4

Classifi cation of log del Pezzo surfaces of index ≤ 2 and applications

4.1. Classification of log del Pezzo surfaces of index ≤ 2

From the results of Chapters 1 — 3 we obtain

Theorem 4.1. *For any log del Pezzo surface Z of index ≤ 2 there exists a unique resolution of singularities $\sigma : Y \to Z$ (called* right*) such that Y is a right DPN surface of elliptic type and σ contracts exactly all exceptional curves of the Du Val and the logarithmic part of $\Gamma(Y)$. Vice versa, if Y is a right DPN surface of elliptic type, then there exists a unique morphism $\sigma : Y \to Z$ of contraction of all exceptional curves corresponding to the Du Val and the logarithmic part of $\Gamma(Y)$ which gives resolution of singularities of log del Pezzo surface Z of index ≤ 2 (it will be automatically the right resolution).*

Thus, classifications of log del Pezzo surfaces of index ≤ 2 and right DPN surfaces of elliptic type are equivalent, and they are given by Theorems 3.18, 3.19 and 3.20.

Proof. Let Z be a log del Pezzo surface of index ≤ 2. In Chapter 1, a "canonical" (i. e. uniquely defined) resolution of singularities $\sigma : Y \to Z$ had been suggested such that Y is a right DPN surface of elliptic type. First, a minimal resolution of singularities $\sigma_1 : Y' \to Z$ is taken, and second, the blow-up of all intersection points of components of curves in preimages of non Du Val singularities K_n of Z is taken. Let us show that σ contracts exactly exceptional curves of $\mathrm{Duv}\,\Gamma(Y)$ and $\mathrm{Log}\,\Gamma(Y)$.

Let E be an exceptional curve of Y corresponding to a vertex of the subgraph $\mathrm{Duv}\,\Gamma(Y)$ or $\mathrm{Log}\,\Gamma(Y)$. Let $\widetilde{C_g} \in |-2K_Z|$ be a non-singular curve of Z which does not contain singular points of Z (it does exist by Theorem 1.5), and $C_g = \sigma^{-1}(\widetilde{C_g})$. Then (see Sections 1.5 and 2) $C_g + E_1 + \cdots + E_k \in |-2K_Y|$ where E_i are all exceptional curves on Y with the square (-4) and C_g a non-singular irreducible curve of genus $g \geq 2$.

By Chapter 2, one has $E \cdot C_g = 0$. If σ does not contract E, then for the curve $\sigma(E)$ on Z we have $\sigma(E) \cdot \widetilde{C_g} = \sigma(E) \cdot (-2K_Z) = 0$. Then $-K_Z$ is not ample. We get a contradiction. Vice versa, by construction, σ contracts only curves from $\mathrm{Duv}\,\Gamma(Y)$ and $\mathrm{Log}\,\Gamma(Y)$. This shows that σ is a right resolution.

Now, let Y be a right DPN surface of elliptic type and $\sigma : Y \to Z$ a contraction of all exceptional curves corresponding to vertices $\mathrm{Duv}\,\Gamma(Y)$ and $\mathrm{Log}\,\Gamma(Y)$ (it does exist analytically because $\mathrm{Duv}\,\Gamma(Y) \cup \mathrm{Log}\,\Gamma(Y)$ is negative, and we show that it does exist algebraically by the direct construction below). To prove Theorem, we should prove that Z is a log del Pezzo surface of index ≤ 2, and σ the right resolution of singularities of Z.

Second statement becomes obvious if one decomposes σ as the composition of contractions of all exceptional curves of 1st kind from $\mathrm{Log}\,\Gamma(Y)$ (they don't intersect each other) and further contraction of the remaining exceptional curves.

To prove the first statement, one can use the double covering $\pi : X \to Y$ with the involution θ (see Chapter 2), the relation between exceptional curves of Y and (X, θ) (see Chapter 2), and that the contraction of A, D and E configurations of (-2) curves on X does exit and gives the corresponding quotient singularities $\mathbb{C}/G_i$ where $G_i \subset SL(2, \mathbb{C})$ are finite subgroups. Using these considerations (i. e. first we consider the corresponding contraction, and second the quotient by involution), and Brieskorn's results [Bri68], we obtain that all non Du Val singularities of Z are $\mathbb{C}/\widetilde{G_i}$ where $\widetilde{G_i} \subset GL(2, \mathbb{C})$ and $\widetilde{G_i} \cap SL(2, \mathbb{C}) = G_i$ have index 2 in $\widetilde{G_i}$, and $\widetilde{G_i}/G_i = \{1, \theta\}$. It follows that Z is a complete algebraic surface with log-terminal singularities of index ≤ 2.

