CHAPTER V

Numerical Kodaira dimension

We give a criterion for an $\mathbb{R}$-divisor to be pseudo-effective in §**1** by applying the Kawamata–Viehweg vanishing theorem. In §**2**, we introduce two invariants, denoted by $\kappa_\sigma(D)$ and $\kappa_\nu(D)$, respectively, both of which seem to be the candidates and deserve to be called the numerical D-dimension for a pseudo-effective divisor D. Both invariants have many properties expected for numerical D-dimension, which we prove using the results in §**1**. In §**3**, we introduce the notion of ω-sheaves, which is useful for the study of direct images of relative pluricanonical sheaves. The notion of weak positivity introduced by Viehweg is refined also in §**3**. We prove some addition theorems for κ and κ_σ and for log-terminal pairs in §**4**. These are slight generalizations of Viehweg's results in [**147**]. In the last part of §**4**, we prove the abundance theorem in a special case where $\kappa_\sigma = 0$, as an application of the addition theorems.

§1. Pseudo-effective $\mathbb{R}$-divisors

§1.a. Base-point freeness.

1.1. Lemma *Let Δ and D be effective $\mathbb{R}$-divisors without common prime components on a normal variety X and let x be a point of X.*

(1) *If (X, bD) and $(X, b/(b-1)\Delta)$ are log-terminal at x for some $b > 1$, then $(X, D + \Delta)$ is log-terminal at x.*
(2) *Suppose that X is non-singular at x and $\operatorname{mult}_x \Delta < 1$. Then (X, Δ) is log-terminal at x.*
(3) *Suppose that X is non-singular at x, (X, bD) is log-terminal at x, and $\operatorname{mult}_x \Delta < (b-1)/b$ for some $b > 1$. Then $(X, \Delta + D)$ is log-terminal at x.*

Proof. (1) Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety such that the union of the exceptional locus $G = \sum G_i$, the proper transform D_Y of D, and the proper transform Δ_Y of Δ is a simple normal crossing divisor. Then we can write

$$K_Y = f^*(K_X + bD) + \sum a_i G_i - bD_Y = f^*\left(K_X + \frac{b}{b-1}\Delta\right) + \sum c_i G_i - \frac{b}{b-1}\Delta_Y$$

for real numbers a_i, c_i. If $x \in f(G_i)$, then a_i, $c_i > -1$. Furthermore, $\llcorner bD_Y \lrcorner = \llcorner (b/(b-1))\Delta_Y \lrcorner = 0$ over a neighborhood of x. Since $1/b + (b-1)/b = 1$, we have

$$K_Y = f^*(K_X + D + \Delta) + \sum \frac{a_i + (b-1)c_i}{b} G_i - D_Y - \Delta_Y.$$

Thus $(X, D + \Delta)$ is log-terminal at x.

(2) Suppose that the bimeromorphic morphism $f\colon Y \to X$ in the proof of (1) is a succession of blowups

$$Y := X_l \xrightarrow{\mu_l} X_{l-1} \to \cdots \to X_1 \xrightarrow{\mu_1} X_0 := X$$

along non-singular centers $W_k \subset X_{k-1}$. Let Δ_k be the proper transform of Δ in X_k and set $w_k := \operatorname{codim} W_k$, $E_k := \mu_k^{-1}(W_k)$, and $r_k := \operatorname{mult}_{W_k} \Delta_{k-1}$. We may assume that the image of W_k in X contains x and that $r_k \leq \operatorname{mult}_x \Delta < 1$ by replacing X with an open neighborhood of x. Then

$$K_{X_k} = \mu_k^*(K_{X_{k-1}} + \Delta_{k-1}) + (w_k - 1 - r_k)E_k - \Delta_k$$

where $w_k - 1 - r_k \geq 1 - r_k > 0$. Therefore,

$$K_Y = f^*(K_X + \Delta) + \sum\nolimits_{k=1}^{l} (w_k - 1 - r_k)\phi_k^* E_k - \Delta_Y,$$

where ϕ_k is the composite $Y = X_l \to X_k$ and $\Delta_Y = \Delta_l$. Thus (X, Δ) is log-terminal at x.

(3) follows from (1) and (2). □

1.2. Proposition *Let x be a point of an n-dimensional non-singular projective variety X and let Δ be an effective $\mathbb{R}$-divisor such that (X, Δ) is log-terminal at x. Let E_x be the exceptional divisor for the blowing-up $\rho_x\colon Z \to X$ at x and let L be a $\mathbb{Z}$-divisor of X. If $\rho_x^*(L - (K_X + \Delta)) - nE_x$ is ample, then $x \notin \operatorname{Bs}|L|$.*

Proof. For the proper transform Δ_Z of Δ in Z, we have

$$K_Z = \rho_x^*(K_X + \Delta) + (n - 1 - \operatorname{mult}_x \Delta)E_x - \Delta_Z.$$

There exists a birational morphism $\mu\colon Y \to Z$ from a non-singular projective variety such that the union E of the exceptional locus for $f := \rho_x \circ \mu\colon Y \to X$ and $f^{-1}(\operatorname{Supp} \Delta)$ is a simple normal crossing divisor. Let $E = \sum_{i=0}^{l} E_i$ be the prime decomposition in which E_0 is the proper transform of E_x. By comparing K_Y with $K_X + \Delta$, we have real numbers a_i for $0 \leq i \leq l$ such that

$$K_Y = f^*(K_X + \Delta) + \sum\nolimits_{i=0}^{l} a_i E_i.$$

Here $a_0 = n - 1 - \operatorname{mult}_x \Delta$. If $x \in f(E_i)$, then $a_i > -1$, since (X, Δ) is log-terminal at x. Now the $\mathbb{R}$-divisor

$$f^*L + \sum\nolimits_{i=0}^{l} a_i E_i - n\mu^* E_x - K_Y$$

is nef and big. We define

$$R := \sum\nolimits_{i=0}^{l} r_i E_i := \sum\nolimits_{i=0}^{l} a_i E_i - n\mu^* E_x.$$

Then $r_0 = -1 - \operatorname{mult}_x \Delta \le -1$. If $x \in f(E_i)$ and if $\{x\} \ne f(E_i)$, then $r_i > -1$. Hence there exist ideal sheaves $\mathcal{J}_0, \mathcal{J}_1$ of $\mathcal{O}_Z$ such that

(1) $\mu_* \mathcal{O}_Y(\ulcorner R \urcorner) = \mathcal{J}_0 \cap \mathcal{J}_1 \subset \mathcal{O}_Z$,
(2) $\operatorname{Supp} \mathcal{O}_Z/\mathcal{J}_0 = E_x$ and $E_x \cap \operatorname{Supp} \mathcal{O}_Z/\mathcal{J}_1 = \emptyset$.

Thus $\mathcal{J} := f_*\mathcal{O}_Y(\ulcorner R \urcorner)$ is an ideal sheaf of $\mathcal{O}_X$ and x is an isolated point of $\operatorname{Supp} \mathcal{O}_X/\mathcal{J}$. On the other hand,

$$\mathrm{H}^1(X, f_*\mathcal{O}_Y(\ulcorner R \urcorner) \otimes \mathcal{O}_X(L)) = 0,$$

by the vanishing theorem **II.5.9**. Therefore, the composite

$$\mathrm{H}^0(X, \mathcal{O}_X(L)) \to \mathrm{H}^0(X, \mathcal{O}_X(L) \otimes (\mathcal{O}_X/\mathcal{J})) \to \mathcal{O}_X(L) \otimes \mathbb{C}(x)$$

is surjective and hence $x \notin \operatorname{Bs}|L|$. □

1.3. Theorem *Let D be a pseudo-effective $\mathbb{R}$-divisor of a non-singular projective variety X. Then there exists an ample divisor A such that*

$$x \notin \operatorname{Bs}\left|\ulcorner tD \urcorner + A\right| \cup \operatorname{Bs}|\llcorner tD \lrcorner + A|$$

for any $t \in \mathbb{R}_{>0}$ and for any point $x \in X$ with $\sigma_x(D) = 0$.

Proof. Let $\rho\colon Z \to X$ be the blowing-up at a point x with $\sigma_x(D) = 0$ and let E_x be the exceptional divisor. If H is a very ample divisor of X, then $|\rho^*H - E_x|$ is base point free. Therefore $\rho^*(kH) - nE_x$ is ample for $k > n := \dim X$. We fix a number $0 < \alpha < 1$. Then we can take an ample divisor A such that

$$\rho^*((1-\alpha)A - K_X + \langle -tD \rangle) - nE_x \quad \text{and} \quad \rho^*((1-\alpha)A - K_X - \langle tD \rangle) - nE_x$$

are both ample for any $t > 0$ and for any $x \in X$, since $\{c_1(\langle tD \rangle)\}$ is bounded in $\mathrm{N}^1(X)$. Then, for any $t > 0$, there exists a member $\Delta \in |tD + \alpha A|_{\text{num}}$ with $\operatorname{mult}_x \Delta < 1$, since $\sigma_x(D) = 0$. Here (X, Δ) is log-terminal at x by **1.1**. We set $L_1 := \ulcorner tD \urcorner + A$ and $L_2 := \llcorner tD \lrcorner + A$. Then

$$\rho^*(L_1 - (K_X + \Delta)) - nE_x \quad \text{and} \quad \rho^*(L_2 - (K_X + \Delta)) - nE_x$$

are both ample by

$$\begin{aligned}
\rho^*(L_1 - (K_X + \Delta)) - nE_x &\approx \rho^*((1-\alpha)A - K_X + \langle -tD \rangle) - nE_x, \\
\rho^*(L_2 - (K_X + \Delta)) - nE_x &\approx \rho^*((1-\alpha)A - K_X - \langle tD \rangle) - nE_x.
\end{aligned}$$

Therefore, $x \notin \operatorname{Bs}|L_1| \cup \operatorname{Bs}|L_2|$ by **1.2**. □

1.4. Corollary *Let A be an ample divisor of X such that $A - K_X - (\dim X)H$ is ample for some very ample divisor H. Then the following two conditions are equivalent for an $\mathbb{R}$-divisor D of X:*

(1) *D is pseudo-effective*;
(2) $\mathrm{h}^0(X, \ulcorner tD \urcorner + A) \ne 0$ *for any $t \in \mathbb{R}_{>0}$*.

PROOF. It is enough to show (1) $\Rightarrow$ (2). In the proof of **1.3**, we choose a point $x \in X$ with $\sigma_x(D) = 0$ and $x \notin \mathrm{Supp}\langle D \rangle$, and choose a number $0 < \alpha < 1$ with $\rho^*((1-\alpha)A - K_X) - nE_x$ being ample. Let us fix $t > 0$ and choose a member $\Delta \in |tD + \alpha A|_{\mathrm{num}}$ with $\mathrm{mult}_x \Delta < 1$. We set $L_1 = \ulcorner tD \urcorner + A$. Then

$$\rho^*(L_1 - (K_X + \Delta + \langle -tD \rangle)) - nE_x \approx \rho^*((1-\alpha)A - K_X) - nE_x$$

is ample. Here $(X, \Delta + \langle -tD \rangle)$ is log-terminal at x by **1.1**. Thus $x \notin \mathrm{Bs}\,|L_1|$ by **1.2**. In particular, $\mathrm{H}^0(X, L_1) \neq 0$. $\square$

We have the following generalization of **III.1.7**-(3):

1.5. Corollary *Suppose that $\sigma_x(D) = 0$ for a pseudo-effective $\mathbb{R}$-divisor D and a point $x \in X$. Then, for any ample $\mathbb{R}$-divisor A, there is an effective $\mathbb{R}$-divisor Δ such that $\Delta \approx D + A$ and $x \notin \mathrm{Supp}\,\Delta$.*

Recall that the numerical base locus $\mathrm{NBs}(D)$ is the set of points with $\sigma_x(D) > 0$. This is a countable union of proper subvarieties. In fact,

$$\mathrm{NBs}(D) = \bigcup\nolimits_{m>0} \mathrm{Bs}\left| \ulcorner mD \urcorner + A \right|$$

by **1.3**. If $N_\sigma(D) = 0$, then $\mathrm{codim}\,\mathrm{NBs}(D) \geq 2$. If $\mathrm{NBs}(P_\sigma(D))$ is not a Zariski-closed subset, then D admits no Zariski-decompositions.

1.6. Corollary *The numerical base locus* $\mathrm{NBs}(D)$ *has no isolated points*: *if $\sigma_x(D) > 0$, then there is a curve $\gamma \subset \mathrm{NBs}(D)$ passing through x.*

PROOF. Assume that x is an isolated point of $\mathrm{NBs}(D)$. Since $\mathrm{NBs}(D)$ depends only on the Chern class $c_1(D)$, we may assume that $\mathrm{Supp}\langle D \rangle \not\ni x$. By **1.3**, x is also an isolated point of $\mathrm{Bs}\,|\llcorner mD \lrcorner + A|$ for an ample divisor A and for infinitely many $m \in \mathbb{N}$. By [**151**], for such m, there exists $k \in \mathbb{N}$ with $x \notin \mathrm{Bs}\,|k(\llcorner mD \lrcorner + A)|$. Since $k(mD + A) = k(\llcorner mD \lrcorner + A) + k\langle mD \rangle$, we have $\sigma_x(mD + A) = 0$. This is a contradiction. $\square$

1.7. Corollary *Let Γ be a prime divisor.*

(1) *For a pseudo-effective $\mathbb{R}$-divisor D, there is an ample divisor A such that $\sigma_\Gamma(tD + A)_\mathbb{Z} \leq t\sigma_\Gamma(D)$ for any $t \in \mathbb{R}_{>0}$ and*

$$\lim_{t\to\infty} \frac{1}{t}\sigma_\Gamma(tD + A)_\mathbb{Z} = \sigma_\Gamma(D).$$

(2) *If B is a big $\mathbb{R}$-divisor, then $\sigma_\Gamma(tB)_\mathbb{Z} - t\sigma_\Gamma(B)$ is bounded for $t > 0$.*

PROOF. (1) By **1.3**, there is an ample divisor A such that $\sigma_\Gamma(tP_\sigma(D) + A)_\mathbb{Z} = 0$ for any $t > 0$. Therefore $\sigma_\Gamma(tD + A)_\mathbb{Z} \leq t\,\mathrm{mult}_\Gamma N_\sigma(D) = t\sigma_\Gamma(D)$. Furthermore,

$$\frac{1}{tk}\sigma_\Gamma(tkD + A)_\mathbb{Z} \geq \frac{1}{tk}\sigma_\Gamma(t(kD + A))_\mathbb{Z}.$$

Therefore

$$\lim_{t\to\infty} \frac{1}{t}\sigma_\Gamma(tD + A)_\mathbb{Z} \geq \varlimsup_{k\to\infty} \sigma_\Gamma(D + (1/k)A) = \sigma_\Gamma(D).$$

(2) By (1), we have an ample divisor A with $\sigma_\Gamma(tB+A)_\mathbb{Z} \le t\sigma_\Gamma(B)$. Since B is big, there exist a positive integer k and an effective $\mathbb{R}$-divisor Δ such that $kB \sim A + \Delta$. Therefore, for $t > k$,

$$\sigma_\Gamma(tB)_\mathbb{Z} \le \sigma_\Gamma((t-k)B + A)_\mathbb{Z} + \mathrm{mult}_\Gamma \Delta \le (t-k)\sigma_\Gamma(B) + \mathrm{mult}_\Gamma \Delta. \qquad \square$$

Remark The author was informed **1.7**-(2) from H. Tsuji, who seemed to have similar results to **1.2** and **1.3** by applying some L^2-vanishing theorem.

1.8. Problem Let D be a pseudo-effective $\mathbb{R}$-divisor, A an ample divisor, and Γ a prime divisor. Then is $t\sigma_\Gamma(D) - \sigma_\Gamma(tD+A)$ bounded for $t > 0$?

Let B be a big $\mathbb{R}$-divisor of X. The set

$$\mathrm{SBs}(B) := \bigcap\nolimits_{\Delta \in |B|_\mathbb{Q}} \mathrm{Supp}\,\Delta$$

is called the *stable base locus* of B. Since $|B|_\mathbb{Q}$ is the set of effective $\mathbb{R}$-divisors $\mathbb{Q}$-linearly equivalent to B, we have

$$\mathrm{SBs}(B) = \bigcap\nolimits_{m=1}^\infty \mathrm{Bs}\,|mB| = \bigcup\nolimits_{\mathrm{mult}_\Gamma B \notin \mathbb{Q}} \Gamma \cup \bigcap\nolimits_{m=1}^\infty \mathrm{Bs}\,\big|\llcorner mB \lrcorner\big|\,.$$

We introduce the following $\mathbb{R}$-version of the stable base locus:

$$\mathrm{SBs}(B)_\mathbb{R} := \bigcap\nolimits_{\Delta \in |B|_\mathbb{R}} \mathrm{Supp}\,\Delta.$$

Note that $\mathrm{SBs}(B)$ and $\mathrm{SBs}(B)_\mathbb{R}$ are Zariski-closed subsets of X containing $\mathrm{NBs}(B)$. For an ample $\mathbb{R}$-divisor A, let us consider the set

$$\mathbb{G}(B, A) := \{t \in \mathbb{R} \mid B + tA \text{ is big and } \mathrm{NBs}(B+tA) \ne \mathrm{SBs}(B+tA)_\mathbb{R}\}.$$

1.9. Lemma

(1) $\mathrm{NBs}(B) = \bigcup_{t>0} \mathrm{SBs}(B+tA)_\mathbb{R}$.
(2) *If $B + tA$ is big, then $(t-\varepsilon, t) \cap \mathbb{G}(B,A) = \emptyset$ for some $\varepsilon > 0$.*
(3) *If $t \in \mathbb{G}(B,A)$ and if $\mathrm{NBs}(B+tA)$ is a Zariski-closed subset, then t is an isolated point of $\mathbb{G}(B,A)$.*

Proof. (1) If $x \in \mathrm{NBs}(B)$, i.e., $\sigma_x(B) > 0$, then $\sigma_x(B+tA) > 0$ for some $t > 0$. Thus $x \in \mathrm{SBs}(B+tA)_\mathbb{R}$. Suppose that $x \notin \mathrm{NBs}(B)$. Then $x \notin \mathrm{SBs}(B+qH)$ for any $q \in \mathbb{Q}_{>0}$ and for an ample $\mathbb{Q}$-divisor H, by **1.3**. For any $t \in \mathbb{R}_{>0}$, we can find $q \in \mathbb{Q}_{>0}$ such that $tA - qH$ is ample. Thus

$$\mathrm{SBs}(B+tA)_\mathbb{R} \subset \mathrm{SBs}(B+qH)_\mathbb{R} \cup \mathrm{SBs}(tA-qH)_\mathbb{R} \subset \mathrm{SBs}(B+qH).$$

Hence $x \notin \mathrm{SBs}(B+tA)_\mathbb{R}$ for $t > 0$.

(2) We consider a sequence $\{\mathrm{SBs}(B+t'A)_\mathbb{R}\}$ of Zariski-closed subsets. If $t_1 < t_2$ and $B + t_1A$ is big, then $\mathrm{SBs}(B+t_1A)_\mathbb{R} \supset \mathrm{SBs}(B+t_2A)_\mathbb{R}$. By the Noetherian condition, we have

$$\bigcap\nolimits_{t>t'} \mathrm{SBs}(B+t'A)_\mathbb{R} = \mathrm{SBs}(B+t_0A)_\mathbb{R}$$

for some $t_0 < t$. Then $\mathrm{SBs}(B+t'A)_\mathbb{R} = \mathrm{NBs}(B+t'A)_\mathbb{R} = \mathrm{SBs}(B+t_0A)_\mathbb{R}$ for $t > t' \ge t_0$.

(3) If $\mathrm{NBs}(B)$ is Zariski-closed, then $\mathrm{NBs}(B) = \mathrm{SBs}(B + t_1 A)_{\mathbb{R}}$ for some $t_1 > 0$ by (1). Hence $\mathrm{SBs}(B + tA)_{\mathbb{R}} = \mathrm{SBs}(B + t_1 A)_{\mathbb{R}}$ for $0 < t < t_1$. Thus the assertion follows from (2). □

Therefore, $\mathrm{NBs}(B)$ is Zariski-closed for 'almost all' big $\mathbb{R}$-divisors B. Note that if t is an accumulation point of $\mathbb{G}(B, A)$, then $B + tA$ admits no Zariski-decomposition.

§1.b. Restriction to general subvarieties. We shall generalize the argument of **1.2**.

1.10. Proposition *Let C be a non-singular projective curve of a non-singular projective variety X of dimension n and let Δ be an effective $\mathbb{R}$-divisor such that (X, Δ) is log-terminal around C and $C \not\subset \operatorname{Supp}\Delta$. Let E_C be the exceptional divisor for the blowing-up $\rho\colon Z \to X$ along C and let L be a $\mathbb{Z}$-divisor of X. If*

$$\rho^*(L - (K_X + \Delta)) - (n-1)E_C$$

is ample, then the restriction homomorphism $\mathrm{H}^0(X, L) \to \mathrm{H}^0(C, L|_C)$ is surjective.

Proof. The proof is similar to that of **1.2**. We have

$$K_Z = \rho^*(K_X + \Delta) + (n-2)E_C - \Delta_Z$$

for the proper transform Δ_Z of Δ. We can take a birational morphism $\mu\colon Y \to Z$ from a non-singular projective variety and a normal crossing divisor $E = \sum E_i$ of Y as the union of the exceptional locus for $f := \rho \circ \mu\colon Y \to X$ and $\operatorname{Supp}(f^*\Delta)$. We may assume that f is an isomorphism over general points of C. Then

$$K_Y = f^*(K_X + \Delta) + \sum a_i E_i,$$

for $a_i \in \mathbb{R}$. If $f(E_i) \cap C \neq \emptyset$, then $a_i > -1$. Now the $\mathbb{R}$-divisor

$$f^*L + \sum a_i E_i - (n-1)\mu^* E_C - K_Y$$

is nef and big. We set $R := \sum r_i E_i = \sum a_i E_i - (n-1)\mu^* E_C$. Then $r_i > -1$ if $f(E_i) \cap C \neq \emptyset$ and $f(E_i) \not\subset C$. Let E_0 be the proper transform of E_C. Then $r_0 = -1$. Therefore $\mu_* \mathcal{O}_Y(\ulcorner R \urcorner) = \mathcal{J}_0 \cap \mathcal{J}_1$ for suitable ideal sheaves $\mathcal{J}_0$ and $\mathcal{J}_1$ such that

(1) $\operatorname{Supp} \mathcal{O}_Z/\mathcal{J}_0 \cap \operatorname{Supp} \mathcal{O}_Z/\mathcal{J}_1 = \emptyset$,
(2) $\operatorname{Supp} \mathcal{O}_Z/\mathcal{J}_0 = E_C$,
(3) $\operatorname{Supp} \mathcal{O}_Z(-E_C)/\mathcal{J}_0$ does not dominate C.

Thus $\mathcal{I}_C/\rho_*\mathcal{J}_0$ is a skyscraper sheaf for the defining ideal $\mathcal{I}_C$ of C. The vanishing theorem **II.5.9** implies $\mathrm{H}^1(X, f_*\mathcal{O}_Y(\ulcorner R \urcorner) \otimes \mathcal{O}_X(L)) = 0$. Thus

$$\mathrm{H}^0(X, \mathcal{O}_X(L)) \to \mathrm{H}^0(X, \mathcal{O}_X(L) \otimes \mathcal{O}_X/\rho_*\mathcal{J}_0)$$

is surjective. Hence $\mathrm{H}^0(X, L) \to \mathrm{H}^0(C, L|_C)$ is surjective by $\mathrm{H}^1(X, \mathcal{I}_C/\rho_*\mathcal{J}_0) = 0$. □

1.11. Theorem *Let D be a pseudo-effective $\mathbb{R}$-divisor of a non-singular projective variety X. Suppose that $D \not\approx 0$ and $N_\sigma(D) = 0$. Then there exist an ample divisor A and a positive number β such that* $\mathrm{h}^0(X, \llcorner mD \lrcorner + A) > \beta m$ *for $m \gg 0$.*

Proof. $\mathrm{NBs}(D)$ is a countable union of subvarieties of codimension greater than one. Thus there is a non-singular curve C as a complete intersection of non-singular ample divisors such that $C \cap \mathrm{NBs}(D) = \emptyset$ and $C \not\subset \mathrm{Supp}\langle D\rangle$. Let $\rho_C : Z \to X$ be the blowing-up along C and let E_C be the exceptional divisor. We fix a number $0 < \alpha < 1$. Then we can find an ample divisor A such that the $\mathbb{R}$-divisor

$$\rho_C^*((1-\alpha)A - K_X - \langle mD\rangle) - (n-1)E_C$$

is ample for any $m \in \mathbb{N}$. We set $L_m := \llcorner mD \lrcorner + A$ for $m \in \mathbb{N}$. Since $\sigma_x(D) = 0$ for $x \in C$, there exists an effective $\mathbb{R}$-divisor $\Delta_m \sim_{\mathbb{R}} mD + \alpha A$ such that (X, Δ_m) is log-terminal around C and $C \not\subset \mathrm{Supp}(\Delta_m)$. The $\mathbb{R}$-linear equivalence

$$L_m - (K_X + \Delta_m) \sim_{\mathbb{R}} (1-\alpha)A - K_X - \langle mD\rangle$$

implies that

$$\rho_C^*(L_m - (K_X + \Delta_m)) - (n-1)E_C$$

is ample. Thus, by **1.10**, the restriction homomorphism

$$\mathrm{H}^0(X, \llcorner mD \lrcorner + A) \to \mathrm{H}^0(C, (\llcorner mD \lrcorner + A)|_C)$$

is surjective for any $m \in \mathbb{N}$. Note that $D \cdot C > 0$ since $D \not\approx 0$. Hence there is a positive number β such that

$$\mathrm{h}^0(X, \llcorner mD \lrcorner + A) \geq \mathrm{h}^0(C, (\llcorner mD \lrcorner + A)|_C) \geq \beta m \quad \text{for } m \gg 0. \qquad \square$$

1.12. Corollary *Let D be a pseudo-effective $\mathbb{R}$-divisor. Then the following three conditions are equivalent*:

(1) $D \approx N_\sigma(D)$;
(2) *For any ample divisor A, the function $t \mapsto \mathrm{h}^0(X, \llcorner tD \lrcorner + A)$ is bounded*;
(3) *For any ample divisor A,* $\lim_{t\to\infty}(1/t)\,\mathrm{h}^0(X, \llcorner tD \lrcorner + A) = 0$.

Proof. The implication (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) follows from **1.11**. We shall show (1) $\Rightarrow$ (2). Now $P := P_\sigma(D)$ is numerically trivial. By the argument of **1.3**, there is an ample divisor A' such that $|A' - A - \ulcorner tP \urcorner| \neq \emptyset$ for any $t > 0$. Thus $\mathrm{h}^0(X, \llcorner tD \lrcorner + A) \leq \mathrm{h}^0(X, \llcorner tN \lrcorner + A')$ for $N := N_\sigma(D)$. Hence we may assume $D = N$. There is a number $k \in \mathbb{R}_{>0}$ such that $\sigma_\Gamma(kN + A) > 0$ for any prime component Γ of N. Thus $\sigma_\Gamma(tN + A) > (t-k)\sigma_\Gamma(N)$ for $t > k$ by **III.1.9**. Hence $\mathrm{h}^0(X, \llcorner tN \lrcorner + A) = \mathrm{h}^0(X, \llcorner kN \lrcorner + A)$ for $t > k$. $\square$

The following result is a partial generalization of **1.10**:

1.13. Proposition *Let $W \subset X$ be a non-singular subvariety of a non-singular projective variety X and let Δ be an effective $\mathbb{R}$-divisor such that (X, Δ) is log-terminal around W and $W \not\subset \mathrm{Supp}\,\Delta$. Let E_W be the exceptional divisor for the* *blowing-up* $\rho\colon Z \to X$ *along* W *and let* L *be a* $\mathbb{Z}$*-divisor of* X*. Suppose further that* $(Z, E_W + \rho^*\Delta)$ *is log-canonical around* E_W *and*

$$\rho^*(L - (K_X + \Delta)) - (\operatorname{codim} W)E_W$$

is ample. Then $\mathrm{H}^0(X, L) \to \mathrm{H}^0(W, L|_W)$ *is surjective.*

Proof. Now $\Delta_Z := \rho^*\Delta$ is the proper transform of Δ. Thus

$$K_Z = \rho^*(K_X + \Delta) + (\operatorname{codim} W - 1)E_W - \Delta_Z.$$

Let us take a birational morphism $\mu\colon Y \to Z$ and let $f\colon Y \to X$ be the composite $\rho \circ \mu$. We may assume that Y is a non-singular projective variety and that there is a normal crossing divisor $E = \sum_{i=0}^k E_i$ satisfying the following conditions:

(1) E_0 is the proper transform of E_W in Y;
(2) $K_Y = f^*(K_X + \Delta) + \sum_{i=0}^k a_i E_i$ for some $a_i \in \mathbb{R}$;
(3) If $f(E_i) \cap W \neq \emptyset$, then $a_i > -1$.

We look at the $\mathbb{R}$-divisor

$$R := \sum\nolimits_{i=0}^k r_i E_i := \sum\nolimits_{i=0}^k a_i E_i - (\operatorname{codim} W)\mu^* E_W.$$

Then $f^*L + R - K_Y$ is nef and big. If $r_i > 0$, then E_i is μ-exceptional. If $f(E_i) \cap W \neq \emptyset$ and if $f(E_i) \not\subset W$, then $r_i = a_i > -1$. If $f(E_i) \subset W$, then $r_i \geq -1$, since $(Z, E_W + \Delta_Z)$ is log-canonical around E_W. Obviously, $r_0 = -1$. For the set

$$I := \{0 \leq i \leq k \mid r_i = -1 \text{ and } f(E_i) \cap W \neq \emptyset\},$$

we have

$$\sum\nolimits_{i \in I} r_i E_i \geq -\mu^* E_W.$$

Thus $\mu_*\mathcal{O}_Y(\ulcorner R \urcorner) = \mathcal{O}_Z(-E_W) \cap \mathcal{J}_1$ for an ideal sheaf $\mathcal{J}_1$ with $E_W \cap \operatorname{Supp} \mathcal{O}_Z/\mathcal{J}_1 = \emptyset$. Therefore,

$$f_*\mathcal{O}_Y(\ulcorner R \urcorner) \simeq I_W \cap \rho_*\mathcal{J}_1.$$

By the vanishing theorem **II.5.9**, we have

$$\mathrm{H}^1(X, f_*\mathcal{O}_Y(\ulcorner R \urcorner) \otimes \mathcal{O}_X(L)) = 0.$$

Thus $\mathrm{H}^0(X, L) \to \mathrm{H}^0(W, L|_W)$ is surjective, since $W \cap \operatorname{Supp} \mathcal{O}_X/\rho_*\mathcal{J}_1 = \emptyset$. □

The following result is a partial generalization of **1.11**:

1.14. Proposition *Let* X *be a non-singular projective variety,* D *a pseudo-effective* $\mathbb{R}$*-divisor, and let* $W \subset X$ *a non-singular subvariety. Assume that*

(1) $\mathrm{NBs}(D) \cap W = \emptyset$,
(2) $W \not\subset \operatorname{Supp}\langle D \rangle$,
(3) $\operatorname{Supp}\langle D \rangle$ *is normal crossing over a neighborhood of* W,
(4) *locally on a neighborhood of* W*, every non-empty intersection of irreducible components of* $\operatorname{Supp}\langle D \rangle$ *intersects* W *transversely.*

Then there exists an ample divisor A such that the restriction homomorphism

$$\mathrm{H}^0(X, \llcorner tD \lrcorner + A) \to \mathrm{H}^0(W, (\llcorner tD \lrcorner + A)|_W)$$

is surjective for any $t \in \mathbb{R}_{>0}$.

