CHAPTER 10

On classification of exceptional complements: case $\delta \geq 1$

Now we study the case $\delta \geq 1$ in details.

10.1. The inequlity $\delta \leq 2$

In this section we show that $\delta \leq 2$. Replace (X, B) with a model $(\widetilde{X}, \widetilde{B})$. By construction, $\delta(X, B) = \delta(\widetilde{X}, \widetilde{B})$. Thus we assume that $\rho(X) = 1$, $B \in \Phi_{\mathbf{m}}$, $K_X + B$ is (1/7)-lt and $-(K_X + B)$ is nef. Moreover, there exists a boundary D defined by (9.1) such that $K_X + D$ is ample and lc. Let $C := \lfloor D \rfloor$. Then $\delta(X, B)$ is the number of components of C. Since $K_X + D$ is lc, C has only nodal singularities. The following is a very important ingredient in the classification.

THEOREM 10.1.1 ([**Sh3**]). *Notation as in 10. Then* $p_a(C) \leq 1$.

SKETCH OF PROOF. Assume that $p_a(C) \geq 2$. Consider the following birational modifications:

(10.1)
$$\begin{array}{ccc} & X^{\min} & \\ {}^{\mu}\swarrow & & \searrow^{\varphi} \\ X & & X' \end{array}$$

where $\mu\colon X^{\min} \to X$ be a minimal resolution and $\varphi\colon X^{\min} \to X'$ is a composition of contractions of -1-curves. Since $K_X + C$ is lc, C has only nodal singularities. By Lemma 9.1.8, X is smooth at Sing$C$. Therefore $C^{\min} \simeq C$. Thus $p_a(C) = p_a(C^{\min}) \geq 2$, $C^{\min}$ is not contracted and $p_a(C') \geq 2$. Take the crepant pull back

$$\mu^*(K_X + B) = K_{X^{\min}} + B^{\min}, \quad \text{with} \quad \mu_* B^{\min} = B$$

and put

$$B' := \varphi_* B^{\min}.$$

Note that both $-(K_{X^{\min}} + B^{\min})$ and $-(K_{X'} + B')$ are nef and big. Since $\rho(X) = 1$ and $C \simeq C^{\min}$, we have

(*) every two irreducible components of $C^{\min}$ intersect each other.

If $X' \simeq \mathbb{P}^2$, then $-(K_{X'} + \frac{6}{7}C')$ is ample. This gives $\frac{6}{7}\deg C' < 3$, $\deg C' \leq 3$ and $p_a(C') \leq 1$. Now we assume that $X' \simeq \mathbb{F}_n$. We claim that $n \geq 2$. Indeed, otherwise $X' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $X' \neq X^{\min}$ (because $\rho(X) = 1$) and we have at least one blowup $X^{\min} \to X'' \to X'$. Contracting another -1-curve on X'' we get $\mathbb{F}_1$ instead of $\mathbb{P}^1 \times \mathbb{P}^1$ and after the next blowdown we get $\mathbb{P}^2$. Thus $n \geq 2$. Let Σ_0 be a negative section of $\mathbb{F}_n$ and F be a general fiber. Since $\frac{6}{7}C' \cdot F \leq -K_{X'} \cdot F = 2$, we have $C' \cdot F \leq 2$. So C' must be generically a 2-section of $\mathbb{F}_n \to \mathbb{P}^1$ (otherwise C' is generically a section and $p_a(C') = 0$).

