Table 1, one can see that K is elliptic and has $\operatorname{rk} S - 2$ elements. By Proposition 2 in [Vin72], it is enough to show that $(L \cup K)^\perp \cap \widetilde{\mathcal{M}^{(2,4)}} = \emptyset$ (it then implies that $L^\perp \cap \widetilde{\mathcal{M}^{(2,4)}} = \emptyset$). Since $\#K = \operatorname{rk} S - 2$, the $K^\perp \cap \widetilde{\mathcal{M}^{(2,4)}}$ is the edge (1-dimensional) r_1 of $\mathcal{M}^{(2,4)}$. There are two more elements $f_1 \in P$ and $f_2 \in P$ such that $K_1 = K \cup \{f_1\}$ and $K_2 = K \cup \{f_2\}$ are elliptic. It follows that the edge r_1 terminates in two vertices A_1 and A_2 of $\mathcal{M}^{(2,4)}$ which are orthogonal to K_1 and K_2 respectively. Any element $\mathbb{R}^+x \in r_1$ then has $x^2 \geq 0$. It follows that $(x, L) \neq 0$ because L is a hyperbolic subset. It follows that $(L \cup K)^\perp \cap \widetilde{\mathcal{M}^{(2,4)}} = \emptyset$. Thus, $\mathcal{M}^{(2,4)}$ has finite volume for $N = 45, 47$ either.

A.4.2. Fundamental chambers $\mathcal{M}_+^{(2,4)}$ of cases $N = 7$ (Table 2).

We use orthogonal basis $h, v_1, \ldots, v_6$ of $S \otimes \mathbb{Q}$ where $h^2 = 8$, $v_1^2 = \cdots = v_6^2 = -2$.

Case 7a. As $P(\mathcal{M}^{(2,4)})$, we can take $f_1 = v_1 - v_2$, $f_2 = v_2 - v_3$, $f_3 = v_3 - v_4$, $f_4 = v_4 - v_5$, $f_5 = v_4 + v_5$ with square (-4) defining the root system D_5, and $e = (-v_1 - v_2 - v_3 - v_4 + v_5)/2 + h/4$ with square (-2). They define the diagram 7a. The Weyl group $W = W(D_5)$ (generated by reflections in $f_1, \dots, f_5$) is the semi-direct product of permutations of v_i, $1 \le i \le 5$, and linear maps $v_i \to (\pm)_i v_i$, $1 \le i \le 5$, where $\prod_i (\pm)_i 1 = 1$, see [Bou68]. It follows that $W(e)$ consists of

$$e_{i_1 i_2 \dots i_k} = (\pm v_1 \pm v_2 \pm \dots \pm v_5)/2 + h/4$$

where $1 \le i_1 < i_2 < \dots < i_k \le 5$ show where the signs $(-)$ are placed, and $k \equiv 0 \mod 2$ (their number is 16 which is the number of exceptional curves on non-singular del Pezzo surface of degree 4), e.g. we have $e = e_{1234}$.

Case 7b. As the basis of the root subsystem $2A_1 \oplus A_3 \subset D_5$, we take $f_1 = v_1 - v_2$, $f_6 = v_1 + v_2$, $f_3 = v_3 - v_4$, $f_4 = v_4 - v_5$, $f_5 = v_4 + v_5$. Only $e_{1234} = (-v_1 - v_2 - v_3 - v_4 + v_5)/2 + h/4$, $e_{1345} = (-v_1 + v_2 - v_3 - v_4 - v_5)/2 + h/4$ (from the orbit $W(e)$) have non-negative pairing with this basis. We obtain the diagram 7b of Table 2.

A.4.3. Fundamental chambers $\mathcal{M}_+^{(2,4)}$ of cases $N = 8$ (Table 2).

We use the orthogonal basis $h, v_1, \dots, v_8$ over $\mathbb{Q}$ with $h^2 = 6$, $v_1^2 = \dots = v_8^2 = -2$. As root system E_6 we can take (see [Bou68]) all roots $\pm v_i \pm v_j$ $(1 \le i < j \le 5)$ and $\pm\frac{1}{2}(\pm v_1 \pm v_2 \pm v_3 \pm v_4 \pm v_5 - v_6 - v_7 + v_8)$ with even number of $(-)$. I. e. $E_6 \subset E_8$ consists of all roots in E_8 which are orthogonal to roots $v_6 - v_7$ and $v_7 + v_8$ (they define A_2). We denote $W = W(E_6)$, the Weyl group of E_6.