Let us show that $-K_Z$ is ample. By Nakai–Moishezon criterion [Nak63], [Moi67] (see also Kleiman's criterion [Kle66]), it is enough to show that $(-K_Z)^2 > 0$ and $(-K_Z) \cdot D > 0$ for any curve D on Z. We have (see Section 1.5)

$$4(-K_Z)^2 = (-2K_Z)^2 = (\sigma^*(-2K_Z))^2 = (C_g)^2 > 0.$$

since Y is a DPN surface of elliptic type. Moreover,

$$-2K_Z \cdot D = -2\sigma^* K_Z \cdot \sigma^* D = C_g \cdot \sigma^* D \geq 0$$

because C_g is irreducible with $(C_g)^2 > 0$ and $\sigma^* D$ is effective. Moreover, we get here zero, only if the effective divisor $\sigma^* D$ consists of exceptional curves F on Y with $C_g \cdot F = 0$. But such curves F correspond to vertices of the logarithmic or the Du Val part of $\Gamma(Y)$. They are contracted by σ into points of Z which is impossible for the divisor $\sigma^* D$. $\square$

Using Theorem 4.1, we can transfer to log del Pezzo surfaces Z of index ≤ 2 *the main invariants* (r, a, δ), *equivalently* (k, g, δ), *the root invariant, the root subsystem, the exceptional curves* which are defined for the surface Y of the right resolution $\sigma : Y \to Z$. In particular, the *Picard number of Z* is

$$(83) \qquad \tilde{r} = \mathrm{rk} \, \mathrm{Pic} \, Z = r - \#V(\mathrm{Duv} \, \Gamma(Y)) - \#V(\mathrm{Log} \, \Gamma(Y)).$$

In Theorem 3.18 we have shown the Picard number $\tilde{r}$ in the extremal (for Y) case. Obviously, surfaces Z with the extremal Y are distinguished by the minimal Picard number $\tilde{r}$ for the fixed main invariants (r, a, δ) (equivalently, (k, g, δ)). Since the $\mathrm{Log} \, \Gamma(Y)$ is prescribed by the main invariants and is then fixed, this is equivalent to have the maximal rank (i. e. the number of (-2)-curves for the minimal resolution of singularities) for Du Val singularities of Z.

In Mori Theory, see [Mor82] and [Rei83], log del Pezzo surfaces Z with $\mathrm{rk} \, \mathrm{Pic} \, Z = 1$ are especially important. They give relatively minimal models in the class of rational surfaces with log-terminal singularities: any rational surface $X \neq \mathbb{P}^1 \times \mathbb{P}^1$ with log-terminal singularities has a contraction morphism onto such a model. From Theorems 4.1 and 3.6, we obtain classification of such models with log-terminal singularities of index ≤ 2. By Theorem 3.18, they correspond to extremal DPN surfaces of elliptic type with

$$\tilde{r} = r - \#V(\mathrm{Duv} \, \Gamma(Y)) - \#V(\mathrm{Log} \, \Gamma(Y)) = 1,$$

and Theorem 3.18 gives the classification of the graphs of exceptional curves on them. This classification can be extended to a fine classification of the surfaces themselves. Here are results for the case of $\mathrm{rk} \, \mathrm{Pic} \, Z = 1$.

Theorem 4.2. *There exist, up to isomorphism, exactly 18 log del Pezzo surfaces Z of index 2 with* $\mathrm{rk} \, \mathrm{Pic} \, Z = 1$. *The DPN surfaces Y of their right resolution of singularities are extremal and correspond to the following cases of Theorem 3.18, where we also show in parentheses the type of singularities of Z:*

$$11(K_1), \quad 15(K_1A_4), \quad 18(K_1A_1A_5), \quad 19(K_1A_7), \quad 20a(K_1D_8),$$

$$20b(K_12A_1D_6), \quad 20c(K_1A_3D_5), \quad 20d(K_12D_4); \quad 21(K_2A_2),$$

$$25(2K_1A_7), \quad 26(K_22A_3), \quad 27(K_2A_7), \quad 30(K_32A_2); \quad 33(K_3A_1A_5);$$

$$40(K_5), \quad 44(K_5A_4); \quad 46(K_6A_2); \quad 50(K_9).$$

In particular, the isomorphism class of Z is defined by its configuration of singularities. The number of non-Du Val singularities is at most one except when the singularities are $2K_1A_7$.

In all other cases 11–50 the surface with maximal Du Val part is also unique.