First we consider the case when Σ_0 is not a component of C'. Then the coefficient of Σ_0 in $C' \leq 2 - \frac{2 \cdot 6}{7} = \frac{2}{7}$. Thus

$$0 \leq -(K_{X'} + B') \cdot \Sigma_0 \leq -\left(K_{X'} + \frac{2}{7}\Sigma_0\right) \cdot \Sigma_0 = 2 - n + \frac{2n}{7}.$$

Hence $n = 2$, $X' \simeq \mathbb{F}_2$. If $X^{\min} \neq X'$, then $X^{\min} \to X'$ contracts at least one -1-curve. But then contracting another -1-curve we obtain either $X' = \mathbb{F}_3$ or $X' = \mathbb{F}_1$, a contradiction with our assumptions. Therefore $X^{\min} = X'$ and X is a quadratic cone in $\mathbb{P}^3$. Since $-(K_X + \frac{6}{7}C)$ is ample, $C \equiv aH$, where H is the ample generator of $\operatorname{Pic}(X)$ and $a < \frac{7}{3}$. By Adjunction we have

$$\deg K_C \leq (K_X + C) \cdot C = 2(a-2)a < 2.$$

Hence $p_a(C) \leq 1$ in this case.

Finally, we consider the case when Σ_0 is a component of C'. Write $C' = \Sigma_0 + \Sigma'$. Then Σ' is generically a section. From $p_a(C') \geq 2$ by genus formula, we have $\Sigma_0 \cdot \Sigma' \geq 3$. But then

$$0 \geq (K_{X'} + B') \cdot \Sigma_0 \geq \left(K_{X'} + \Sigma_0 + \frac{6}{7}\Sigma'\right) \cdot \Sigma_0 \geq -2 + \frac{6}{7} \cdot 3 \geq \frac{4}{7},$$

a contradiction. □

COROLLARY 10.1.2 ([**Sh3**]). *Notation as in 10. Then $\delta(X, B) \leq 2$.*

PROOF. Let $C = \sum_{i=1}^{\delta} C_i$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \oplus \mathcal{O}_{C_i} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{F}$ is a sheaf with $\operatorname{Supp}\mathcal{F} = \operatorname{Sing}C$, we have

$$1 \geq p_a(C) = 1 - \delta + \#\{C_i \cap C_j \mid i \neq j\} + \sum p_a(C_i). \tag{10.2}$$

On the other hand, by (*) we have $\#\{C_i \cap C_j \mid i \neq j\} \geq \frac{1}{2}\delta(\delta - 1)$. This yields

$$0 \geq \frac{1}{2}\delta(\delta - 3) + \sum p_a(C_i). \tag{10.3}$$

In particular, $\delta \leq 3$. Assume that $\delta = 3$. Then C is a wheel of smooth rational curves and in (10.3) the equality holds. Let H be an ample generator of $\operatorname{Pic}(X)$. We have $-K_X \equiv rH$, $C_i \equiv \gamma_i H$ for some positive rational r, γ_1, γ_2, γ_3. Since every C_i intersects C_j transversally at a (unique) nonsingular point, $1 = C_i \cdot C_j = \gamma_i \gamma_j H^2$. Hence

$$\gamma_1\gamma_2 = \gamma_1\gamma_3 = \gamma_2\gamma_3 = \frac{1}{H^2}.$$

This implies

$$\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{\sqrt{H^2}} \leq 1. \tag{10.4}$$

Since $-(K_X + B)$ is ample,

$$r > \frac{6}{7}\gamma_1 + \frac{6}{7}\gamma_2 + \frac{6}{7}\gamma_3 = \frac{18}{7}\gamma_1.$$

Therefore $K_X + C_1 + C_2 + \frac{4}{7}C_3 \equiv -\left(r - \frac{18}{7}\gamma_1\right) H$ is antiample (and lc). We claim that X is smooth along C_1. Indeed, otherwise $\operatorname{Diff}_{C_1}(0) \geq \frac{1}{2}P$, where $P \notin C_2, C_3$. On the other hand, by Adjunction we have

$$2 > \deg \operatorname{Diff}_{C_1}\left(C_2 + \frac{4}{7}C_3\right) = 1 + \frac{4}{7} + \frac{1}{2} > 2.$$