Case 8a. As $P(\mathcal{M}^{(2,4)})$, we can take $f_1 = (v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 + v_8)/2$, $f_2 = v_1 + v_2$, $f_3 = -v_1 + v_2$, $f_4 = -v_2 + v_3$, $f_5 = -v_3 + v_4$, $f_6 = -v_4 + v_5$ (with square -4) defining the basis of the root system E_6, and

$$e = -v_5 + \frac{1}{3}v_6 + \frac{1}{3}v_7 - \frac{1}{3}v_8 + \frac{1}{3}h = -v_5 + v_6 - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h$$

(with square -2). They define the diagram 8a.

We have

$$W(e) = W(-v_5 + v_6) - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h$$

where $W(-v_5 + v_6)$ consists of all roots α of E_8 with the properties: $(\alpha, v_6 - v_7) = -2$ and $(\alpha, v_7 + v_8) = 0$. Thus, $W(e)$ consists of all elements

$$e_{\pm i} = \pm v_i + v_6 - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h,\ 1 \le i \le 5;$$

$$e_{i_1 i_2 \dots i_k} =$$

$\frac{1}{2}(\pm v_1 \pm v_2 \pm v_3 \pm v_4 \pm v_5 + v_6 - v_7 + v_8) - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h$
where $1 \le i_1 < \dots < i_k \le 5$ show where are $(-)$, and $k \equiv 1 \mod 2$;

$$e_{78} = -v_7 + v_8 - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h.$$

(Their number is 27, the number of lines on a non-singular cubic.)

Case 8b. As a basis of $A_5 \oplus A_1 \subset E_6$ we can take $f_1 = (v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 + v_8)/2$, $f_3 = -v_1 + v_2$, $f_4 = -v_2 + v_3$, $f_5 = -v_3 + v_4$, $f_6 = -v_4 + v_5$ and $f_7 = (-v_1 - v_2 - v_3 - v_4 - v_5 + v_6 + v_7 - v_8)/2$. Only $e_{+1} = v_1 + v_6 - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h$ and $e_5 = (v_1 + v_2 + v_3 + v_4 - v_5 + v_6 - v_7 + v_8)/2 - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h$ have non-negative pairing with this basis. They define the diagram 8b of Table 2.

Case 8c. As a basis of $3A_2 \subset E_6$ we can take $f_1 = (v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 + v_8)/2$, $f_3 = -v_1 + v_2$; $f_5 = -v_3 + v_4$, $f_6 = -v_4 + v_5$; $f_2 = v_1 + v_2$, $f_7 = (-v_1 - v_2 - v_3 - v_4 - v_5 + v_6 + v_7 - v_8)/2$. Only $e_{+3} = v_3 + v_6 - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h$, $e_{125} = (-v_1 - v_2 + v_3 + v_4 - v_5 + v_6 - v_7 + v_8)/2 - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h$, $e_2 = (v_1 - v_2 + v_3 + v_4 + v_5 + v_6 - v_7 + v_8)/2 - \frac{2}{3}(v_6 - v_7) - \frac{1}{3}(v_7 + v_8) + \frac{1}{3}h$ have non-negative pairing with this basis. We obtain the diagram 8c of Table 2.

A.4.4. Fundamental chambers $\mathcal{M}_+^{(2,4)}$ of cases $N = 9$

We use the orthogonal basis $h, v_1, \dots, v_8$ over $\mathbb{Q}$ with $h^2 = 4$, $v_1^2 = \dots = v_8^2 = -2$. As a root system E_7 we can take (see [Bou68]) all roots $\pm v_i \pm v_j$ $(1 \le i < j \le 6)$, $\pm(v_7 - v_8)$, and $\pm\frac{1}{2}(\pm v_1 \pm v_2 \pm v_3 \pm v_4 \pm v_5 \pm v_6 + v_7 - v_8)$ with even number of $(-)$. I. e. $E_7 \subset E_8$ consists of all roots in E_8 which are orthogonal to the root $v_7 + v_8$. We denote $W = W(E_7)$, the Weyl group of E_7.