Proof. By **1.3**, there is an ample divisor H of X such that $W \cap \mathrm{Bs}\,|\llcorner tD \lrcorner + H| = \emptyset$ for any $t > 0$. For a number $0 < \varepsilon \ll 1$, we choose a general member F of $|\llcorner (t/\varepsilon)D \lrcorner + H|$. Then, for the $\mathbb{R}$-divisor $\Delta = \varepsilon F + \varepsilon \langle (t/\varepsilon)D \rangle$, we have

- $\Delta \sim_{\mathbb{R}} tD + \varepsilon H$,
- $W \not\subset \operatorname{Supp} \Delta$,
- (X, Δ) is log-terminal around W,

by **1.1**. Let $\rho\colon Z \to X$ be the blowing-up along W and let E_W be the exceptional divisor. By construction, $\rho^* F + (\rho^* \langle D \rangle)_{\mathrm{red}} + E_W$ is a normal crossing divisor around E_W. Hence $(Z, E_W + \rho^* \Delta)$ is log-canonical around E_W. Let us consider $L := \llcorner tD \lrcorner + A$ for an ample divisor A with $\rho^*(A - \varepsilon H - K_X - \langle tD \rangle) - (\operatorname{codim} W) E_W$ being ample. Then $\rho^*(L - (K_X + \Delta)) - (\operatorname{codim} W) E_W$ is ample. Thus, by **1.13**, we have the surjection $\mathrm{H}^0(X, L) \twoheadrightarrow \mathrm{H}^0(W, L|_W)$. □

§2. Numerical D-dimensions

§2.a. Numerical D-dimensions for nef $\mathbb{R}$-divisors. We recall an invariant $\nu(D) = \nu(D, X)$ called the *numerical D-dimension* defined for a nef $\mathbb{R}$-divisor D of an n-dimensional non-singular projective variety X. The Chern class $c_1(D)$ is considered as an element of $\mathrm{H}^{1,1}(X, \mathbb{R}) = \mathrm{H}^2(X, \mathbb{R}) \cap \mathrm{H}^{1,1}(X)$. Suppose that

$$D^k \cdot A^{n-k} = c_1(D)^k \cup c_1(A)^{n-k}[X] = 0$$

for an integer $k \geq 1$ and for an ample divisor A. Then $c_1(D)^k \in \mathrm{H}^{k,k}(X, \mathbb{R})$ is zero by **II.6.3**. The invariant $\nu(D)$ is defined to be the largest integer $k \geq 0$ with $c_1(D)^k \neq 0$ in $\mathrm{H}^{k,k}(X, \mathbb{R})$. This is also the largest integer k with $D^k \cdot A^{n-k} \neq 0$ for an ample divisor A. For a nef $\mathbb{R}$-Cartier divisor of a projective variety, its ν is defined by the pullback to a desingularization.

Remark Let $\pi\colon X \to S$ be a flat projective surjective morphism of varieties and let D be a π-nef $\mathbb{R}$-divisor of X. Suppose that any fiber $X_s = \pi^{-1}(s)$ is irreducible. Then $\nu(D|_{X_s})$ is constant.

The following lemma is well-known for $\mathbb{Q}$-divisors and proved by the same argument as usual.

2.1. Lemma *Let D be a nef $\mathbb{R}$-divisor of a non-singular projective variety X of dimension n. Then the following properties hold*:

(1) *$\kappa(D) \leq \nu(D)$;*
(2) *$\kappa(D) = n$ if and only if $\nu(D) = n$;*
(3) *If $\nu(D) = n$, then there is an effective $\mathbb{R}$-divisor Δ such that $D - \varepsilon\Delta$ is ample for any $0 < \varepsilon \leq 1$.*

2.2. Definition An $\mathbb{R}$-divisor D is called *nef and abundant* if D is nef and $\nu(D) = \kappa(D)$.

2.3. Lemma *Let D be a nef $\mathbb{R}$-divisor of a non-singular projective variety X of dimension n. Then the following properties hold*:

(1) *If D is nef and abundant, then there exist a birational morphism $\mu \colon Y \to X$, a surjective morphism $f \colon Y \to Z$ of non-singular projective varieties, and a nef and big $\mathbb{R}$-divisor B of Z such that $\mu^* D \sim_{\mathbb{Q}} f^* B$;*

(2) *Let $\pi \colon X \to S$ be a fibration onto a normal variety and let F be a general fiber. Then $\nu(D|_F) \leq \nu(D) \leq \nu(D|_F) + \dim S$.*

Proof. (1) This is also well-known for $\mathbb{Q}$-divisors (cf. [**55**]). By the same argument, we can find a birational morphism $\lambda \colon V \to X$, an equi-dimensional surjective morphism $q \colon V \to Z$, a birational morphism $\varphi \colon Y \to V$, a semi-ample big $\mathbb{Q}$-divisor L of Z and an effective $\mathbb{R}$-divisor E of Y satisfying the following conditions:

- Y and Z are non-singular projective varieties;
- V is a normal projective variety;
- q is birational to the Iitaka fibration for D;
- $\mu^* D \sim_{\mathbb{Q}} f^* L + E$, where $\mu := \lambda \circ \varphi$ and $f := q \circ \varphi$.

Let A be an ample divisor of Y. Then, for $\nu = \nu(D) = \dim Z$, we have

$$0 = (\mu^* D)^{\nu+1} \cdot A^{n-\nu-1} \geq (f^* L)^{\nu} \cdot E \cdot A^{n-\nu-1} \geq 0.$$

Therefore, $f(\operatorname{Supp} E) \neq Z$. Thus $E = f^* \Delta$ for an effective $\mathbb{R}$-divisor Δ, by **III.5.9**. Hence $\mu^* D \sim_{\mathbb{Q}} f^* B$ for the nef and big $\mathbb{R}$-divisor $B = L + \Delta$.

(2) We may assume that S is projective. Let A and H be very ample divisors of X and S, respectively. We set $d := \dim S$, $\nu := \nu(D)$, and $\nu' := \nu(D|_F)$. Then $D^{\nu'} \cdot f^* H^d \cdot A^{n-d-\nu'} > 0$. Hence $\nu \geq \nu'$. In order to show the other inequality, we may assume that $\nu' < n - d$ and $\nu > d$. If D is big, then $D - \varepsilon\Delta$ is ample for $0 < \varepsilon < 1$ for some effective $\mathbb{R}$-divisor Δ. Hence $(D - \varepsilon\Delta)|_F$ is also ample and $D|_F$ is big. In particular, $\nu = \nu' + d$. Suppose that $\nu < n$. Let $V = \bigcap A_i$ be the complete intersection of $(n - \nu)$-general members $A_1, A_2, \cdots, A_{n-\nu}$ of $|A|$. Then V is a non-singular projective variety and $D|_V$ is a nef and big $\mathbb{R}$-divisor. Thus $D^{\nu-d} \cdot f^* H^d \cdot A^{n-\nu} > 0$. In particular, $\nu' \geq \nu - d$. □

§2.b. κ_σ. Let X be a non-singular projective variety of dimension n.

2.4. Lemma *Let D be an $\mathbb{R}$-divisor and let A be an ample divisor of X. Then*

$$\overline{\lim_{t \to \infty}} \frac{\mathrm{h}^0(X, A + \ulcorner tD \urcorner)}{t^n} < +\infty.$$

Proof. We can take an effective $\mathbb{R}$-divisor Δ and an ample divisor H such that $D + \Delta \sim H$. Thus $\mathrm{h}^0(X, A + \ulcorner tD \urcorner) \leq \mathrm{h}^0(X, A + \ulcorner t \urcorner H)$. Hence we are done by the Riemann-Roch formula. □

2.5. Definition Let D be a pseudo-effective $\mathbb{R}$-divisor and A a divisor. If $\mathrm{H}^0(X, A + \llcorner mD \lrcorner) \neq 0$ for infinitely many $m \in \mathbb{N}$, then we define:

$$\sigma^+(D; A) := \min\{k \in \mathbb{Z}_{\geq 0} \mid \overline{\lim}_{m\to\infty} m^{-k}\, \mathrm{h}^0(X, A + \llcorner mD \lrcorner) < +\infty\},$$
$$\sigma(D; A) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \overline{\lim}_{m\to\infty} m^{-k}\, \mathrm{h}^0(X, A + \llcorner mD \lrcorner) > 0\},$$
$$\sigma^-(D; A) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \underline{\lim}_{m\to\infty} m^{-k}\, \mathrm{h}^0(X, A + \llcorner mD \lrcorner) > 0\}.$$

If $\mathrm{H}^0(X, A + \llcorner mD \lrcorner) \neq 0$ only for finitely many $m \in \mathbb{N}$, then we define $\sigma^+(D; A) = \sigma(D; A) = \sigma^-(D; A) = -\infty$. We define the following numerical versions of D-dimension of X:

$$\kappa_\sigma(D) = \kappa_\sigma(D, X) := \max\{\sigma(D; A) \mid A \text{ is a divisor}\},$$
$$\kappa_\sigma^+(D) = \kappa_\sigma^+(D, X) := \max\{\sigma^+(D; A) \mid A \text{ is a divisor}\},$$
$$\kappa_\sigma^-(D) = \kappa_\sigma^-(D, X) := \max\{\sigma^-(D; A) \mid A \text{ is a divisor}\}.$$

2.6. Remark

(1) $\sigma(D; 0) = \sigma^+(D; 0) = \kappa(D)$.
(2) The definition of $\sigma^+(D; A)$ is similar to Fujita's definition [**23**] of $\kappa(L, \mathcal{F})$ for a line bundle L and a coherent sheaf $\mathcal{F}$.
(3) In the original version [**104**], $\sigma(D; A)$ was defined as $\sigma^-(D; A)$ and κ_σ was defined as κ_σ^-.
(4) There are inequalities

$$\sigma^-(D; A) \leq \sigma(D; A) \leq \sigma^+(D; A) \leq \sigma(D; A) + 1,$$
$$\kappa_\sigma^-(D, X) \leq \kappa_\sigma(D, X) \leq \kappa_\sigma^+(D, X) \leq \kappa_\sigma(D, X) + 1.$$

(5) An $\mathbb{R}$-divisor D is pseudo-effective if and only if $\kappa_\sigma^-(D) \geq 0$, by **1.4**.
(6) By replacing $\llcorner\ \lrcorner$ by $\ulcorner\ \urcorner$, we define

$$\sigma(D; A)' := \max\{k \in \mathbb{Z}_{\geq 0} \cup \{-\infty\} \mid \overline{\lim_{m\to\infty}}\, m^{-k}\, \mathrm{h}^0(X, A + \ulcorner mD \urcorner) > 0\}.$$

Since $c_1(\ulcorner mD \urcorner - \llcorner mD \lrcorner)$ are bounded in $\mathrm{N}^1(X)$, we have

$$\kappa_\sigma(D) = \max\{\sigma(D; A)' \mid A \text{ is a divisor}\}.$$

In the definition of $\kappa_\sigma^\pm$, we can also replace $\llcorner\ \lrcorner$ by $\ulcorner\ \urcorner$.

2.7. Proposition *Let D be a pseudo-effective $\mathbb{R}$-divisor of a non-singular projective variety X of dimension n.*

(1) *If D' is an $\mathbb{R}$-divisor with $D' - D$ being pseudo-effective, then $\kappa_\sigma(D') \geq \kappa_\sigma(D)$, $\kappa_\sigma^+(D') \geq \kappa_\sigma^+(D)$, and $\kappa_\sigma^-(D') \geq \kappa_\sigma^-(D)$. In particular, $\kappa_\sigma(D)$, $\kappa_\sigma^+(D)$, and $\kappa_\sigma^-(D)$ depend only on the first Chern class $c_1(D) \in \mathrm{N}^1(X)$.*
(2) *Suppose that $\llcorner kD \lrcorner$ is pseudo-effective for some $k \in \mathbb{N}$. Then $\kappa_\sigma(D) = \max_{k\in\mathbb{N}} \kappa_\sigma(\llcorner kD \lrcorner)$, $\kappa_\sigma^+(D) = \max_{k\in\mathbb{N}} \kappa_\sigma^+(\llcorner kD \lrcorner)$, and $\kappa_\sigma^-(D) = \max_{k\in\mathbb{N}} \kappa_\sigma^-(\llcorner kD \lrcorner)$. In particular, $\kappa_\sigma^-(D) \geq \kappa(D)$.*
(3) *$\kappa_\sigma^+(D) = n$ if and only if D is big. In this case, $\kappa_\sigma(D) = \kappa_\sigma^-(D) = n$.*

(4) $\kappa_\sigma(f^*D) = \kappa_\sigma(D)$, $\kappa_\sigma^+(f^*D) = \kappa_\sigma^+(D)$, *and* $\kappa_\sigma^-(f^*D) = \kappa_\sigma^-(D)$ *hold for any proper surjective morphism* $f\colon Y \to X$ *from a non-singular projective variety.*
(5) *If* $H \subset X$ *is a non-singular ample prime divisor and if* $\kappa_\sigma(D) < \dim X$, *then* $\kappa_\sigma(D) \le \kappa_\sigma(D|_H)$, $\kappa_\sigma^+(D) \le \kappa_\sigma^+(D|_H)$, *and* $\kappa_\sigma^-(D) \le \kappa_\sigma^-(D|_H)$.
(6) *If* D *is nef, then* $\kappa_\sigma^-(D) = \kappa_\sigma^+(D) = \kappa_\sigma(D) = \nu(D)$.
(7) *Let* $f\colon X \to Y$ *be a generically finite surjective morphism onto a projective variety and let* E *be an effective* $\mathbb{R}$*-divisor such that* $N_\sigma(D; X/Y) \ge E$. *Then* $\kappa_\sigma(D) = \kappa_\sigma(D-E)$, $\kappa_\sigma^+(D) = \kappa_\sigma^+(D-E)$, *and* $\kappa_\sigma^-(D) = \kappa_\sigma^-(D-E)$.
(8) $\kappa_\sigma^-(D) = 0$ *if and only if* $D \approx N_\sigma(D)$. *In this case,* $\kappa_\sigma^+(D) = \kappa_\sigma(D) = 0$.
(9) (Easy addition): *Let* $f\colon X \to Y$ *be a fiber space and let* F *be a 'general' fiber. Then* $\kappa_\sigma(D) \le \kappa_\sigma(D|_F) + \dim Y$, $\kappa_\sigma^+(D) \le \kappa_\sigma^+(D|_F) + \dim Y$, *and* $\kappa_\sigma^-(D) \le \kappa_\sigma^-(D|_F) + \dim Y$.

PROOF. (1) By **1.3**, there is an ample divisor A such that

$$\mathrm{H}^0(X, \llcorner m(D' - D) \lrcorner + A) \ne 0$$

for any $m > 0$. Hence $\mathrm{h}^0(X, \llcorner mD' \lrcorner + 2A) \ge \mathrm{h}^0(X, \llcorner mD \lrcorner + A)$.

(2) Let l be a positive integer such that $\mathrm{Supp}\langle lD \rangle$ coincides with the union of prime components Γ of $\langle D \rangle$ with $\mathrm{mult}_\Gamma \langle D \rangle \notin \mathbb{Q}$. There is a constant c with $\langle lD \rangle \le c\langle kD \rangle$. We can choose the integer l above with $l > ck + 1$. Then there is an ample divisor A such that

$$\mathrm{H}^0\left(X, \llcorner m((l - ck - 1)D + c\llcorner kD \lrcorner) \lrcorner + A\right) \ne 0$$

for any $m > 0$ by **1.3**. Since

$$\begin{aligned} m\llcorner lD \lrcorner + 2A &= mD + 2A + m(l-1)D - m\langle lD \rangle \\ &\ge mD + A + m(l - ck - 1)D + mc\llcorner kD \lrcorner + A, \end{aligned}$$

we have

$$\mathrm{h}^0(X, m\llcorner lD \lrcorner + 2A) \ge h^0(X, \llcorner mD \lrcorner + A),$$

which implies the expected equalities.

(3) If D is big, then $\kappa_\sigma^-(D) = n$ by (2). Conversely, assume that $\kappa_\sigma^+(D) = n$. Let A be a very ample divisor such that $\sigma^+(D; A) = n$. Let H be another non-singular very ample divisor such that $H - A$ is ample. There is an exact sequence

$$0 \to \mathrm{H}^0(X, \llcorner mD \lrcorner + A - H) \to \mathrm{H}^0(X, \llcorner mD \lrcorner + A) \to \mathrm{H}^0(H, (\llcorner mD \lrcorner + A)|_H).$$

We note that

$$\varlimsup_{m\to\infty} m^{-n+1}\,\mathrm{h}^0(H, (\llcorner mD \lrcorner + A)|_H) < +\infty, \quad \varlimsup_{m\to\infty} m^{-n+1}\,\mathrm{h}^0(X, \llcorner mD \lrcorner + A) = +\infty.$$

Hence $mD - (H - A)$ is pseudo-effective for some $m > 0$. Thus D is big.

(4) Let H be an ample divisor of Y. Then $f_*\mathcal{O}_Y(H) \subset \mathcal{O}_X(A)^{\oplus k}$ for some ample divisor A of X and a positive integer k. Hence $\mathrm{h}^0(Y, \llcorner mf^*D \lrcorner + H) \le k\,\mathrm{h}^0(X, \ulcorner mD \urcorner + A)$. Thus $\kappa_\sigma(f^*D) \le \kappa_\sigma(D)$, and the same inequalities for κ_σ^+ and

κ_σ^- hold. For the converse, it is enough to take an ample divisor H of Y such that $H - f^*A$ is very ample for a given ample divisor A of X.

(5) We may assume that $H \not\subset \operatorname{Supp}\langle D\rangle$. For an ample divisor A, let us consider the exact sequences:

$$0 \to \mathcal{O}_X(\llcorner mD \lrcorner + A - (j+1)H) \to \mathcal{O}_X(\llcorner mD \lrcorner + A - jH) \to \mathcal{O}_H(\llcorner mD \lrcorner + A - jH) \to 0$$

for integers $j \geq 0$. There is an integer k such that $kH - A$ is ample. Then $\mathrm{h}^0(X, \llcorner mD \lrcorner + A - kH) = 0$, since D is not big. Therefore

$$\mathrm{h}^0(X, \llcorner mD \lrcorner + A) \leq k\, \mathrm{h}^0(H, (\llcorner mD \lrcorner + A)|_H)$$

for any $m > 0$. Therefore $\kappa_\sigma(D) \leq \kappa_\sigma(D|_H)$, and the same inequalities for κ_σ^+ and κ_σ^- hold.

(6) We may assume that D is not big. Let $\nu := \nu(D) < n$. Let $A_1, A_2, \ldots, A_{n-\nu}$ be general non-singular ample prime divisors of X. Then the intersections $V_j := \bigcap_{i \leq j} A_i$ are non-singular, $D|_{V_j}$ is not big for $j < n - \nu$, and $D|_{V_{n-\nu}}$ is big. Then, by (5), $\kappa_\sigma^+(D) \leq \kappa_\sigma^+(D|_{V_j}) \leq \dim V_{n-\nu} = \nu(D)$. The converse inequality $\kappa_\sigma^-(D) \geq \nu(D)$ follows from **1.14**, since we may replace D so that $\operatorname{Supp}\langle D\rangle$ is a simple normal crossing divisor.

(7) Let H be an ample divisor of Y. Then

$$mE \leq N_\sigma(mD; X/Y) = N_\sigma(mD + f^*H; X/Y) \leq N_\sigma(mD + f^*H)$$

for any $m > 0$. Therefore $\mathrm{H}^0(X, \llcorner mD \lrcorner + f^*H) \simeq \mathrm{H}^0(X, \llcorner m(D - E) \lrcorner + f^*H)$.

(8) follows from **1.12**.

(9) Let A be an ample divisor of X. We shall prove the following assertion by induction on $\dim Y$: there is a constant $c > 0$ such that

$$\mathrm{h}^0(X, \llcorner mD \lrcorner + A) \leq c m^{\dim Y}\, \mathrm{h}^0(F, (\llcorner mD \lrcorner + A)|_F)$$

for $m \gg 0$. Let $H \subset Y$ be a 'general' ample divisor of Y. Then there is a positive integer l such that $D - lf^*H$ is not pseudo-effective. Thus $\mathrm{h}^0(X, \llcorner mD \lrcorner + A - mlf^*H) = 0$ for $m \gg 0$. Hence

$$\begin{aligned}\mathrm{h}^0(X, \llcorner mD \lrcorner + A) &\leq \sum\nolimits_{i=0}^{ml-1} \mathrm{h}^0(f^*H, (\llcorner mD \lrcorner + A - if^*H)|_{f^*H}) \\ &\leq ml\, \mathrm{h}^0(f^*H, (\llcorner mD \lrcorner + A)|_{f^*H}).\end{aligned}$$

Thus we are done by induction. □

§2.c. Numerical domination.

2.8. Definition Let D_1 and D_2 be two $\mathbb{R}$-divisors of a non-singular projective variety X. We say that D_1 *dominates* D_2 if $tD_1 - D_2$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{R}$-divisor for some $t \in \mathbb{Q}_{>0}$. In this case, we write $D_1 \succeq D_2$ or $D_2 \preceq D_1$.

2.9. Remark

(1) If $D_1 \succeq D_2$ and $D_2 \succeq D_3$, then $D_1 \succeq D_3$.
(2) If D_1 and D_2 are effective $\mathbb{R}$-divisors with $\operatorname{Supp} D_1 \supset \operatorname{Supp} D_2$, then $D_1 \succeq D_2$.

(3) If $D_1 \succeq D_2$, then $\kappa(D_1) \geq \kappa(D_2)$.

2.10. Definition Let D be an $\mathbb{R}$-divisor and let W be a Zariski-closed proper subset of a non-singular projective variety X. We say that D *dominates* W and write $D \succeq W$ or $W \preceq D$ if the following condition is satisfied: let $\mu\colon Y \to X$ be a birational morphism from a non-singular projective variety such that $\mu^{-1}(W)$ is the support of an effective divisor E. Then $\mu^* D \succeq E$. Note that this condition does not depend on the choices of $\mu\colon Y \to X$ and E.

2.11. Lemma *For any $\mathbb{R}$-divisor D with $0 \leq \kappa(D) < \dim X$,*

$$\kappa(D) = \min\{\dim W \mid W \not\preceq D\}.$$

Proof. If $\kappa(D) = 0$, then $\{x\} \not\preceq D$ for a point $x \notin \bigcup_{m>0} \mathrm{Bs}\left|\ulcorner mD \urcorner\right|$. Thus, we may assume $0 < \kappa(D) < \dim X$. Let $\Phi\colon X \dashrightarrow Z$ be the Iitaka fibration for D. If $W \subset X$ is a general subvariety of $\dim W = \kappa(D) = \dim Z$, then $\mu^* D - \beta E$ is not pseudo-effective for any $\beta > 0$, for a birational morphism $\mu\colon Y \to X$, and for an effective divisor E with $\operatorname{Supp} E = \mu^{-1}(W)$. On the other hand, if $\dim W < \dim Z$, then $\mu^{-1} W$ is contained the pullback of an ample divisor H of Z and H is dominated by $\mu^* D$. Hence $D \succeq W$. □

We shall give a numerical version of the notion of domination as follows:

2.12. Definition Let D_1 and D_2 be two $\mathbb{R}$-divisors of a non-singular projective variety. If the following condition is satisfied, we say that D_1 *dominates* D_2 *numerically* and write $D_1 \succcurlyeq D_2$ or $D_2 \preccurlyeq D_1$: for an ample divisor A and for any positive number $b > 0$, there exist real numbers $x > b$ and $y > b$ such that $xD_1 - yD_2 + A$ is pseudo-effective.

For an ample divisor A and for a number $x \in \mathbb{R}_{\geq 0}$, we consider the set

$$\mathcal{D}(x) := \{y \in \mathbb{R}_{\geq 0} \mid xD_1 - yD_2 + A \text{ is pseudo-effective}\}$$

and define a function

$$\varphi(x) = \begin{cases} \sup\{y \in \mathcal{D}(x)\}, & \text{if } \mathcal{D}(x) \neq \emptyset, \\ -\infty, & \text{otherwise} \end{cases}$$

with values in $\{\pm\infty\} \cup \mathbb{R}_{\geq 0}$. Then $D_1 \succcurlyeq D_2$ if and only if $\lim_{x \to +\infty} \varphi(x) = +\infty$.

2.13. Lemma

(1) *If $D_1 \succcurlyeq D_2$ and $D_2 \succcurlyeq D_3$ and if D_1 is pseudo-effective, then $D_1 \succcurlyeq D_3$.*
(2) *If $D_1 \succeq D_2$, then $D_1 \succcurlyeq D_2$.*
(3) *If D_1 and D_2 are nef $\mathbb{R}$-divisors with $D_1 \succcurlyeq D_2$, then $\nu(D_1) \geq \nu(D_2)$.*

Proof. (1) For a given positive number b, we choose numbers u, $v \in \mathbb{R}_{>b}$ so that $uD_2 - vD_3 + A$ is pseudo-effective. Let c be a positive number with $vc/(u+c) > b$ and we choose numbers x, $y \in \mathbb{R}_{>c}$ so that $xD_1 - yD_2 + A$ is pseudo-effective. Then

$$\frac{ux}{u+y}D_1 - \frac{yv}{u+y}D_3 + A = \frac{u}{u+y}(xD_1 - yD_2 + A) + \frac{y}{u+y}(uD_2 - vD_3 + A)$$

is pseudo-effective. Since $y > c$, we have $yv/(u+y) > b$. Since D_1 is pseudo-effective, we can choose x to satisfy $ux/(u+y) > b$. Thus $D_1 \succcurlyeq D_3$.

(2) Let t be a positive number such that $tD_1 - D_2$ is pseudo-effective. Then, for any $b > 0$, there is a number m such that $mt > b$ and $m > b$. Then $(mt)D_1 - mD_2 + A$ is pseudo-effective.

(3) Let b be an arbitrary positive integer. Then there exist real numbers $x > b$ and $y > b$ such that $xD_1 - yD_2 + A$ is pseudo-effective. Then, for any $0 \le k \le \nu := \nu(D_1)$, we have inequalities

$$xD_1^{\nu+1-k}D_2^kA^{n-\nu-1} + A^{n-\nu}D_1^{\nu-k}D_2^k \ge yD_1^{\nu-k}D_2^{k+1}A^{n-\nu-1},$$

since D_1 and D_2 are nef. Hence, we infer that if $D_1^{\nu+1-k}D_2^k$ is numerically trivial, then $D_1^{\nu-k}D_2^{k+1}$ is also numerically trivial by **II.6.3**. Therefore $D_2^{\nu+1}$ is numerically trivial since $D_1^{\nu+1}$ is so. Thus $\nu \ge \nu(D_2)$. □

2.14. Lemma *Let X be a non-singular projective variety, D a nef and abundant $\mathbb{R}$-divisor, and E an effective $\mathbb{R}$-divisor. If $D \succcurlyeq E$, then $D \succeq E$.*

Proof. We can reduce to the following situation by **2.3**-(1): there is a fibration $f\colon X \to Y$ onto a non-singular projective variety such that $D \sim_{\mathbb{Q}} f^*B$ for a nef and big $\mathbb{R}$-divisor B. Let F be a 'general' fiber of f. Then $E|_F \preccurlyeq D|_F \sim_{\mathbb{Q}} 0$. It follows that $-E$ is relatively pseudo-effective over Y. Thus $f(\operatorname{Supp} E) \neq Y$. Hence, there is a positive integer l such that $lf^*B - E$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{R}$-divisor. □

2.15. Corollary *Let $f\colon X \to Y$ be a surjective morphism from a non-singular projective variety onto a projective variety, D a nef and abundant $\mathbb{R}$-divisor of X, and A an ample divisor of Y. Then the following conditions are equivalent*:

(1) $D \succcurlyeq f^*A$;
(2) $D \succeq f^*A$;
(3) *f is the composite of the Iitaka fibration $X \dashrightarrow Z$ for D and a rational map $Z \dashrightarrow Y$.*

2.16. Definition Let D be an $\mathbb{R}$-divisor and let W be a proper Zariski-closed subset of a non-singular projective variety X. If the following condition is satisfied, then we say that D *dominates* W *numerically* and write $D \succcurlyeq W$ or $W \preccurlyeq D$: let $\mu\colon Y \to X$ be a birational morphism from a non-singular projective variety such that $\mu^{-1}(W)$ is the support of an effective divisor E. Then $\mu^*D \succcurlyeq E$. Note that this condition does not depend on the choices of $\mu\colon Y \to X$ and E.

2.17. Lemma *Let D be an $\mathbb{R}$-divisor of a non-singular projective variety X, $W \subset X$ a Zariski-closed proper subset with $W \preccurlyeq D$, and $Z \subset X \times U$ a dominant family of closed subvarieties of X parameterized by a complex analytic variety U such that general members $Z_u \subset X$ are non-singular. Then the restriction $D|_{Z_u}$ numerically dominates $W \cap Z_u$ for a 'general' member Z_u.*

Proof. Let $\rho\colon X' \to X$ be a birational morphism from a non-singular projective variety such that $\rho^{-1}(W)$ is an effective reduced divisor E. Let $Z' \to Z$ be a bimeromorphic morphism from a non-singular variety such that the induced meromorphic map $p\colon Z' \cdots\to X'$ from the first projection $Z \to X$ is a holomorphic. For an ample divisor A of X' and for any positive number b, there exist x, $y \in \mathbb{R}_{>b}$ such that $x\rho^*D - yE + A$ is pseudo-effective. Then $p^*(x\rho^*D - yE + A)$ is relatively pseudo-effective over U. Therefore, $D|_{Z_u} \succcurlyeq W \cap Z_u$ for a 'general' member Z_u. $\square$

2.18. Lemma *Let $\pi\colon X \to S$ be a flat projective surjective morphism of complex analytic varieties and let $W \subset X$ be a proper closed analytic subspace such that*

(1) *every fiber $X_s = \pi^{-1}(s)$ is irreducible and reduced,*
(2) *the sheaf $\mathcal{O}_X/\mathcal{I}_W^k$ is flat over S for any $k \geq 1$ for the defining ideal $\mathcal{I}_W$ of W.*

Let D be an $\mathbb{R}$-Cartier divisor of X such that $D|_{X_s} \succcurlyeq W \cap X_s$ for a 'general' fiber X_s. Then $D|_{X_s} \succcurlyeq W \cap X_s$ for any $s \in S$.

Proof. We may assume that S is a non-singular curve. Let $\rho\colon Y \to X$ be the blowing-up along W and let E be the effective Cartier divisor such that $\mathcal{O}_Y(-E) \simeq \rho^*\mathcal{I}_W/(\mathrm{tor})$. Note that, for the composite $f := \pi \circ \rho\colon Y \to S$, every fiber $Y_s := f^{-1}(s)$ is irreducible and reduced, and $Y_s \to X_s$ is the blowing-up along the defining ideal of $W_s := W \times_S \{s\}$. For an f-ample divisor A of Y and for positive numbers x, y, suppose that $x\rho^*D - yE + A$ is f-pseudo-effective. Then the restriction $(x\rho^*D - yE + A)|_{Y_s}$ to any fiber Y_s is also pseudo-effective. Hence $D|_{X_s} \succcurlyeq W_s$ for any $s \in S$. $\square$

2.19. Lemma *Let D be a pseudo-effective $\mathbb{R}$-divisor of a non-singular projective variety X, $H \subset X$ a non-singular ample prime divisor, and $W \subset H$ a Zariski-closed subset with $D \succcurlyeq W$. Then $D|_H \succcurlyeq W$.*

Proof. Let $\rho\colon Y \to X$ be a birational morphism from a non-singular projective variety such that $\rho^{-1}(W)$ is a reduced divisor E and that the proper transform H' of H is non-singular. For an ample divisor A of Y, we consider

$$\sigma(x, y) := \sigma_{H'}(x\rho^*D - yE + A)$$

as a function on

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2_{\geq 0} \mid x\rho^*D - yE + A \text{ is pseudo-effective}\}.$$

Note that $x\rho^*D - yE + A - \sigma(x, y)H'$ is pseudo-effective for $(x, y) \in \mathcal{D}$, and that $\mathcal{D}_b := \{(x, y) \in \mathcal{D} \mid x, y > b\}$ is non-empty for any $b > 0$.