Proof. For each of the graphs of Table 3 it is straightforward to pick a subgraph such that contracting the corresponding curves realizes Y as a sequence of blowups starting from $V = \mathbb{P}^2$ or $\mathbb{F}_n$, $n \leq 4$. The images of the remaining curves give a configuration of curves on V.

By Theorem 3.20 we are guaranteed that, vice versa, starting with such a configuration, the corresponding series of blowups leads to a right resolution of singularities $\widetilde{Z}$ of a log del Pezzo surface Z of index ≤ 2.

So, to compute the number of isomorphism classes, one has to find the orbits of the G-action on the parameter space for the choices of the blowups. Finally, one has to take into account the action of the symmetry group of the graph and the (finitely many) choices for the contractions to V.

In all the cases this is a straightforward computation which gives precisely one orbit.

A typical case is that of case 48. The configuration of curves can be contracted to a ruled surface $\mathbb{F}_1$ so that the images of non-contracted curves are two distinct fibres, the exceptional section s_1 and an infinite section $s_\infty \sim s_1 + f$. In other words, they are the $(\mathbb{C}^*)^2$-invariant divisors on the toric variety $\mathbb{F}_1$. The blowups $Y \to \mathbb{F}_1$ are uniquely determined except for the two last blowups corresponding to the two white end-vertices. One easily sees that these two blowups correspond to a choice of two points P_1, P_2 lying on two torus orbits O_1, O_2 on a toric surface $Y' \to \mathbb{F}_1$ with $\mathrm{rk}\,\mathrm{Pic}\,Y' = 4$. The surface Y' corresponds to a polytope obtained from the polytope of $\mathbb{F}_1$ by cutting two corners, which adds two new sides. These sides are obviously not parallel. Hence, the torus $(\mathbb{C}^*)^2$ acts transitively on $O_1 \times O_2$, so the surface Y is unique.

The only cases where a similar toric argument does not work are 39, 45 and 47. In case 39 the surface Y can be contracted to $\mathbb{P}^1 \times \mathbb{P}^1$ with 6 curves, 3 sections and 3 fibres. This configuration is unique and the blowups are uniquely determined, so the surface Y is unique. In case 45 the surface Y is similarly contracted to $\mathbb{P}^1 \times \mathbb{P}^1$ with 6 curves, sections s, s', fibres f, f' and curves $C \sim C' \sim s + f$ so that C passes through $s \cap f$ and $s' \cap f'$ and C' through $s \cap f'$ and $s' \cap f$. This configuration is unique as well.

In the most difficult case 47, Y can be contracted to $\mathbb{P}^2$ with the following configuration:

(1) three non-collinear points P_1, P_2, P_3,

(2) three lines, l_1, l_2, l_3 each passing through two of the three points so that $P_i \notin l_i$.

(3) two conics q_1, q_2 such that q_1 is tangent to lines l_2 and l_3 respectively at the points P_3 and P_2; and q_2 is tangent to lines l_1 and l_3 respectively at the points P_3 and P_1.

It is easy to see that this configuration is rigid as well. $\square$

The Gorenstein case is "well known" to experts but we were unable to find a complete and accurate description of the isomorphism classes in the literature. Therefore, we include the following theorem for completeness. Here we use the **degree** d of a del Pezzo surface Z which is $d = K_Z^2$.

Theorem 4.3. *(a) There exist 28 configurations of singularities of Gorenstein log del Pezzo surfaces of Picard number 1, and each type determines the corresponding surface up to a deformation. The types (and the cases N in Table 3) are as follows:*

(1) $d = 9$: $\emptyset$ *(case 1)*

(2) $d = 8$: A_1 *(2)*

(3) $d = 6$: $A_2 A_1$ *(5)*

(4) $d = 5$: A_4 *(6)*

(5) $d = 4$: D_5 *(7a)* $A_3 2A_1$ *(7b)*

(6) $d = 3$: E_6 *(8a)*, $A_5 A_1$ *(8b)*, $3A_2$ *(8c)*

(7) $d = 2$: E_7 *(9a)*, A_7 *(9b)*, $A_5 A_2$ *(9c)*, $2A_3 A_1$ *(9d)*, $D_6 A_1$ *(9e)*, $D_4 3A_1$ *(9f)*

(8) $d = 1$: E_8 *(10a)*, A_8 *(10b)*, $A_7 A_1$ *(10c)*, $A_5 A_2 A_1$ *(10d)*, $2A_4$ *(10e)*, D_8 *(10f)*, $D_5 A_3$ *(10g)*, $E_6 A_2$ *(10h)*, $E_7 A_1$ *(10i)*, $D_6 2A_1$ *(10j)*, $2D_4$ *(10k)*, $2A_3 2A_1$ *(10l)*, $4A_2$ *(10m)*.