Case 9a. As $P(\mathcal{M}^{(2,4)})$ we can take $f_1 = (v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 + v_8)/2$, $f_2 = v_1 + v_2$, $f_3 = -v_1 + v_2$, $f_4 = -v_2 + v_3$, $f_5 = -v_3 + v_4$, $f_6 = -v_4 + v_5$, $f_7 = -v_5 + v_6$ (with square -4) defining the basis of the root system E_7, and

$$e = -v_6 + \frac{1}{2}v_7 - \frac{1}{2}v_8 + \frac{1}{2}h = -v_6 + v_7 - \frac{1}{2}(v_7 + v_8) + \frac{1}{2}h$$

(with square -2). They define the diagram 9a of Table 2.

The orbit $W(e) = W(-v_6 + v_7) - \frac{1}{2}(v_7 + v_8) + \frac{1}{2}h$ where $W(-v_6 + v_7)$ consists of all roots α in E_8 with the property $(\alpha, v_7 + v_8) = -2$. It follows that $W(e)$ consists of

$$e_{\pm i7} = \pm v_i + v_7 - \frac{1}{2}(v_7 + v_8) + \frac{1}{2}h,\ 1 \le i \le 6;$$

$$e_{\pm i8} = \pm v_i + v_8 - \frac{1}{2}(v_7 + v_8) + \frac{1}{2}h, \ 1 \le i \le 6;$$

$$e_{i_1 \dots i_k} = \frac{1}{2}(\pm v_1 \pm v_2 \pm \cdots \pm v_6 + v_7 + v_8) - \frac{1}{2}(v_7 + v_8) + \frac{1}{2}h$$

where $1 \le i_1 < \cdots < i_k \le 6$ show $(-)$ and $k \equiv 0 \mod 2$. Their number is 56, the number of exceptional curves on a non-singular del Pezzo surface of degree 2.

Case 9b. As a basis of $A_7 \subset E_7$ we can take $f_8 = v_7 - v_8$, f_1, f_3, f_4, f_5, f_6, f_7. Only $e_0 = \frac{1}{2}(v_1 + v_2 + \cdots + v_6 + v_7 + v_8) - \frac{1}{2}(v_7 + v_8) + \frac{1}{2}h$ and e_{56} have non-negative pairing with the basis. We obtain the diagram 9b of Table 2.

Case 9c. As a basis of $A_2 \oplus A_5 \subset E_7$ we can take $f_8 = v_7 - v_8$, f_1 and f_2, f_4, f_5, f_6, f_7. Only e_{-18}, e_{16}, e_{1456} have non-negative pairing with this basis. We obtain the diagram 9c of Table 2.

Case 9d. As a basis of $A_3 \oplus A_1 \oplus A_3 \subset E_7$ we can take $f_8 = v_7 - v_8$, f_1, f_3, and f_2, and f_5, f_6, f_7. Only e_{+38}, e_{26}, e_{12}, e_{1256} have non-negative pairing with this basis. We obtain the diagram 9d of Table 2.

Case 9e. As a basis of $A_1 \oplus D_6 \subset E_7$ we can take $f_8 = v_7 - v_8$ and f_2, f_3, f_4, f_5, f_6, f_7. Only e_{-68} and e_{123456} have non-negative pairing with the basis. We obtain the diagram 9e of Table 2.

Case 9f. As a basis of $D_4 \oplus 3A_1 \subset E_7$ we can take f_2, f_3, f_4, f_5 and f_7, $f_8 = v_7 - v_8$, $f_9 = -v_5 - v_6$. Only e_{-48}, e_{+58}, e_{1234}, e_{2346} have non-negative pairing with the basis. We obtain the diagram 9f of Table 2.

Case 9g. As a basis of $7A_1 \subset E_7$ we can take $u_1 = v_1 + v_2$, $u_2 = -v_1 + v_2$, $u_3 = v_3 + v_4$, $u_4 = -v_3 + v_4$, $u_5 = v_5 + v_6$, $u_6 = -v_5 + v_6$, $u_7 = v_7 - v_8$. Only e_{-28}, e_{-48}, e_{-68}, e_{2456}, e_{2346}, e_{1246}, e_{123456} have non-negative pairing with the basis. We obtain the digram 9g described in Section 3.4.4.