Suppose that $\sup\{\sigma(x, y) \mid (x, y) \in \mathcal{D}_b\} = \infty$ for any $b > 0$. Then $\rho^*D \succcurlyeq E + H' = \rho^{-1}H$. Hence $D \succcurlyeq H$ and D is big. Since H is ample, $D|_H$ is still big. Thus $D|_H \succcurlyeq W$.

Next suppose that $\beta := \sup\{\sigma(x, y) \mid (x, y) \in \mathcal{D}_b\} < +\infty$ for some $b > 0$. Let c be a positive number with $cA + \beta H'$ being ample. Then $cA + \sigma(x, y)H'$ is ample for $(x, y) \in \mathcal{D}_b$. Since

$$x\rho^*D - yE + (1+c)A = (x\rho^*D - yE + A - \sigma(x,y)H') + cA + \sigma(x,y)H',$$

$x\rho^*D - yE + (1+c)A$ and its restriction $(x\rho^*D - yE + (1+c)A)|_{H'}$ are pseudo-effective. Therefore, $D|_H \succcurlyeq W$. $\square$

§2.d. κ_ν.

2.20. Definition For an $\mathbb{R}$-divisor D, we define $\kappa_\nu(D) = \kappa_\nu(D, X)$ as follows:

(1) If D is not pseudo-effective, then $\kappa_\nu(D) := -\infty$;
(2) If D is big, then $\kappa_\nu(D) := \dim X$;
(3) In the other case, $\kappa_\nu(D) := \min\{\dim W \mid D \not\succcurlyeq W\}$.

2.21. Lemma *If $\kappa_\nu(D) = k < \dim X = n$, then, for any ample divisor A, there exist a positive integer m and 'general' members $A_1, A_2, \ldots, A_{n-k} \in |mA|$ such that $D \not\succcurlyeq A_1 \cap \cdots \cap A_{n-k}$.*

Proof. Let W be a subvariety of X of dimension k with $D \not\succcurlyeq W$. Then there exist a positive integer m and members $A_1^0, A_2^0, \ldots, A_{n-k}^0 \in |mA|$ such that $V^0 := A_1^0 \cap A_2^0 \cap \cdots \cap A_{n-k}^0$ is a k-dimensional subspace with $W \subset V^0$. Hence $D \not\succcurlyeq V^0$. Let $\pi\colon Z \to U$ be a flat family of closed subspaces of X whose fibers are complete intersections $V = A_1 \cap \cdots \cap A_{n-k}$ for some members $A_1, \ldots, A_{n-k} \in |mA|$. Suppose that V^0 is the fiber $\pi^{-1}(0)$ for a point $0 \in U$. By applying **2.18** to the flat morphism $X \times U \to U$ and the closed subspace $Z \subset X \times U$, we infer that D does not dominate numerically a 'general' fiber V of π. $\square$

In particular, if D is a non-big pseudo-effective $\mathbb{R}$-divisor, then $\kappa_\nu(D)$ is the minimum of $\dim W$ for non-singular complete intersections W with $D \not\succcurlyeq W$.

The following is an example of pseudo-effective divisor D such that $\kappa_\nu(D)$ is not the maximum of $\kappa(L)$ for semi-ample $\mathbb{Q}$-divisors L of non-singular projective varieties Y with birational morphisms $\mu\colon Y \to X$ such that $\mu^*D \succcurlyeq L$ (cf. **2.22**-(5)).

Example Let L be a divisor of degree zero of an elliptic curve E such that $mL \not\sim 0$ except for $m = 0$. Let $X \to E$ be the $\mathbb{P}^1$-bundle associated with $\mathcal{O}_E \oplus \mathcal{O}_E(L)$ and H a tautological divisor. Then H is nef and $\nu(H) = 1$. Suppose that there exist a birational morphism $\mu\colon Y \to X$ and a fiber space $f\colon Y \to Z$ such that $\mu^*H \succcurlyeq f^*B$ for an ample divisor B of Z. Then we can show that Z is a point as follows: Assume the contrary. Then Z is a curve. Let F be a fiber of f. If $\mu^*(xH + A) - yF$ is pseudo-effective for an ample divisor A and for positive numbers x, $y \gg 0$, then $A \cdot H \geq yF \cdot \mu^*H$ and hence $F \cdot \mu^*H = 0$. There is a surjection

$$\mathcal{O}_F \oplus \mathcal{O}_F(\tau^*L) \twoheadrightarrow \mathcal{O}_F(\mu^*H|_F)$$

for $\tau\colon F \to E$. Since $\mathcal{O}_E(L)$ is not a torsion element in $\operatorname{Pic}(E)$, the surjection above factors through the first projection or the second projection. Therefore, $\mu(F)$ is contained in one of two sections of $X \to E$ corresponding to the splittings of $\mathcal{O}_E \oplus \mathcal{O}_E(L)$. This is a contradiction.

2.22. Proposition *Let D be a pseudo-effective $\mathbb{R}$-divisor of a non-singular projective variety X of dimension n.*

(1) $\kappa_\nu(D) \geq \kappa_\sigma(D)$.
(2) $\kappa_\nu(D) = 0$ *if and only if* $D \approxeq N_\sigma(D)$.
(3) *Let $H \subset X$ be a non-singular ample prime divisor. If $\kappa_\nu(D) < n$, then* $\kappa_\nu(D) \leq \kappa_\nu(D|_H)$.
(4) $\kappa_\nu(f^*D) = \kappa_\nu(D)$ *for any proper surjective morphism* $f\colon Y \to X$.
(5) *If D is nef, then* $\kappa_\nu(D) = \nu(D)$.
(6) *Let $f\colon X \to Y$ be a generically finite surjective morphism onto a projective variety and let E be an effective $\mathbb{R}$-divisor with $N_\sigma(D; X/Y) \geq E$. Then* $\kappa_\nu(D) = \kappa_\nu(D - E)$.
(7) (Easy Addition): *For a fiber space $\pi\colon X \to S$, $\kappa_\nu(D) \leq \kappa_\nu(D|_{X_s}) + \dim S$ holds for a 'general' fiber $X_s = \pi^{-1}(s)$.*

Proof. (1) Let A be a very ample divisor of X and let $W \subset X$ be a non-singular subvariety of dimension $w < \kappa_\sigma(D)$ that is the complete intersection $\bigcap A_j$ of $(n-w)$-general members of $|A|$. It is enough to show that $D \succcurlyeq W$ by **2.21**. The conormal bundle $N^\vee_{W/X}$ is isomorphic to $\mathcal{O}_X(-A)^{\oplus(n-w)}$. We consider the exact sequence:

$$0 \to \mathrm{H}^0(X, \mathcal{I}_W^{q+1}\mathcal{O}_X(A + \llcorner mD \lrcorner)) \to \mathrm{H}^0(X, \mathcal{I}_W^{q}\mathcal{O}_X(A + \llcorner mD \lrcorner)) \to \\ \to \mathrm{H}^0(W, \mathrm{Sym}^q(N^\vee_{W/X}) \otimes \mathcal{O}_W(A + \llcorner mD \lrcorner)),$$

for positive integers q, where $\mathcal{I}_W$ is the defining ideal sheaf of W. Thus

$$\mathrm{h}^0(X, \mathcal{I}_W^{q+1}\mathcal{O}(A + \llcorner mD \lrcorner)) \geq \mathrm{h}^0(X, A + \llcorner mD \lrcorner) - \binom{n-w+q}{n-w}\mathrm{h}^0(W, (A + \llcorner mD \lrcorner)|_W).$$

Let us consider a function $q\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_{m\to+\infty} q(m) = +\infty$ and

$$\log q(m) \leq \frac{\kappa_\sigma(D) - \varepsilon - w}{n-w}\log m$$

for a fixed positive number ε. Then the boundedness of $m^{-w}\,\mathrm{h}^0(W, (A + \llcorner mD \lrcorner)|_W)$ implies that there is a constant c such that

$$\binom{n-w+q(m)}{n-w}\mathrm{h}^0(W, (A + \llcorner mD \lrcorner)|_W) < cm^{\kappa_\sigma(D)-\varepsilon}$$

for $m \gg 0$. Hence $\mathrm{H}^0(X, \mathcal{I}_W^{q(m)+1}\mathcal{O}(A + \llcorner mD \lrcorner)) \neq 0$ for $m \gg 0$, since

$$\overline{\lim}_{m\to\infty}\, m^{-\kappa_\sigma(D)}\,\mathrm{h}^0(X, A + \llcorner mD \lrcorner) > 0.$$

Therefore, $D \succcurlyeq W$.

(2) By (1) and **1.12**, $D \approxeq N_\sigma(D)$ if $\kappa_\nu(D) = 0$. Conversely, assume that $D \approxeq N_\sigma(D)$. We may assume that $D = N_\sigma(D)$, since κ_ν depends on the numerical equivalence class. Let x be a point of $X \setminus \mathrm{Supp}\, D$, $\rho\colon Z \to X$ the blowing-up at

x, E the exceptional divisor, and A a sufficiently ample divisor of X. Suppose that $D \succcurlyeq \{x\}$. Then, by **1.4**, there is a function $l\colon \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{h}^0(Z, \rho^*(\llcorner mD \lrcorner + A) - l(m)E) \neq 0 \quad \text{and} \quad \lim_{m\to\infty} l(m) = +\infty.$$

Since $E \not\subset \mathrm{Bs}\,|\rho^*(\llcorner mD \lrcorner + A)|$, we have $\mathrm{h}^0(X, \llcorner mD \lrcorner + A) > l(m)$ contradicting **1.12**.

(3) Let $W \subset H$ be a non-singular subvariety of $\dim W < \kappa_\nu(D)$. Then $D \succcurlyeq W$. By **2.19**, $D|_H \succcurlyeq W$. Hence $\kappa_\nu(D|_H) > \dim W$ by **2.21**.

(4) Let $W \subset Y$ be a non-singular subvariety of dimension $w < \kappa_\nu(D)$ that is the complete intersection of general ample divisors. Then $\dim f(W) = w$. Thus $f^*D \succcurlyeq W$ by the same argument as in **2.21**. Hence $\kappa_\nu(f^*D) \geq \kappa_\nu(D)$. By (3) above, if $\dim Y > \dim X$, then $\kappa_\nu(f^*D|_H) \geq \kappa_\nu(f^*D)$ for a general ample divisor H. Therefore, in order to show the equality: $\kappa_\nu(f^*D) = \kappa_\nu(D)$, we may assume that f is generically finite. Let $V \subset X$ be a general non-singular subvariety of dimension $v < \kappa_\nu(f^*D)$, $\rho\colon X' \to X$ the blowing-up along V, and E the exceptional divisor. Let $\rho_W\colon Y' \to Y$ be the blowing-up along $W := f^{-1}(V)$, E_W the exceptional divisor, and $\tau\colon Y' \to X'$ the induced generically finite morphism. Note that $Y' \simeq Y \times_X X'$ and $E_W \simeq Y \times_X E$. There exist an ample divisor H on X' and positive numbers x, $y \gg 0$ such that $\tau^*(x\rho^*D + H) - yE_W$ is pseudo-effective. Thus $x\rho^*D + H - yE$ is pseudo-effective. Hence $D \succcurlyeq W$ and we have the equality.

(5) Let $W \subset X$ be a general non-singular subvariety of dimension $w = \nu(D)$, $\rho\colon Z \to X$ the blowing-up along W, and E_W the exceptional divisor. We take an ample divisor A with $\rho^*A - E_W$ being ample. If $\rho^*(xD + A) - yE_W$ is pseudo-effective for some x, $y > 0$, then

$$\begin{aligned} 0 &\leq (\rho^*(xD + A) - yE_W) \cdot (\rho^*D)^w \cdot (\rho^*A - E_W)^{n-1-w} \\ &= \rho^*A \cdot (\rho^*D)^w \cdot (\rho^*A - E_W)^{n-1-w} - yc(D|_W)^w \end{aligned}$$

for a positive constant c. Hence y is bounded. Therefore, $D \not\succcurlyeq W$ and $\kappa_\nu(D) \leq \nu(D)$. The other inequality follows from (1) and **2.7**-(6).

(6) Let $W \subset X$ be a non-singular subvariety of dimension $w < \kappa_\nu(D)$, $\rho\colon Z \to X$ the blowing-up along W, E_W the exceptional divisor, and H an ample divisor of Y. Then there exist positive numbers x, $y \gg 0$ such that $\rho^*(xD + f^*H) - yE_W$ is pseudo-effective. Let Γ be a prime component of E and let Γ' be the proper transform of Γ. Note that $\Gamma' = \rho^*\Gamma$. We have

$$\begin{aligned} x \operatorname{mult}_\Gamma E &\leq \sigma_\Gamma(xD; X/Y) = \sigma_\Gamma(xD + f^*H; X/Y) \\ &\leq \sigma_{\Gamma'}(\rho^*(xD + f^*H) - yE_W). \end{aligned}$$

Therefore, the $\mathbb{R}$-divisor

$$\rho^*(xD + f^*H) - yE_W - x\rho^*E$$

is pseudo-effective. Thus $D - E \succcurlyeq W$.

(7) Suppose that $\kappa_\nu(D) > \kappa_\nu(D|_{X_s}) + \dim S$ for 'general' $s \in S$. Let $W \subset X$ be a non-singular subvariety of dimension $\kappa_\nu(D|_{X_s}) + \dim S$. Since $D \succcurlyeq W$, we have

$D|_{X_s} \succcurlyeq W \cap X_s$ for 'general' $s \in S$ by **2.17**. Thus $\kappa_\nu(D|_{X_s}) > \dim W - \dim S$. This is a contradiction. $\square$

Problem

(1) $\kappa_\sigma(D) = \kappa_\sigma^{\pm}(D) = \kappa_\nu(D)$ for all pseudo-effective $\mathbb{R}$-divisors D?
(2) $\kappa_\sigma(D) = \kappa_\sigma(P_\sigma(D))$? $\kappa_\nu(D) = \kappa_\nu(P_\sigma(D))$?

The affirmative answer to **1.8** implies the expected equalities in (2).

§2.e. Geometrically abundant divisors.

2.23. Definition Let X be a non-singular projective variety and let D be an $\mathbb{R}$-divisor.

(1) D is called *abundant* if $\kappa_\nu(D) = \kappa(D)$.
(2) D is called *geometrically abundant* if the following conditions are satisfied:
  (a) $\kappa(D) \geq 0$;
  (b) let $X \dashrightarrow Z$ be the Iitaka fibration for D and let $\mu\colon Y \to X$ be a birational morphism from a non-singular projective variety such that the composite $f\colon Y \to X \dashrightarrow Z$ is holomorphic. Then
$$\kappa_\sigma(\mu^* D|_{Y_z}) = 0$$
  for a 'general' fiber $Y_z = f^{-1}(z)$.

A geometrically abundant $\mathbb{R}$-divisor is abundant by **2.7** and **2.22**. A nef and abundant $\mathbb{R}$-divisor is geometrically abundant by **2.3**-(1). The Zariski-decomposition problem for a geometrically abundant $\mathbb{R}$-divisor D is reduced to that of a big $\mathbb{R}$-divisor of the base variety of the Iitaka fibration for D.

2.24. Notation Let $f\colon X \to Y$ be a projective morphism from a normal complex analytic space into a complex analytic space and let $X \to Y' \to Y$ be the Stein factorization. Let F be a 'general' fiber of $X \to Y'$. Note that F is a connected component of a 'general' fiber of $X \to f(X)$. For an $\mathbb{R}$-Cartier divisor D of X, we denote
$$\kappa_\nu(D; X/Y) = \kappa_\nu(D|_F) \quad \text{and} \quad \kappa_\sigma(D; X/Y) = \kappa_\sigma(D|_F).$$
If $D|_F$ is abundant, then D is called f-*abundant*. If $D|_F$ is geometrically abundant, then D is called *geometrically* f-*abundant*. Let D' be another $\mathbb{R}$-divisor of X. If $D_1|_F \succcurlyeq D_2|_F$ (resp. $D_1|_F \succeq D_2|_F$), then we write $D_1 \succcurlyeq_f D_2$ (resp. $D_1 \succeq_f D_2$).

2.25. Lemma *Let $f\colon X \to Y$ be a projective surjective morphism of non-singular varieties with connected fibers. Let D be an $\mathbb{R}$-divisor of X with $\kappa(D; X/Y) = \kappa_\sigma(D; X/Y) = 0$. Then there exist a positive integer m, a reflexive $\mathbb{R}$-sheaf $\boldsymbol{\Xi}$ of rank one of Y, and an f-exceptional effective $\mathbb{R}$-divisor E of X such that*
$$mP_\sigma(D; X/Y) \sim f^*\boldsymbol{\Xi} - E.$$
If $E \neq 0$, then $\operatorname{Supp} E \not\subset \operatorname{Supp} N_\sigma(D; X/Y)$. *If D is a $\mathbb{Q}$-divisor, then $\boldsymbol{\Xi}$ is also a reflexive $\mathbb{Q}$-sheaf.*

If every reflexive sheaf of rank one on Y admits a meromorphic section, then we can take $\mathbf{\Xi}$ above as an $\mathbb{R}$-divisor.

Proof. We can consider the relative σ-decomposition with respect to f by **III.4.3** since $f_*\mathcal{O}_X(\llcorner mD \lrcorner) \neq 0$ for a positive integer m. Suppose that mD is linearly equivalent to an effective divisor Δ. This is satisfied, for example, if Y is Stein. Here we have $N_\sigma(\Delta; X/Y) = mN_\sigma(D; X/Y)$ and the effective $\mathbb{R}$-divisor $P_\sigma(\Delta; X/Y) = \Delta - N_\sigma(\Delta; X/Y)$ is linearly equivalent to $mP_\sigma(D; X/Y)$. By **1.12**, $P_\sigma(D; X/Y)|_{X_y} \approx 0$ for a 'general' point $y \in Y$. Thus $f(\operatorname{Supp} P_\sigma(\Delta; X/Y)) \neq Y$ and hence $P_\sigma(\Delta; X/Y) = f^*\Xi_0 - E$ for an $\mathbb{R}$-divisor Ξ_0 of Y and an f-exceptional effective $\mathbb{R}$-divisor E of X by **III.5.8**. Even if mD is not linearly equivalent to any effective divisor, we can patch E locally defined over Y to the globally defined effective $\mathbb{R}$-divisor E of X. Thus $mP_\sigma(D; X/Y) - E \sim f^*\mathbf{\Xi}$ for some $\mathbf{\Xi} \in \operatorname{Ref}_1(Y)_\mathbb{R}$.

Suppose that $E \neq 0$ and let Γ be an irreducible component of E. Then $\sigma_\Gamma(-E; X/Y) = 0$ and $m\sigma_\Gamma(D; X/Y) = \sigma_\Gamma(-E + mN_\sigma(D; X/Y); X/Y)$ by the formula

$$mD \sim f^*\mathbf{\Xi} - E + mN_\sigma(D; X/Y).$$

In particular, for $0 < \alpha < 1$, we have

$$\sigma_\Gamma(-\alpha E + mN_\sigma(D; X/Y); X/Y) = m\sigma_\Gamma(D; X/Y)$$

from the triangle inequality

$$\begin{aligned}\sigma_\Gamma(-E + mN_\sigma(D; X/Y); X/Y) \le \sigma_\Gamma(-\alpha E &+ mN_\sigma(D; X/Y); X/Y) \\ &+ (1-\alpha)\sigma_\Gamma(-E; X/Y).\end{aligned}$$

Suppose that $\operatorname{Supp} E \subset \operatorname{Supp} N_\sigma(D; X/Y)$. Then $mN_\sigma(D; X/Y) \geq \alpha E$ for some $0 < \alpha < 1$ and

$$\sigma_\Gamma(-\alpha E + mN_\sigma(D; X/Y); X/Y) = m\sigma_\Gamma(D; X/Y) - \alpha \operatorname{mult}_\Gamma E$$

by **III.1.8**. This is a contradiction.

We shall show that $\mathbf{\Xi} \in \operatorname{Ref}_1(Y)_\mathbb{Q}$ if D is a $\mathbb{Q}$-divisor. It is enough to consider locally on Y. Hence we have only to show that Ξ_0 above is a $\mathbb{Q}$-divisor. For any prime divisor $Q \subset Y$, there is a prime divisor $\Theta \subset X$ with $\Theta \not\subset \operatorname{Supp} N_\sigma(D; X/Y)$ and $f(\Theta) = Q$. Thus

$$\operatorname{mult}_\Theta \Delta = \operatorname{mult}_\Theta f^*\Xi_0 = \operatorname{mult}_\Theta f^*Q \operatorname{mult}_Q \Xi_0.$$

Hence Ξ_0 is a $\mathbb{Q}$-divisor. □

2.26. Corollary *Under the same situation as* **2.25**, *let $\mu\colon Z \to Y$ be a bimeromorphic morphism flattening f, $f'\colon X' \to Z$ a bimeromorphic transform of f by μ from a non-singular variety, and $\nu\colon X' \to X$ the induced bimeromorphic morphism. Then there exists a reflexive $\mathbb{R}$-sheaf $\mathbf{\Xi}_Z$ of rank one on Z such that*

$$\nu^*D \sim_\mathbb{Q} f'^*\mathbf{\Xi}_Z + N_\sigma(\nu^*D; X'/Z).$$

If D is a $\mathbb{Q}$-divisor, then $\mathbf{\Xi}_Z$ is a reflexive $\mathbb{Q}$-sheaf of rank one.

Proof. By **2.25**, there exist a positive integer m, a reflexive $\mathbb{R}$-sheaf $\mathbf{\Xi}_Z$ of rank one on Z, and an f'-exceptional effective $\mathbb{R}$-divisor E' of X' such that

$$m\nu^*D \sim mf'^*\mathbf{\Xi}_Z - E' + mN_\sigma(\nu^*D; X'/Z).$$

If D is a $\mathbb{Q}$-divisor, then $\mathbf{\Xi}_Z \in \mathrm{Ref}_1(Z)_\mathbb{Q}$ by **2.25**. Let X_1 be the normalization of the main component of $X \times_Y Z$ and let $\lambda\colon X' \to X_1$ be the induced morphism. Then $\lambda_*E = 0$. In particular,

$$0 \le m\lambda^*\lambda_*N_\sigma(\nu^*D; X'/Z) = mN_\sigma(\nu^*D; X'/Z) - E'.$$

Hence $E' = 0$ by **2.25**. □

2.27. Lemma *Let $f\colon X \to Y$ be a surjective morphism of normal projective varieties and let D be a pseudo-effective geometrically f-abundant $\mathbb{R}$-Cartier divisor of X. Then $D + f^*H$ is geometrically abundant for any big $\mathbb{R}$-Cartier divisor H of Y. More generally, if $D \succcurlyeq f^*H$, then $D - \varepsilon f^*H$ is geometrically abundant for some $\varepsilon > 0$.*

Proof. We may assume that X and Y are non-singular and that there exist morphisms $h\colon X \to Z$ and $g\colon Z \to Y$ such that Z is a non-singular projective variety, $f = g \circ h$, and that h is the relative Iitaka fibration for D. Let P be the positive part $P_\sigma(D; X/Z)$ of the relative σ-decomposition of D over Z. Then P is pseudo-effective, since $N_\sigma(D; X/Z) \le N_\sigma(D)$. By **2.25** and **2.26**, we may assume that $P \sim_\mathbb{Q} h^*\Xi$ for a pseudo-effective g-big $\mathbb{R}$-divisor Ξ of Z. Here, $\Xi - \Delta$ is g-ample for some effective $\mathbb{R}$-divisor Δ of Z. Hence, for any big $\mathbb{R}$-divisor H of Y, $\Xi - \Delta + kg^*H$ is big for some $k \in \mathbb{N}$. Thus $m\Xi + kg^*H$ is big for any $m \ge 1$. Therefore, $D + f^*H$ is geometrically abundant.

Next, suppose that $D \succcurlyeq f^*H$. It is enough to show that the $\mathbb{R}$-divisor Ξ above is big. For an ample divisor A of X and for any $b > 0$, there exist rational numbers x, $y > b$ such that $xD - yf^*H + A$ is pseudo-effective. Thus

$$xh^*\Xi - yf^*H + cN_\sigma(D; X/Z) + A$$

is pseudo-effective for a constant c by **III.1.9**. Hence, by the same argument as in **II.5.6**-(2), we infer that $\Xi \succcurlyeq g^*H$. Since $\Xi + g^*H$ is big,

$$x\Xi - yg^*H + (\Xi + g^*H)$$

is pseudo-effective for x, $y > b \gg 0$. Thus Ξ is big. □

Applying **2.27** to the case where D is nef, we have:

2.28. Corollary *Let $f\colon X \to Y$ be a surjective morphism of normal projective varieties and let D be a nef and f-abundant $\mathbb{R}$-Cartier divisor. Then $D + f^*H$ is nef and abundant for any nef and big $\mathbb{R}$-Cartier divisor H of Y. More generally, if $D \succcurlyeq f^*H$ in addition, then D is nef and abundant.*

2.29. Definition Let X be a non-singular projective variety. The *numerical Kodaira dimensions* of X of type σ and of type ν, respectively, are defined to be the following numbers:

$$\kappa_\sigma(X) := \kappa_\sigma(K_X) \quad \text{and} \quad \kappa_\nu(X) := \kappa_\nu(K_X).$$

These are birational invariants by **2.7**-(7) and **2.22**-(6). Thus, even for a projective variety V with singularities, we define $\kappa_\sigma(V) := \kappa_\sigma(X)$ and $\kappa_\nu(V) := \kappa_\nu(X)$ for a non-singular model X of V.

Remark If a non-singular projective variety X admits a minimal model $X_{\min}$, then $\kappa_\nu(X) = \kappa_\sigma(X) = \nu(K_{X_{\min}})$.

Conjecture (abundance) K_X is abundant: $\kappa(X) = \kappa_\nu(X)$.

In **4.2** below, we shall show that if K_X is abundant, then K_X is geometrically abundant.

§3. Direct images of canonical sheaves

§3.a. Variation of Hodge structure. A (pure) *Hodge structure* (cf. [**10**]) consists of a free abelian group H of finite rank, a descending filtration

$$\cdots \supset F^p(H_\mathbb{C}) \supset F^{p+1}(H_\mathbb{C}) \supset \cdots$$

of vector subspaces of $H_\mathbb{C} = H \otimes \mathbb{C}$, and an integer w such that

(1) $F^p(H_\mathbb{C}) = H_\mathbb{C}$ for $p \ll 0$ and $F^p(H_\mathbb{C}) = 0$ for $p \gg 0$,
(2) $F^p(H_\mathbb{C}) \oplus \overline{F^{w-p+1}(H_\mathbb{C})} = H_\mathbb{C}$ for any p,

where $^-$ denotes the complex conjugate. If we set $H^{p,q} := F^p(H_\mathbb{C}) \cap \overline{F^q(H_\mathbb{C})}$, then $H^{p,q} = 0$ unless $p + q \neq w$, $H_\mathbb{C} = \bigoplus_{p+q=w} H^{p,q}$, and $F^p(H_\mathbb{C}) = \bigoplus_{i \geq p} H^{i,w-i}$. The filtration $\{F^p(H_\mathbb{C})\}$ is called the *Hodge filtration* and w is called the *weight*. A *polarization* (defined over $\mathbb{Q}$) of the Hodge structure is a non-degenerate bilinear form $Q\colon H \times H \to \mathbb{Q}$ satisfying the following conditions:

(1) Q is symmetric if w is even, and is skew-symmetric if w is odd;
(2) $Q(F^p(H_\mathbb{C}), F^{w-p+1}(H_\mathbb{C})) = 0$;
(3) $(\sqrt{-1})^{p-q} Q(x, \overline{x}) > 0$ for any $0 \neq x \in H^{p,q}$.

The map $C\colon H_\mathbb{C} \to H_\mathbb{C}$ defined by $Cx = (\sqrt{-1})^{p-q}x$ for $x \in H^{p,q}$ is called the *Weil operator*, which is defined over $\mathbb{R}$.

An example of Hodge structure is the cohomology group $H^w(M, \mathbb{Z})$ modulo torsion for a compact Kähler manifold M. It is of weight w and the Hodge filtration is given by the hyper-cohomology group

$$F^p(H^w(M, \mathbb{C})) = \mathbb{H}^w(M, \sigma_{\geq p}\Omega^\bullet_M) \simeq \bigoplus_{i \geq p} H^{w-i}(M, \Omega^i_M)$$

for the complex

$$\sigma_{\geq p}\Omega^\bullet_M := [\cdots \to 0 \to 0 \to \Omega^p_M \xrightarrow{\mathrm{d}} \Omega^{p+1}_M \xrightarrow{\mathrm{d}} \Omega^{p+2}_M \to \cdots]$$

for p. If M is a projective variety and if $l = c_1(A) \in H^2(M, \mathbb{Z})$ is the Chern class of an ample divisor A, then we have the Hard Lefschetz theorem: the homomorphism

$$L^i := (\cup l)^i \colon H^{n-i}(M, \mathbb{Q}) \to H^{n+i}(M, \mathbb{Q})$$

given by the cup-product with $l^i = l \cup l \cup \cdots \cup l$ is isomorphic for $0 \le i \le n$. For $w \le n$, the primitive cohomology group $P^w(M, \mathbb{Z})$ is defined as the kernel of

$$L^{n-w+1} \colon H^w(M, \mathbb{Z}) \to H^{2n-w+2}(M, \mathbb{Z})$$

modulo torsion. Then we have the Lefschetz decomposition

$$H^w(M, \mathbb{Q}) = \bigoplus\nolimits_{i \ge 0} L^i P^{w-2i}(M, \mathbb{Q}).$$

The primitive cohomology $P^w(M, \mathbb{Z})$ has a Hodge structure by

$$P^{p,q}(M, \mathbb{Z}) = P^{p+q}(M, \mathbb{Z}) \cap H^{p,q}(M)$$

and has a polarization given by

$$Q_w(x, y) = (-1)^{w(w-1)/2} x \cup y \cup l^{n-w}[M].$$

Thus $H^w(M, \mathbb{Q})$ also has a polarization as the direct sum of the polarizations on $P^{w-2i}(M, \mathbb{Q})$.

Let S be a complex analytic manifold. A *variation of Hodge structure* (cf. [**32**]) of weight w on S consists of a locally constant system H of free abelian groups of finite rank on S and a descending filtration

$$\cdots \supset \mathcal{F}^p(\mathcal{H}) \supset \mathcal{F}^{p+1}(\mathcal{H}) \supset \cdots$$

of holomorphic subbundles of $\mathcal{H} = H \otimes_{\mathbb{Z}} \mathcal{O}_S$ such that

(1) H_s and $F^p_s = \mathcal{F}^p(\mathcal{H}) \otimes \mathbb{C}(s)$ form a Hodge structure of weight w for any point $s \in S$,
(2) the connection $\nabla \colon \mathcal{H} \to \Omega^1_S \otimes \mathcal{H}$ associated with H induces
$$\nabla(\mathcal{F}^p(\mathcal{H})) \subset \Omega^1_S \otimes \mathcal{F}^{p-1}(\mathcal{H})$$
for any p.

The second condition is called the Griffiths transversality condition. A polarization of the variation of Hodge structure is a locally constant bilinear from $Q \colon H \times H \to \mathbb{Q}_S$ whose fiber $Q_s \colon H_s \times H_s \to \mathbb{Q}$ is a polarization of the Hodge structure H_s. An example of variation of Hodge structure is the higher direct image sheaf $R^w f_* \mathbb{Z}_X$ modulo torsion for a proper smooth surjective morphism $f \colon X \to S$ from a Kähler manifold X. If f is projective and if $l \in H^0(S, R^2 f_* \mathbb{Q}_X)$ is induced from an f-ample line bundle, then the primitive part of $R^w f_* \mathbb{Z}_X$ for $w \le \dim X - \dim S$ admits a polarization.