(b) In each type there is exactly one isomorphism class, with the following exceptions: in types E_8, $E_7 A_1$, $E_6 A_2$ there are two isomorphism classes; and in type $2D_4$ there are infinitely many isomorphism classes parameterized by A^1.

(c) The three extra surfaces of type E_8, $E_7 A_1$, $E_6 A_2$ and all surfaces of type $2D_4$ are distinguished by the fact that their automorphism groups are 1-dimensional and contain $\mathbb{C}^*$. All other surfaces with $d = 1$ have finite automorphism groups.

1st proof. The first case to consider is $d = 1$. Choosing an appropriate subgraph in the graph of exceptional curves on $\widetilde{Z}$, one picks a sequence of blowups $\widetilde{Z} \to \mathbb{P}^2$. These contractions and images of (-2)-curves are listed in [BBD84]. In addition, one has to compute the images of (-1)-curves. The result is a configuration of lines, conics and cubics on $\mathbb{P}^2$, and in most cases cubics can be avoided.

Again, by theorem 3.20 we are guaranteed that, vice versa, starting with such a configuration, the corresponding series of blowups leads to a minimal resolution of singularities $\widetilde{Z}$ of a Gorenstein log del Pezzo surface Z.

To compute the automorphism groups and the number of isomorphism classes, one has to compute the stabilizer $G \subset \mathrm{PGL}(3)$ of a projective configuration on $\mathbb{P}^2$ and the orbits of the G-action on the parameter space for the configurations and the choices for the blowups; and to take into

account the action of the symmetry group of the graph and the (finitely many) choices for the contractions to $\mathbb{P}^2$.

In the case E_8, the projective configuration is a line and a point on it, the group G is the subgroup of upper-triangular matrices, and the parameter space is $\mathbb{C}^* \times \mathbb{C}^4$ which can be identified with the set of power series

$$y = \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6 + \alpha_7 x^7 \quad \mathrm{mod}\ x^8 \quad \text{with} \quad \alpha_3 \neq 0.$$

The G-action has two orbits: those of $y = x^3$ and of $y = x^3 + x^7$. The first orbit is in the closure of the second. The stabilizer of $y = x^3$ is isomorphic to $\mathbb{C}^*$ and consists of diagonal matrices $(1, c, c^3)$, the second stabilizer is finite. The model for the moduli stack is $[\mathbb{A}^1 : \mathbb{G}_m]$ with $\mathbb{C}^*$-action $\lambda.a = \lambda^4 a$.

In the case $E_7 A_1$, the projective configuration is a line l_1, a conic q tangent to it, and another line l_2. There are two cases: when l_2 intersects q at 2 distinct points, and when they are tangent. One case is a degeneration of another, and in the degenerate case the stabilizer of the configuration contains $\mathbb{C}^*$.

In the case $E_6 A_2$, the projective configuration consists of 4 lines and 3 of them either pass through the same point or they do not. Once again, the local model is $[\mathbb{A}^1 : \mathbb{G}_m]$ with the standard action, one configuration degenerates into another, and the degenerate configuration has stabilizer $\mathbb{C}^*$.

In the case $2D_4$, the projective configuration consists of 4 lines through a point P and the 5th line $l_5 \not\ni P$. The parameter space for such configuration is $\mathbb{P}^1 \backslash (3 \text{ points})$. Dividing by the symmetry group $\mathbb{Z}/2 \times \mathbb{Z}/2$ gives $\mathbb{A}^1$. Every configuration has $\mathbb{C}^*$ as the stabilizer group.

In all other cases for $d = 1$ the computation gives one isomorphism class.

For $d = 2$, the surfaces $\widetilde{Z}_2$ are obtained from the surfaces $\widetilde{Z}_1$ of $d = 1$ by contracting one (-1)-curve. So, the cases where more than one isomorphism class is possible are the ones that come from the four exceptional cases above.

The only contraction of the E_8-case is the case E_7. In this case, the group of upper triangular matrices acts on the polynomials $y = \alpha_3 x^3 + \cdots + \alpha_6 x^6 \mathrm{\ mod\ } x^7$ with $\alpha_3 \neq 0$ transitively; so there is only one isomorphism class.

The case $E_7 A_1$ produces $D_6 A_1$ and E_7. In each of these, the surface is unique because it can also be obtained by contracting a surface of type $D_6 2A_1$ and E_8, respectively.