A.4.5. Fundamental chambers $\mathcal{M}_+^{(2,4)}$ of cases $N = 10$

We use the orthogonal basis $h, v_1, \dots, v_8$ of $S \otimes \mathbb{Q}$ with $h^2 = 2$, $v_1^2 = \cdots = v_8^2 = -2$. As a root system E_8 we can take (see [Bou68]) all roots $\pm v_i \pm v_j$ $(1 \le i < j \le 8)$ and $\frac{1}{2}(\pm v_1 \pm v_2 \pm \cdots \pm v_8)$ with even number of $(-)$. We denote $W = W(E_8)$, the Weyl group of E_8.

Case 10a. As $P(\mathcal{M}^{(2,4)})$ we can take $f_1 = (v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 + v_8)/2$, $f_2 = v_1 + v_2$, $f_3 = -v_1 + v_2$, $f_4 = -v_2 + v_3$, $f_5 = -v_3 + v_4$, $f_6 = -v_4 + v_5$, $f_7 = -v_5 + v_6$, $f_8 = -v_6 + v_7$ (with square -4) defining the basis of the root system E_8, and $e = -v_7 - v_8 + h$. They define the diagram 10a of Table 2.

The orbit $W(e) = W(-v_7 - v_8) + h$ where $W(-v_7 - v_8)$ consists of all roots α in E_8. It follows that $W(e)$ consists of

$$e_{\pm i, \pm j} = \pm v_i \pm v_j + h, \ 1 \le i < j \le 8;$$

$$e_{i_1\dots i_k} = \frac{1}{2}(\pm v_1 \pm v_2 \pm \cdots \pm v_8) + h$$

where $1 \le i_1 < \cdots < i_k \le 8$ show $(-)$ and $k \equiv 0 \mod 2$. Their number is 240, the number of exceptional curves on a non-singular del Pezzo surface of degree 1.

Case 10b. As a basis of $A_8 \subset E_8$ we can take f_1, f_3, f_4, f_5, f_6, f_7, f_8, $f_9 = -v_7 - v_8$. Only $e_{+1,+2}$, $e_0 = (v_1 + \cdots + v_8)/2 + h$, e_{67} have non-negative pairing with the basis. We obtain the diagram 10b of Table 2.

Case 10c. As a basis of $A_1 \oplus A_7 \subset E_8$ we can take f_1 and f_2, f_4, f_5, f_6, f_7, f_8, $f_9 = -v_7 - v_8$. Only $e_{-1,+2}$, $e_{-1,+8}$, e_{18}, e_{17}, e_{1567} have non-negative pairing with the basis. We obtain the diagram 10c of Table 2.

Case 10d. As a basis of $A_2 \oplus A_1 \oplus A_5 \subset E_8$ we can take f_1, f_3, and f_2, and f_5, f_6, f_7, f_8, $f_9 = -v_7 - v_8$. Only $e_{-1,-2}$, $e_{-2,+3}$, $e_{+3,+4}$, $e_{+3,+8}$, e_{12}, e_{27}, e_{28}, e_{1267} have non-negative pairing with the basis. We obtain the diagram 10d of Table 2.

Case 10e. As a basis of $A_4 \oplus A_4 \subset E_8$ we can take f_1, f_3, f_4, f_2 and f_6, f_7, f_8, $f_9 = -v_7 - v_8$. Only $e_{-3,+4}$, $e_{+4,+5}$, $e_{+4,+8}$, e_{1237}, e_{1238}, e_{23} have non-negative pairing with the basis. We obtain the diagram 10d of Table 2.

Case 10f. As a basis of $D_8 \subset E_8$ we can take f_2, f_3, f_4, f_5, f_6, f_7, f_8, $f_9 = -v_7 - v_8$. Only $e_{-7,+8}$, e_{234567} have non-negative pairing with the basis. We obtain the diagram 10f of Table 2.

Case 10g. As a basis of $D_5 \oplus A_3 \subset E_8$ we can take f_1, f_2, f_3, f_4, f_5 and f_7, f_8, $f_9 = -v_7 - v_8$. Only $e_{-4,+5}$, $e_{+5,+6}$, $e_{+5,+8}$, e_{1234}, e_{2348} have non-negative pairing with the basis. We obtain the diagram 10g of Table 2.