The contradiction shows that X is smooth along C_1, and similarly X is smooth along C_2 and C_3. Thus C_1, C_2, C_3 are Cartier. In particular, $\gamma_i \in \mathbb{N}$. By (10.4), $\gamma_1 = 1$ and $H^2 = 1$. Since $\operatorname{Pic}(X) \simeq \mathbb{Z} \cdot H$, $C_1, C_2, C_3 \in |H|$. The linear subsystem of $|H|$ generated by C_1, C_2, C_3 is base point free and determines a morphism $X \to \mathbb{P}^2$ of degree one (see also Lemma 10.2.4 below). Therefore $X \simeq \mathbb{P}^2$ and C_1, C_2, C_3 are lines in the general position. Simple computations show that B has no other components. Finally, $K_X + C$ is an 1-complements of $K_X + B$, a contradiction proves the corollary. □

10.2. Case $\delta = 2$

Following Shokurov [**Sh3**] we describe the case $\delta = 2$:

THEOREM 10.2.1. *Let (X, B) be a log surface such that $K_X + B$ is $(1/7)$-lt, $-(K_X + B)$ is nef, $B \in \Phi_{\mathbf{m}}$, $\delta(X, B) = 2$ and $\rho(X) = 1$. Assume that (X, B) is exceptional. Let H be a positive generator of* $\operatorname{Pic}(X)$. *Write*

$$\begin{aligned} B &= b_1C_1 + b_2C_2 + F, \quad F = \sum(1 - 1/m_i)F_i, \\ b_1, b_2 &\geq 6/7, \quad m_i \in \{1, 2, 3, 4, 5, 6\}, \end{aligned}$$

where C_1 and C_2 are irreducible curves. Then $C := C_1 + C_2$ has only normal crossings at smooth points of X, $\operatorname{Supp}F$ *does not pass $C_1 \cap C_2$ and $b_1 + b_2 < 13/7$. We have one of the following possibilities:*

- (A_2^1) *$X = \mathbb{P}^2$, $B = b_1C_1 + b_2C_2 + \frac{1}{2}F_1 + \frac{2}{3}F_2$, where C_1, C_2, F_1, F_2 are lines such that no three of them intersect at a point and $b_1 + b_2 \leq 11/6$;*
- ($A_2^{1\prime}$) *$X = \mathbb{P}^2$, $B = b_1C_1 + b_2C_2 + \frac{1}{2}F_1 + \frac{3}{4}F_2$, where C_1, C_2, F_1, F_2 are lines such that no three of them intersect at a point and $b_1 + b_2 \leq 7/4$;*
- (A_2^2) *X is a quadratic cone in $\mathbb{P}^3$, $B = b_1C_1 + b_2C_2 + \frac{2}{3}F_1$, where C_1 is its generator, C_2, F_1 are its smooth hyperplane sections, $b_1 + 2b_2 \leq 8/3$;*
- (A_2^3) *X is a rational cubic cone in $\mathbb{P}^4$, $B = b_1C_1 + b_2C_2 + \frac{1}{2}F_1$, where C_1 is its generator, C_2, F_1 are its smooth hyperplane sections, $b_1 + 3b_2 \leq 7/2$ and $\#C_2 \cap F_1 \geq 2$;*

(A_2^4) $X = \mathbb{P}(1,2,3)$, $B = b_1C_1+b_2C_2+\frac{1}{2}F_1$, *where* $C_1 = \{x_2 = 0\}$, $C_2 = \{x_3 = 0\}$ *(i.e.,* $3C_1 \sim H$, $2C_2 \sim H$*),* F_1 *is a smooth rational curve* $\equiv \frac{1}{2}H$, $F_1 \neq C_2$ *which is given by the equation* $x_3 = x_1^3 + x_1x_2$, $2b_1 + 3b_2 \leq 9/2$*;*

(A_2^5) $X = \mathbb{P}(1,3,4)$, $B = \frac{6}{7}(C_1+C_2)+\frac{1}{2}F_1$, *where* $C_1 = \{x_2 = 0\}$, $C_2 = \{x_3 = 0\}$ *(i.e.,* $4C_1 \sim H$, $3C_2 \sim H$*),* F_1 *is a smooth rational curve* $\equiv \frac{1}{3}H$, $F_1 \neq C_2$ *which is given by the equation* $x_3 = x_1^4 + x_1x_2$*, in this case* $14(K_X+B) \sim 0$*;*