The case $E_6 A_2$ produces $A_5 A_1$, which also comes from the type $A_5 A_2 A_1$ with a unique isomorphism class. Similarly, the case $2D_4$ produces $D_4 3A_1$, which also comes from the type $D_6 2A_1$. For $d \geq 3$, moreover, there is only one isomorphism class for each configuration of singularities. $\square$

2nd proof for the $d = 1$ case. By Theorem 1.5 and Remark 1.7, a general element of the linear system $|-K_Z|$ is smooth. By Riemann-Roch theorem, $h^0(-K_Z) = 2$. Hence, $|-K_Z|$ is a pencil with a unique, nonsingular base point P. The blowup of Z at P is an elliptic surface with a fibration $\pi : Z' \to \mathbb{P}^1$ and a section. The condition $\operatorname{rk} \operatorname{Pic} Z = 1$ implies that the minimal resolution of singularities $\widetilde{Z}' \to \mathbb{P}^1$ is an *extremal rational elliptic surface*, as defined in [MP86].

Vice versa, given an extremal relatively minimal (with no (-1)-curves in fibres of π) surface $\widetilde{Z}'$ with a section, contracting the (-2)-curves not meeting the section and then the section gives a Gorenstein del Pezzo surface with Du Val singularities and $\operatorname{rk} \operatorname{Pic} Z = 1$. The finitely many choices of a section differ by the action of the Mordell-Weil group of the elliptic fibration, and hence give isomorphic Z's.

Hence, the classification of Gorenstein log del Pezzo surfaces of degree 1 and rank 1 is equivalent to the classification of extremal rational elliptic fibrations with a section. The latter was done by Miranda and Persson in [MP86], and we just need to translate it to del Pezzo surfaces.

On the level of graphs of exceptional curves, the transition from $\widetilde{Z}$ to $\widetilde{Z}'$ consists of inserting an extra (-1)-curve and changing (-1)-curves through P to (-2)-curves. The graphs A_n, D_n, E_n turn into the corresponding extended Dynkin graphs $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_n$. In addition, A_1 and A_2 can turn into graphs $*\widetilde{A}_1$, $*\widetilde{A}_2$ respectively. The elliptic fibres $\widetilde{A}_0$ and $*\widetilde{A}_0$ are not seen in the graphs of Z.

According to [MP86, Thm 4.1] there are 16 types of elliptic fibrations. The four special types $*\widetilde{A}_0\widetilde{E}_8$, $*\widetilde{A}_1\widetilde{E}_7$, $*\widetilde{A}_2\widetilde{E}_6$ and $2\widetilde{D}_4$ are distinguished by the fact that the induced modular j-function $j : \mathbb{P}^1 \to \mathbb{P}^1_j$ is constant and there are exactly two singular fibres.

The subgroup $\operatorname{Aut}_j Y$ of automorphisms commuting with j is always finite. Hence, in the four exceptional cases $\operatorname{Aut} Y$ has dimension one and contains $\mathbb{C}^*$. In all other cases j-map is surjective, and hence the automorphism group is finite.

By [MP86, Thm 5.4], in fifteen of the sixteen cases the elliptic surface is unique. In the case $2\widetilde{D}_4$, there are infinitely many isomorphism classes, one for each value $j \in \mathbb{A}^1_j$. $\square$

If we consider log del Pezzo surfaces Z of index ≤ 2 and without Du Val singularities, we get an opposite case to the previous one, every singularity of Z must have index 2. This case includes and is surprisingly similar to the classical case of non-singular del Pezzo surfaces when there are no singularities at all. Applying Theorem 4.1 we get the following

Theorem 4.4. *Up to deformation, there exist exactly 50 types of log del Pezzo surfaces Z with singularities of index exactly 2 (if a singularity does exist). The DPN surfaces Y of their right resolution of singularities have empty Du Val part $\mathrm{Duv}\,\Gamma(Y)$, zero root invariants, and are defined by their main invariants (r, a, δ) (equivalently (k, g, δ)), up to deformation (the moduli are irreducible and connected). The diagram $\Gamma(Y)$ can be obtained from the diagram Γ of cases $1 - 50$ of Table 3 (with the same main invariants) as follows: $\Gamma(Y)$ consists of $\mathrm{Log}\,\Gamma(Y) = \mathrm{Log}\,\Gamma$ and*

$$(84) \qquad\qquad \mathrm{Var}\,\Gamma(Y) = W(\mathrm{Var}\,\Gamma)$$

where W is generated by reflections in all vertices of $\mathrm{Duv}\,\Gamma$ (i. e. one should take $D = \emptyset$ in Theorem 3.19). In cases 7, 8, 9, 10, 20 one can consider only diagrams Γ of cases 7a, 8a, 9a, 10a and 20a (diagrams 7b, 8b,c, 9b—f, 10b—m, 20b—d give the same).