Case 10h. As a basis of $E_6 \oplus A_2 \subset E_8$ we can take f_1, f_2, f_3, f_4, f_5, f_6, and f_8, and $f_9 = -v_7 - v_8$. Only $e_{-5,+6}$, $e_{+6,+7}$, $e_{+6,+8}$, e_{123458} have non-negative pairing with the basis. We obtain the diagram 10h of Table 2.

Case 10i. As a basis of $E_7 \oplus A_1 \subset E_8$ we can take f_1, f_2, f_3, f_4, f_5, f_6, f_7, and $f_9 = -v_7 - v_8$. Only $e_{-6,+7}$, $e_{+7,-8}$, $e_{+7,+8}$ have non-negative pairing with the basis. We obtain the diagram 10i of Table 2.

Case 10j. As a basis of $2A_1 \oplus D_6 \subset E_8$ we can take f_2 and f_3, and f_5, f_6, f_7, f_8, $f_9 = -v_7 - v_8$, $f_{10} = -v_7 + v_8$. Only $e_{-1,-2}$, $e_{+1,-2}$, $e_{-2,+3}$, $e_{+3,+4}$, e_{12}, e_{28} have non-negative pairing with the basis. We obtain the diagram 10j of Table 2.

Case 10k. As a basis of $2D_4 \subset E_8$ we can take f_2, f_3, f_4, f_5 and f_7, f_8, $f_9 = -v_7 - v_8$, $f_{10} = -v_7 + v_8$. Only $e_{-3,-4}$, $e_{-4,+5}$, $e_{+5,+6}$, e_{1234}, e_{2348} have non-negative pairing with the basis. We obtain the diagram 10k of Table 2.

Case 10l. As a basis of $2A_1 \oplus 2A_3 \subset E_8$ we can take f_2; f_3; f_5, f_6, $f_{11} = v_3 + v_4$; $f_9 = -v_7 - v_8$, f_8, $f_{10} = -v_7 + v_8$. Only $e_{-1,-2}$, $e_{+1,-2}$,

$e_{-2,-5}$, $e_{-2,+6}$, $e_{-4,-5}$, $e_{-5,+6}$, $e_{+6,+7}$, e_{123458}, e_{1245}, e_{2345}, e_{2458} have non-negative pairing with the basis. We obtain the diagram 10l of Table 2.

Case 10m. As a basis of $4A_2 \subset E_8$ we can take f_1, f_3; f_2, $f_{10} = \frac{1}{2}(-v_1 - v_2 - v_3 - v_4 - v_5 + v_6 + v_7 - v_8)$; f_5, f_6; f_8, $f_9 = -v_7 - v_8$. Only $e_{-2,+3}$, $e_{+3,+4}$, $e_{+3,-5}$, $e_{+3,+6}$, $e_{+3,+8}$, $e_{+6,+8}$, e_{12}, e_{1257}, e_{1267}, e_{25}, e_{27}, e_{28} have non-negative pairing with the basis. We obtain the diagram 10m of Table 2.

Case 10n. As a basis of $D_4 \oplus 4A_1 \subset E_8$ we can take $u_1 = v_1 + v_2$, $u_2 = -v_1 + v_2$, $u_3 = -v_3 + v_4$, $u_4 = -v_2 + v_3$, $u_5 = v_5 + v_6$, $u_6 = -v_5 + v_6$, $u_7 = -v_7 + v_8$, $u_8 = v_7 + v_8$ (this basis agrees with the one used in case 10n of Lemma 3.12 if one replaces f_i by u_i). Only $e_{-3,-4}$, $e_{-4,-6}$, $e_{-4,-8}$, $e_{-5,-6}$, $e_{+5,-6}$, $e_{-6,-8}$, $e_{-7,-8}$, $e_{+7,-8}$, $e_{12345678}$, e_{123468}, e_{234568}, e_{234678} have non-negative pairing with the basis. We obtain the diagram 10n of Figure 1.