Let $H = (H, F^p(H_{\mathbb{C}}), Q)$ be a polarized Hodge structure of weight w. We consider groups

$$G_{\mathfrak{K}} := \{g \in \operatorname{Aut}(H_{\mathfrak{K}}) \mid Q(gx, gy) = Q(x, y)\}$$

for $\mathfrak{K} = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. Then $G_{\mathbb{C}}$ is a complex algebraic group and $G_{\mathbb{Z}}$ is a discrete subgroup. Let $\check{\mathcal{D}}$ and $\mathcal{D}$ be the following sets of descending filtrations $\{F^p\}$ of vector subspaces of $H_{\mathbb{C}}$:

$$\check{\mathcal{D}} : = \big\{\{F^p\} \mid \dim F^p = \dim F^p(H_{\mathbb{C}}),\ Q(F^p, F^{w-p+1}) = 0\big\},$$
$$\mathcal{D} : = \big\{\{F^p\} \in \check{\mathcal{D}} \mid (\sqrt{-1})^{p-q} Q(x, \overline{x}) > 0 \text{ for non-zero } x \in F^p \cap \overline{F^{w-p}}\big\}.$$

Then the Hodge filtration $\{F^p(H_{\mathbb{C}})\}$ defines an element o of $\mathcal{D}$. We write $F^p_o = F^p(H_{\mathbb{C}})$, $H^{p,q}_o := F^p_o \cap \overline{F^q_o}$, and $H_o := (H, F^p_o)$. The set $\check{\mathcal{D}}$ has a structure of complex projective manifold and $\mathcal{D}$ is an open subset, which is regarded as the classifying space of Hodge structures on the abelian group H with the polarization Q. There are a natural transitive action of $G_{\mathbb{C}}$ on $\check{\mathcal{D}}$ and that of $G_{\mathbb{R}}$ on $\mathcal{D}$. Let B be the stabilizer of $G_{\mathbb{C}}$ at o:

$$B := \{g \in G_{\mathbb{C}} \mid g(F^p_o) = F^p_o\}.$$

Then B is an algebraic subgroup and $\check{\mathcal{D}}$ is regarded as the homogeneous space $G_{\mathbb{C}}/B$. The intersection $V = B \cap G_{\mathbb{R}}$ preserves the Hodge structure H_o. Thus Q and the Weil operator C_o of H_o are preserved. Hence V is contained in a unitary group and is compact. The tangent space of the homogeneous space $\mathcal{D} = G_{\mathbb{R}}/V$ at o is isomorphic to $\mathfrak{g}/\mathfrak{b}$ for

$$\mathfrak{g} := \{T \in \operatorname{End}(H_{\mathbb{C}}) \mid Q(Tx, y) + Q(x, Ty) = 0\}, \quad \mathfrak{b} := \{T \in \mathfrak{g} \mid T(F^p_o) \subset F^p_o\},$$

where $\mathfrak{g}$ and $\mathfrak{b}$ are the Lie algebras of $G_{\mathbb{C}}$ and B, respectively. We have the decomposition

$$\mathfrak{g} = \bigoplus\nolimits_{p \in \mathbb{Z}} \mathfrak{g}^{p,-p} \quad \text{and} \quad \mathfrak{b} = \bigoplus\nolimits_{p \geq 0} \mathfrak{g}^{p,-p},$$

for subspaces

$$\mathfrak{g}^{p,-p} := \{T \in \mathfrak{g} \mid TH^{r,s}_o \subset H^{r+p,s-p}_o\}.$$

We also have an injection

$$\mathfrak{g}/\mathfrak{b} \hookrightarrow \bigoplus\nolimits_{p \geq 0} \operatorname{Hom}(F^p_o, H_{\mathbb{C}}/F^p_o).$$

Let $H = (H, \mathcal{F}^\bullet(\mathcal{H}), Q)$ be a polarized variation of Hodge structure defined on a complex analytic manifold S. Let $\tau \colon \widetilde{S} \to S$ be the universal covering map. Let us fix a point $s \in S$ and denote the polarized Hodge structure $(H_s, F^p_s = \mathcal{F}^p(\mathcal{H}) \otimes \mathbb{C}(s), Q_s)$ by (H_o, F^p_o, Q_o). Then $\tau^{-1}H \simeq H_o \otimes \mathbb{Z}_{\widetilde{S}}$, $\tau^{-1}(\mathcal{F}^p(\mathcal{H}))$ is a subbundle of the trivial vector bundle $H_o \otimes \mathcal{O}_{\widetilde{S}}$, and we have a *period* map $p \colon \widetilde{S} \to \mathcal{D}$ into the classifying space $\mathcal{D}$ of Hodge structures on H_o compatible with Q_o.

We have also a *monodromy* representation $\rho \colon \pi_1(S, s) \to G_{\mathbb{Z}}$ compatible with p: it satisfies $p(\gamma z) = \rho(\gamma)p(z)$ for $z \in \widetilde{S}$ and $\gamma \in \pi_1(S, s)$. For a point $\tilde{s} \in \widetilde{S}$ over s, the tangent map of p at $\tilde{s}$ is written as

$$\Theta_{\widetilde{S}, \tilde{s}} \simeq \Theta_{S,s} \to (\mathfrak{b} \oplus \mathfrak{g}^{-1,1})/\mathfrak{b} \subset \mathfrak{g}/\mathfrak{b}$$

by the Griffith transversality. The composite

$$\Theta_{S,s} \to \mathfrak{g}/\mathfrak{b} \to \operatorname{Hom}(F^p_o, H_{\mathbb{C}}/F^p_o)$$

is given by the $\mathcal{O}_S$-linear map

$$\mathcal{F}^p(\mathcal{H}) \xrightarrow{\nabla} \Omega^1_S \otimes \mathcal{H} \to \Omega^1_S \otimes (\mathcal{H}/\mathcal{F}^p(\mathcal{H})).$$

Suppose that S is isomorphic to the Zariski-open subset $M \smallsetminus D$ for a complex analytic manifold M and a normal crossing divisor D. Then the local monodromies of H around D is quasi-unipotent by a lemma of Borel (cf. [**126**, 4.5]). Let ${}^{\ell}\mathcal{H}^{\mathrm{can}}$ be the *lower-canonical extension* (cf. [**71**], [**92**]) of $\mathcal{H}$, which is called the canonical extension in the sense of Deligne. The *upper-canonical extension* ${}^{u}\mathcal{H}^{\mathrm{can}}$ is defined as the dual of the lower-canonical extension of the dual $\mathcal{H}^{\vee}$. If the local monodromies of H are unipotent, then two canonical extensions coincide with each other, and are denoted by $\mathcal{H}^{\mathrm{can}}$. For $\bullet = \ell$ and u, ∇ extends to a *logarithmic connection* (cf. [**13**]):

$$\nabla\colon {}^{\bullet}\mathcal{H}^{\mathrm{can}} \to \Omega^1(\log D) \otimes {}^{\bullet}\mathcal{H}^{\mathrm{can}}.$$

We set

$$\mathcal{F}^p({}^{\bullet}\mathcal{H}^{\mathrm{can}}) := j_*\mathcal{F}^p(\mathcal{H}) \cap {}^{\bullet}\mathcal{H}^{\mathrm{can}} \subset j_*\mathcal{H}$$

for the open immersion $j\colon S \hookrightarrow M$. Then $\mathcal{F}^p({}^{\bullet}\mathcal{H}^{\mathrm{can}})$ are locally free $\mathcal{O}_M$-modules and are subbundles of ${}^{\bullet}\mathcal{H}^{\mathrm{can}}$. This is a consequence of the nilpotent orbit theorem by Schmid [**126**, 4.12].

3.1. Definition A locally free sheaf of a projective variety is called *numerically semi-positive* if its tautological line bundle is nef.

Kawamata [**50**, §4] has proved the following semi-positivity for variations of Hodge structure:

3.2. Theorem *Let M be a compact Kähler manifold, D a normal crossing divisor, and let H be a polarized variation of Hodge structure of weight w defined on $S = M \smallsetminus D$. Suppose that $\mathcal{F}^0(\mathcal{H}) = \mathcal{H}$, $\mathcal{F}^{w+1}(\mathcal{H}) = 0$, and that H has only unipotent local monodromies along D. Then $\mathcal{F}^w(\mathcal{H}^{\mathrm{can}}) \otimes \mathcal{O}_C$ is a numerically semi-positive vector bundle for any compact curve $C \subset M$. In particular, if M is projective, then $\mathcal{F}^w(\mathcal{H}^{\mathrm{can}})$ is numerically semi-positive.*

For the proof of **3.2**, we may assume that D is a simple normal crossing divisor by a suitable blowing-up of M, since the canonical extension is compatible with pulling back for variations of Hodge structure with unipotent local monodromies.

Kawamata [**53**, Theorem 3] has proved another positivity:

3.3. Theorem *Under the same situation as* **3.2**, *if*

$$\Theta_{S,s} \to \mathrm{Hom}(F^w_s, F^{w-1}_s/F^w_s)$$

is injective at a point $s \in S$, then $\det(\mathcal{F}^w(\mathcal{H}^{\mathrm{can}}))^n > 0$, where $n = \dim S = \dim M$.

Kollár [**72**] pointed out a gap in the proof of [**53**, Theorem 3] and gave a modification. Kawamata's original modification was mentioned there, but it does not seem to be published yet. These modifications are applications of SL_2-orbit theorem of several complex variables (cf. [**7**], [**48**], [**49**]).

It is natural to consider the following:

3.4. Conjecture In the situation of **3.2**, if M is projective, then the line bundle $\det(\mathcal{F}^w(\mathcal{H}^{\mathrm{can}}))$ is nef and abundant.

This is considered as a version of the abundance conjecture. We have a partial answer as follows:

3.5. Proposition *The conjecture* **3.4** *is true if* $w \leq 2$.

PROOF. By assumption, the natural homomorphism

$$(\mathfrak{b} \oplus \mathfrak{g}^{-1,1})/\mathfrak{b} \to \operatorname{Hom}(F^w_o, F^{w-1}_o/F^w_o)$$

is injective. In fact, if F^w is given, then F^1 is determined by

$$F^1 = \{x \in H_{\mathbb{C}} \mid Q(x, F^w) = 0\}.$$

We may assume that D is a simple normal crossing divisor by the same reason as above. If the local monodromy around a prime component D_i is trivial, then H extends to $M \smallsetminus \operatorname{Supp}(D - D_i)$ as a variation of Hodge structure. Hence we may assume that all the local monodromies around any prime component D_i are non-trivial. Let Γ be the image of the monodromy representation $\rho\colon \pi_1(S, s) \to G_{\mathbb{Z}}$. Then Γ is a discrete subgroup of $G_{\mathbb{R}}$ and the quotient $\Gamma\backslash\mathcal{D}$ exists as a normal complex analytic space, since $V \cap \Gamma$ is a finite group. The period mapping $p\colon \widetilde{S} \to \mathcal{D}$ induces $\pi\colon S \to \Gamma\backslash\mathcal{D}$. We infer that π is a proper morphism by [**32**, III, 9.6] or by the nilpotent orbit theorem [**126**, 4.12]. By [**53**, Theorem 11], there exist a birational morphism $\nu\colon M' \to M$ from a non-singular projective variety, a fiber space $\pi'\colon M' \to Z$ onto a non-singular projective variety, an open subset $Z^\star \subset Z$, and a generically finite proper surjective morphism $\tau\colon Z^\star \to \Gamma\backslash\mathcal{D}$ such that

$$\nu^{-1}(S) = \pi'^{-1}(Z^\star) \quad \text{and} \quad \tau \circ \pi'|_{\nu^{-1}(S)} = \pi \circ \nu|_{\nu^{-1}(S)}.$$

Let W be the image of $(\nu, \pi')\colon M' \to M \times Z$. By considering the flattening of π', we may assume that any π'-exceptional divisor is exceptional for $M' \to W$. Let F be a general fiber of π. For the numerically semi-positive vector bundle $\mathcal{F}^w(\mathcal{H}^{\mathrm{can}})$, the restriction $\mathcal{F}^w(\mathcal{H}^{\mathrm{can}}) \otimes \mathcal{O}_F$ is a flat vector bundle with only finite monodromies, since it is associated with a constant variation of Hodge structure. Hence

$$\nu^* \det(\mathcal{F}^w(\mathcal{H}^{\mathrm{can}}))^{\otimes m} \simeq \pi'^*\mathcal{L} \otimes \mathcal{O}_{M'}(-E)$$

for an invertible sheaf $\mathcal{L}$ of Z, a π'-exceptional effective divisor E, and a positive integer m. Since E is exceptional for $M' \to W$, we have

$$\nu^* \det(\mathcal{F}^w(\mathcal{H}^{\mathrm{can}}))^{\otimes m} \simeq \pi'^*\mathcal{L}$$

and thus $\mathcal{L}$ is nef. Let $Y \subset M$ be the complete intersection of general smooth ample divisors with $\dim Y = \dim Z$. Then $p\colon Y \cap S \to \Gamma\backslash\mathcal{D}$ is generically finite. Thus

$$\Theta_{Y,y} \to \operatorname{Hom}(\mathcal{F}^w(\mathcal{H}) \otimes \mathbb{C}(y), (\mathcal{F}^{w-1}(\mathcal{H})/\mathcal{F}^w(\mathcal{H})) \otimes \mathbb{C}(y))$$

is injective for a general point $y \in Y \cap S$. Hence

$$\det(\mathcal{F}^w(\mathcal{H}^{\mathrm{can}}))^{\dim Y} \cdot Y > 0 \quad \text{and} \quad \mathcal{L}^{\dim Z} > 0,$$

by **3.3**. Therefore, $\det(\mathcal{F}^w(\mathcal{H}^{\mathrm{can}}))$ is nef and abundant. $\square$

By applying a similar argument to the Kuranishi space of a compact complex manifold, we have:

3.6. Proposition *Under the same situation as* **3.2**, *assume that M is projective and the variation of Hodge structure H is isomorphic to $R^w f_* \mathbb{Z}_X$ modulo torsion for a projective smooth morphism $f: X \to S$ with $w = \dim X - \dim S$. Assume in addition that, for the fiber $F = f^{-1}(s)$, the homomorphism*

$$H^1(F, \Theta_F) \to \operatorname{Hom}(H^0(F, \Omega_F^w), H^1(F, \Omega_F^{w-1}))$$

given by the cup-product is injective. Then $\det(\mathcal{F}^w(\mathcal{H}^{\mathrm{can}}))$ *is nef and abundant.*

§3.b. ω-sheaves. Let $f: X \to Y$ be a proper surjective morphism from a normal variety into a non-singular variety. For the dualizing sheaves ω_X and ω_Y, we denote $\omega_{X/Y} := \omega_X \otimes f^* \omega_Y^{-1}$ and call it the *relative dualizing sheaf*. For the twisted inverse $f^!$ (cf. [**37**], [**116**], [**117**]), we have

$$f^! \mathcal{O}_Y \sim_{\mathrm{qis}} \omega_X^{\bullet}[-\dim Y] \overset{\mathrm{L}}{\otimes} f^* \omega_Y^{-1}.$$

In particular, $\mathcal{H}^{-d}(f^! \mathcal{O}_Y) \simeq \omega_{X/Y}$ for $d = \dim X - \dim Y$ and $\mathcal{H}^{-k}(f^! \mathcal{O}_Y) = 0$ for $k > d$.

We recall the following results on the higher direct images of dualizing sheaves proved by [**71**], [**97**], [**13**], [**92**], [**121**], [**122**], [**135**].

3.7. Theorem *Let $f: X \to Y$ be a proper surjective morphism of complex analytic varieties with $d := \dim X - \dim Y$. Suppose that X is a Kähler manifold. Then the following properties hold*:

(1) (Torsion-freeness) $\mathrm{R}^i f_* \omega_X$ *is a torsion free sheaf for any i*;
(2) (Vanishing) *Let $g: Y \to Z$ be a projective surjective morphism and let $\mathcal{H}$ be a g-ample invertible sheaf. Then, for any $p > 0$ and $i \geq 0$,*

$$\mathrm{R}^p g_*(\mathcal{H} \otimes \mathrm{R}^i f_* \omega_X) = 0;$$

(3) (Injectivity) *In the situation of* (2) *above, suppose that Z is Stein. Let $s \in \mathrm{H}^0(Y, \mathcal{H}^{\otimes l})$ be a non-zero section for an integer $l > 0$. Then the induced homomorphism*

$$\mathrm{H}^p(X, \omega_X \otimes f^* \mathcal{H}) \xrightarrow{\otimes f^* s} \mathrm{H}^p(X, \omega_X \otimes f^* \mathcal{H}^{\otimes (l+1)})$$

is injective for any $p \geq 0$;
(4) (Hodge filtration) *Suppose that Y is non-singular and f is smooth outside a normal crossing divisor $D \subset Y$. For $i \geq 0$, let ${}^u\mathcal{H}^{d+i}$ be the upper-canonical extension for the variation of Hodge structure*

$$H^{d+i} = (\mathrm{R}^{d+i} f_* \mathbb{Z}_X)|_{Y \setminus D}.$$

Then there is an isomorphism

$$\mathrm{R}^i f_* \omega_{X/Y} \simeq \mathcal{F}^d({}^u\mathcal{H}^{d+i}).$$

(5) (Splitting) *Suppose that $d > 0$ and let $Z \subset X$ be an f-ample non-singular divisor. Then the surjection*

$$f_*\omega_Z \twoheadrightarrow \mathrm{R}^1 f_*\omega_X$$

derived from the short exact sequence

$$0 \to \omega_X \to \omega_X(Z) = \omega_X \otimes \mathcal{O}_X(Z) \to \omega_Z \to 0$$

admits a splitting;

(6) (Decomposition) *In the derived category $D_c(\mathcal{O}_Y)$ of $\mathcal{O}_Y$-modules with coherent cohomologies,*

$$\mathrm{R}\, f_*\omega_X \sim_{\mathrm{qis}} \bigoplus\nolimits_{i=0}^{d} \mathrm{R}^i f_*\omega_X[-i].$$

Remark Kawamata [**50**] showed (4) for $i = 0$ by applying some results of Schmid [**126**]. Kollár [**71**] proved **3.7** in the case: X and Y are projective. The argument in [**71**, I] implies (1) and (3) also in the case: X is compact Kähler and Y is projective. Esnault–Viehweg [**13**] gave simple proofs of (1), (2), and (3) in the same case. The assertion (4) in the algebraic case was proved by a different argument in [**97**], which is effective for other cases. Moriwaki [**92**] proved (1) in the case: f is a projective morphism, by applying the relative Kodaira vanishing theorem **II.5.12**. The assertion (5) is derived from (4) by the same argument as [**71**, II]. If X is projective, then (6) follows from (5). On the other hand, Saito [**119**] developed the theory of Hodge modules and proved **3.7** in the case: f is a projective morphism, in [**120**] (cf. [**122**]), where (6) is derived from the decomposition of related perverse sheaves. In the case: f is a Kähler morphism, **3.7** is proved implicitly in [**122**]. Takegoshi [**135**] also proved the Kähler case by an L^2-method and by analyzing the Hodge $*$-operator. Takegoshi's result is more general than **3.7**; in the most statements, ω_X can be replaced with $\omega_X \otimes \mathcal{E}$ for a Nakano-semi-positive vector bundle $\mathcal{E}$.

3.8. Definition A coherent sheaf $\mathcal{F}$ of a complex analytic variety Y is called an *ω-sheaf* if there exists a proper morphism $f\colon X \to Y$ from a non-singular Kähler space such that $\mathcal{F}$ is a direct summand of $\mathrm{R}^i f_*\omega_X$ for some i.

An ω-sheaf $\mathcal{F}$ is a torsion-free $\mathcal{O}_Y$-module if $\operatorname{Supp}\mathcal{F} = Y$.

Remark (cf. [**71**]) If $f\colon X \to Y$ is a morphism from a non-singular projective variety, then $\mathrm{R}^i f_*\omega_X$ is a direct summand of $h_*\omega_Z$ for another morphism $h\colon Z \to Y$ from a non-singular projective variety. This is shown as follows: let $Z \subset X$ be a non-singular ample divisor and let us consider the exact sequence

$$0 \to \omega_X \to \omega_X(Z) \to \omega_Z \to 0.$$

By **3.7**-(5), $\mathrm{R}^1 f_*\omega_X$ is a direct summand of $f_*\omega_Z$. We have $\mathrm{R}^i f_*\omega_X(Z) = 0$ for $i > 0$ by the relative Kodaira vanishing theorem **II.5.12**. Hence $\mathrm{R}^{i-1} f_*\omega_Z \simeq \mathrm{R}^i f_*\omega_X$ for $i \geq 2$. Thus we are done by induction.

Remark It may be possible to generalize the notion of ω-sheaves in terms of Hodge modules, etc. For example, it is expected that we can include in "ω-sheaves" the sheaves of the form $\mathcal{F}^d({}^u\mathcal{H})\otimes\omega_M$, where M is a Kähler manifold and $\mathcal{F}^d({}^u\mathcal{H})$ is the bottom filtration of the upper canonical extension ${}^u\mathcal{H}$ of an abstract polarized variation of Hodge structure defined outside a normal crossing divisor of M.

3.9. Corollary *Let $f\colon X \to Y$ be a morphism of complex analytic varieties and let $\mathcal{F}$ be an ω-sheaf on X. Then the following properties hold*:

(1) (Torsion-freeness) $\mathrm{R}^i f_*\mathcal{F}$ *is an ω-sheaf for any i*;
(2) (Vanishing) *Let $g\colon Y \to Z$ be a projective morphism and let $\mathcal{H}$ be a g-ample invertible sheaf. Then, for any $p > 0$ and $i \geq 0$,*
$$\mathrm{R}^p g_*(\mathcal{H}\otimes \mathrm{R}^i f_*\mathcal{F}) = 0;$$
(3) (Decomposition) *In the derived category $D_c(\mathcal{O}_Y)$,*
$$\mathrm{R} f_*\mathcal{F} \sim_{\mathrm{qis}} \bigoplus\nolimits_{i\geq 0} \mathrm{R}^i f_*\mathcal{F}[-i].$$

Proof. Suppose that $\mathcal{F}$ is a direct summand of $\mathrm{R}^j h_*\omega_M$ for a morphism $h\colon M \to X$ from a Kähler manifold. Then $\mathrm{R}^i f_*\mathcal{F}$ is a direct summand of $\mathrm{R}^{i+j}(f\circ h)_*\omega_M$ by **3.7**-(6). Hence (1) and (2) hold for $\mathcal{F}$. By **3.7**-(6) for $\mathrm{R}\, h_*\omega_M$ and by a projection $\mathrm{R}^j h_*\omega_M \to \mathcal{F}$, we have a projection
$$\mathrm{R}^{i+j}(f\circ h)_*\omega_M \to \mathrm{R}^i f_*(\mathrm{R}^j h_*\omega_M) \to \mathrm{R}^i f_*\mathcal{F}$$
for any i such that the composite
$$\mathrm{R}^i f_*\mathcal{F} \to \mathrm{R}^i f_*(\mathrm{R}^j h_*\omega_M) \to \mathrm{R}^{i+j}(f\circ h)_*\omega_M \to \mathrm{R}^i f_*\mathcal{F}$$
is identical. Hence we have a quasi-isomorphism
$$\mathrm{R} f_*\mathcal{F} \to \mathrm{R}(f\circ h)_*\omega_M[j] \to \bigoplus\nolimits_{i\geq 0} \mathrm{R}^i(f\circ h)_*\omega_M[-i+j] \to \bigoplus\nolimits_{i\geq 0} \mathrm{R}^i f_*\mathcal{F}[-i]. \ \square$$

3.10. Lemma *Let X be a non-singular variety and let L be a $\mathbb{Q}$-divisor with* $\mathrm{Supp}\langle L\rangle$ *being normal crossing. Suppose either that $mL \sim 0$ or that $mL \sim D$ for a non-singular divisor D for some $m \geq 2$ in which any component of D is not contained in* $\mathrm{Supp}\langle L\rangle$ *and $D \cup \mathrm{Supp}\langle L\rangle$ is a normal crossing divisor. Then there exists a generically finite proper surjective morphism $M \to X$ from a non-singular variety M such that $\omega_X(\ulcorner L\urcorner) = \mathcal{O}_X(K_X + \ulcorner L\urcorner)$ is a direct summand of $f_*\omega_M$. In particular, if X is Kähler, then $\omega_X(\ulcorner L\urcorner)$ restricted to a relatively compact open subset is an ω-sheaf.*

Proof. First we consider the case: $mL \sim 0$. By applying **II.5.10**, we have a cyclic covering $\tau\colon V \to X$ from a normal analytic space with only quotient singularities such that
$$\tau_*\omega_V \simeq \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X\left(K_X + \ulcorner iL\urcorner\right).$$
Let $\mu\colon Y \to V$ be a Hironaka's desingularization. It is a finite succession of blowups over a relatively compact open subset of V. Let $f\colon Y \to X$ be the composite. Then

$\omega_X(\ulcorner L \urcorner)$ is a direct summand of $f_*\omega_Y$, since $\mu_*\omega_Y \simeq \omega_V$. Since $\omega_X(\ulcorner L \urcorner)$ is of rank one, it is a direct summand of $f_*\omega_M$ for a connected component M of Y. If X is Kähler, then $f^{-1}U$ is Kähler for any relatively compact open subset $U \subset X$, since f is a projective morphism over U. Thus $\omega_X(\ulcorner L \urcorner)|_U$ is an ω-sheaf.

Next, we consider the other case. Then $\operatorname{Supp}\langle L' \rangle$ is normal crossing and $mL' \sim 0$ for the $\mathbb{Q}$-divisor $L' := L - (1/m)D$. Thus $\omega_X(\ulcorner L \urcorner) = \omega_X(\ulcorner L' \urcorner)$ and the assertion follows from the first case. $\square$

3.11. Proposition *Let $\pi\colon X \to S$ be a proper surjective morphism from a non-singular variety into a Stein space and let L be an $\mathbb{R}$-divisor of X such that* $\operatorname{Supp}\langle L \rangle$ *is a normal crossing divisor. Suppose either*

(1) *L is a π-semi-ample $\mathbb{Q}$-divisor, or*
(2) *π is a projective morphism and L is π-nef and π-abundant.*

Then, for a relatively compact open subset $S_c \subset S$ and for the pullback $X_c = \pi^{-1}S_c$, there exist

- *a generically finite proper surjective morphism $\phi\colon M \to X_c$ from a non-singular variety*
- *a projective surjective morphism $h\colon Z \to S_c$ from a non-singular variety with* $\dim Z = \dim S + \kappa(L; X/S)$*,*
- *a proper surjective morphism $f\colon M \to Z$ over S_c, and*
- *an h-ample divisor H of Z*

such that $\omega_{X_c}(\ulcorner L \urcorner)$ is a direct summand of $\phi_*\omega_M(f^*H)$. In particular, if X is Kähler, then $\omega_X(\ulcorner L \urcorner)$ restricted to any relatively compact open subset of X is an ω-sheaf.

Proof. In the proof, we replace S by a relatively compact open subset freely without mentioning it. By **II.4.3**, we may replace X and L with X' and L', respectively by a generically finite proper surjective morphism $\rho\colon X' \to X$ and $L' = \rho^*L$. In fact, $\mathcal{O}_X$ is a direct summand of $\rho_*\mathcal{O}_{X'}(R_\rho)$ for the ramification divisor R_ρ and **II.4.3** implies that $\omega_X(\ulcorner L \urcorner)$ is a direct summand of $\rho_*\omega_{X'}(\ulcorner L' \urcorner)$.

In the case (2), we may assume that there exist a projective morphism $h_1\colon Z_1 \to S$ from a non-singular variety, a surjective morphism $f_1\colon X \to Z_1$ over S, and an h_1-nef and h_1-big $\mathbb{R}$-divisor D of Z_1 such that $L \sim_{\mathbb{Q}} f_1^*D$ by the same argument as **2.3-**(1). In the case (1), we also have the same morphisms $h_1\colon X \to Z_1$, $f_1\colon X \to Z_1$, and the same $\mathbb{R}$-divisor D with $L \sim_{\mathbb{Q}} f_1^*D$, where D is a $\mathbb{Q}$-divisor.

We may also assume that there is an effective $\mathbb{R}$-divisor B of Z_1 such that

- $H_1 := D - B$ is an h_1-ample $\mathbb{Q}$-divisor,
- $\ulcorner L - f_1^*B \urcorner = \ulcorner L \urcorner$,
- $\operatorname{Supp} f_1^*(B) \cup \operatorname{Supp}\langle L \rangle$ is a normal crossing divisor.

Then $L_1 := L - f_1^*B \sim_{\mathbb{Q}} f_1^*(H_1)$ is a π-semi-ample $\mathbb{Q}$-divisor such that $\operatorname{Supp}\langle L_1 \rangle$ is normal crossing and $\ulcorner L \urcorner = \ulcorner L_1 \urcorner$.

Let $\lambda\colon Z \to Z_1$ be a finite surjective morphism from a non-singular variety such that $H := \lambda^*(H_1)$ is a $\mathbb{Z}$-divisor (cf. **II.5.11**). Let $Y \to X \times_{Z_1} Z$ be a bimeromorphic morphism from a non-singular variety into the main component of $X \times_{Z_1} Z$ and let $\psi\colon Y \to X$ and $f_2\colon Y \to Z$ be the induced morphisms. Then $\psi^* L \sim_{\mathbb{Q}} f_2^* H$. We can take Y so that $\operatorname{Supp} \psi^*\langle L\rangle$ is normal crossing. Let $m > 1$ be an integer such that $\psi^*(mL)$ is Cartier and $\psi^*(mL) \sim f_2^*(mH)$. Then, by **II.5.10**, we have a cyclic covering $\tau\colon V \to Y$ from a normal complex analytic space V with only quotient singularities such that

$$\tau_*\omega_V \simeq \omega_Y \otimes \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_Y\left(\ulcorner i\psi^* L \urcorner - i f_2^* H\right).$$

Thus $\omega_Y(\ulcorner \psi^* L \urcorner)$ is a direct summand of $\tau_*\omega_V(\tau^* f_2^* H)$. Since V has only rational singularities, ω_V is isomorphic to the direct image of the dualizing sheaf of a desingularization $M \to V$. Let $\phi\colon M \to X$ and $f\colon M \to Z$ be the induced morphisms. Then $\omega_X(\ulcorner L \urcorner)$ is a direct summand of $\phi_*\omega_M(f^* H)$. Since $\omega_X(\ulcorner L \urcorner)$ is of rank one, we can replace M by a connected component. □

3.12. Corollary *Let $\pi\colon X \to S$ be a projective surjective morphism from a normal variety into a Stein space. Let Δ and L be an effective $\mathbb{R}$-divisor and a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor, respectively, on X. Suppose that (X, Δ) is log-terminal and $L - (K_X + \Delta)$ is π-nef and π-abundant. Then the reflexive sheaf $\mathcal{O}_X(L)$ restricted to any relatively compact open subset of X is an ω-sheaf. Furthermore, for a relatively compact open subset $S_c \subset S$ and for the pullback $X_c = \pi^{-1} S_c$, there exist*

- *a generically finite surjective morphism $\phi\colon M \to X_c$ from a non-singular variety,*
- *a projective morphism $h\colon Z \to S_c$ from a non-singular variety with* $\dim Z - \dim S = \kappa(L - (K_X + \Delta); X/S)$,
- *a surjective morphism $f\colon M \to Z$ over S_c, and*
- *an h-ample divisor H of Z*

such that $\mathcal{O}_{X_c}(L)$ is a direct summand of $\phi_*\omega_M(f^* H)$.