(A_2^6) $X = \mathbb{P}(1,2,3)$, $B = \frac{6}{7}(C_1 + C_2)$*, where* C_1 *is a line* $\{x_1 = 0\}$, $C_2 \in |-K_X|$ *(i.e.,* $6C_1 \sim H$, $C_2 \sim H$*),* $\mathrm{Sing} X \subset C_1$*, in this case* $7(K_X + B) \sim 0$*;*

(I_2^1) X *is a quadratic cone in* $\mathbb{P}^3$, $B = b_1C_1 + b_2C_2 + \frac{1}{2}F_1$*, where* C_1, C_2 *are two smooth hyperplane sections,* F_1 *is a generator of the cone,* $b_1 + b_2 \leq 7/4$*;*

(I_2^2) $X = \mathbb{P}(1,2,3)$, $B = \frac{6}{7}C_1 + \frac{6}{7}C_2$*, where* $C_1 = \{x_3 = 0\}$, $C_2 = \{x_2^2 = \alpha_1x_1^4 + \alpha_2x_1^2x_2 + x_1x_3\}$, $\alpha_1,\alpha_2 \in \mathbb{C}$, $(\alpha_1,\alpha_2) \neq (0,0)$, $2C_1 \sim H$, $3C_2 \sim 2H$*, in this case* $7(K_X + B) \sim 0$.

REMARK. Note that in all cases $\mathrm{Weil}_{\mathrm{lin}}(X) \simeq \mathbb{Z}$. Therefore we can verify (i) in the definition of complements 4.1.3 numerically, i.e., we need to check only that nB^+ is integral and $K_X + B^+ \equiv 0$. By the Inductive Theorem 8.3.1, (ii) of 4.1.3 holds automatically whenever (X, B) is exceptional.

Shokurov's proof is based on a detailed analysis of the minimal resolution, cf. (10.1). Our proof uses computations of Fano indices of X (as in the proof of Corollary 10.1.2). We use slightly 5.2.3. Note that one can avoid using of 5.2.3, but then computations become a little more complicated.

The important property is that $K_X + D$ is analytically dlt except for one case:

LEMMA 10.2.2 ([**Sh3**]). *Let* $(S \ni o, B = \sum b_iB_i)$ *be a log surface germ, where* $B \in \Phi_{\mathbf{m}}$*. Assume that* $K_S + B$ *is (1/7)-lt. As in* (9.1), put

$$C := \left\lfloor \frac{7}{6}B \right\rfloor = \sum_{b_i \geq 6/7} B_i, \qquad F := \sum_{b_i < 6/7} b_iB_i \quad \text{and} \quad D := C + F.$$

Then one of the following holds:

(i) $K_S + D$ is analytically dlt at o;

(ii) $o \in S$ is smooth and near o we have $D = C + \frac{1}{2}L$, where $(S, C + L) \simeq_{\mathrm{an}} (\mathbb{C}^2, \{y(y - x^2) = 0\})$.

PROOF. Clearly, we may assume that $K_S + D$ is not plt (otherwise we have case (i)). By Theorem 6.0.6 there is a regular complement K_S+B^+. Since $B \in \Phi_{\mathbf{m}}$, $B^+ \geq D$. In particular, $K_S + D$ is lc and $C = \lfloor D \rfloor$ has at most two (analytic) components passing through o (see Theorem 2.1.3). If C has exactly two components, then $S \ni o$ is smooth by Lemma 9.1.8. Obviously, $K_S + D$ is analytically dlt at o in this case. From now on we assume that C is analytically irreducible at o. Write $B = bC + F$, where $b \geq 6/7$. Recall that $F \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$.