The type of Dynkin diagram $\mathrm{Duv}\,\Gamma$ can be considered as analogous to the type of root system which one usually associates to non-singular del Pezzo surfaces. Its actual meaning is to give the type of the Weyl group W describing the varying part $\mathrm{Var}(\Gamma(Y))$ by (84). In cases $7 - 10$, 20, one should (or can) take graphs Γ of cases 7a—10a, 20a.

Proof. This case corresponds to Y with empty $D \subset \mathrm{Duv}\,\Gamma$ of Theorem 3.19. Then the root invariant is 0. Thus, all cases 7, 8, 9, 10 or 20 give the isomorphic root invariants and the same diagrams, and we can consider only the corresponding cases 7a, 8a, 9a, 10a and 20a to calculate the diagrams.

Let us show that moduli spaces of DPN surfaces Y with the same main invariants (r, a, δ) and zero root invariant (i. e. $D = 0$) are irreducible.

It is enough to show irreducibility of the moduli space of the corresponding right DPN pairs (Y, C) where $C \in |-2K_Y|$ is non-singular. Taking double covering $\pi : X \to Y$ ramified in C, it is enough to prove irreducibility of moduli $\mathrm{Mod}_{(r,a,\delta)}$ of K3 surfaces with non-symplectic involutions (X, θ) and $(S_X)_+ = S$ where S has invariants (r, a, δ). General such pairs have zero root invariant, as we want, because general (X, θ) have $S_X = S$. Irreducibility of $\mathrm{Mod}_{(r,a,\delta)}$ had been discussed in Section 2.3 with proofs in Appendix: Section A.2. $\qquad\square$

We remark that the result equivalent to Theorem 4.4 was first obtained in [Nik83].

We remark that cases $1 - 10$ of Theorem 4.4 give classical non-singular del Pezzo surfaces. Therefore, Theorem 4.4 and all results of this work show that log del Pezzo surfaces of index ≤ 2 are very similar to classical non-singular del Pezzo surfaces.

4.2. Example: Enumeration of all possible types for $N = 20$

Let us consider enumeration of all types of singularities and graphs of exceptional curves of log del Pezzo surfaces of index 2 of type $N = 20$, i. e. with the main invariants $(r, a, \delta) = (10, 8, 1)$.

From Theorem 4.1 and Table 3, cases 20a—d, we obtain that all of them have one singularity K_1 of index 2 and Du Val singularities which correspond to a subgraph of one of graphs D_8, $D_6 2A_1$, $D_5 A_3$ and $2D_4$. It follows that their Du Val singularities are exactly of one of 52 types listed below:

$2D_4$;

D_8;

D_7;

$D_6 2A_1$, $D_6 A_1$, D_6;

$D_5 A_3$, $D_5 A_2$, $D_5 2A_1$, $D_5 A_1$, D_5;

$D_4 A_3$, $D_4 A_2$, $D_4 3A_1$, $D_4 2A_1$, $D_4 A_1$, D_4;

A_7;

A_6;

$A_5 2A_1$, $A_5 A_1$, A_5;

$A_4 A_3$, $A_4 A_2$, $A_4 2A_1$, $A_4 A_1$, A_4;

$2A_3 A_1$, $2A_3$, $A_3 A_2 2A_1$, $A_3 A_2 A_1$, $A_3 A_2$, $A_3 4A_1$, $A_3 3A_1$, $A_3 2A_1$, $A_3 A_1$, A_3;

$2A_2 2A_1$, $2A_2 A_1$, $2A_2$, $A_2 4A_1$, $A_2 3A_1$, $A_2 2A_1$, $A_2 A_1$, A_2;

$6A_1$, $5A_1$, $4A_1$, $3A_1$; $2A_1$, A_1;

$\emptyset$.

Using calculations of root invariants of Lemma 3.12, it is easy to calculate the root invariant for any of the subgraphs. One can see that it is defined uniquely by the type of Du Val singularities except the following 15 types of Du Val parts of singularities for which we show all differences in their root invariants.

$D_4 A_3$: There are exactly *two possibilities* for the root invariant (and then for the dual graph of exceptional curves). The first one can be obtained by taking $D_4 A_3$ as a subdiagram in D_8 (case 20a), and the second by taking $D_4 A_3$ as a subdiagram in $D_5 A_3$ (case 20c). In the second case, the characteristic element can be written using elements of the component A_3, and this is impossible in the first case.