Case 10o. As a basis of $8A_1 \subset E_8$ we can take $u_1 = -v_1 + v_2$, $u_2 = v_1 + v_2$, $u_3 = -v_3 + v_4$, $u_4 = v_3 + v_4$, $u_5 = -v_5 + v_6$, $u_6 = v_5 + v_6$, $u_7 = -v_7 + v_8$, $u_8 = v_7 + v_8$. The set of indices $I = 1, \ldots, 8$ has the structure of a 3-dimensional affine space over F_2 with (affine) planes $J \subset I$ determined by the property $\frac{1}{2}\sum_{j \subset J} u_j \in E_8$. It is the same as the one used in case 10o of Lemma 3.12. For $i \in I$ we set $w_i = e_{+i,-(i+1)}$ if i is odd, and $w_i = e_{-(i-1),-i}$ if i is even. For a plane $\pi \subset I$ we set $w_\pi = -\frac{1}{2}\sum_{i \in \pi} u_i + h$. The introduced elements w_i, $i \in I$, and w_π, $\pi \subset I$ is a plane, are the only elements (from the orbit $W(e)$) which have non-negative pairing with the basis. We obtain the graph 10o described in Section 3.4.4.

Case of $7A_1 \subset E_8$. As a basis of $7A_1 \subset E_8$ we can take $u_2 = v_1 + v_2$, $u_3 = -v_3 + v_4$, $u_4 = v_3 + v_4$, $u_5 = -v_5 + v_6$, $u_6 = v_5 + v_6$, $u_7 = -v_7 + v_8$, $u_8 = v_7 + v_8$. We denote $u_1 = -v_1 + v_2$ (the roots $\pm u_1$ are the only roots of E_8 which are orthogonal to $7A_1 \subset E_8$. The set of indices $I = 1, \ldots, 8$ has the structure of a 3-dimensional affine space over F_2 with (affine) planes $J \subset I$ determined by the property $\frac{1}{2}\sum_{j \in J} u_j \in E_8$. Taking $1 \in I$ as an origin, makes the set I to be a 3-dimensional vector space over F_2. As in the previous case, we define w_i for $i \in I$, and w_π for an affine plane $\pi \subset I$. We set $w_1^{(+)} = w_1 = e_{+1,-2}$ and $w_1^{(-)} = e_{-1,+2}$. If $1 \in \pi$, we set $w_\pi^{(+)} = w_\pi$, and $w_\pi^{(-)} = w_\pi + u_1$. The introduced elements w_i, $i \in I - \{1\}$, $w_0^{(+)}$, $w_0^{(-)}$, w_π for planes $\pi \subset I - \{1\}$, and $w_\pi^{(+)}$, $w_\pi^{(-)}$ for planes $\pi \subset I$ containing 1 are the only elements (from the orbit $W(e)$) which have non-negative pairing with the basis. We obtain the graph described and used in Section 3.4.7 (cases 10n and 10o).

A.4.6. Fundamental chambers $\mathcal{M}_+^{(2,4)}$ of cases $N = 20$

This case had been partly described (including cases 20a and 20b below) at the end of Section 3.4.4; here we add further details of calculations for readers' convenience. We use the orthogonal basis $h, \alpha, v_1, \dots, v_8$ of $S\otimes\mathbb{Q}$ with $h^2 = 2$, $\alpha^2 = v_1^2 = \cdots = v_8^2 = -2$. As a root system D_8 we can take (see [Bou68]) all roots $\pm v_i \pm v_j$ $(1 \le i < j \le 8)$. We denote $W = W(D_8)$, the Weyl group of D_8.

Case 20a. As $P(\mathcal{M}^{(2,4)})$ we can take $f_1 = v_1 - v_2$, $f_2 = v_2 - v_3$, $f_3 = v_3 - v_4$, $f_4 = v_4 - v_5$, $f_5 = v_5 - v_6$, $f_6 = v_6 - v_7$, $f_7 = v_7 - v_8$, $f_8 = v_7 + v_8$ (with square -4) defining the basis of the root system D_8, and α, $b = \frac{h}{2} - \frac{\alpha}{2} - v_1$, $c = h - \frac{1}{2}(v_1 + v_2 + \cdots + v_8)$ (with square -2). They define the diagram 20a of Table 2.