First we consider the case when $K_S + C$ is not plt. Then $D = C$ and (S, C, o) is such as in (ii) of 2.1.3. In particular, $2(K_S + C) \sim 0$ and $K_S + C \not\sim 0$. Let $f\colon (\widetilde{S}, E) \to S$ be an inductive blowup of (S, D) and $\widetilde{C}$ the proper transform of C.

Write

$$f^*(K_S + C) = K_{\widetilde{S}} + \widetilde{C} + E,$$
$$f^*(K_S + bC) = K_{\widetilde{S}} + b\widetilde{C} + \alpha E,$$

where $\alpha < 6/7$. Here $2\left(K_{\widetilde{S}} + \widetilde{C} + E\right) \sim 0$. By Adjunction, $K_E + \mathrm{Diff}_E(\widetilde{C})$ is not klt and $\deg \mathrm{Diff}_E(\widetilde{C}) = 2$. Moreover, $K_E + \mathrm{Diff}_E(\widetilde{C})$ is not 1-complementary (because neither is $K_S + C$). Therefore we have (cf. Lemma 6.1.1)

$$\mathrm{Diff}_E(\widetilde{C}) = \frac{1}{2}P_1 + \frac{1}{2}P_2 + P_3, \qquad \mathrm{Diff}_E(0) = \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{m-1}{m}P_3$$

for some points $P_1, P_2, P_3 \in E$ and some $m \in \mathbb{N}$. From this we have

$$(K_{\widetilde{S}} + E) \cdot E + b\widetilde{C} \cdot E + (-1 + \alpha)E^2 = 0.$$

By Adjunction

$$(K_{\widetilde{S}} + E) \cdot E = -2 + \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{m} = -\frac{1}{m}.$$

Since $\widetilde{C} \cap E$ is a point of type $\frac{1}{m}(1, q)$, $\widetilde{C} \cdot E \geq 1/m$. This yields

$$\frac{1}{7}\left(-E^2\right) < (-1 + \alpha)E^2 \leq \frac{1}{7m}.$$

Thus $0 < -E^2 < 1/m$ and $-1/m < K_{\widetilde{S}} \cdot E < 0$. On the other hand, $mK_{\widetilde{S}}$ is Cartier near E. Therefore $mK_{\widetilde{S}} \cdot E \in \mathbb{Z}$, a contradiction.

Now we may assume that $K_S + C$ is plt. By Theorem 6.0.6, $K_S + D$ is 2-complementary and $D^+ \geq D$, so $2(K_S + D) \sim 0$ and $2F$ is integral. We claim that $(S \ni o)$ is smooth. Assume the opposite. Then

$$(S, C) \simeq (\mathbb{C}^2, \{y = 0\})/\mathbb{Z}_m(1, q), \quad \gcd(q, m) = 1, \ m \geq 2, \ 1 \leq q \leq m - 1.$$

Consider the weighted blowup with weights $\frac{1}{m}(1, q)$. By Lemma 3.2.1 we get the exceptional divisor E with

$$a(E, D) = -1 + \frac{1+q}{m} - \frac{q}{m} - \frac{\mu}{2} = -1 + \frac{1}{m} - \frac{\mu}{2},$$

where $\mu = \mathrm{mult}_E(2F) \in \frac{1}{m}\mathbb{N}$. Since $2(K_S + D) \sim 0$, we have $a(E, D) = -1$ or $-1/2$. But in the second case $\mu = 2/m - 1 \leq 0$, a contradiction. Therefore $a(E, C + F) = -1$ and $\mu = 2/m$. Further,

$$-1 + \frac{1}{7} < a(E, B) = -1 + \frac{1+q}{m} - b\frac{q}{m} - \frac{\mu}{2} = -1 + \frac{q(1-b)}{m} < -1 + \frac{1}{7}.$$

The contradiction shows that $(S \ni o)$ is smooth. Now we claim that $\lceil F \rceil$ is a smooth curve. As above, consider the blowup of $o \in S$. For the exceptional divisor E, we have

$$-1 + \frac{1}{7} < a(E, B) = 1 - b - \frac{\mu}{2},$$

where $\mu = \mathrm{mult}_E(2F) \in \mathbb{N}$. Hence $\mu = 1$ and $L = \lceil F \rceil$ is smooth. Finally, $K_S + C + (\frac{1}{2} - \varepsilon)L$ is plt for any $\varepsilon > 0$. By Adjunction, $\lfloor \mathrm{Diff}_C\left((\frac{1}{2} - \varepsilon)L\right) \rfloor \leq 0$.