$D_4 2A_1$: There are exactly *two possibilities* for the root invariant. The first one can be obtained by taking $D_4 2A_1$ as a subdiagram in D_8 (case 20a), and the second one by taking $D_4 2A_1$ as a subdiagram in $D_6 2A_1$ (case 20b). In the second case, the characteristic element can be written using elements of the components $2A_1$, and it is impossible in the first case.

A_7: There are exactly *two possibilities* for the root invariant. The group $H = \{0\}$ or $H \cong \mathbb{Z}/2$. Both cases can be obtained by taking subdiagrams in D_8 (case 20a).

$A_5 A_1$: There are exactly *two possibilities* for the root invariant: the group $H = \{0\}$ or $H \cong \mathbb{Z}/2$. Both cases can be obtained by taking subdiagrams in D_8 (case 20a).

$2A_3$: There are exactly *four possibilities* for the root invariant. For the group $H = \{0\}$ the characteristic element can be written using elements either of one component A_3 or only by both components A_3. For the group $H \cong \mathbb{Z}/2$ either $\alpha = 1$ or $\alpha = 0$. Three of these cases can be obtained by taking subdiagrams in D_8 (case 20a). The remaining case $H \cong \mathbb{Z}/2$ and $\alpha = 0$ can be obtained by taking a subdiagram in $D_5 A_3$ (case 20c).

$A_3 A_2$: There are exactly *two possibilities* for the root invariant: $\alpha = 0$ or $\alpha = 1$. Both cases can be obtained by taking subdiagrams in D_8 (case 20a).

$A_3 3A_1$: There are exactly *two possibilities* for the root invariant: In the first case the characteristic element cannot be written using elements of the components A_3 (it can be obtained by taking a subdiagram in D_8, i. e. for the case 20a). For the second case it can be written using elements of the component A_3 (it can be obtained by taking a subdiagram in $D_6 2A_1$, i. e. for the case 20b).

$A_3 2A_1$: There are exactly *five possibilities* for the root invariant. For the group $H = \{0\}$ the characteristic element can be written using elements either of one component A_3, or by components $2A_1$, or using all three components $A_3 2A_1$. For the group $H \cong \mathbb{Z}/2$ either $\alpha = 1$ or $\alpha = 0$. Four of these cases can be obtained by taking subdiagrams in D_8 (case 20a). The remaining case $H \cong \mathbb{Z}/2$ and $\alpha = 0$ can be obtained considering a subdiagram in $D_6 2A_1$ (case 20b).

$A_3 A_1$: There are exactly *two possibilities* for the root invariant: $\alpha = 0$ or $\alpha = 1$. Both cases can be obtained by taking subdiagrams in D_8 (case 20a).

A_3: There are exactly *two possibilities* for the root invariant: $\alpha = 0$ or $\alpha = 1$. Both cases can be obtained by taking subdiagrams in D_8 (case 20a).

$A_2 2A_1$: There are exactly *two possibilities* for the root invariant: $\alpha = 0$ or $\alpha = 1$. Both cases can be obtained by taking subdiagrams in D_8 (case 20a).

$5A_1$: There are exactly *two possibilities* for the root invariant. For the first one the characteristic element can be written using two pairs of components of $5A_1$. For the second one the characteristic element can be written using only one pair of components of $5A_1$.

$4A_1$: There are exactly *four possibilities* for the root invariant. For the group $H = \{0\}$ the characteristic element can be written using elements

either of two components A_1 or by only four components A_1. For the group $H \cong \mathbb{Z}/2$ either $\alpha = 1$ or $\alpha = 0$. All four cases can be obtained by taking subdiagrams in $2D_4$ (case 20c).

$3A_1$: There are exactly *two possibilities* for the root invariant: $\alpha = 0$ or $\alpha = 1$. Both cases can be obtained by considering subdiagrams in D_8 (case 20a).

$2A_1$: There are exactly *two possibilities* for the root invariant: $\alpha = 0$ or $\alpha = 1$. Both cases can be obtained by considering subdiagrams in D_8 (case 20a).

Thus, for the types of Du Val singularities shown above (together with the singularity K_1 of index two) we obtain the number shown above of different types of log del Pezzo surfaces: their right resolution of singularities can have that number of different graphs of exceptional curves. By taking the corresponding sequence of contractions of -1 curves, one can further investigate these surfaces in details; in particular, one can enumerate irreducible components of their moduli.

Thus, there are exactly $52 + 12 + 3 \cdot 2 + 4 = 74$ different graphs of exceptional curves on the right resolution of singularities of log del Pezzo surfaces of index 2 with the main invariants $(r, a, \delta) = (10, 8, 1)$ (i. e. $N = 20$).