The orbit $W(\alpha)$ consists of only α; the orbit $W(b)$ consists of all

$$b_{\pm i} = \frac{h}{2} - \frac{\alpha}{2} \pm v_i,\ 1 \le i \le 8;$$

the orbit $W(c)$ consists of all

$$c_{i_1\dots i_k} = h + \frac{1}{2}(\pm v_1 \pm v_2 \pm \cdots \pm v_8)$$

where $1 \le i_1 < i_2 < \cdots < i_k \le 8$ show where are $(-)$, and $k \equiv 0 \mod 2$. Thus, $P(\mathcal{M}^{(2)})$ has $1 + 2\cdot 8 + 2^7 = 81$ elements. This is the number of exceptional curves on the right DPN surface with the main invariants $(r, a, \delta) = (10, 8, 1)$ and the zero root invariant. One of them (corresponding to α) has square (-4), all other are (-1)-curves.

Case 20b. As a basis of $2A_1 \oplus D_6 \subset D_8$ we can take f_1; $f_9 = -v_1 - v_2$; $f_3, f_4, f_5, f_6, f_7, f_8$. Only α, b_{+2}, b_{-3}, c_{134567}, c_{345678} have non-negative pairing with the basis. We obtain the diagram 20b of Table 2.

Case 20c. As a basis of $A_3 \oplus D_5 \subset D_8$ we can take $f_1, f_2, f_9 = -v_1 - v_2$ and f_4, f_5, f_6, f_7, f_8. Only α, b_{+3}, b_{-4}, c_{145678}, c_{4567} have non-negative pairing with the basis. We obtain the diagram 20c of Table 2.

Case 20d. As a basis of $2D_4 \subset D_8$ we can take $f_1, f_2, f_3, f_9 = -v_1 - v_2$ and f_5, f_6, f_7, f_8. Only α, b_{+4}, b_{-5}, c_{1567}, c_{5678} have non-negative pairing with the basis. We obtain the diagram 20d of Table 2.

Case 20e. As a basis of $2A_1 \oplus 2A_3 \subset D_8$ we can take f_1; $f_9 = -v_1 - v_2$; $f_3, f_4, f_{10} = -v_3 - v_4$; f_6, f_7, f_8. Only α, b_{+2}, b_{+5}, b_{-6}, c_{1367}, c_{1678}, c_{3678}, c_{67} have non-negative pairing with the basis. We obtain the diagram 20e of Figure 2.

Case 20f. As a basis of $4A_1 \oplus D_4 \subset D_8$ we can take f_1; $f_9 = v_1 + v_2$; f_3; $f_{10} = v_3 + v_4$; f_5, f_6, f_7, f_8. Only α, b_{-1}, b_{-3}, b_{-5}, $c_{12345678}$, c_{123567}, c_{134567}, c_{135678} have non-negative pairing with the basis. We obtain the diagram 20f of Figure 3.

Case 20g. As a basis of $8A_1 \subset D_8$ we can take $u_1 = v_1 - v_2$; $u_2 = v_1 + v_2$; $u_3 = v_3 - v_4$; $u_4 = v_3 + v_4$; $u_5 = v_5 - v_6$; $u_6 = v_5 + v_6$; $u_7 = v_7 - v_8$; $u_8 = v_7 + v_8$. Only α, b_{-1}, b_{-3}, b_{-5}, b_{-7}, c_{1357}, c_{135678}, c_{134578}, c_{134567}, c_{123578}, c_{123567}, c_{123457}, $c_{12345678}$ have non-negative pairing with the basis. We obtain the diagram 20g of Figure 4.

Case $4A_1 \oplus A_3 \subset D_8$. As a basis of $4A_1 \oplus A_3 \subset D_8$ we can take $u_1 = v_1 - v_2$; $u_2 = -v_1 - v_2$; $u_3 = v_3 - v_4$; $u_4 = -v_3 - v_4$; $u_5 = v_5 - v_6$, $u_6 = -v_5 - v_6$, $u_7 = v_6 + v_7$. Only α, b_{+2}, b_{+4}, b_{-7}, b_{-8}, b_{+8} and

$$c_{(\mu_1,\mu_3,\mu_5)} = \frac{1}{2}\big((-1)^{\mu_1}v_1 + v_2 + (-1)^{\mu_3}v_3 + v_4 + (-1)^{\mu_5}v_5 + v_6 -$$

$$-v_7 + (-1)^{(\mu_1+\mu_3+\mu_5+1)}v_8\big) + h,$$

$(\mu_1, \mu_3, \mu_5) \in (\mathbb{Z}/2\mathbb{Z})^3$, have non-negative pairing with the basis. We obtain the diagram which had been described in Case 20e of Section 3.4.7.