Proof. We also replace S by a relatively open subset freely. Let $\mu\colon X' \to X$ be a bimeromorphic morphism from a non-singular variety projective over S such that the union of the proper transform of Δ and the μ-exceptional locus is a normal crossing divisor. Then $\ulcorner R \urcorner$ is a μ-exceptional effective divisor for $R := K_{X'} - \mu^*(K_X + \Delta)$. Now

$$\mu^* L + R - K_Y = \mu^*(L - (K_X + \Delta))$$

is $(\pi \circ \mu)$-nef and $(\pi \circ \mu)$-abundant. Therefore, by **3.11**, $\mathcal{O}_{X'}(\ulcorner \mu^* L + R \urcorner)$ is an ω-sheaf. Since $\ulcorner \mu^* L + R \urcorner \geq \llcorner \mu^* L \lrcorner$, we have

$$\mathcal{O}_X(L) \simeq \mu_* \mathcal{O}_{X'}(\ulcorner \mu^* L + R \urcorner). \qquad \square$$

The following is a generalization of **3.7**-(3) and also is that of a similar injectivity obtained in [**55**]:

3.13. Proposition (Injectivity) *Let $\pi\colon X \to S$ be a proper surjective morphism from a Kähler manifold and let L and D be $\mathbb{R}$-divisors X such that D is effective, and* $\mathrm{Supp}\langle L\rangle$ *and* $\mathrm{Supp}\langle L+D\rangle$ *are normal crossing. Suppose that one of the following two conditions is satisfied*:

(1) *L is a π-semi-ample $\mathbb{Q}$-divisor and $\kappa(aL - D; X/S) \geq 0$;*
(2) *π is a projective morphism and L is a π-nef and π-abundant $\mathbb{R}$-divisor with $L \succcurlyeq_\pi D$.*

Then the natural homomorphism

$$\mathrm{R}^i\,\pi_*\mathcal{O}_X(K_X + \ulcorner L \urcorner) \to \mathrm{R}^i\,\pi_*\mathcal{O}_X(K_X + \ulcorner L + D \urcorner)$$

is injective for any i.

Proof. Since the statement is local, we may assume that S is Stein. Furthermore, we replace S by an open subset freely without mentioning it. By **2.14** and by the proof of **3.11**, we may assume that there exist a projective morphism $h\colon Z \to S$ from a non-singular variety, a surjective morphism $f\colon X \to Z$, and an h-ample divisor H of Z such that $L = f^*H$ and that $aL - D$ is linearly equivalent to an effective $\mathbb{R}$-divisor for some $a \in \mathbb{N}$. Then the result follows from **3.7**-(3). □

We have also the following generalization of **3.7**-(2):

3.14. Proposition (Vanishing) *Let $f\colon X \to Y$ and $g\colon Y \to S$ be proper surjective morphisms such that g is projective and X is a Kähler manifold. Let π be the composite $g \circ f$ and let L be an $\mathbb{R}$-divisor of X with* $\mathrm{Supp}\langle L\rangle$ *being normal crossing. Suppose that one of the following conditions is satisfied*:

(1) *L is a π-semi-ample $\mathbb{Q}$-divisor with $\kappa(aL - f^*A; X/S) \geq 0$ for some g-ample divisor A of Y;*
(2) *f is a projective morphism, L is a π-nef and f-abundant $\mathbb{R}$-divisor such that $L \succcurlyeq_\pi f^*A$ for a g-ample divisor A of Y.*

Then $\mathrm{R}^p\,g_*(\mathrm{R}^i\,f_*\omega_X(\ulcorner L \urcorner)) = 0$ for any $p > 0$ and for any $i \geq 0$.

Proof. Similarly to the above, we replace S by an open subset freely. We note that L is π-abundant in the case (2), by **2.28**. We may assume that there is an effective g-ample divisor H of Y. Then

$$\mathrm{R}^p\,\pi_*\omega_X(\ulcorner L \urcorner) \to \mathrm{R}^p\,\pi_*\omega_X(\ulcorner L \urcorner + f^*H)$$

is injective for any $p \geq 0$, by **3.13**. Applying **3.9**-(2) and **3.9**-(3), we infer that if $p > 0$, then

$$\mathrm{R}^p\,g_*\left(\mathrm{R}^i\,f_*\omega_X(\ulcorner L \urcorner)\right) \hookrightarrow \mathrm{R}^p\,g_*\left(\mathcal{O}_Y(H) \otimes \mathrm{R}^i\,f_*\omega_X(\ulcorner L \urcorner)\right) = 0. \qquad \square$$

3.15. Corollary *Let $f\colon X \to Y$ and $g\colon Y \to S$ be projective surjective morphisms where X is normal, and let π be the composite $g \circ f$. Let Δ be an effective $\mathbb{R}$-divisor and L be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor of X satisfying the following conditions*:

(1) *(X, Δ) is log-terminal;*
(2) *$L - (K_X + \Delta)$ is π-nef and π-abundant;*

(3) *$L - (K_X + \Delta) \succcurlyeq_\pi f^*A$ for a g-ample divisor A on Y.*

Then $\mathrm{R}^i f_*\mathcal{O}_X(L)$ restricted to any relatively compact open subset of Y is an ω-sheaf for any i. If $p > 0$, then

$$\mathrm{R}^p g_*(\mathrm{R}^i f_*\mathcal{O}_X(L)) = 0.$$

3.16. Definition Let $f\colon X \to Y$ be a surjective morphism of normal projective varieties.

(1) An ω-sheaf $\mathcal{F}$ on X is called *ω-big over Y* if there exist surjective morphisms $\phi\colon M \to X$, $p\colon M \to Z$, and $q\colon Z \to Y$ satisfying the following conditions:
   (a) M is a compact Kähler manifold and Z is a non-singular projective variety;
   (b) $f \circ \phi = q \circ p$;
   (c) $\mathcal{F}$ is a direct summand of $\mathrm{R}^i \phi_*\omega_M(p^*A)$ for some i and for some ample divisor A of Z.

(2) A coherent torsion-free sheaf $\mathcal{F}$ of X is called an *$\hat{\omega}$-sheaf* if there exist an ω-sheaf $\mathcal{G}$ and a generically isomorphic injection $\mathcal{G} \hookrightarrow \mathcal{F}^\wedge$ into the double-dual $\mathcal{F}^\wedge$ of $\mathcal{F}$.

(3) An $\hat{\omega}$-sheaf $\mathcal{G}$ on X is called ω-big over Y if there is a generically isomorphic injection $\mathcal{F} \hookrightarrow \mathcal{G}^\wedge$ from an ω-sheaf $\mathcal{F}$ that is ω-big over Y.

By **3.9** and **3.14**, we have:

3.17. Corollary *Let $f\colon X \to Y$ be a surjective morphism of normal projective varieties and let $\mathcal{F}$ be an ω-sheaf on X that is ω-big over Y. Then any higher direct image sheaf $\mathrm{R}^i f_*\mathcal{F}$ is ω-big over Y and $\mathrm{H}^p(Y, \mathrm{R}^i f_*\mathcal{F}) = 0$ for $p > 0$.*

3.18. Lemma *Let $\mathcal{F}$ be an ω-sheaf of a non-singular projective variety X of dimension n and let A be an ample divisor of X. Suppose that $\rho_x^*(A) - nE_x$ is ample for a general point $x \in X$, where $\rho_x\colon Q_x(X) \to X$ is the blowing-up at x and E_x is the exceptional divisor. Then $\mathcal{F} \otimes \mathcal{O}_X(A)$ is generically generated by global sections.*

Proof. We may assume that $\mathcal{F} = \mathrm{R}^p h_*\omega_Z$ for a surjective morphism $h\colon Z \to X$ from a non-singular projective variety and for some $p \geq 0$. For a general point $x \in X$, set $X' := Q_x(X)$, $Z' := Z \times_X X'$, and let $h'\colon Z' \to X'$ be the induced morphism. Then $\mathrm{R}^p h'_*\omega_{Z'/X'} \simeq \rho_x^*(\mathrm{R}^p h_*\omega_{Z/X})$, since h is smooth along $h^{-1}(x)$. Hence

$$\rho_x^*(\mathrm{R}^p h_*\omega_Z) \otimes \mathcal{O}_{X'}(\rho_x^*A - E_x) \simeq \mathrm{R}^p h'_*\omega_{Z'} \otimes \mathcal{O}_{X'}(\rho_x^*A - nE_x)$$

is an ω-big ω-sheaf and

$$\mathrm{H}^1(X', \rho_x^*(\mathrm{R}^p h_*\omega_Z) \otimes \mathcal{O}_{X'}(\rho_x^*A - E_x)) = 0$$

by **3.14**. Thus we have the surjection

$$\mathrm{H}^0(X, \mathrm{R}^p h_*\omega_Z \otimes \mathcal{O}_X(A)) \twoheadrightarrow \mathrm{R}^p h_*\omega_Z \otimes \mathcal{O}_X(A) \otimes \mathbb{C}(x). \qquad \square$$

The following result is similar to **3.18**:

3.19. Lemma *Let $\pi\colon X \to S$ be a projective morphism from a normal variety into a Stein variety. Let $\mathcal{F}$ be a coherent sheaf on X such that*

$$\mathrm{R}^p\,\pi_*(\mathcal{F} \otimes \mathcal{O}_X(A')) = 0$$

for any $p > 0$ and for any π-ample divisor A'. Then $\mathcal{F} \otimes \mathcal{O}_X(A)$ is π-generated for a divisor A such that $A - (\dim \operatorname{Supp} \mathcal{F})H$ is π-ample for a π-very ample divisor H.

Proof. By the same argument as [**71**, I, 3.1], we shall prove by induction on $\dim \operatorname{Supp} \mathcal{F}$. Let $x \in \operatorname{Supp} \mathcal{F}$ be an arbitrary point. Suppose first that the local cohomology sheaf $\mathcal{F}' := \mathcal{H}^0_{\{x\}}(\mathcal{F})$ is not zero. Then, for the quotient sheaf $\mathcal{F}'' := \mathcal{F}/\mathcal{F}'$, we have $\mathcal{H}^0_{\{x\}}(\mathcal{F}'') = 0$. Since $\mathcal{F}'$ is a coherent skyscraper sheaf, we have only to show the surjectivity of

$$\pi_*(\mathcal{F}'' \otimes \mathcal{O}(A)) \to \mathcal{F}'' \otimes \mathcal{O}(A) \otimes \mathbb{C}(x).$$

Therefore we can reduce to the case $\mathcal{H}^0_{\{x\}}(\mathcal{F}) = 0$ and $\dim \operatorname{Supp} \mathcal{F} > 0$. Let $X_1 \in |H|$ be a general member containing x. Then the homomorphism

$$\mathcal{F} \otimes \mathcal{O}_X(-X_1) \to \mathcal{F}$$

is injective. Let $\mathcal{F}_1 := \mathcal{F} \otimes \mathcal{O}_{X_1}(H)$. Then $A - H - (\dim \operatorname{Supp} \mathcal{F}_1)H$ is π-ample, since $\dim \operatorname{Supp} \mathcal{F}_1 \leq \dim \operatorname{Supp} \mathcal{F} - 1$. We have a surjective homomorphism

$$\pi_*(\mathcal{F} \otimes \mathcal{O}_X(A)) \twoheadrightarrow \pi_*(\mathcal{F}_1 \otimes \mathcal{O}_{X_1}(A - H))$$

and a vanishing

$$\mathrm{R}^p\,\pi_*(\mathcal{F}_1 \otimes \mathcal{O}_X(A')) \simeq \mathrm{R}^{p+1}\,\pi_*(\mathcal{F} \otimes \mathcal{O}_X(A')) = 0$$

for $p > 0$ for any π-ample divisor A'. Thus, by induction, the homomorphism

$$\pi_*(\mathcal{F} \otimes \mathcal{O}_X(A)) \to \mathcal{F} \otimes \mathcal{O}_X(A) \otimes \mathbb{C}(x)$$

is surjective. □

§3.c. Weak positivity and pseudo-effectivity.

3.20. Definition Let $\mathcal{F}$ be a torsion-free coherent sheaf of a non-singular projective variety Y.

(1) For a point $y \in Y$, $\mathcal{F}$ is called (*globally*) *generated at* y or *generated by global sections at* y if the evaluation homomorphism $\mathrm{H}^0(Y, \mathcal{F}) \otimes \mathcal{O}_Y \to \mathcal{F}$ is surjective at y.
(2) $\widehat{\mathrm{S}}^m(\mathcal{F})$ denotes the double-dual of the symmetric tensor product $\mathrm{Sym}^m(\mathcal{F})$ for $m \geq 0$, where $\widehat{\mathrm{S}}^0(\mathcal{F}) = \mathcal{O}_X$.
(3) $\widehat{\otimes}^m(\mathcal{F})$ denotes the double-dual of the tensor product $\mathcal{F}^{\otimes m}$ for $m \geq 0$, where $\widehat{\otimes}^0(\mathcal{F}) = \mathcal{O}_X$.
(4) $\widehat{\det}(\mathcal{F})$ denotes the double-dual of $\bigwedge^r \mathcal{F}$ for $r = \operatorname{rank} \mathcal{F} > 0$.

Let Q be an $\mathbb{R}$-divisor and let y be a point of $Y \setminus \bigcap_{m \in \mathbb{Z}} \operatorname{Supp}\langle mQ \rangle$. We introduce the symbol $\mathcal{F}[\![Q]\!]$. If Q is a $\mathbb{Z}$-divisor, then we identify $\mathcal{F}[\![Q]\!]$ with the double-dual of $\mathcal{F} \otimes \mathcal{O}_Y(Q)$.

(5) $\mathcal{F}[\![Q]\!]$ is called *dd-ample* at y if $\widehat{\mathrm{S}}^m(\mathcal{F}) \otimes \mathcal{O}_Y(\llcorner mQ \lrcorner - A)$ is generated by global sections at y for an ample divisor A and $m > 0$ with $y \notin \operatorname{Supp}\langle mQ \rangle$. (Here, "dd-ample" is an abbreviation for "ample modulo double-duals.")

(6) If $\mathcal{F}[\![Q]\!]$ is dd-ample at some point y as above, then it is called *big*.

(7) Let A be an ample divisor. $\mathcal{F}[\![Q]\!]$ is called *weakly positive at* y if for any $a \in \mathbb{N}$, there is $b \in \mathbb{N}$ such that $y \notin \operatorname{Supp}\langle abQ \rangle$ and

$$\widehat{\mathrm{S}}^{ab}(\mathcal{F}) \otimes \mathcal{O}_Y(\llcorner abQ \lrcorner + bA)$$

is generated by global sections at y. Note that the condition does not depend on the choice of A.

(8) If $\mathcal{F}[\![Q]\!]$ is weakly positive at a point of Y, then $\mathcal{F}[\![Q]\!]$ is called *weakly positive*.

Remark

(1) Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism of torsion free coherent sheaves that is surjective over an open neighborhood of y. Then, if $\mathcal{F}$ is generated by global sections at y, then $\mathcal{G}$ is so. Thus if $\mathcal{F}$ is dd-ample at y and weakly positive at y, respectively, then so is $\mathcal{G}$. In particular, if $\mathcal{F}$ is generated by global sections at y, then $\mathcal{F}$ is weakly positive at y.

(2) If $\mathcal{F}[\![Q]\!]$ is dd-ample at y, then $\mathcal{F}[\![Q]\!]$ is weakly positive at y. Conversely, if $\mathcal{F}[\![Q]\!]$ is weakly positive at y, then $\mathcal{F}[\![Q + A]\!]$ is dd-ample at y for any ample $\mathbb{Q}$-divisor A.

(3) $\mathcal{F}[\![Q]\!]$ is dd-ample at y if and only if $\mathcal{F}[\![Q - A]\!]$ is weakly positive at y for an ample $\mathbb{Q}$-divisor A.

(4) The set of points at which $\mathcal{F}$ is generated by global sections is a Zariski-open subset. In fact, its complement is the support of the cokernel of

$$\mathrm{H}^0(Y, \mathcal{F}) \otimes \mathcal{O}_Y \to \mathcal{F}.$$

In particular, the set of points y at which $\mathcal{F}[\![Q]\!]$ is dd-ample is also Zariski-open. However, the set of points at which $\mathcal{F}[\![Q]\!]$ is weakly positive is only an intersection of countable Zariski-open subsets. A weakly positive sheaf in the sense of Viehweg [**147**] is a sheaf that is weakly positive at every point of some dense Zariski-open subset.

3.21. Lemma *Let $f\colon X \to Y$ be a surjective morphism from a non-singular projective variety onto a projective variety, L an $\mathbb{R}$-divisor of X, and $F = f^{-1}(y)$ the fiber over a point $y \in Y$ such that f is smooth along F and* $\operatorname{Supp}\langle L \rangle \cap F = \emptyset$. *If one of the following conditions is satisfied, then there is an ample divisor H of Y such that*

$$\mathrm{H}^0(X, \llcorner lL \lrcorner + f^*H) \to \mathrm{H}^0(F, lL|_F)$$

is surjective for any $l \gg 0$:

(1) *$\kappa(L) \geq 0$, $\mathrm{SBs}(L) \cap F = \emptyset$, and the evaluation homomorphism*

$$f^* f_* \mathcal{O}_X(\llcorner maL \lrcorner - mK_X) \to \mathcal{O}_X(\llcorner maL \lrcorner - mK_X)$$

is surjective along F for some positive integers m and a;

(2) *L is pseudo-effective, $\mathrm{NBs}(L) \cap F = \emptyset$, and $L|_F$ is ample.*

Proof. We may replace X by a blowing-up $X' \to X$ such that $X' \to Y$ is still smooth over y. Let H be an ample divisor of Y.

(1) By replacing m, we may assume that $F \cap \mathrm{Bs}\,|mL| = \emptyset$ and $F \cap \mathrm{Bs}\,|m(aL + bf^*H - K_X)| = \emptyset$ for some $b \in \mathbb{N}$. Hence we may assume that there are effective $\mathbb{R}$-divisors Δ_1, Δ_2 such that $\mathrm{Supp}(\Delta_1 + \Delta_2)$ is a normal crossing divisor and

$$\mathrm{Bs}\,|mL - \Delta_1| = \mathrm{Bs}\,|m(aL + bf^*H - K_X) - \Delta_2| = F \cap \mathrm{Supp}(\Delta_1 + \Delta_2) = \emptyset.$$

Since f is flat along F, y is a non-singular point of Y. Let $\mu\colon Y' \to Y$ be the blowing-up at y and let $\nu\colon X' \to X$ be the blowing-up along F. Then $X' \simeq X \times_Y Y'$. Let $f'\colon X' \to Y'$ be the induced morphism and let $E = \mu^{-1}(y)$ and $G = \nu^{-1}(F)$ be exceptional divisors. Then $c\mu^*H - E$ is ample for $c \gg 0$. We set

$$D_l := lL - \frac{l-a}{m}\Delta_1 - \frac{1}{m}\Delta_2 + (b+c)f^*H.$$

Then, for any $l \geq a$,

$$\begin{aligned}\nu^*D_l - G - K_{X'} = {} & \frac{l-a}{m}\nu^*(mL - \Delta_1) + \nu^*\left(aL - \frac{1}{m}\Delta_2 + bf^*H - K_X\right) \\ & + {f'}^*(c\mu^*H - (\dim Y)E)\end{aligned}$$

is semi-ample and

$$\mathrm{H}^1(X', \ulcorner \nu^*D_l \urcorner - G) \to \mathrm{H}^1(X', \ulcorner \nu^*D_l \urcorner)$$

is injective by **3.7**-(3). Therefore,

$$\mathrm{H}^0(X, \llcorner lL \lrcorner + (b+c)f^*H) \to \mathrm{H}^0(F, lL|_F)$$

is surjective.

(2) For some ample divisor A of X, the restriction homomorphism

$$\mathrm{H}^0(X, \llcorner lL \lrcorner + A) \to \mathrm{H}^0(F, (\llcorner lL \lrcorner + A)|_F)$$

is surjective for any $l > 0$ by **1.14**. Since $L|_F$ is ample, $L + bf^*H$ is big and $c(L + bf^*H) - A - \Delta$ is ample for some b, $c \in \mathbb{N}$, and an effective $\mathbb{R}$-divisor Δ with $F \cap \mathrm{Supp}\,\Delta = \emptyset$. By the proof of **1.14**,

$$\mathrm{H}^0(X, \llcorner mL + c(L + bf^*H) \lrcorner) \to \mathrm{H}^0(F, (m+c)L|_F)$$

is also surjective for any $m > 0$. □

3.22. Lemma *Let $f\colon X \to Y$ be a surjective morphism from a non-singular projective variety onto a projective variety, L an $\mathbb{R}$-divisor of X, and $F = f^{-1}(y)$ the fiber over a point $y \in Y$ such that f is smooth along F and $\mathrm{Supp}\langle L \rangle \cap F = \emptyset$. Suppose that $f^* f_* \mathcal{O}_X(\llcorner mL \lrcorner) \to \mathcal{O}_X(\llcorner mL \lrcorner)$ is surjective along F for some $m > 0$.*

Let H be an ample divisor of Y. Then (1) $\Rightarrow$ (2)*,* (3) $\Rightarrow$ (4)*, and* (4) $\Rightarrow$ (5) *hold for the following conditions*:

(1) $f_*\mathcal{O}_X(\llcorner aL \lrcorner)\otimes\mathcal{O}_Y(-H)$ *is generated by global sections at y for some $a > 0$;*
(2) $\kappa(L) = \kappa(L, X/Y) + \dim Y$ *and* $\mathrm{SBs}(L) \cap F = \emptyset$;
(3) *There is a positive integer b such that $f_*\mathcal{O}_X(\llcorner aL \lrcorner)\otimes\mathcal{O}_Y(bH)$ is generated by global sections at y for any $a > 0$;*
(4) *For any $a > 0$, there is a positive integer b such that $f_*\mathcal{O}_X(\llcorner abL \lrcorner) \otimes \mathcal{O}_Y(bH)$ is generated by global sections at y;*
(5) *L is pseudo-effective and* $\mathrm{NBs}(L) \cap F = \emptyset$.

If $L|_F$ is ample, then (2) $\Rightarrow$ (1) *and* (5) $\Rightarrow$ (3) *also hold.*

Proof. (1) $\Rightarrow$ (2): The equality for κ follows from **II.3.13**, since $\mathrm{h}^0(X, \llcorner aL \lrcorner - f^*H) \neq 0$. Let $\Psi = \Phi_{m/Y} \colon X \cdots\to P = \mathbb{P}_Y(f_*\mathcal{O}_X(\llcorner mL \lrcorner))$ be the meromorphic mapping associated with $f^*f_*\mathcal{O}_X(\llcorner mL \lrcorner) \to \mathcal{O}_X(\llcorner mL \lrcorner)$ which is surjective along F. Then Ψ is holomorphic along F. We may assume that Ψ is holomorphic by replacing X by a blowing-up and that Ψ induces the Iitaka fibration for L restricted to a general fiber of f. Then, for the tautological line bundle $\mathcal{O}_P(1)$, we have $\Psi^*\mathcal{O}_P(1) \simeq \mathcal{O}_X(mL - \Delta)$ for an effective $\mathbb{R}$-divisor Δ with $F \cap \operatorname{Supp}\Delta = \emptyset$. On the other hand, $\mathcal{O}_P(1) \otimes p^*\mathcal{O}_Y(bH)$ is very ample for the structure morphism $p \colon P \to Y$ for some $b \in \mathbb{N}$. By assumption, $\mathrm{Bs}\,|m(aL - f^*H)| \cap F = \emptyset$. Thus $\kappa(L) = \kappa(L, X/Y) + \dim Y$ and $\mathrm{SBs}(L) \cap F = \emptyset$.

(2) $\Rightarrow$ (1): Here, we assume $L|_F$ is ample. Let $\Phi = \Phi_k \colon X \cdots\to |kL|^\vee$ be the Iitaka fibration for L associated with the linear system $|kL|$ for some $k \in \mathbb{I}(L)$. Then Φ and Ψ are birational to each other, since $\kappa(L; X/Y) = \dim X - \dim Y$ and $\kappa(L) = \dim X$. Furthermore, Φ is holomorphic along F and is an embedding near F. By replacing X by a blowing-up with center away from F, we may assume that $kL - \Delta_k$ is ample for an effective $\mathbb{R}$-divisor Δ_k with $F \cap \operatorname{Supp}\Delta_k = \emptyset$. Then $c(kL - \Delta_k) - f^*H$ is ample and free for some $c > 0$. By **3.21**, there is a positive integer b such that

$$\mathrm{H}^0(X, \llcorner lL \lrcorner + bf^*H) \to \mathrm{H}^0(F, lL|_F)$$

is surjective for $l \gg 0$. By the proof of **3.21**,

$$\mathrm{H}^0(X, \llcorner (l + (b+1)ck)L \lrcorner - f^*H) \to \mathrm{H}^0(F, (l + (b+1)ck)L|_F)$$

is also surjective. In particular, $f_*\mathcal{O}_X(\llcorner lL \lrcorner - f^*H) \otimes \mathbb{C}(y) \simeq \mathrm{H}^0(F, lL|_F)$ and $f_*\mathcal{O}_X(\llcorner lL \lrcorner - f^*H)$ is generated by global sections at y for $l \gg 0$.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (5): For any $a > 0$, we can choose $b > 0$ so that $F \cap \mathrm{Bs}\,|b(aL + f^*H)| = \emptyset$. Thus (5) follows.

(5) $\Rightarrow$ (3) follows from **3.21** under the assumption: $L|_F$ is ample. $\square$

Let $\mathcal{F}$ be a non-zero torsion-free coherent sheaf on a non-singular projective variety Y and let $p \colon \mathbb{P}(\mathcal{F}) = \mathbb{P}_Y(\mathcal{F}) \to Y$ be the associated projective morphism defined as $\operatorname{Projan}\operatorname{Sym}(\mathcal{F})$. Let U be the maximum open subset of Y over which $\mathcal{F}$ is locally free. Let $\mathbb{P}'(\mathcal{F}) \to \mathbb{P}(\mathcal{F})$ be the normalization of the component of $\mathbb{P}(\mathcal{F})$

containing $p^{-1}(U)$ and let $X \to \mathbb{P}'(\mathcal{F})$ be a birational morphism from a non-singular projective variety that is an isomorphism over U. We assume that $X \setminus f^{-1}U$ is a divisor for the composite $f\colon X \to \mathbb{P}(\mathcal{F}) \to Y$. Let $\mathcal{O}_{\mathcal{F}}(1)$ be the tautological line bundle of $\mathbb{P}(\mathcal{F})$ associated with $\mathcal{F}$ and let L_0 be a Cartier divisor of X linearly equivalent to the pullback of $\mathcal{O}_{\mathcal{F}}(1)$. There is a natural inclusion $\mathcal{F} \hookrightarrow f_*\mathcal{O}_X(L_0)$ which is an isomorphism over U. By **III.5.10**-(3), there is an f-exceptional effective divisor E such that $f_*\mathcal{O}_X(a(L_0+E)) \simeq \widehat{\mathrm{S}}^a(\mathcal{F})$ for any $a \in \mathbb{N}$. We now fix the divisor E above and set $L := L_0 + E$. Note that $N_\sigma(L + E'; X/Y) \geq E'$, for another f-exceptional effective divisor E'. In particular, if $L + E'$ is pseudo-effective, then L is so and $\mathrm{NBs}(L + E') = \mathrm{NBs}(L) \cup \operatorname{Supp} E'$.

By applying **3.22**, we have the following criterion.

3.23. Theorem *In the situation above, let y be a point of U and let Q be an $\mathbb{R}$-divisor of Y with $y \notin \operatorname{Supp}\langle Q\rangle$. Then the equivalences* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3)*, and* (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) *hold for the following conditions*:

(1) $\widehat{\mathrm{S}}^a(\mathcal{F})[\![aQ - H]\!]$ *is weakly positive at y for some $a > 0$ for an ample divisor H*;
(2) $\mathcal{F}[\![Q]\!]$ *is dd-ample at y*;
(3) $L + f^*Q$ *is big and* $\mathrm{SBs}(L + f^*Q) \cap f^{-1}(y) = \emptyset$;
(4) *There is an ample divisor H of Y such that $\widehat{\mathrm{S}}^m(\mathcal{F}) \otimes \mathcal{O}_Y(\llcorner mQ \lrcorner + H)$ is globally generated at y for any $m > 0$*;
(5) $\mathcal{F}[\![Q]\!]$ *is weakly positive at y*;
(6) $L + f^*Q$ *is pseudo-effective and* $\mathrm{NBs}(L + f^*Q) \cap f^{-1}(y) = \emptyset$.

Proof. (1) $\Rightarrow$ (2): There is a surjection $\mathrm{Sym}^m(\mathrm{Sym}^a(\mathcal{F})) \twoheadrightarrow \mathrm{Sym}^{ma}(\mathcal{F})$. Hence $\widehat{\mathrm{S}}^m(\widehat{\mathrm{S}}^a(\mathcal{F})) \to \widehat{\mathrm{S}}^{ma}(\mathcal{F})$ is induced and it is surjective over the open subset U where $\mathcal{F}$ is locally free. Hence, by definition,

$$\widehat{\mathrm{S}}^{2am}(\mathcal{F}) \otimes \mathcal{O}_Y(\llcorner 2mQ \lrcorner - 2mH) \otimes \mathcal{O}_Y(mH) \simeq \widehat{\mathrm{S}}^{2am}(\mathcal{F}) \otimes \mathcal{O}_Y(\llcorner 2mQ \lrcorner - mH)$$

is generated by global sections at y for some $m > 0$.

(2) $\Rightarrow$ (1) is trivial.

(2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) are shown in **3.22**. □

Remark A numerically semi-positive vector bundle on Y is a locally free sheaf that is weakly positive at every point of Y.

3.24. Corollary *Let $\mathcal{F}$ be a torsion-free coherent sheaf of Y and let Q be an $\mathbb{R}$-divisor.*

(1) *If $\mathcal{F}[\![Q']\!]$ is weakly positive for an $\mathbb{R}$-divisor Q' with $Q - Q'$ being pseudo-effective, then $\mathcal{F}[\![Q]\!]$ is weakly positive.*
(2) *Let Q_k $(k = 1, 2, \ldots)$ be a sequence of $\mathbb{R}$-divisors such that $c_1(Q) = \lim_{k\to\infty} c_1(Q_k)$ in $\mathrm{N}^1(Y)$. If $\mathcal{F}[\![Q_k]\!]$ are all weakly positive, then $\mathcal{F}[\![Q]\!]$ is weakly positive.*

Proof. We consider the morphism $f\colon X \to Y$ and L above and apply (6) $\Leftrightarrow$ (5) of **3.23**.

(1) $L+f^*Q$ is pseudo-effective. If y is a 'general' point, then $\mathrm{NBs}(L+f^*Q')\cap f^{-1}(y)=\emptyset$ and $y\not\in\mathrm{NBs}(Q-Q')$. Thus $\mathrm{NBs}(L+f^*Q)\cap f^{-1}(y)=\emptyset$.