Of course, similar calculations can be done for all 50 types of main invariants of log del Pezzo surfaces of index ≤ 2. The considered case $N = 20$ is one of the richest and most complicated.

4.3. Application: Minimal projective compactifications of affine surfaces in $\mathbb{P}^2$ by relatively minimal log del Pezzo surfaces of index ≤ 2.

This is similar to [BBD84] in the Gorenstein case.

Let us consider one of the 45 relatively minimal surfaces of Theorems 4.2, 4.3 which are different from $\mathbb{P}^2$ (i. e. except the case 1). Let σ : $Y \to Z$ be its right resolution of singularities, and $v_1, \ldots, v_{r-1}$ a sequence of vertices of $\Gamma(Y)$ such that the corresponding exceptional curves on Y give a contraction of the sequence of curves of the 1st kind τ : $Y \to \mathbb{P}^2$. Let $C \subset \mathbb{P}^2$ be the union of images by τ of all exceptional curves E_v, $v \in V(\mathrm{Duv}(\Gamma(Y))) \cup V(\mathrm{Log}(\Gamma(Y)))$. Then, the embedding $f = (\sigma\tau^{-1})$: $W \to Z$ gives a compactification of the affine surface $W = \mathbb{P}^2 - C$ of $\mathbb{P}^2$. The morphism f is minimal in the sense that f cannot be extended through components of C (see [BBD84] for details). The description of all such affine surfaces W and such their compactifications is then reduced to the

description of subdiagrams of $\Gamma(Y)$ (defined by $v_1, \ldots, v_{r-1}$)) which were described by their connected components in Section 3.6.

4.4. Dimension of the moduli space

For each triple of invariants

$$(85) \qquad\qquad (k, \ g, \ \delta, \ \text{the root invariant}),$$

equivalently,

$$(86) \qquad (r, \ a, \ \delta, \ \text{the dual diagram of exceptional curves } \Gamma(Y))$$

one has the moduli space of pairs $\mathcal{M}_{(Z,C)}$ of log del Pezzo surfaces together with a smooth curve $C \in |-2K_Z|$. We have established the equivalence between pairs (Z, C) and K3 surfaces (X, θ) with a non-symplectic involution. Hence instead of moduli $\mathcal{M}_{(Z,C)}$ of pairs (Z, C) we can consider moduli $\mathcal{M}_{(X,\theta)}$ of pairs (X, θ).

By (63),

$$(87) \quad \dim \mathcal{M}_{(X,\theta)} = 20 - r - \#V(\mathrm{Duv}(\Gamma)) = 9 + g - k - \#V(\mathrm{Duv}(\Gamma)).$$

Moreover,

$$(88) \qquad\qquad \dim |-2K_Z| = \dim |C_g| = 3g - 3$$

(see Sections 1.4 and 1.5). It follows that the dimension of the parameter space $\mathcal{M}_{(r,a,\delta),\Gamma(Y)}$ of *generic* surfaces Z of type (r, a, δ) and with the graph $\Gamma(Y)$ of exceptional curves on the right resolution Y of singularities (or with the corresponding root invariant) is equal to

$$(89) \quad \begin{aligned} \dim \mathcal{M}_{(r,a,\delta),\Gamma(Y)} &= 12 - 2g - k - \#V(\mathrm{Duv}(\Gamma(Y))) + \dim \mathrm{Aut}\, Z = \\ & \frac{r+3a}{2} - 10 - \#V(\mathrm{Duv}(\Gamma(Y))) + \dim \mathrm{Aut}\, Z \end{aligned}$$

Note that this formula may fail for non-generic surfaces. For example, by Theorem 4.3 there are exactly two isomorphism classes of Gorenstein surfaces with a single E_8-singularity. The formula above gives

$$\dim \mathcal{M}_{(r,a,\delta),\Gamma(Y)} = \dim \mathrm{Aut}\, Z$$

which is true for the generic surface that has trivial isomorphism group and fails for the second surface which has $\mathrm{Aut}\, Z = \mathbb{C}^*$.

4.5. Some open questions

4.5.1. Finite characteristic

It would be very interesting to generalize results of this work to finite characteristic. As we had mentioned in Remark 3.16, it seems, the main problem is to generalize Theorem 1.5. We think that our results are valid in characteristic ≥ 3. As we have seen, in characteristic 2 the number of cases increases.

4.5.2. Arithmetic of log del Pezzo surfaces of index ≤ 2

There are many results (e.g. see [Man86], [MT86] and [CT88]) where the arithmetic of classical non-singular del Pezzo surfaces is studied. What is the arithmetic of log del Pezzo surfaces of index ≤ 2 and equivalent DPN surfaces of elliptic type?