In a usual way, we identify $(\mathbb{Z}/2\mathbb{Z})^3$ with the set of vertices $V(K)$ of a 3-dimensional cube K. Thus the last set is c_v, $v \in V(K)$. Each u_i, $1 \le i \le 6$, defines a 2-dimensional face γ_i of the cube K which consists of c_v such that $(u_i, c_v) = 2$. Therefore, we further write $u_i = u_{\gamma_i}$, where γ_i belongs to the set $\gamma(K)$ of 2-dimensional faces of K. The u_7 defines two distinguished opposite faces $\gamma_5, \gamma_6 \in \gamma(K)$ characterized by $(u_5, u_7) = (u_6, u_7) = 2$ (i. e. u_5, u_7, u_6 define the component A_3).

We identify b_{-7} with the pair $\{\gamma_5, \gamma_6\} \subset \gamma(K)$ of distinguished opposite faces of K. We identify b_{+2} and b_{+4} with the pairs $\{\gamma_1, \gamma_2\}$ and $\{\gamma_3, \gamma_4\}$ of opposite faces of K. Thus, we further numerate them as $b_{\overline{\gamma}}$, where $\overline{\gamma}$ belongs to the set $\overline{\gamma(K)}$ of pairs of opposite 2-dimensional faces of K. We have $(u_{\gamma'}, b_{\overline{\gamma}}) = 2$ if $\gamma' \in \overline{\gamma}$ and $\overline{\gamma} \in \overline{\gamma(K)}$ is different from the pair of two distinguished opposite faces of K; otherwise it is 0.

We can identify b_{-8} (respectively b_{+8}) with the four vertices $c_{(\mu_1,\mu_3,\mu_5)}$ where $\mu_1 + \mu_3 + \mu_5 + 1 \equiv 0 \mod 2$ (respectively $\equiv 1 \mod 2$). Each of these four vertices contains one vertex from any two opposite vertices of K, no three of its vertices belong to a face of K. Thus, we can further denote $b_{\pm 8}$ by b_t where t belongs to the set $\overline{V(K)}$ of these two fours. We have $(c_v, b_t) = 2$ if $v \in t \in \overline{V(K)}$. Otherwise, it is 0.

Case $7A_1 \subset D_8$. As a basis of $7A_1 \subset D_8$ we can take $u_1 = v_1 - v_2$; $u_2 = -v_1 - v_2$; $u_3 = v_3 - v_4$; $u_4 = -v_3 - v_4$; $u_5 = v_5 - v_6$; $u_6 = -v_5 - v_6$; $u_7 = v_7 - v_8$. Only α, b_{+2}, b_{+4}, b_{+6}, b_{-7}, b_{+8} and

$$c_{j_1 j_2 j_3(\pm 2)} = \frac{1}{2}\big((-1)^{j_1}v_1 + v_2 + (-1)^{j_2}v_3 + v_4 + (-1)^{j_3}v_5 +$$

$$+v_6 \pm (v_7 + v_8)\big) + h$$

where $j_i = 1$ or 2, $1 \le i \le 3$, and $j_1 + j_2 + j_3 + (\pm 2) \equiv 0 \mod 2$, and

$$c_{j_1 j_2 j_3 1} = \frac{1}{2}\left((-1)^{j_1} v_1 + v_2 + (-1)^{j_2} v_3 + v_4 + (-1)^{j_3} v_5 + \right.$$
$$\left. + v_6 - v_7 + v_8\right) + h$$

where $j_i = 1$ or 2, $1 \le i \le 3$, and $j_1 + j_2 + j_3 + 1 \equiv 0 \mod 2$, have non-negative pairing with the basis. We obtain the diagram which had been described in Case 20f,g of Section 3.4.7.

One can denote $f_{11} = u_1$, $f_{12} = u_2$, $f_{21} = u_3$, $f_{22} = u_4$, $f_{31} = u_5$, $f_{32} = u_6$, $f_{41} = u_7$, $b_1 = b_{+2}$, $b_2 = b_{+4}$, $b_3 = b_{+6}$, $b_{4(-)} = b_{-7}$, $b_{4(+)} = b_{+8}$