(2) $L+f^*Q$ is pseudo-effective since it is a limit of pseudo-effective $\mathbb{R}$-divisors. Let A be an ample divisor of X. Then, for any positive integer m, there is a number k_m such that $mf^*(Q-Q_k)+A$ is ample for any $k\geq k_m$. For a point $x\in X$, we have

$$\sigma_x(m(L+f^*Q)+A)\leq\sigma_x(m(L+f^*Q_k))+\sigma_x(mf^*(Q-Q_k)+A)=\sigma_x(m(L+f^*Q_k))$$

for $k\geq k_m$. Hence, if $\mathrm{NBs}(L+f^*Q_k)\cap f^{-1}(y)=\emptyset$ for any k, then $\mathrm{NBs}(L+f^*Q)\cap f^{-1}(y)=\emptyset$. □

3.25. Lemma (cf. [**36**, Theorem 5.2], [**148**, Lemma 3.2]) *Let $\mathcal{F}$ and $\mathcal{G}$ are torsion-free coherent sheaves on Y, Q an $\mathbb{R}$-divisor, and y a point of $Y\setminus\mathrm{Supp}\langle Q\rangle$ such that $\mathcal{F}$ and $\mathcal{G}$ are locally free at y.*

(1) *If $\mathcal{F}[\![Q]\!]$ and $\mathcal{G}[\![Q]\!]$ are weakly positive (resp. dd-ample) at y, then so is $(\mathcal{F}\oplus\mathcal{G})[\![Q]\!]$.*
(2) *If $\mathcal{F}[\![Q]\!]$ is weakly positive (resp. dd-ample) at y and if $\mathcal{G}$ is generated by global sections at y, then $(\mathcal{F}\otimes\mathcal{G})[\![Q]\!]$ is weakly positive (resp. dd-ample) at y.*
(3) *If $\mathcal{F}[\![Q]\!]$ is weakly positive (resp. dd-ample) at y, then*
$$\widehat{\mathrm{S}}^a(\mathcal{F})[\![aQ]\!],\quad \widehat{\otimes}^a(\mathcal{F})[\![aQ]\!],\quad \text{and}\quad \widehat{\det}(\mathcal{F})[\![(\operatorname{rank}\mathcal{F})Q]\!]$$
are weakly positive (resp. dd-ample) at y, for $a>0$.
(4) *If $\mathcal{F}[\![Q]\!]$ and $\mathcal{G}[\![Q]\!]$ are weakly positive (resp. dd-ample) at y, then so is $(\mathcal{F}\otimes\mathcal{G})^\wedge[\![2Q]\!]$.*
(5) *If $\widehat{\mathrm{S}}^a(\mathcal{F})[\![aQ]\!]$ is weakly positive (resp. dd-ample) at y for some $a>0$, then $\mathcal{F}[\![Q]\!]$ is weakly positive (resp. dd-ample) at y.*
(6) *Let $\tau\colon Y'\to Y$ be a morphism (resp. generically finite morphism) from a non-singular projective variety such that τ is smooth along $\tau^{-1}(y)$. Let E be a τ-exceptional effective divisor. Then $\mathcal{F}[\![Q]\!]$ is weakly positive (resp. dd-ample) at y if and only if $\tau^*\mathcal{F}\otimes\mathcal{O}_{Y'}(E)[\![\tau^*Q]\!]$ is so at any point of $\tau^{-1}(y)$.*

PROOF. (1) Suppose that $\mathcal{F}[\![Q]\!]$ and $\mathcal{G}[\![Q]\!]$ are weakly positive at y. By **3.23**, there exist an ample divisor H of Y and $k_0\in\mathbb{N}$ such that $\widehat{\mathrm{S}}^k(\mathcal{F})\otimes\mathcal{O}_Y(\llcorner kQ\lrcorner+H)$ and $\widehat{\mathrm{S}}^k(\mathcal{G})\otimes\mathcal{O}_Y(\llcorner kQ\lrcorner+H)$ are generated by global sections at y for any $k\geq k_0$. Let b be a positive integer such that $\widehat{\mathrm{S}}^i(\mathcal{F})\otimes\mathcal{O}_Y(\llcorner iQ\lrcorner+bH)$ and $\widehat{\mathrm{S}}^j(\mathcal{G})\otimes\mathcal{O}_Y(\llcorner jQ\lrcorner+bH)$ are generated by global sections at y for any $0\leq i,j\leq k_0-1$. For integers $m\geq 2k_0$ and $0\leq n\leq m$, if $n<k_0$, then $m-n\geq k_0$. Hence

$$\Big(\widehat{\mathrm{S}}^{m-n}(\mathcal{F})\otimes\mathcal{O}_Y(\llcorner(m-n)Q\lrcorner)\otimes\widehat{\mathrm{S}}^n(\mathcal{G})\otimes\mathcal{O}_Y(\llcorner nQ\lrcorner)\Big)^\wedge\otimes\mathcal{O}_Y((b+1)H)$$

is generated by global sections at y. Since

$$\widehat{\mathrm{S}}^m(\mathcal{F}\oplus\mathcal{G})\simeq\bigoplus\nolimits_{n=0}^m\Big(\widehat{\mathrm{S}}^{m-n}(\mathcal{F})\otimes\widehat{\mathrm{S}}^n(\mathcal{G})\Big)^\wedge,$$

$\widehat{\mathrm{S}}^m(\mathcal{F} \oplus \mathcal{G}) \otimes \mathcal{O}_Y(\llcorner mQ \lrcorner + (b+1)H)$ is generated by global sections at y. Hence $(\mathcal{F} \oplus \mathcal{G})[\![Q]\!]$ is weakly positive at y.

The case in which $\mathcal{F}[\![Q]\!]$ and $\mathcal{G}[\![Q]\!]$ are dd-ample at y is reduced to the case above by the following property: $\mathcal{F}[\![Q]\!]$ is dd-ample at y if and only if $\mathcal{F}[\![Q - A]\!]$ is weakly positive at y for some ample $\mathbb{Q}$-divisor A with $y \notin \operatorname{Supp} A$.

(2) There is a homomorphism $\mathcal{O}_Y^{\oplus r} \to \mathcal{G}$ surjective at y. Thus $\mathcal{F}^{\oplus r} \to \mathcal{F} \otimes \mathcal{G}$ is also surjective at y. Since $\mathcal{F}^{\oplus r}[\![Q]\!]$ is weakly positive or dd-ample at y by (1), so is $\mathcal{F} \otimes \mathcal{G}[\![Q]\!]$.

(3) This is proved by the same argument as [**36**, Theorem 5.2] with properties obtained in (1), (2), and **3.23**-(1), -(4).

(4) $\widehat{\mathrm{S}}^2(\mathcal{F} \oplus \mathcal{G})[\![2Q]\!]$ is weakly positive (resp. dd-ample) at y and $(\mathcal{F} \otimes \mathcal{G})^\wedge$ is a direct summand of $\widehat{\mathrm{S}}^2(\mathcal{F} \oplus \mathcal{G})$. Thus (4) follows.

(5) It is derived from the homomorphism $\widehat{\mathrm{S}}^m(\widehat{\mathrm{S}}^a(\mathcal{F})) \to \widehat{\mathrm{S}}^{ma}(\mathcal{F})$.

(6) Let $X' \to X \times_Y Y'$ be a birational morphism from a non-singular projective variety into the main component. Then we can define a divisor L' on X' for $\tau^*\mathcal{F}$ similarly to L for $\mathcal{F}$. Let $\lambda \colon X' \to X$ and $f' \colon X' \to Y'$ be the induced morphisms. Then we can write $E' - G = L' - \lambda^* L$ for effective divisors E' and G which are exceptional for $X' \to Y$. If $L' + f'^*(\tau^* Q + E)$ is pseudo-effective, then $L' + f'^*(\tau^*Q + E) + G$ is pseudo-effective and

$$\operatorname{NBs}(L' + f'^*(\tau^*Q + E)) \cap \lambda^{-1} f^{-1}(y) \supset \operatorname{NBs}(L' + f'^*(\tau^*Q + E) + G) \cap \lambda^{-1} f^{-1}(y).$$

There is an f-exceptional effective divisor E'' of X such that $E' + f'^*E \leq \lambda^* E''$, since $X \setminus f^{-1}U$ is a divisor. Thus

$$\begin{aligned} \lambda^{-1} \operatorname{NBs}(L + f^*Q) &\subset \operatorname{NBs}(\lambda^*(L + f^*Q + E'')) \\ &\subset \operatorname{NBs}(\lambda^*(L + f^*Q) + E' + f'^*E) \cup \lambda^{-1}(\operatorname{Supp} E'') \\ &= \operatorname{NBs}(L' + f'^*(\tau^*Q + E) + G) \cup \lambda^{-1}(\operatorname{Supp} E''). \end{aligned}$$

Hence if $\tau^*(\mathcal{F}) \otimes \mathcal{O}_{Y'}(E)[\![\tau^*Q]\!]$ is weakly positive at any point of $\tau^{-1}(y)$, then $L + f^*Q$ is pseudo-effective and $\operatorname{NBs}(L + f^*Q) \cap f^{-1}(y) = \emptyset$. Thus $\mathcal{F}[\![Q]\!]$ is weakly positive at y. The inverse implication is trivial. We can reduce the case of dd-ample to the case of weakly positive above by replacing Q by $Q - A$ for some ample $\mathbb{Q}$-divisor A. □

§3.d. ω-sheaves and weak positivity.

3.26. Lemma *Let H be a polarized variation of Hodge structure of weight $w \geq 0$ defined on $M \setminus D$ for a non-singular projective variety M and a normal crossing divisor D. Suppose that $\mathcal{F}^0(\mathcal{H}) = \mathcal{H}$ and $\mathcal{F}^{w+1}(\mathcal{H}) = 0$ for the Hodge filtration $\mathcal{F}^\bullet(\mathcal{H})$ of $\mathcal{H} = H \otimes \mathcal{O}_{M \setminus D}$. Then, without the assumption of monodromies, $\mathcal{F}^w({}^u\mathcal{H}^{\mathrm{can}})$ is weakly positive at every point of $M \setminus D$.*

Proof. We may assume that D is a simple normal crossing divisor. By Kawamata's covering lemma **II.5.11**, we have a finite Galois morphism $\tau \colon Y \to M$ from a non-singular projective variety such that $\tau^{-1}D$ is also a simple normal crossing divisor and $\tau^{-1}H$ has only unipotent local monodromies along $\tau^{-1}D$. Let $\mathcal{F}_M$ be the w-th filter $\mathcal{F}^w({}^u\mathcal{H}^{\mathrm{can}})$ and let $\mathcal{F}_Y$ be the corresponding w-th filter to the canonical extension of $\tau^{-1}H$. Then there is a natural injection

$$\mathcal{F}_Y \hookrightarrow \tau^*\mathcal{F}_M$$

which is isomorphic outside $\tau^{-1}D$. Since $\mathcal{F}_Y$ is numerically semi-positive by **3.2**, $\mathcal{F}_M$ is weakly positive at every point of $M \setminus D$. □

3.27. Corollary *For a torsion-free ω-sheaf $\mathcal{F}$ on a non-singular projective variety Y, $\mathcal{F} \otimes \omega_Y^{-1}$ is weakly positive at every point of a dense Zariski-open subset of Y.*

Proof. We may assume $\mathcal{F} = \mathrm{R}^i f_*\omega_X$ for a surjective morphism $f\colon X \to Y$ from a compact Kähler manifold and for some $i \geq 0$. Let $\mu\colon Y' \to Y$ be a birational morphism from a non-singular projective variety such that $X \times_Y Y' \to Y'$ is smooth outside a normal crossing divisor E of Y'. Then there is a bimeromorphic morphism $X' \to X \times_Y Y'$ into the main component from a compact Kähler manifold such that $f'\colon X' \to Y'$ is smooth outside E. Then $\mathrm{R}^i f'_*\omega_{X'/Y'}$ is weakly positive at every point of $Y' \setminus E$ by **3.7**-(4) and **3.26**. Since μ is birational, $\mathrm{R}^p \mu_*(\mathrm{R}^i f'_*\omega_{X'}) = 0$ for any $p > 0$, by **3.14**. Thus there is a natural injection

$$\mu_*(\mathrm{R}^i f'_*\omega_{X'/Y'}) \hookrightarrow \mu_*(\mathrm{R}^i f'_*\omega_{X'/Y}) \simeq \mathrm{R}^i(\mu \circ f')_*\omega_{X'/Y} \simeq \mathrm{R}^i f_*\omega_{X/Y}.$$

Therefore, $\mathcal{F} \otimes \omega_Y^{-1} = \mathrm{R}^i f_*\omega_{X/Y}$ is weakly positive at every point of a dense Zariski-open subset. □

We shall give in §**3.e** below a generalization of the following weak positivity theorem by Viehweg [**147**]:

3.28. Theorem *Let $f\colon X \to Y$ be a surjective morphism of non-singular projective varieties. Then $f_*(\omega_{X/Y}^{\otimes m})$ is weakly positive for any $m \geq 1$.*

Here, the case $m = 1$ is derived from **3.27** (cf. [**50**, Theorem 5]).

We recall the following lemma by Viehweg [**147**, 3.2] which is important for the proof of **3.28**: let $f\colon X \to Y$ be a proper surjective morphism of non-singular varieties, $\tau\colon Y' \to Y$ a finite surjective morphism from a non-singular variety, $\sigma\colon V \to X \times_Y Y'$ the normalization map, and $\delta\colon X' \to V$ a bimeromorphic morphism from a non-singular variety. Let $f'\colon X' \to Y'$ be the induced morphism and let p_1, p_2 be the projections from $X \times_Y Y'$.

$$\begin{array}{ccccccc}
X' & \xrightarrow{\delta} & V & \xrightarrow{\sigma} & X \times_Y Y' & \xrightarrow{p_1} & X \\
 & & & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\
 & & & & Y' & \xrightarrow{\tau} & Y.
\end{array}$$

3.29. Lemma *Suppose that f is smooth over an open subset $U_0 \subset Y$ and τ is étale over an open subset $U_1 \subset Y$. Let $U_2 \subset Y$ be an open subset such that*

(1) *f is flat over U_2,*

(2) *$f^{-1}(y)$ is reduced for any $y \in U_2$,*
(3) *the branch divisor of $V \to X$ is a normal crossing divisor over U_2.*

Then, for any $m \in \mathbb{N}$, there exist injections

$$\sigma_*\delta_*(\omega_{X'/Y'}^{\otimes m}) \hookrightarrow p_1^*\omega_{X/Y}^{\otimes m} \quad \textit{and} \quad f'_*(\omega_{X'/Y'}^{\otimes m}) \hookrightarrow \tau^*\left(f_*(\omega_{X/Y}^{\otimes m})\right)$$

which are isomorphic over $\tau^{-1}(U_0 \cup U_1 \cup U_2)$.

Proof. Since τ is flat, we have isomorphisms

$$\omega_{X\times_Y Y'/Y'} \simeq p_1^*\omega_{X/Y} \quad \text{and} \quad \omega_{X\times_Y Y'/X} \simeq p_2^*\omega_{Y'/Y},$$

by [**37**], [**145**]. Thus $X \times_Y Y'$ is Gorenstein. Since $f^{-1}U_2 \to U_2$ is smooth outside a Zariski-closed subset of $f^{-1}U_2$ of codimension greater than one, $X \times_Y Y'$ is normal over U_2. Therefore, σ is isomorphic over $\tau^{-1}(U_0 \cup U_1 \cup U_2)$. There is a trace map $\sigma_*\omega_V \to \omega_{X\times_Y Y'}$, where $\omega_V = \mathcal{O}_V(K_V)$. Since σ is finite and bimeromorphic, $\sigma^*\sigma_*\omega_V \to \omega_V$ is surjective and its kernel is a torsion sheaf. Hence the trace map induces an injection

$$\omega_{V/Y'} = \omega_V \otimes \sigma^* p_2^*\omega_{Y'}^{-1} \hookrightarrow \sigma^*\omega_{X\times_Y Y'/Y'} \simeq \sigma^* p_1^*\omega_{X/Y}.$$

For $m \in \mathbb{N}$, let $\omega_{V/Y'}^{[m]}$ denote the double-dual of $\omega_{V/Y'}^{\otimes m}$. Then we have

$$\omega_{V/Y'}^{[m]} \hookrightarrow \omega_{V/Y'} \otimes \sigma^* p_1^*\omega_{X/Y}^{\otimes(m-1)}$$

and the composite

$$\sigma_*\omega_{V/Y'}^{[m]} \hookrightarrow \sigma_*\omega_{V/Y'} \otimes p_1^*\omega_{X/Y}^{\otimes(m-1)} \hookrightarrow p_1^*\omega_{X/Y}^{\otimes m}.$$

There is a natural injection

$$\delta_*(\omega_{X'/Y'}^{\otimes m}) \hookrightarrow \omega_{V/Y'}^{[m]}$$

given by the double-dual. This is also isomorphic over $\tau^{-1}(U_0 \cup U_1 \cup U_2)$, since V has only rational singularities over $\tau^{-1}U_2$. Thus we have the first injection. The second injection is derived from the flat base change

$$p_{2*}\left(p_1^*\omega_{X/Y}^{\otimes m}\right) \simeq \tau^*\left(f_*(\omega_{X/Y}^{\otimes m})\right). \qquad \square$$

3.30. Lemma *Under the same situation as* **3.29**, *there is an injection*

$$\mathrm{R}^p f'_*\omega_{X'/Y'} \hookrightarrow \tau^*\left(\mathrm{R}^p f_*\omega_{X/Y}\right)$$

for any p, which is an isomorphism over $\tau^{-1}(U_0 \cup U_1 \cup U_2)$.

Proof. The composite of trace maps

$$\sigma_*\delta_*\omega_{X'/Y'} \to \sigma_*\omega_{V/Y'} \to \omega_{X\times_Y Y'/Y'} \simeq p_1^*\omega_{X/Y}$$

is an isomorphism over $\tau^{-1}(U_0 \cup U_1 \cup U_2)$. The vanishing $R^q(\sigma\circ\delta)_*\omega_{X'/Y'} = 0$ for $q > 0$ by [**30**] (cf. **3.14**, **II.5.12**) induces the expected injection

$$\mathrm{R}^p f'_*\omega_{X'/Y'} \simeq \mathrm{R}^p p_{2*}\left(\sigma_*\delta_*\omega_{X'/Y'}\right) \hookrightarrow \mathrm{R}^p p_{2*}p_1^*\omega_{X/Y} \simeq \tau^*\,\mathrm{R}^p f_*\omega_{X/Y}. \qquad \square$$

3.31. Proposition (cf. **3.27**) *Let $\mathcal{F}$ be an ω-big $\hat{\omega}$-sheaf on a non-singular projective variety Y. Then $\mathcal{F} \otimes \omega_Y^{-1}$ is big.*

PROOF. Let $f\colon X \to Z$ and $g\colon Z \to Y$ be surjective morphisms of non-singular varieties in which X is compact Kähler and Z is projective. Let A be an ample divisor of Z and set $h = g \circ f$. It is enough to show that $\mathrm{R}^p h_*\omega_{X/Y}(f^*A)$ is big for any $p \geq 0$. Let H be an ample divisor of Y and let us take $m \in \mathbb{N}$ with $mA - g^*H$ being ample. Then there exist a finite surjective morphism $\tau\colon Y' \to Y$ a non-singular projective variety and an ample divisor H' of Y' with $\tau^*H \sim mH'$ by **II.5.11**. Let X' and Z' be desingularizations of the main components of the fiber products $X \times_Y Y'$ and $Z \times_Y Y'$, respectively. Let $h'\colon X' \to Y'$, $f'\colon X' \to Z'$, $g'\colon Z' \to Y'$, and $\tau_Z\colon Z' \to Z$ be the induced morphisms. By **3.30**, we have a generically isomorphic injection

$$\mathrm{R}^p h'_* \left(\omega_{X'/Y'}({f'}^*\tau_Z^*A)\right) \hookrightarrow \tau^* \left(\mathrm{R}^p h_*\omega_{X/Y}(f^*A)\right).$$

In particular, the tensor product $\tau^*(\mathrm{R}^p h_*\omega_{X/Y}(f^*A)) \otimes \mathcal{O}_{Y'}(-H')$ contains a sheaf $\mathrm{R}^p h'_*\omega_{X'/Y'}({f'}^*(\tau_Z^*A - {g'}^*H'))$, which is weakly positive by **3.27**. Hence $\mathrm{R}^p h_*\omega_{X/Y}(f^*A)$ is big. □

3.32. Theorem *Let Y be a normal projective variety and let $\mathcal{L}$ be an invertible ω-sheaf. Then there exist a birational morphism $\varphi\colon M \to Y$ from a non-singular projective variety M and a nef $\mathbb{Q}$-divisor D of M such that* $\mathrm{Supp}\langle D\rangle$ *is a normal crossing divisor and*

$$\mathcal{L} \simeq \varphi_*\omega_M(\ulcorner D \urcorner).$$

PROOF. Let $\mu\colon Z \to Y$ be a birational morphism from a non-singular projective variety, $f\colon X \to Z$ a surjective morphism from a compact Kähler manifold, and L a Cartier divisor of Y such that

(1) $\mathcal{L} \simeq \mathcal{O}_Y(L)$ is a direct summand of $\mu_*(\mathrm{R}^j f_*\omega_X)$ for some j,
(2) f is smooth outside a simple normal crossing divisor $E = \sum E_i$, and
(3) the μ-exceptional locus is contained in E.

The sheaf $\mathrm{R}^j f_*\omega_{X/Z}$ is isomorphic to the upper-canonical extension of the d-the Hodge filtration of the variation of Hodge structures associated with $\mathrm{R}^{d+j} f_*\mathbb{C}_X$, where $d = \dim X - \dim Y$. Let $\tau\colon Z' \to Z$ be a finite Galois morphism from a non-singular projective variety Z' that is a unipotent reduction for the variation of Hodge structure; here, the local monodromies of the pullback are unipotent. We may assume that the branch locus of τ is contained in a normal crossing divisor as in **II.5.11**. Then we have the canonical extension $\mathcal{E}$ of the d-th filtration of the induced variation of Hodge structure. This is numerically semi-positive by **3.2**. For the Galois group G of τ, the G-invariant part of $\tau_*\mathcal{E}$ is the lower-canonical extension and that of $\tau_*(\mathcal{E} \otimes \omega_{Z'})$ is isomorphic to $\mathrm{R}^j f_*\omega_X$. Now we have an injection $\mathcal{O}_Z(\mu^*L) \hookrightarrow \mathrm{R}^j f_*\omega_X$ and a generic surjection $\mathrm{R}^j f_*\omega_X \to \mathcal{O}_Z(\mu^*L + E_1)$ for a μ-exceptional effective divisor E_1, which is surjective outside a Zariski-closed subset of codimension greater than one. Since $\mathrm{R}^j f_*\omega_X$ is the G-invariant part of

$\tau_*(\mathcal{E} \otimes \omega_{Z'})$, we have an injection $\mathcal{O}_{Z'}(\tau^*\mu^*L) \hookrightarrow \mathcal{E} \otimes \omega_{Z'}$. Similarly, we have an effective divisor $\Delta' \subset Z'$ such that there is a generic surjection

$$\mathcal{E} \otimes \omega_{Z'} \to \mathcal{O}_{Z'}(\tau^*\mu^*L + \tau^*E_1 + \Delta')$$

whose cokernel is supported on a Zariski-closed subset of codimension greater than one. Then $\Delta' \leq R_\tau$ for the ramification divisor $R_\tau = K_{Z'} - \tau^*K_Z$, since there is an injection $\mathcal{E} \hookrightarrow \tau^*\left(\mathrm{R}^j f_*\omega_{X/Z}\right)$ by **3.30**. There exist effective $\mathbb{Q}$-divisors Δ and R_Z of Z such that $\Delta' = \tau^*\Delta$ and $R_\tau = \tau^*R_Z$, since Δ' and R_τ are G-invariant. Note that $\llcorner R_Z \lrcorner = 0$. Let $\nu\colon Z'' \to Z'$ be a birational morphism from a non-singular projective variety such that there exist a ν-exceptional effective divisor E_2' and a surjection

$$\nu^*(\mathcal{E} \otimes \omega_{Z'}) \twoheadrightarrow \mathcal{O}_{Z''}(\nu^*\tau^*(\mu^*L + E_1 + \Delta) - E_2').$$

Since $\mathcal{E}$ is numerically semi-positive, the divisor

$$\nu^*\tau^*(\mu^*L + E_1 + \Delta) - \nu^*K_{Z'} - E_2' = \nu^*\tau^*(\mu^*L + E_1 - (R_Z - \Delta) - K_Z) - E_2'$$

is nef. Furthermore, $\nu^*\tau^*(E_1 + \Delta) - E_2'$ is an effective Cartier divisor. We may assume that the Galois group G acts holomorphically on Z''. Since E_2' is also G-invariant, there is an effective $\mathbb{Q}$-Cartier divisor E_2 on the quotient variety Z''/G such that $E_2' = \lambda^*E_2$, where $\lambda\colon Z'' \to Z''/G$ is the quotient morphism. Let $\rho\colon Z''/G \to Z$ be the induced morphism. Then $\rho^*(\mu^*L + E_1 + \Delta - K_Z - R_Z) - E_2$ is nef and $\rho^*(E_1 + \Delta) - E_2$ is an effective $\mathbb{Q}$-divisor. Let $\delta\colon M \to Z''/G$ be a birational morphism from a non-singular projective variety such that the union of the exceptional locus for $\varphi := \mu \circ \rho \circ \delta\colon M \to Y$ and the proper transform of $E \subset Z$ is a normal crossing divisor. Let R_M be the $\mathbb{Q}$-divisor $K_M - \delta^*\rho^*(K_Z + R_Z)$. Then $\ulcorner R_M \urcorner \geq 0$. We know the $\mathbb{Q}$-divisor

$$D := \varphi^*L + \delta^*\rho^*(E_1 + \Delta) - \delta^*E_2 - K_M + R_M$$

is nef. We shall consider the $\mathbb{Q}$-divisor

$$E_M := \delta^*\rho^*(E_1 + \Delta) - \delta^*E_2 + R_M.$$

Let Γ be a prime component of of E_M. Since $\rho^*(E_1 + \Delta) - E_2$ is effective, $c := \operatorname{mult}_\Gamma E_M \geq \operatorname{mult}_\Gamma R_M > -1$. On the other hand, if Γ is not φ-exceptional, then $c = c_1 - c_2$, where $c_1 := \operatorname{mult}_\Gamma \delta^*\rho^*\Delta$ and $c_2 := \operatorname{mult}_\Gamma \delta^*\rho^*R_Z$. Since $\Delta \leq R_Z$, $c \leq 0$. Hence $\ulcorner E_M \urcorner$ is a φ-exceptional effective divisor on M. Therefore

$$\varphi_*\mathcal{O}_M(K_M + \ulcorner D \urcorner) = \varphi_*\mathcal{O}_M(\varphi^*L + \ulcorner E_M \urcorner) \simeq \mathcal{O}_Z(L). \qquad \square$$

§3.e. Direct images of relative pluricanonical sheaves. Let $f\colon X \to Y$ be a proper surjective morphism from a normal variety onto a non-singular variety. We denote the relative canonical divisor $K_X - f^*K_Y$ by $K_{X/Y}$. Then $\mathcal{O}_X(K_{X/Y}) \simeq \omega_{X/Y}$. For a Cartier divisor D of X, we denote $\omega_{X/Y}(D) = \omega_{X/Y} \otimes \mathcal{O}_X(D)$ and $\omega_X(D) = \omega_X \otimes \mathcal{O}_X(D)$, for short.

3.33. Lemma *Let Δ be an effective $\mathbb{R}$-divisor of X, L a Cartier divisor of X, and k a positive integer. Suppose that $K_X + \Delta$ is $\mathbb{R}$-Cartier.*

(1) *Let* $\rho\colon \widetilde{X} \to X$ *be a bimeromorphic morphism from a non-singular variety. For the* $\mathbb{R}$*-divisor* $R := K_{\widetilde{X}} - \rho^*(K_X + \Delta)$, *let* $\ulcorner R \urcorner = \widetilde{R}_+ - \widetilde{R}_-$ *be the decomposition into the positive and the negative parts of the prime decomposition and set*

$$\widetilde{\Delta} := \langle -R \rangle + \widetilde{R}_- \quad \text{and} \quad \widetilde{L} := \rho^* L + k\widetilde{R}_+.$$

Then

$$\widetilde{L} - k\left(K_{\widetilde{X}/Y} + \widetilde{\Delta}\right) = \rho^*(L - k(K_{X/Y} + \Delta))$$

and there is an isomorphism

$$\rho_* \mathcal{O}_{\widetilde{X}}(\widetilde{L}) \simeq \mathcal{O}_X(L).$$

(2) *Suppose that* X *is non-singular and* $\operatorname{Supp}\Delta$ *is a normal crossing divisor. Let* $\tau\colon Y' \to Y$ *be a generically finite surjective morphism from a non-singular variety and let* $\delta\colon X' \to X \times_Y Y'$ *be a bimeromorphic morphism from a non-singular space. Let* $f'\colon X' \to Y'$ *and* $\lambda\colon X' \to X$ *be the induced morphisms. For the* $\mathbb{R}$*-divisor* $R_\Delta := K_{X'} - \lambda^*(K_X + \Delta)$, *let* $\ulcorner R_\Delta \urcorner = R'_+ - R'_-$ *be the decomposition into the positive and the negative parts of the prime decomposition, and set* $R_\tau := K_{Y'} - \tau^* K_Y$,

$$\Delta' := \langle -R_\Delta \rangle + R'_-, \quad \text{and} \quad L' := \lambda^* L + kR'_+ - k{f'}^* R_\tau.$$

Then

$$L' - k(K_{X'/Y'} + \Delta') = \lambda^*(L - k(K_{X/Y} + \Delta)).$$

(3) *Under the situation of* (2), *suppose that* τ *is finite. Then there is a generically isomorphic injection*

$$f'_* \mathcal{O}_{X'}(L') \hookrightarrow \tau^* \left(f_* \mathcal{O}_X(L)\right).$$

(4) *Under the situation of* (2), *suppose that* τ *is bimeromorphic and the morphism from the main component of* $X \times_Y Y'$ *to* Y' *is flat. Then*

$$\tau_*(f'_* \mathcal{O}_{X'}(L'))^\wedge \subset f_* \mathcal{O}_X(L).$$

(5) *Under the situation of* (2), *there exist a* τ*-exceptional effective divisor* E *and a generically isomorphic injection*

$$f'_* \mathcal{O}_{X'}(L') \hookrightarrow (\tau^* f_* \mathcal{O}_X(L))^\wedge \otimes \mathcal{O}_{Y'}(E).$$

Proof. (1) The equality is straightforward and the isomorphism follows from that $\widetilde{R}_+$ is ρ-exceptional.

(2) The equality is also straightforward.

(3) For the ramification divisor $R_\lambda := K_{X'} - \lambda^* K_X$, we have $R_\Delta = R_\lambda - \lambda^*\Delta$. Hence $\ulcorner R_\Delta \urcorner \leq R_\lambda$ and $R'_+ \leq R_\lambda$. We have an injection

$$\delta_*(\omega_{X'/Y'}^{\otimes m}) \hookrightarrow p_1^* \omega_{X/Y}^{\otimes m}$$

for any $m \in \mathbb{N}$ by **3.29**, where p_1 is the first projection $X \times_Y Y' \to X$. The injection is isomorphic over a dense Zariski-open subset of Y'. Hence we also have an injection

$$\delta_* \mathcal{O}_{X'}(k(R'_+ - f'^* R_\tau)) \hookrightarrow \delta_* \mathcal{O}_{X'}(k(R_\lambda - f'^* R_\tau)) \hookrightarrow \mathcal{O}_{X \times_Y Y'},$$

which is an isomorphism over a dense open subset of Y', equivalently, the injection $\delta_* \mathcal{O}_{X'}(kR'_+) \hookrightarrow \delta_* \mathcal{O}_{X'}(kR_\lambda)$ is so. In fact, it follows from that $\mathcal{O}_{X \times_Y Y'} \to \delta_* \mathcal{O}_{X'}(kR_\lambda)$ is an isomorphism over a dense open subset of Y' along which τ is étale. Thus we have the expected generically isomorphic injection

$$f'_* \mathcal{O}_{X'}(L') \hookrightarrow p_{2*}\left(p_1^* \mathcal{O}_X(L)\right) \simeq \tau^*\left(f_* \mathcal{O}_X(L)\right)$$

by a flat base change.

(4) $(f'_* \mathcal{O}_{X'}(L'))^\wedge \simeq f'_* \mathcal{O}_{X'}(L' + E)$ for an f'-exceptional divisor E. On the other hand, $\lambda_* \mathcal{O}_{X'}(L' + E) \subset \mathcal{O}_X(L)$, since E is also λ-exceptional.

(5) Let $Y' \to V \to Y$ be the Stein factorization of τ, where we write $\mu\colon Y' \to V$ and $\phi\colon V \to Y$. Then there is a Zariski-open subset $U \subset Y$ such that $\operatorname{codim}(Y \setminus U) \geq 2$ and $\phi^{-1}U$ is non-singular. Hence we have a generically isomorphic injection

$$\mu_* f'_* \mathcal{O}_{X'}(L') \hookrightarrow \phi^*\left(f_* \mathcal{O}_X(L)\right)^\wedge$$

by (3) and by taking j_* for the open immersion $j\colon \phi^{-1}U \hookrightarrow V$. Let $\mathcal{G}$ be the cokernel of

$$\mu^* \mu_* f'_* \mathcal{O}_{X'}(L') \to f'_* \mathcal{O}_{X'}(L') \oplus \mu^*\left(\phi^*\left(f_* \mathcal{O}_X(L)\right)^\wedge\right).$$

Then $f'_* \mathcal{O}_{X'}(L') \subset \mathcal{G}/(\mathrm{tor})$ and

$$\mathcal{G}/(\mathrm{tor}) \subset \left(\tau^* f_* \mathcal{O}_X(L)\right)^\wedge \otimes \mathcal{O}_{Y'}(E)$$

for a μ-exceptional effective divisor E. Thus we are done. $\square$

3.34. Lemma (cf. [**147**, 5.2]) *Suppose that X and Y are projective varieties. Let L be a Cartier divisor of X, Δ an effective $\mathbb{R}$-divisor of X, and let k be an integer greater than one satisfying the following conditions*:

(1) *$K_X + \Delta$ is $\mathbb{R}$-Cartier*;
(2) *(X, Δ) is log-terminal over a non-empty open subset of Y*;
(3) *$L - k(K_{X/Y} + \Delta)$ is nef and f-abundant.*

Let H be an ample divisor of Y and let l be a positive integer such that

$$\mathcal{O}_Y(lH) \otimes f_* \mathcal{O}_X(L)$$

is big in the sense of **3.20**. *Then*

$$\omega_Y((l - \llcorner l/k \lrcorner)H) \otimes f_* \mathcal{O}_X(L)$$

is an ω-big $\hat{\omega}$-sheaf. In particular,

$$\omega_Y((k-1)H) \otimes f_* \mathcal{O}_X(L)$$

is an ω-big $\hat{\omega}$-sheaf for any ample divisor H of Y.

PROOF. By **3.33**-(1), we may assume that X is non-singular and $\operatorname{Supp}\Delta$ is normal crossing. We can replace X by a further blowing-up. There is an f-exceptional effective divisor E such that $f_*\mathcal{O}_X(mL+mE)$ is the double-dual of $f_*\mathcal{O}_X(mL)$ for any $m \in \mathbb{N}$, by **III.5.10**-(3). Replacing X by a blowing-up, we may assume that the image of

$$f^*f_*\mathcal{O}_X(L+E) \to \mathcal{O}_X(L+E)$$

is an invertible sheaf which is written as $\mathcal{O}_X(L+E-B)$ for an effective divisor B of X. There is a positive integer a such that the sheaf

$$\mathcal{O}_Y((al-1)H) \otimes \widehat{\mathrm{S}}^a(f_*\mathcal{O}_X(L))$$

is generically generated by global sections. Note that the inequality

$$\frac{(al-1)(k-1)}{ak} < \frac{\ulcorner l(k-1) \urcorner}{k} = l - \llcorner l/k \lrcorner$$

holds. The natural homomorphism

$$\mathrm{Sym}^a(f_*\mathcal{O}_X(L+E)) \to f_*\mathcal{O}_X(a(L+E))$$

factors through $\widehat{\mathrm{S}}^a(f_*\mathcal{O}_X(L))$ and the image of the composite

$$f^*\,\mathrm{Sym}^a(f_*\mathcal{O}_X(L+E)) \to f^*f_*\mathcal{O}_X(a(L+E)) \to \mathcal{O}_X(a(L+E))$$

is $\mathcal{O}_X(a(L+E-B))$. Therefore, if we replace X by a further blowing-up, then there exist an f-exceptional effective divisor E' and an f-vertical effective divisor C of X such that $\mathcal{O}_X(a(L+E-B)+E')$ is the image of

$$f^*\widehat{\mathrm{S}}^a(f_*\mathcal{O}_X(L)) \to \mathcal{O}_X(a(L+E))$$

and $\mathcal{O}_X(P')$ is the image of

$$\mathrm{H}^0\left(Y, \mathcal{O}_Y((al-1)H) \otimes \widehat{\mathrm{S}}^a(f_*\mathcal{O}_X(L))\right) \otimes \mathcal{O}_X \to \mathcal{O}_X(a(L+E)+(al-1)f^*H)$$

for the divisor

$$P' := a(L+E-B) + E' - C + (al-1)f^*H.$$

Here, $\mathrm{Bs}\,|P'| = \emptyset$. We may assume that $\operatorname{Supp}(E+B+E'+C+\Delta)$ is a normal crossing divisor. For any $\varepsilon > 0$, $L - k(K_{X/Y}+\Delta) + \varepsilon f^*H$ is nef and abundant by **2.28**. Let us consider an $\mathbb{R}$-divisor

$$P := L - (K_{X/Y}+\Delta) + \frac{k-1}{k}(E-B) + \frac{k-1}{ak}(E'-C) + (l - \llcorner l/k \lrcorner)f^*H.$$

Then

$$P - \frac{k-1}{ak}P' - \frac{1}{k}(L - k(K_{X/Y}+\Delta) + \varepsilon f^*H) = \alpha f^*H$$

for some ε, $\alpha > 0$. Thus P is nef and abundant, and $P \succeq f^*H$. Hence $f_*\mathcal{O}_X(K_X + \ulcorner P \urcorner)$ is an ω-big ω-sheaf and there is a generically isomorphic injection

$$\begin{aligned} f_*\mathcal{O}_X(K_X + \ulcorner P \urcorner) &\hookrightarrow \omega_Y((l - \llcorner l/k \lrcorner)H) \otimes f_*\mathcal{O}_X\Big(L - \llcorner \frac{k-1}{k}B + \Delta \lrcorner\Big)^\wedge \\ &\hookrightarrow \omega_Y((l - \llcorner l/k \lrcorner)H) \otimes f_*\mathcal{O}_X(L - B)^\wedge \\ &= \omega_Y((l - \llcorner l/k \lrcorner)H) \otimes f_*\mathcal{O}_X(L)^\wedge. \end{aligned}$$

Thus the first assertion is proved. Let l_0 be the minimum of $l \in \mathbb{N}$ such that $\omega_Y(l_0H) \otimes f_*\mathcal{O}_X(L)$ is an ω-big $\hat{\omega}$-sheaf. Then $\mathcal{O}_Y(l_0H) \otimes f_*\mathcal{O}_X(L)$ is big by **3.31**. Thus $l_0 - \llcorner l_0/k \lrcorner \geq l_0$, equivalently, $l_0 \leq k - 1$. Thus we are done. □

3.35. Theorem *Let $f\colon X \to Y$ be a surjective morphism from a normal projective variety onto a non-singular projective variety. Let Δ be an effective $\mathbb{R}$-divisor of X, L a Cartier divisor of X, Q an $\mathbb{R}$-divisor of Y, and k a positive integer satisfying the following conditions*:

(1) *$K_X + \Delta$ is $\mathbb{R}$-Cartier*;
(2) *(X, Δ) is log-terminal over a non-empty open subset of Y*;
(3) *$L + f^*Q - k(K_{X/Y} + \Delta)$ is nef and f-abundant.*

Then $f_*\mathcal{O}_X(L)[\![Q]\!]$ is weakly positive. Suppose the following condition is also satisfied:

(4) *$L + f^*Q - k(K_{X/Y} + \Delta) \succcurlyeq f^*H$ for an ample divisor H of Y.*

Then

$$\omega_Y(\ulcorner Q \urcorner) \otimes f_*\mathcal{O}_X(L)$$

is an ω-big $\hat{\omega}$-sheaf.

Proof. *Step 1. A reduction step.* We can replace X by a blowing-up by **3.33**-(1). Thus we may assume that X is non-singular and $\operatorname{Supp} f^*Q \cup \operatorname{Supp}\Delta$ is normal crossing. Furthermore, we may assume that $\operatorname{Supp} Q$ is normal crossing, which is related to the proof of the second assertion. In fact, for a suitable birational morphism $\tau\colon Y' \to Y$ from a non-singular projective variety, we may assume that $X \to Y$ factors through Y', and $\tau^{-1}(\operatorname{Supp} Q)$ is normal crossing. Then

$$L + f^*Q - k(K_{X/Y'} + \Delta) - kR_\tau \succcurlyeq f^*H$$

for the effective divisor $R_\tau = K_{Y'} - \tau^*K_Y$. Thus $X \to Y'$ and τ^*Q satisfy the conditions above. For the morphism $f'\colon X \to Y'$, we have a generically isomorphic injection

$$\tau_*\left(\omega_{Y'}(\ulcorner \tau^*Q \urcorner) \otimes f'_*\mathcal{O}_X(L)\right) \subset \left(\omega_Y(\ulcorner Q \urcorner) \otimes f_*\omega_X(L)\right)^\wedge.$$

Thus we may assume that $\operatorname{Supp} Q$ is normal crossing.

Step 2. The first assertion in the case $Q = 0$. We fix an ample divisor H of Y. Let $\tau\colon Y' \to Y$ be a finite Galois surjective morphism from a non-singular projective variety such that $\tau^*H = mH'$ for a divisor H' of Y' for $m \gg 0$. Let $X' \to X \times_Y Y'$, $\lambda\colon X' \to X$, $f'\colon X' \to Y'$, R_Δ, R_τ, Δ', and L' be the same objects as in **3.33**-(2). Here we assume that $\operatorname{Supp}\Delta'$ is a normal crossing divisor. Then (X', Δ') is log-terminal over a non-empty open subset of Y',

$$L' - k(K_{X'/Y'} + \Delta') = \lambda^*(L - k(K_{X/Y} + \Delta))$$

is a nef and f'-abundant $\mathbb{R}$-divisor, and there is a generically isomorphic injection

$$f'_*\mathcal{O}_{X'}(L') \hookrightarrow \tau^*(f_*\mathcal{O}_X(L)).$$

If $k = 1$, then $\omega_{Y'}(H') \otimes f'_*\mathcal{O}_{X'}(L' - \llcorner\Delta'\lrcorner)$ is an ω-big ω-sheaf by **3.12** and **2.28**. Since $f'_*\mathcal{O}_{X'}(L' - \llcorner\Delta'\lrcorner) \hookrightarrow f'_*\mathcal{O}_{X'}(L')$ is generically isomorphic, $\omega_{Y'}(H') \otimes \tau^*(f_*\mathcal{O}_X(L))$ is an ω-big $\hat{\omega}$-sheaf. If $k \geq 2$, then $\omega_{Y'}((k-1)H') \otimes \tau^*(f_*\mathcal{O}_X(L))$ is an ω-big $\hat{\omega}$-sheaf by **3.34**. Hence, by **3.27**, $\tau^*(f_*\mathcal{O}_X(L)) \otimes \mathcal{O}_{Y'}(kH')$ is a big weakly positive sheaf in the both cases above. Thus $f_*\mathcal{O}_Y(L)[\![(k/m)H]\!]$ is big for $m \gg 0$ and hence $f_*\mathcal{O}_Y(L)$ is weakly positive.

Step 3 The second assertion in the case $Q = 0$. Assume that $L - k(K_{X/Y} + \Delta) \succcurlyeq f^*H$. Then we may assume that there are surjective morphisms $p\colon X \to Z$ and $q\colon Z \to Y$ with $f = q \circ p$ for a non-singular projective variety Z, and a nef and big $\mathbb{R}$-divisor A' of Z such that

$$(1/k)L - (K_{X/Y} + \Delta) \sim_{\mathbb{Q}} p^*A'$$

by **2.3**, **2.15**, and **2.28**. There is an effective $\mathbb{R}$-divisor G of Z such that $A' - G$ is an ample $\mathbb{Q}$-divisor and $(X, \Delta + p^*G)$ is log-terminal over a non-empty open subset of Y. Therefore, we may assume that Δ is a $\mathbb{Q}$-divisor and

$$(1/k)L - (K_{X/Y} + \Delta) \sim_{\mathbb{Q}} p^*A$$

for an ample $\mathbb{Q}$-divisor A. We can find a rational number $\alpha > 0$ such that $L - k(K_{X/Y} + \Delta) - \alpha f^*H$ is semi-ample. Let $\tau\colon Y' \to Y$ be the finite Galois surjective morphism in *Step 2* for $m > (k-1)/\alpha$ and let H' be the same ample divisor. Then the $\mathbb{Q}$-divisor

$$L' - k(K_{X'/Y'} + \Delta') - (k-1)f'^*H' = \lambda^*\left(L - k(K_{X/Y} + \Delta) - \frac{k-1}{m}f^*H\right)$$

is semi-ample. Thus $\omega_{Y'} \otimes f'_*\mathcal{O}_{X'}(L')$ is an ω-big $\hat{\omega}$-sheaf by **3.34**. By the proof of **3.34**, we have an ω-big ω-sheaf $\mathcal{F}'$ with a $\operatorname{Gal}(\tau)$-linearization and a generically isomorphic injection

$$\mathcal{F}' \hookrightarrow \omega_{Y'} \otimes (f'_*\mathcal{O}_{X'}(L'))^{\wedge}$$

which is compatible with $\operatorname{Gal}(\tau)$-linearizations. Hence there is a generically isomorphic injection

$$\mathcal{F} \hookrightarrow \omega_Y \otimes (f_*\mathcal{O}_X(L))^{\wedge}$$

from a direct summand $\mathcal{F}$ of $\tau_*\mathcal{F}'$. Hence $\omega_Y \otimes f_*\mathcal{O}_X(L)$ is an ω-big $\hat{\omega}$-sheaf.

Step 4 The case $Q \neq 0$. By *Step 1*, we assume $\operatorname{Supp} Q$ and $\operatorname{Supp}\Delta \cup \operatorname{Supp} f^*Q$ are normal crossing divisors. We set $\Delta_Q := \Delta + \langle -(1/k)f^*Q \rangle$. Then $\llcorner\Delta_Q\lrcorner$ is f-vertical and

$$L + k(\ulcorner \tfrac{1}{k}f^*Q \urcorner) - k(K_{X/Y} + \Delta_Q) = L + f^*Q - (K_{X/Y} + \Delta)$$

is nef and f-abundant. Thus

$$f_*\mathcal{O}_X\left(L + k(\ulcorner \tfrac{1}{k} f^*Q \urcorner)\right)$$

is weakly positive by *Step 2*. If the condition (4) is satisfied, then

$$\omega_Y \otimes f_*\mathcal{O}_Y\left(L + k(\ulcorner \tfrac{1}{k} f^*Q \urcorner)\right)$$

is an ω-big $\hat{\omega}$-sheaf by *Step 3*. Since $\ulcorner (1/k)f^*Q \urcorner \leq f^*(\ulcorner (1/k)Q \urcorner)$,

$$\mathcal{F}_1 := f_*\mathcal{O}_X(L) \otimes \mathcal{O}_Y\left(k(\ulcorner \tfrac{1}{k} Q \urcorner)\right)$$

is weakly positive. If the condition (4) is satisfied, then $\omega_Y \otimes \mathcal{F}_1$ is an ω-big $\hat{\omega}$-sheaf. For a positive integer $m > 0$, let $\tau \colon Y' \to Y$ be a finite surjective Galois morphism from a non-singular projective variety such that $\tau^*(\ulcorner mQ \urcorner) = mkQ'$ for a Cartier divisor Q' with $\operatorname{Supp} Q'$ being normal crossing. Let $X' \to X \times_Y Y'$, $f' \colon X' \to Y'$, R_Δ, Δ', and L' be the same objects as in **3.33**-(2). Since

$$L' + f'^*\tau^*Q - k(K_{X'/Y'} + \Delta') = \lambda^*(L + f^*Q - k(K_{X/Y} + \Delta))$$

is nef and f'-abundant, and since $\ulcorner (1/k)\tau^*Q \urcorner \leq Q'$,

$$\mathcal{F}_2 := f'_*\mathcal{O}_{X'}(L') \otimes \mathcal{O}_{Y'}(kQ')$$

is weakly positive. If the condition (4) is satisfied, then $\omega_{Y'} \otimes \mathcal{F}_2$ is an ω-big $\hat{\omega}$-sheaf. By the injection of **3.33**-(3),

$$f_*\mathcal{O}_X(L)[\![\tfrac{1}{m}(\ulcorner mQ \urcorner)]\!]$$

is weakly positive for any $m > 0$. Thus so is $f_*\mathcal{O}_X(L)[\![Q]\!]$ by **3.24**-(2). If the condition (4) is satisfied, then we have a generically isomorphic $\operatorname{Gal}(\tau)$-linearized injection

$$\mathcal{F}'' \hookrightarrow (\omega_{Y'}(kQ') \otimes \tau^*(f_*\mathcal{O}_X(L)))^\wedge$$

from an ω-big ω-sheaf $\mathcal{F}''$. Hence, by the same argument as above, $\omega_Y(\ulcorner Q \urcorner) \otimes f_*\mathcal{O}_X(L)$ is an ω-big $\hat{\omega}$-sheaf. $\square$

3.36. Corollary *Suppose that X is non-singular. Let Δ and D be $\mathbb{R}$-divisors of X and let Q be an $\mathbb{R}$-divisor of Y satisfying the following conditions*:

(1) $\operatorname{Supp}\Delta \cup \operatorname{Supp}\langle D \rangle$ *is a normal crossing divisor*;
(2) $\llcorner \Delta \lrcorner$ *is f-vertical*;
(3) $D + f^*Q - (K_{X/Y} + \Delta)$ *is nef and f-abundant.*

Let k be a positive integer such that

$$\llcorner \Delta + \frac{1}{k}\langle -kD \rangle \lrcorner$$

is f-vertical. Then $f_*\mathcal{O}_X(\ulcorner kD \urcorner)[\![kQ]\!]$ is weakly positive, and $\omega_Y(H + \ulcorner kQ \urcorner) \otimes f_*\mathcal{O}_X(\ulcorner kD \urcorner)$ is an ω-big $\hat{\omega}$-sheaf for any ample divisor H of Y.

Proof. We have

$$\ulcorner kD \urcorner - k(K_{X/Y} + \Delta + \frac{1}{k}\langle -kD \rangle) = kD - k(K_{X/Y} + \Delta).$$

Apply **3.35** to the divisors $L = \ulcorner kD \urcorner$ and $L = \ulcorner kD \urcorner + f^*H$. □

3.37. Corollary *For a big divisor H of Y, there is a positive integer a satisfying the following condition: if a Cartier divisor L of X, an effective $\mathbb{R}$-divisor Δ of X, an $\mathbb{R}$-divisor Q of Y, and a positive integer k satisfy the conditions* (1)–(3) *of* **3.35**, *then*

$$\mathcal{O}_Y(aH + \ulcorner Q \urcorner) \otimes f_*\mathcal{O}_X(L)$$

is generically generated by global sections.

Proof. $\omega_Y(H + \ulcorner Q \urcorner) \otimes f_*\mathcal{O}_X(L)$ is an $\hat{\omega}$-sheaf by **3.35**. Thus we can find a positive integer a such that

$$\mathcal{O}_Y(aH + \ulcorner Q \urcorner) \otimes (f_*\mathcal{O}_X(L))^\wedge$$

is generically generated by global sections by **3.18**.

Let $\tau\colon Y' \to Y$ be a birational morphism from a non-singular projective variety flattening f such that $\tau^{-1}(\operatorname{Supp} Q)$ is a normal crossing divisor. Let $X' \to X \times_Y Y'$, $\lambda\colon X' \to X$, $f'\colon X' \to Y'$, R_Δ, R_τ, Δ', and L' be the same objects defined in **3.33**-(4). Then L', Δ', k, and τ^*Q satisfy the same conditions as (1)–(3) of **3.35** for the morphism $f'\colon X' \to Y'$. Therefore, there is a positive integer a such that

$$\mathcal{O}_{Y'}(a\tau^*H + \ulcorner \tau^*Q \urcorner) \otimes f'_*\mathcal{O}_{X'}(L')^\wedge$$

is generically generated by global sections. Since $\ulcorner \mu^*Q \urcorner \leq \mu^*(\ulcorner Q \urcorner)$,

$$\mathcal{O}_Y(aH + \ulcorner Q \urcorner) \otimes f_*\mathcal{O}_X(L)$$

is generically generated by global sections by **3.33**-(4). □

3.38. Corollary *Suppose that X is non-singular. Let Δ and D be $\mathbb{R}$-divisors of X and let Q be an $\mathbb{R}$-divisor of Y satisfying the following conditions*:

(1) $\operatorname{Supp}\Delta \cup \operatorname{Supp}\langle D \rangle$ *and* $\operatorname{Supp} Q$ *are normal crossing divisors*;
(2) $\llcorner \Delta \lrcorner$ *is f-vertical*;
(3) $D + f^*Q - (K_{X/Y} + \Delta)$ *is nef and f-abundant*;
(4) $D + f^*Q - (K_{X/Y} + \Delta) \succcurlyeq f^*H$.

Then, for any big divisor H of Y, there exist positive integers b and d such that

$$f_*\mathcal{O}_X(\ulcorner mbD \urcorner) \otimes \mathcal{O}_Y(\ulcorner mbQ \urcorner - (m-d)H)$$

is generically generated by global sections for any $m > 0$.

Proof. The $\mathbb{R}$-divisor $P := D + f^*Q - (K_{X/Y} + \Delta)$ is nef and abundant by **2.28**. Furthermore, by **2.27**, there exist a positive integer c and an effective $\mathbb{R}$-divisor G on X such that $cP - f^*H \sim_{\mathbb{Q}} G$. We may assume that $\operatorname{Supp}(\Delta + \langle -D \rangle + G) \cup \operatorname{Supp} f^*Q$ is a normal crossing divisor. For m, $b > 0$, we set

$$\Delta_{m,b} := \Delta + \frac{1}{mb}\langle -mbD \rangle + \frac{1}{b}G.$$

Then, for any $m > 0$, there is an integer $b > c$ such that $(X, \Delta_{m,b})$ is log-terminal over a non-empty open subset of Y and

$$\begin{aligned}(b-c)mP &\sim_{\mathbb{Q}} mbD + f^*(mbQ) - mb(K_{X/Y} + \Delta) - mf^*H - mG \\ &= \ulcorner mbD \urcorner + f^*(mbQ - mH) - mb(K_{X/Y} + \Delta_{m,b})\end{aligned}$$

is nef and abundant. Thus there is a constant d such that

$$\mathcal{O}_Y(dH) \otimes f_*\mathcal{O}_X(\ulcorner mbD \urcorner) \otimes \mathcal{O}_Y(\ulcorner mbQ \urcorner - mH)$$

is generically generated by global sections by **3.37**. $\square$

3.39. Lemma *Let L be a Cartier divisor of X, Δ an effective $\mathbb{R}$-divisor of X, Θ a divisor of Y, and $k \geq 2$ an integer satisfying the following conditions*:

(1) $K_X + \Delta$ *is $\mathbb{R}$-Cartier*;
(2) (X, Δ) *is log-terminal over a non-empty open subset of Y*;
(3) $L - k(K_{X/Y} + \Delta)$ *is nef and f-abundant*;
(4) *there is an injection $\mathcal{O}_Y(\Theta) \hookrightarrow f_*\mathcal{O}_X(L)^\wedge$.*

Then there is a number $\alpha \in \mathbb{Q}_{>0}$ such that $f_*\mathcal{O}_X(L)[\![-\alpha\Theta]\!]$ is weakly positive and

$$\omega_Y(H - \llcorner \alpha\Theta \lrcorner) \otimes f_*\mathcal{O}_X(L)$$

is an ω-big $\hat{\omega}$-sheaf for any ample divisor H.

Proof. We follow the proof of **3.34** and fix an ample divisor H of Y. We may assume that X is non-singular and $\operatorname{Supp}\Delta$ is normal crossing. We can replace X by a further blowing-up. Let E and B be effective divisors appearing in the proof of **3.34**. Then, after replacing X by a blowing-up, we have an effective divisor D such that

$$D + f^*\Theta \sim L + E - B.$$

We may assume $\operatorname{Supp}(\Delta + E + B + D)$ is a normal crossing divisor. We fix a positive integer $b > 1$ such that $\llcorner (1/b)D + \Delta \lrcorner$ is f-vertical. Now $f_*\mathcal{O}_X(L)$ is weakly positive by **3.35**. We have a positive integer d such that

$$\widehat{\mathrm{S}}^{a(b-1)}(f_*\mathcal{O}_X(L)) \otimes \mathcal{O}_Y(dH)$$

is generically generated by global sections for $a \gg 0$ by **3.23**. We fix such an integer a. As in the proof of **3.34**, we may assume that $\mathcal{O}_X(a(b-1)(L + E - B) + E')$ is the image of

$$f^*\widehat{\mathrm{S}}^{a(b-1)}(f_*\mathcal{O}_X(L)) \to \mathcal{O}_X(a(b-1)(L + E))$$

for an f-exceptional effective divisor E' and that $\mathcal{O}_X(P')$ is the image of

$$\mathrm{H}^0\big(Y, \widehat{\mathrm{S}}^{a(b-1)}(f_*\mathcal{O}_X(L) \otimes \mathcal{O}_Y(dH))\big) \otimes \mathcal{O}_X \to \mathcal{O}_X(a(b-1)(L + E - B) + E')$$

for the divisor

$$P' := a(b-1)(L + E - B) + E' - C + df^*H$$

for an f-vertical effective divisor C. Moreover, we may assume $\mathrm{Supp}(C + E' + B + E + \Delta + D)$ is a normal crossing divisor. Let D^h and D^v be the f-horizontal and the f-vertical parts of D, respectively. Note that $\mathrm{Bs}\,|P'| = \emptyset$ and

$$P' = a(b(L + E - B) - D^v) - af^*\Theta - aD^h + E' - C + df^*H.$$

We set

$$\begin{aligned} P := L - (K_{X/Y} + \Delta) + \frac{k-1}{k}(E - B) - \frac{k-1}{bk}D^h \\ + \frac{k-1}{abk}(E' - C - aD^v) + f^*\left(\delta H - \frac{k-1}{bk}\Theta\right) \end{aligned}$$

for $\delta > (k-1)d/(abk)$. Then

$$P - \frac{k-1}{abk}P' - \varepsilon f^*H \sim_{\mathbb{Q}} \frac{1}{k}L - (K_{X/Y} + \Delta) + \left(\delta - \frac{(k-1)d}{abk} - \varepsilon\right) f^*H$$

is nef and abundant for some $\varepsilon > 0$ by **2.28**. We can take $\delta \ll 1$ if $a \gg 0$. Since

$$\llcorner \frac{k-1}{k}B + \frac{k-1}{bk}D^h + \Delta \lrcorner \leq B + \llcorner \frac{1}{b}D + \Delta \lrcorner,$$

we can write

$$-\Delta + \frac{k-1}{k}(E - B) - \frac{k-1}{bk}D^h + \frac{k-1}{abk}(E' - C - aD^v) = -\Delta' + E'' - G - B'$$

for an effective $\mathbb{R}$-divisor Δ' with $\llcorner \Delta' \lrcorner = 0$, an f-exceptional effective divisor E'', an f-vertical effective divisor G, and an effective divisor B' with $f_*\mathcal{O}_X(L - B')^\wedge \simeq f_*\mathcal{O}_X(L)^\wedge$. We set $\overline{L} := L + E'' - G - B'$ and $\alpha := (k-1)/(bk)$. Then there is an inclusion $f_*\mathcal{O}_X(\overline{L})^\wedge \subset f_*\mathcal{O}_X(L)^\wedge$ and

$$\overline{L} + f^*(\delta H - \alpha\Theta) - (K_{X/Y} + \Delta') = P \succcurlyeq f^*H.$$

Hence, $f_*\mathcal{O}_X(L)[\![\delta H - \alpha\Theta]\!]$ is big and

$$\omega_Y(H - \llcorner \alpha\Theta \lrcorner) \otimes f_*\mathcal{O}_X(L)$$

is an ω-big $\hat{\omega}$-sheaf, by **3.35**. Taking $\delta \to 0$, we infer that $f_*\mathcal{O}_X(L)[\![-\alpha\Theta]\!]$ is also weakly positive by **3.24**-(2). $\square$

Let $f\colon X \to Y$ be a surjective morphism of non-singular projective varieties. The morphism f is called a *semi-stable reduction in codimension one* or a *semi-stable morphism in codimension one* if there is a Zariski-open subset $Y^\circ \subset Y$ with $\mathrm{codim}(Y \setminus Y^\circ) \geq 2$ such that, for any prime divisor $\Gamma \subset Y$, $f^*\Gamma$ is a reduced and normal crossing divisor over $f^{-1}(Y^\circ)$. Even though f is not a semi-stable reduction in codimension one, there exist a finite surjective morphism $\tau\colon Y' \to Y$ from a non-singular projective variety and a desingularization $X' \to X \times_Y Y'$ such that the induced morphism $f'\colon X' \to Y'$ is a semi-stable reduction in codimension one (cf. [**62**], [**147**, Proposition 6.1], [**88**, 4.6]). This (f', τ) is called also a semi-stable reduction of f in codimension one.

3.40. Lemma *Let $f\colon X \to Y$ be a surjective morphism of non-singular projective varieties that is a semi-stable reduction in codimension one. Let L be a divisor of X, Δ an $\mathbb{R}$-divisor, and k a positive integer satisfying the following conditions*:

(1) *(X, Δ) is log-terminal over a non-empty open subset of Y;*
(2) *$L - k(K_{X/Y} + \Delta)$ is nef and f-abundant.*

Then, for any positive integer m, there is a positive number α such that

$$f_*\mathcal{O}_X(mL)[\![-\alpha\,\widehat{\det}(f_*\mathcal{O}_X(L))]\!]$$

is weakly positive.

Proof. Let r be the rank of $f_*\mathcal{O}_X(L)$. Let $X^{[r]}$ be the r-fold fiber product $X \times_Y \cdots \times_Y X$ over Y. Then $X^{[r]}$ has only toroidal singularities over a Zariski-open subset $Y^\circ \subset Y$ with $\operatorname{codim}(Y \setminus Y^\circ) \geq 2$. Let $p_i\colon X^{[r]} \to X$ be the i-th projection for $1 \leq i \leq r$. Then

$$\omega_{X^{[r]}/Y} \simeq \bigotimes\nolimits_{i=1}^{r} p_i^*\omega_{X/Y}$$

over Y°. Let $\delta\colon X^{(r)} \to X^{[r]}$ be a birational morphism from a projective non-singular space which is an isomorphism over a dense Zariski-open subset of Y. Let $f^{(r)}\colon X^{(r)} \to Y$ and $\pi_i = p_i \circ \delta\colon X^{(r)} \to X$ be the induced morphisms. We can write

$$E - G = K_{X^{(r)}/Y} - \sum\nolimits_{i=1}^{r} \pi_i^* K_{X/Y}$$

for effective divisors E and G such that E is δ-exceptional over Y° and $f^{(r)}(\operatorname{Supp} G)$ is contained in $Y \setminus Y^\circ$. We set

$$L^{(r)} := \sum\nolimits_{i=1}^{r} \pi_i^* L + kE \quad \text{and} \quad \Delta^{(r)} := \sum\nolimits_{i=1}^{r} \pi_i^*\Delta + G.$$

Then $(X^{(r)}, \Delta^{(r)})$ is log-terminal over a non-empty open subset of Y and

$$L^{(r)} - k(K_{X(r)/Y} + \Delta^{(r)}) = \sum\nolimits_{i=1}^{r} \pi_i^*(L - k(K_{X/Y} + \Delta)).$$

Thus $f_*^{(r)}\mathcal{O}_{X^{(r)}}(L^{(r)})$ is weakly positive by **3.35** and we have an isomorphism

$$\left(f_*^{(r)}\mathcal{O}_{X^{(r)}}(L^{(r)})\right)^\wedge \simeq \widehat{\otimes}^r (f_*\mathcal{O}_X(L)).$$

Since $\widehat{\det}(f_*\mathcal{O}_X(L))$ is a subsheaf of the right hand side, we have an injection

$$\widehat{\det}(f_*\mathcal{O}_X(L))^{\otimes m} \hookrightarrow \left(f_*^{(r)}\mathcal{O}_{X^{(r)}}(mL^{(r)})\right)^\wedge$$

for $m > 0$. Note that, for $m > 0$, $f_*^{(r)}\mathcal{O}_{X^{(r)}}(mL^{(r)})$ is weakly positive and there is an isomorphism

$$\left(f_*^{(r)}\mathcal{O}_{X^{(r)}}(mL^{(r)})\right)^\wedge \simeq \widehat{\otimes}^r (f_*\mathcal{O}_X(mL)).$$

Hence, by **3.39**,

$$f_*^{(r)}\mathcal{O}_{X^{(r)}}(mL^{(r)})[\![-\alpha\,\widehat{\det}(f_*\mathcal{O}_X(L))]\!]$$

is weakly positive for some $\alpha > 0$. From the generically surjective homomorphism

$$\widehat{\otimes}^r(f_*\mathcal{O}_X(mL)) \to \widehat{\mathrm{S}}^r(f_*\mathcal{O}_X(mL)),$$

we infer that $f_*\mathcal{O}_X(mL)[\![-(\alpha/r)\widehat{\det}(f_*\mathcal{O}_X(L))]\!]$ is also weakly positive. □

§4. Abundance and Addition

§4.a. Addition Theorem.

4.1. Theorem *Let $f\colon X \to Y$ be a fiber space from a normal projective variety into a non-singular projective variety, Δ an effective $\mathbb{R}$-divisor of X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and (X, Δ) is log-canonical over a non-empty open subset of Y. Let D be an $\mathbb{R}$-Cartier divisor of X such that $D - (K_{X/Y} + \Delta)$ is nef.*

(1) *For any $\mathbb{R}$-divisor Q of Y,*

$$\kappa_\sigma(D + f^*Q) \geq \kappa_\sigma(D; X/Y) + \kappa_\sigma(Q).$$

In particular, for a 'general' fiber $X_y = f^{-1}(y)$,

$$\kappa_\sigma(K_X + \Delta) \geq \kappa_\sigma(K_{X_y} + \Delta|_{X_y}) + \kappa_\sigma(K_Y).$$

(2) *Suppose that (X, Δ) is log-terminal over a non-empty open subset of Y and that $D - (K_{X/Y} + \Delta)$ is f-abundant. Then*

$$\sigma(D; f^*H)' \geq \kappa(D; X/Y)$$

for some ample divisor H of Y, where $\sigma(\ \ ;\ \ )'$ is defined in **2.6**. *If $D - (K_{X/Y} + \Delta) \succcurlyeq f^*H$, then*

$$\kappa(D, X) = \kappa(D; X/Y) + \dim Y.$$

In particular, if Y is of general type, then

$$\kappa(K_X + \Delta) = \kappa(K_{X_y} + \Delta|_{X_y}) + \dim Y.$$

Proof. By **3.33**-(1), we may assume that X is non-singular and $\operatorname{Supp}\Delta \cup \operatorname{Supp}\langle D\rangle$ is normal crossing. For a divisor A of X and for $m \in \mathbb{N}$, we set

$$r(mD; A) := \operatorname{rank} f_*\mathcal{O}_X\left(\ulcorner mD \urcorner + A\right).$$

Then we have

$$\sigma(D|_{X_y}; A|_{X_y})' = \max\left\{k \in \mathbb{Z}_{\geq 0} \cup \{-\infty\} \;\middle|\; \varlimsup_{m\to\infty} \frac{r(mD; A)}{m^k} > 0\right\}$$

for a 'general' fiber $X_y = f^{-1}(y)$. Note that

$$\kappa_\sigma(D; X/Y) = \max\{\sigma(D|_{X_y}; A|_{X_y})' \mid A \text{ is ample}\}.$$

If $\kappa(D; X/Y) \geq 0$, then, by **3.9**,

$$\kappa(D; X/Y) = \varlimsup_{m\to\infty} \frac{\log r(mD; 0)}{\log m}$$

(1) Let A be an ample divisor of X such that $(1/2)A + \langle -mD\rangle$ is ample for any $m \in \mathbb{Z}$. Since $D + (1/2)A - (K_{X/Y} + \Delta)$ is ample, we can find a positive rational

number $\varepsilon \ll 1$ such that $D + (1/2)A - (K_{X/Y} + (1-\varepsilon)\Delta)$ is also ample. Then $(X, (1-\varepsilon/m)\Delta)$ is log-terminal over a non-empty open subset, and

$$\begin{aligned} &\ulcorner mD \urcorner + A - m(K_{X/Y} + (1-\varepsilon/m)\Delta) \\ &\quad = (m-1)\big(D - (K_{X/Y} + \Delta)\big) + \big(D + (1/2)A - (K_{X/Y} + (1-\varepsilon)\Delta)\big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + ((1/2)A + \langle -mD \rangle) \end{aligned}$$

is ample for any $m \in \mathbb{N}$. There exists an ample divisor H of Y such that $\mathcal{O}_Y(H) \otimes f_*\mathcal{O}_X(\ulcorner mD \urcorner + A)$ is generically generated by global sections for any $m \in \mathbb{N}$, by **3.37**. In particular, there exists a generically isomorphic injection

$$\mathcal{O}_Y^{\oplus r(mD;A)} \hookrightarrow \mathcal{O}_Y(H) \otimes f_*\mathcal{O}_X(\ulcorner mD \urcorner + A),$$

which induces the injection

$$\mathcal{O}_Y(\llcorner mQ \lrcorner + H)^{\oplus r(mD;A)} \hookrightarrow \mathcal{O}_Y(\llcorner mQ \lrcorner + 2H) \otimes f_*\mathcal{O}_X(\ulcorner mD \urcorner + A).$$

Therefore,

$$\begin{aligned} &\mathrm{h}^0\big(X, \ulcorner m(D + f^*Q) \urcorner + A + 2f^*H\big) \\ &\quad \geq \mathrm{h}^0\big(X, \ulcorner mD \urcorner + f^*(\llcorner mQ \lrcorner) + A + 2f^*H\big) \geq r(mD;A) \cdot \mathrm{h}^0(Y, \llcorner mQ \lrcorner + H). \end{aligned}$$

Varying $m \in \mathbb{N}$, we have the expected inequality.

(2) We may assume that $\kappa(D; X/Y) \geq 0$. By **3.36** and **3.37**, we have an ample divisor H of Y such that, for each $m \gg 0$ with $r(mD;0) > 0$, there exists a generically isomorphic injection

$$\mathcal{O}_Y^{\oplus r(mD;0)} \hookrightarrow \mathcal{O}_Y(H) \otimes f_*\mathcal{O}_X(\ulcorner mD \urcorner).$$

Therefore,

$$\mathrm{h}^0(X, \ulcorner mD \urcorner + 2f^*H) \geq r(mD;0) \cdot \mathrm{h}^0(Y, H).$$

By varying m and H we have the first inequality. Next, suppose that $D - (K_{X/Y} + \Delta) \succcurlyeq f^*H$. By **3.38**, there exist positive integers b and d such that a generically isomorphic injection

$$\mathcal{O}_Y^{\oplus r(mbD;0)} \hookrightarrow \mathcal{O}_Y(-(m-d)H) \otimes f_*\mathcal{O}_X(\ulcorner mbD \urcorner)$$

exists for any $m > 0$. Therefore,

$$\mathrm{h}^0(X, \ulcorner mbD \urcorner) \geq r(mbD;0) \cdot \mathrm{h}^0(Y, (m-d)H).$$

By varying m and by the easy addition for κ, we have the expected equality. If Y is of general type, then the equality above for $D = K_{X/Y} + \Delta + f^*H$ for an ample divisor H of Y and the property $K_Y \succeq H$ imply the last equality. $\square$

4.2. Corollary *Let X be a normal projective variety, Δ an effective $\mathbb{R}$-divisor, and D an $\mathbb{R}$-divisor such that (X, Δ) is log-canonical and $D - (K_X + \Delta)$ is nef. Then the following three conditions are equivalent*:

(1) *D is abundant*: $\kappa_\nu(D) = \kappa(D)$;
(2) $\kappa_\sigma(D) = \kappa(D)$;

(3) D *is geometrically abundant.*

Proof. It is enough to show (2) $\Rightarrow$ (3). Let m be an integer in $\mathbb{I}(D)$. We may assume the Iitaka fibration $\Phi = \Phi_{mD} \colon X \dashrightarrow Y$ is holomorphic. Then $D \succeq \Phi^* H$ for an ample divisor H of Y. By **4.1**, we have

$$\begin{aligned}\kappa(D) = \kappa_\sigma(D) &= \kappa_\sigma(D + \Phi^* H) \\ &= \kappa_\sigma(D; X/Y) + \dim Y = \kappa_\sigma(D; X/Y) + \kappa(D)\end{aligned}$$

Hence $\kappa_\sigma(D; X/Y) = 0$. □

Remark (1) The abundance conjecture: $\kappa(X) = \kappa_\nu(X)$ for projective varieties X is reduced to the following conjecture by **4.2**: if $\kappa_\nu(X) > 0$, then $\kappa(X) > 0$.

(2) By the abundance theorem [**59**] (cf. [**83**], [**84**]) and the existence of minimal models [**89**] for threefolds, the abundance conjecture is true for a projective variety X with $\dim X \le 3$ or with $\kappa(X) \ge \dim X - 3$.

Let $f \colon X \to Y$ be a fiber space from a normal projective variety onto a non-singular projective variety. Let D be a $\mathbb{Q}$-Cartier divisor of X and let Δ be an effective $\mathbb{R}$-divisor of X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and that (X, Δ) is log-terminal over a non-empty open subset of Y. Let b be a positive integer with bD being Cartier.

Let $\tau \colon Y' \to Y$ be a generically finite morphism from a non-singular projective variety. Let $X' \to X \times_Y Y'$ be a birational morphism from a projective non-singular variety and let $\lambda \colon X' \to X$ and $f' \colon X' \to Y'$ be the induced morphisms. We assume that the union of the non-étale locus of λ and $\lambda^{-1}(\operatorname{Supp} \Delta)$ is a normal crossing divisor. As in **3.33**, we set $R_\Delta := K_{X'} - \lambda^*(K_X + \Delta)$ and $R_\tau := K_{Y'} - \tau^* K_Y$. Let $\ulcorner R_\Delta \urcorner = R'_+ - R'_-$ be the decomposition into the positive and the negative parts of the prime decomposition, and set

$$\Delta' := R'_- + \langle -R_\Delta \rangle \quad \text{and} \quad D' := \lambda^* D + R'_+ - f'^* R_\tau.$$

Then (X', Δ') is log-terminal over a non-empty open subset of Y' and the equalities

$$\begin{aligned}K_{X'} + \Delta' &= \lambda^*(K_X + \Delta) + R'_+, \\ D' - (K_{X'/Y'} + \Delta') &= \lambda^*(D - (K_{X/Y} + \Delta))\end{aligned}$$

hold. Here, bD' is also Cartier.

4.3. Claim If (X, Δ) is log-terminal, then $f'_* \mathcal{O}_{X'}(bD')$ is independent of the choice of birational morphisms $X' \to X \times_Y Y'$.

Proof. $R'_- = 0$ by assumption. For a birational morphism $\varphi \colon X'' \to X'$ from a projective non-singular space such that the composite $X'' \to X' \to X \times_Y Y'$ satisfies the same conditions as $X' \to X \times_Y Y'$, if we set

$$\begin{aligned}R''_\Delta &:= K_{X''} - \varphi^* \lambda^*(K_X + \Delta), \quad \ulcorner R''_\Delta \urcorner = R''_+ - R''_-, \\ \Delta'' &:= R''_- + \langle -R''_\Delta \rangle, \quad D'' := \varphi^* \lambda^* D + R''_+ - \varphi^* f'^* R_\tau,\end{aligned}$$

then $R''_\Delta = R_\varphi + \varphi^* R_\Delta$ and $R''_- = 0$. Hence, by **II.4.3**-(2),

$$R''_+ = R_\varphi + \ulcorner \varphi^* R_\Delta \urcorner \geq \varphi^*(\ulcorner R_\Delta \urcorner) = \varphi^* R'_+.$$

Since $R''_+ - \varphi^* R'_+$ is φ-exceptional, we have an isomorphism

$$\mathcal{O}_{X'}(bD') \simeq \varphi_* \mathcal{O}_{X''}(bD''). \qquad \square$$

There exists a Zariski-open subset $Y^\circ \subset Y$ such that $\operatorname{codim}(Y \smallsetminus Y^\circ) \geq 2$ and τ is a finite morphism over Y°. Thus there exist a τ-exceptional effective divisor $\widehat{E}_b$ and a generically isomorphic injection

$$f'_* \mathcal{O}_{X'}(bD') \hookrightarrow \Bigl(\tau^*(f_* \mathcal{O}_X(bD)) \otimes \mathcal{O}_{Y'}(\widehat{E}_b)\Bigr)$$

by **3.33**-(5). In particular, we have inequalities

$$\kappa(\widehat{\det}(f'_* \mathcal{O}_{X'}(bD')), Y') \leq \kappa(\widehat{\det}(f_* \mathcal{O}_X(bD)), Y),$$
$$\kappa_\sigma(\widehat{\det}(f'_* \mathcal{O}_{X'}(bD')), Y') \leq \kappa_\sigma(\widehat{\det}(f_* \mathcal{O}_X(bD)), Y).$$

We note that, if f is a semi-stable reduction in codimension one and if $D-(K_{X/Y}+\Delta)$ is nef and f-abundant, then

$$\kappa_\sigma(\widehat{\det}(f_* \mathcal{O}_X(mbD)), Y) \geq \kappa_\sigma(\widehat{\det}(f_* \mathcal{O}_X(bD)), Y)$$

for $m > 0$, by **3.40**.

4.4. Definition

$$\kappa_\sigma(D, \det f; Y') := \max_{b>0} \kappa_\sigma(\widehat{\det}(f'_* \mathcal{O}_{X'}(bD')), Y'),$$
$$\kappa_\sigma(D, \det f) := \min_{Y' \to Y} \kappa_\sigma(D, \det f; Y').$$

4.5. Theorem *Let $f\colon X \to Y$ be a fiber space from a normal projective variety onto a non-singular projective variety. Let D be a $\mathbb{Q}$-Cartier divisor on X and let Δ be an effective $\mathbb{R}$-divisor such that*

(1) *$K_X + \Delta$ is $\mathbb{R}$-Cartier,*
(2) *(X, Δ) is log-terminal over a non-empty open subset of Y,*
(3) *$D - (K_{X/Y} + \Delta)$ is nef and f-abundant.*

Then, for an ample divisor H and for $b \in \mathbb{N}$ with bD being Cartier,

$$\sigma(bD; f^* H) \geq \kappa(D; X/Y) + \kappa_\sigma(D, \det f).$$

If $\kappa_\sigma(D, \det f) = \dim Y$, then

$$\kappa(D, X) = \kappa(D; X/Y) + \dim Y.$$

Proof. We may assume that $\kappa(D; X/Y) \geq 0$ and X is non-singular.

Suppose first that f is a semi-stable reduction in codimension one. Let b be a positive integer such that bD is Cartier and $b \in \mathbb{I}(D|_{X_y})$ for a 'general' fiber X_y. For $m > 0$, let $\mathcal{G}_m$ be the image of the multiplication mapping

$$\widehat{\mathrm{S}}^m(f_* \mathcal{O}_X(bD)) \to f_* \mathcal{O}_X(mbD)^\wedge.$$

Then

$$\varlimsup_{m\to\infty} m^{-\kappa(D;X/Y)} \operatorname{rank} \mathcal{G}_m > 0.$$

By **3.40**, we infer that

$$f_*\mathcal{O}_X(bD)[\![-\alpha\,\widehat{\det}(f_*\mathcal{O}_X(bD))]\!]$$

is weakly positive for some $\alpha \in \mathbb{Q}_{>0}$. Thus there is a big divisor H on Y such that

$$\mathcal{G}_m^\wedge \otimes \widehat{\det}(f_*\mathcal{O}_X(bD)))^{\otimes(-m\alpha)} \otimes \mathcal{O}_Y(H)$$

is generically generated by global sections for a large integer m with $m\alpha \in \mathbb{Z}$ by **3.23**. In particular, there is an injection

$$\mathcal{O}_Y^{\oplus \operatorname{rank}\mathcal{G}_m} \otimes \widehat{\det}(f_*\mathcal{O}_X(bD)))^{\otimes m\alpha} \otimes \mathcal{O}_Y(H) \hookrightarrow f_*\mathcal{O}_X(mbD)^\wedge \otimes \mathcal{O}_Y(2H).$$

Therefore,

$$\sigma(bD+E; f^*H) \geq \kappa(D;X/Y) + \kappa_\sigma(\widehat{\det}(f_*\mathcal{O}_X(bD))), Y)$$

for an f-exceptional effective divisor E of X. If $\widehat{\det}(f_*\mathcal{O}_X(bD)))$ is big, then there is a positive integer d such that $\mathcal{G}_m^\wedge \otimes \mathcal{O}_Y(-(m-d)H)$ is generically generated by global sections for $m \gg 0$. Thus there is an injection

$$\mathcal{O}_Y^{\oplus \operatorname{rank}\mathcal{G}_m} \otimes \mathcal{O}_Y((m-d)H) \hookrightarrow f_*\mathcal{O}_X(mbD)^\wedge.$$

Therefore,

$$\kappa(bD+E) = \kappa(D;X/Y) + \dim Y.$$

Next, we consider the general case. Let $\widetilde{Y} \to Y$ be a birational morphism from a non-singular projective variety flattening f. Let $Y' \to \widetilde{Y}$ be a finite surjective morphism from a non-singular projective variety and let $X' \to X \times_Y Y'$ be a birational morphism from a non-singular projective variety into the main component such that the induced morphism $f'\colon X' \to Y'$ is a semi-stable reduction in codimension one. Let $\lambda\colon X' \to X$ and $\tau\colon Y' \to Y$ be the induced morphisms. We consider $\mathbb{R}$-divisors R_Δ, R'_+, Δ', and D' as before. Then we have

$$\begin{aligned}\sigma(bD'+E; {f'}^*\tau^*H) &\geq \kappa(D;X/Y) + \kappa_\sigma(\widehat{\det}(f'_*\mathcal{O}_{X'}(bD')), Y')\\ &\geq \kappa(D;X/Y) + \kappa_\sigma(D, \det f)\end{aligned}$$

for a λ-exceptional effective divisor E. Since $bR'_+ + E$ is λ-exceptional, from the inequality $bD' + E \leq \lambda^*(bD) + bR'_+ + E$, we have

$$\sigma(bD'+E; {f'}^*\tau^*H) \leq \sigma(bD; f^*H).$$

Therefore,

$$\sigma(bD; f^*H) \geq \kappa(D;X/Y) + \kappa_\sigma(D, \det f).$$

If $\kappa_\sigma(D, \det f) = \dim Y$, then

$$\kappa(D,X) \geq \kappa(bD'+E) = \kappa(D;X/Y) + \dim Y \geq \kappa(D,X). \qquad \square$$

§4.b. Abundance theorem for $\kappa_\sigma = 0$.

4.6. Theorem *Let $f\colon X \to Y$ be a fiber space from a normal projective variety onto a non-singular projective variety. Let L be a Cartier divisor of X and let Δ be an effective $\mathbb{R}$-divisor of X such that*

(1) *$K_X + \Delta$ is $\mathbb{R}$-Cartier,*
(2) *(X, Δ) is log-terminal over a non-empty open subset of Y,*
(3) *$L - (K_{X/Y} + \Delta)$ is nef and abundant.*

Suppose that $\operatorname{rank} f_*\mathcal{O}_X(L) = 1$ *and $\kappa_\sigma(f_*\mathcal{O}_X(L)^\wedge) = 0$. Then $\kappa(f_*\mathcal{O}_X(L)^\wedge) = 0$.*

Proof. We may assume that X is non-singular, $\operatorname{Supp}\Delta$ is normal crossing by **3.33**-(1). Since $f_*\mathcal{O}_X(L - \llcorner\Delta\lrcorner)$ is weakly positive by **3.35**, we may assume that $\llcorner\Delta\lrcorner = 0$. Furthermore, we can replace Y by a generically finite morphism $Y' \to Y$, by **3.33**. Then $\omega_Y \otimes f_*\mathcal{O}_X(L)$ is an ω-sheaf and moreover, there exists a surjective morphism $h\colon M \to Y$ from a non-singular projective variety M such that $\omega_Y \otimes f_*\mathcal{O}_X(L)$ is a direct summand of $h_*\omega_M$ by **3.10**, **3.11**. Replacing Y by a generically finite morphism $Y' \to Y$, we may assume that

- h is smooth outside a normal crossing divisor $B \subset Y$,
- the local monodromies of the locally constant system $H = R^d h_*\mathbb{C}_M|_{Y \setminus B}$ along B are unipotent, where $d = \dim M - \dim Y$.

Then the d-th filter $\mathcal{F}^d(\mathcal{H}^{\mathrm{can}})$ of the canonical extension $\mathcal{H}^{\mathrm{can}}$ of $\mathcal{H} = H \otimes \mathcal{O}_{Y\setminus B}$ is a numerically semi-positive vector bundle by **3.2** ([**50**, Theorem 17]). Since $h_*\omega_{M/Y} \simeq \mathcal{F}^d(\mathcal{H}^{\mathrm{can}})$, $f_*\mathcal{O}_X(L)$ is a nef line bundle. Therefore, $f_*\mathcal{O}_X(L)$ is numerically trivial, since $\kappa_\sigma(f_*\mathcal{O}_X(L)) = 0$. The metric induced on $\mathcal{F}^d(\mathcal{H}^{\mathrm{can}})$ has only logarithmic singularities along B and is semi-positive on $Y \setminus B$. Hence $f_*\mathcal{O}_X(L)$ is a flat subbundle of $\mathcal{H}$ over $Y \setminus B$ (cf. [**22**], [**126**], [**52**], [**53**], [**72**]). Then

$$(f_*\mathcal{O}_X(L))^{\otimes k} \simeq \mathcal{O}_Y$$

for some $k \in \mathbb{N}$ by a result [**10**, 4.2.8.(iii)(b)] of Deligne concerning with the semi-simplicity of monodromies. Thus $\kappa(f_*\mathcal{O}_X(L)) = 0$. □

4.7. Proposition *Let $f\colon X \to Y$ be a fiber space from a normal projective variety onto a non-singular projective variety. Let D be a $\mathbb{Q}$-Cartier divisor of X, and Δ an effective $\mathbb{R}$-divisor of X such that*

(1) *$K_X + \Delta$ is $\mathbb{R}$-Cartier,*
(2) *(X, Δ) is log-terminal over a non-empty open subset of Y,*
(3) *$D - (K_{X/Y} + \Delta)$ is nef and abundant.*

Suppose that $\kappa(D; X/Y) = 0$ and $\kappa_\sigma(D, \det f) = 0$. Then $\kappa(D) \geq 0$.

Proof. We may assume that X is non-singular, $\operatorname{Supp}\Delta \cup \operatorname{Supp}\langle D\rangle$ is normal crossing, and that f is semi-stable in codimension one. There is an f-effective divisor E of X such that $f_*\mathcal{O}_X(m(D+E))$ is isomorphic to the invertible sheaf $f_*\mathcal{O}_X(mD)^\wedge$ for any $m > 0$ with mD being Cartier. Let $\mathbb{N}(D, f)$ be the set of natural numbers $m \in \mathbb{N}$ with mD being Cartier and $f_*\mathcal{O}_X(mD) \neq 0$. Let

$F_m := |m(D+E)|_{\mathrm{fix},f}$ be the relative fixed divisor of mD with respect to f for $m \in \mathbb{N}(D,f)$, which is determined by:

$$f^*f_*\mathcal{O}_X(m(D+E)) \simeq \mathcal{O}_X(m(D+E)-F_m) \subset \mathcal{O}_X(m(D+E)).$$

Let G_m be the maximum effective $\mathbb{Q}$-divisor of Y satisfying $F_m \geq f^*G_m$. Here, for a prime divisor B of Y,

$$\operatorname{mult}_B G_m = \min\Big\{\frac{\operatorname{mult}_\Gamma F_m}{\operatorname{mult}_\Gamma f^*B} \;\Big|\; \Gamma \text{ is a prime divisor with } f(\Gamma)=B\Big\}.$$

We have an injection

$$f^*\big(f_*\mathcal{O}_X(m(D+E))\big)^{\otimes l} \hookrightarrow \mathcal{O}_X(lm(D+E))$$

for $l>0$. Thus $lF_m - F_{ml}$ is the pullback of an effective divisor of Y. Therefore,

$$lF_m - F_{ml} = f^*(lG_m - G_{ml}).$$

In particular, the $\mathbb{Q}$-divisor

$$N := N_{\mathbb{Q}}(D+E) := \frac{1}{m}(F_m - f^*G_m)$$

does not depend on $m \in \mathbb{N}(D,f)$. If lG_m is Cartier, then so is G_{ml} and hence $G_{ml}=0$, since $f_*\mathcal{O}_X(F_{ml}) \simeq \mathcal{O}_Y$. Thus N coincides with the negative part

$$N = \lim_{\mathbb{N}(D,f)\ni m\to\infty} \frac{1}{m}F_m$$

of the f-sectional decomposition of $D+E$. Then we can take a $\mathbb{Q}$-divisor Ξ on Y such that $m\Xi$ is Cartier and

$$f_*\mathcal{O}_X(mD)^\wedge \simeq \mathcal{O}_Y(m\Xi)$$

for $m \in \mathbb{N}(D,f)$ with $G_m=0$ (cf. [**88**, §5 Part II]). In particular, $D+E-N \sim_{\mathbb{Q}} f^*\Xi$. We have $\kappa_\sigma(\Xi)=0$, since $\kappa_\sigma(D,\det f)=0$. We fix a positive integer $m\in\mathbb{N}(D,f)$ with $G_m=0$. Then mN and $m\Xi$ are Cartier, and $m(D+E-N)\sim f^*(m\Xi)$.

Let $\tau\colon Y'\to Y$ be a finite Galois surjective morphism from a non-singular projective variety such that $\tau^*\Xi$ is Cartier. For a birational morphism $\delta\colon X'\to X\times_Y Y'$ from a non-singular projective variety into the main component, let $\lambda\colon X'\to X$ and $f'\colon X'\to Y'$ be the induced morphisms. We consider the same $\mathbb{R}$-divisors R_Δ, R'_+, R'_-, R_τ, Δ', and $D'=\lambda^*D+R'_+-f'^*R_\tau$ as before. We may assume that the union of $\operatorname{Supp}\lambda^{-1}\langle D\rangle$, $\operatorname{Supp}R'_+$, $\operatorname{Supp}R'_-$, $\operatorname{Supp}\Delta'$, and $\operatorname{Supp}f^*R_\tau$ is a normal crossing divisor. We define

$$C := m\lambda^*D + (m-1)\lambda^*E - (m-1)\lambda^*N - (m-1){f'}^*\tau^*\Xi + R'_+ - {f'}^*R_\tau.$$

Then C is a $\mathbb{Q}$-divisor and

$$C-(K_{X'/Y'}+\Delta') = D'-(K_{X'/Y'}+\Delta') + (m-1)\lambda^*(D+E-N-f^*\Xi)$$

is nef and abundant. We set $L = \ulcorner C - \Delta' \urcorner$. Then

$$\begin{aligned}\delta_*\mathcal{O}_{X'}(L) &= \delta_*\mathcal{O}_{X'}(\ulcorner -\lambda^*E + \lambda^*N - \Delta' \urcorner + R'_+ - f'^*R_\tau) \otimes p_2^*\mathcal{O}_{Y'}(\tau^*\Xi) \\ &\hookrightarrow \delta_*\mathcal{O}_{X'}(\ulcorner \lambda^*N \urcorner) \otimes p_2^*\mathcal{O}_{Y'}(\tau^*\Xi)\end{aligned}$$

for projections $p_1 \colon X \times_Y Y' \to X'$ and $p_2 \colon X \times_Y Y' \to Y'$.

We shall show the natural injection

$$\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{X'}(\ulcorner \lambda^*N \urcorner)$$

is isomorphic as follows: Assume the contrary. Then there exists a prime divisor B' of Y' such that $\operatorname{Supp}\lambda^*N$ contains all the prime divisors Γ' of X' with $f'(\Gamma') = B'$. The same property hold for the prime divisors conjugate to B' over Y. Hence $\operatorname{Supp} N$ contains all the prime divisors Γ of X with $f(\Gamma) = \tau(B')$. This contradicts $G_m = 0$.

Therefore, we have an injection

$$f'_*\mathcal{O}_{X'}(L) \subset \mathcal{O}_{Y'}(\tau^*\Xi).$$

Here $L - (K_{X'/Y'} + \langle -C + \Delta' \rangle)$ is nef and abundant. Thus $\kappa_\sigma(f'_*\mathcal{O}_{X'}(L)^\wedge) = 0$ and hence $\kappa(f'_*\mathcal{O}_{X'}(L)^\wedge) = 0$ by **4.6**. Therefore $\kappa(\Xi) = 0$ and $\kappa(D + E) \geq 0$. By an argument using a flattening of f, we infer that $\kappa(D) \geq 0$. □

4.8. Theorem *Let X be a normal projective variety and let Δ be an effective $\mathbb{R}$-divisor such that (X, Δ) is log-terminal. Let D be a $\mathbb{Q}$-divisor such that $D - (K_X + \Delta)$ is nef and abundant. If $\kappa_\sigma(D) = 0$, then $\kappa(D) = 0$.*

Proof. We may assume that X is non-singular and $\operatorname{Supp}\Delta$ is a normal crossing divisor by **3.33**-(1). Let $D = P_\sigma(D) + N_\sigma(D)$ be the σ-decomposition. Then $P_\sigma(D) \approx 0$ by **1.12**. Then $N_\sigma(D) \cdot C \in \mathbb{Q}$ for any irreducible curve $C \subset X$. Since the prime components of $N_\sigma(D)$ are numerically linearly independent, $N_\sigma(D)$ is an effective $\mathbb{Q}$-divisor.

Suppose that the irregularity $q(X) = 0$. Then any divisor numerically equivalent to zero is $\mathbb{Q}$-linearly equivalent to zero. Thus $P_\sigma(D) \sim_{\mathbb{Q}} 0$ and $\kappa(D) = 0$.

Thus we may assume that $q(X) > 0$. Let $\alpha \colon X \to \operatorname{Alb} X$ be the Albanese mapping and let $X \to Y \to \operatorname{Alb} X$ be the Stein factorization. Then, by **4.1**,

$$0 = \kappa_\sigma(D) \geq \kappa_\sigma(D|_{X_y}) + \kappa_\sigma(Y) \geq 0$$

for a 'general' fiber X_y of $y \in Y$. Thus $0 = \kappa_\sigma(D|_{X_y}) = \kappa(Y)$. Therefore, by [**50**, Theorem 13], $Y \to \operatorname{Alb} X$ is isomorphic and hence the Albanese mapping α is a fibration. In particular $q(X) \leq \dim X$. Since α induces an isomorphism $\alpha^* \colon \operatorname{Pic}^0(Y) \to \operatorname{Pic}^0(X)$, there exist an integer $b \in \mathbb{N}$ and a numerically trivial divisor L of $Y = \operatorname{Alb} X$ such that $bN_\sigma(D)$ and bD are Cartier with $bN_\sigma(D) - bD \sim \alpha^*(bL)$. Thus $\kappa(D|_{X_y}) = 0$. Then we have $\kappa_\sigma(D, \det f) = 0$ by **4.5**. Since $K_Y = 0$, we have $\kappa(D) \geq 0$ by **4.7**. □

4.9. Corollary *Let X be a normal projective variety and let Δ be a $\mathbb{Q}$-divisor such that (X, Δ) is log-terminal. If $\kappa_\sigma(K_X + \Delta) = 0$, then $\kappa(K_X + \Delta) = 0$.*

Remark The abundance **4.8** was proved for $L = K_X$ for a non-singular projective variety X admitting a minimal model, by Kawamata [**56**]. The idea of applying Iitaka's addition formula for κ to the Albanese map is originally by Tsunoda (cf. [**114**]).