Hence $\lfloor \mathrm{Diff}_C(\frac{1}{2}L) \rfloor$ is reduced. This means that $C \cdot L = 2$, i.e., C and L have a simple tangency at o. The rest is obvious. □

We need some (well known) facts about Fano indices of log del Pezzo surfaces.

DEFINITION 10.2.3. Let (X, D) be a log del Pezzo surface. Define the *Fano index* $r(X, D)$ of (X, D) by

$$r(X, D) = \sup\{t \mid -(K_X + D) \equiv tH, \quad \text{for some} \quad H \in \mathrm{Pic}(X)\}.$$

If $K_X + D$ is klt or $K_X + D$ is dlt and $-(K_X + D)$ is ample, then by Lemma 5.1.3, $r(X, D) \in \mathbb{Q}$ and $-(K_X + D) \equiv r(X, D)H$ for some (primitive and ample) element $H \in \mathrm{Pic}(X)$ (recall that we consider only $\mathbb{Q}$-divisors). In the case $D = 0$ we write $r(X)$ instead of $r(X, 0)$.

The following is an easy consequece of Riemann-Roch, Kawamata-Viehweg vanishing and [**Fuj**].

LEMMA 10.2.4. *Let X be a log del Pezzo with klt singularities of Fano index $r = r(X)$. Assume that $-K_X$ is ample and write $-K_X \equiv rH$, where H is a primitive (ample) element of* $\mathrm{Pic}(X)$. *Then*

(i) $\dim |H| = \frac{1}{2}(1 + r)H^2$, *hence* $r = \frac{2l}{H^2} - 1$, *where* $l := \dim |H|$;

(ii) $H^2 \geq \dim |H| - 1$, *hence* $r \leq 1 + \frac{2}{H^2}$;

(iii) *if* $r > 1$, *then*

$$\dim |H| = H^2 + 1, \quad \text{and} \quad r = 1 + \frac{2}{H^2}.$$

Moreover, X is one of the following $X \simeq \mathbb{P}^2$ ($r = 3$), $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ ($r = 2$), $X \subset \mathbb{P}^{d+1}$ is a cone over a rational normal curve of degree $d = H^2$ ($r = 1 + 2/d$).

PROOF. By Kawamata-Viehweg vanishing [**KMM**, 1-2-5] one has $H^i(X, \mathcal{O}_X(H)) = H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Therefore by Riemann-Roch we obtain

$$\dim |H| = \frac{H \cdot (H - K_X)}{2} = \frac{(1 + r)H^2}{2}.$$

This proves (i). Recall (see [**Fuj**]) that for any polarized variety (X, H) the following equality holds:

$$\dim X + H^{\dim X} - h^0(X, \mathcal{O}_X(H)) \geq 0. \tag{10.5}$$

Combining this with (i) we obtain (ii). Finally, assume $r > 1$. Then by (i), $\dim |H| > H^2$. From (ii) we have $H^2 = \dim |H| - 1$. Moreover, in (10.5) the equality holds. Such polarized varieties (of arbitrary dimension) are classified in [**Fuj**]. In particular, it is proved that H is very ample and $X \subset \mathbb{P}^{\dim |H|}$ are varieties of *minimal degree.* In the two-dimensional case from [**Fuj**] we obtain possibilities as in (iii). □

Log del Pezzo surfaces with $r(X) = 1$ are special cases of the so-called Fujita varieties:

LEMMA 10.2.5. *Let X be a log del Pezzo with klt singularities of Fano index 1. Assume that $-K_X$ is ample and H an ample primitive element of* $\mathrm{Pic}(X)$ *such that $-K_X \equiv H$. Then*

(i) $\dim|H| = H^2$ *and* $H^2 \leq 8$;
(ii) *if $H^2 \geq 4$, then X has only Du Val singularities;*
(iii) *if $H^2 = 6$ and $\rho(X) = 1$, then X has exactly two singular points which are Du Val of types A_1 and A_2; in this case, X is isomorphic to the weighted projective plane* $\mathbb{P}(1, 2, 3)$.

SKETCH OF PROOF. Note that by Lemma 5.4.1, X is rational. As in Lemma 10.2.4, the first part of (i) follows by Riemann-Roch and Kawamata-Viehweg vanishing. Set $D := H + K_X$. If $D \sim 0$, then X has only DuVal singularities. In this case, by Noether's formula,

$$K_{\widetilde{X}}^2 + \rho(\widetilde{X}) = K_X^2 + \rho(\widetilde{X}) = 10,$$

where $\widetilde{X} \to X$ is the minimal resolution. This yields $K_X^2 = H^2 \leq 8$ (because $X \not\simeq \mathbb{P}^2$).

If $D \not\sim 0$, then by Lemma 5.1.3, $nD \sim 0$ for some $n \in \mathbb{N}$. Considering a cyclic cover trick, we get a cyclic étale in codimension one cover $\varphi\colon X' \to X$. Moreover, on X' one has $-K_{X'} \sim H'$, where $H' := \varphi^* H$. Therefore X' is a del Pezzo surface with only DuVal singularities. Further, by the above arguments,

$$K_{X'}^2 = (\deg \varphi) K_X^2 \leq 9.$$

Hence $K_X^2 \leq 4$. If $K_X^2 = 4$, then $K_{X'}^2 = 8$ and X is a quotient of X' by an involution τ. In this case, X' cannot be smooth (otherwise X has only singularities of type A_1 and $-K_X \sim H$). Let $\widetilde{X}' \to X'$ be the minimal resolution. As above, by Noether's formula, $\rho(\widetilde{X}') = 10 - K_{\widetilde{X}'}^2 = 10 - K_{X'}^2 = 2$. Therefore, $\widetilde{X}' \to X'$ contracts a single -2-curve. From this, we have only one possibility: $\widetilde{X}' \simeq \mathbb{F}_2$ and X' is a quadratic cone in $\mathbb{P}^3$. Since $\mathrm{Pic}(X') = \mathbb{Z} \cdot \mathcal{O}_{X'}(1)$, one has that τ acts linearly in $\mathbb{P}^3$. Recall that the quotient of the vertex of the cone is nonGorenstein. The action of τ on $\mathbb{P}^3$ is free in codimension one (because so is the action of τ on X'). Therefore in some coordinate system,

$$\tau = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and X' is given by

$$q(x_1, x_2) + q'(x_3, x_4) = 0,$$

where $q(x_1, x_2)$ and $q'(x_3, x_4)$ are quadratic forms such that $\mathrm{rk}(q + q') = 3$. Changing coordinates we may assume that X' is given by $x_1^2 + x_2^2 + x_3^2 = 0$. But then the quotient of the vertex is a complete intersection singularity $y_1 + y_2 + x_3^2 = 0$,

$y_1y_2 = y_0^2$, where $y_1 = x_1^2$, $y_2 = x_2^2$ and $y_0 = x_1x_2$. In particular, it is Gorenstein, a contradiction.

Assume now that $H^2 = 6$. Then by the above, X is Gorenstein and $\rho(\widetilde{X}) = 4$, where $\widetilde{X} \to X$ is the minimal resolution. Therefore $\widetilde{X} \to X$ contracts exactly three -2-curves and the configuration of singular points on X is either A_3 or A_1A_2. By [**Fu**] the only second case is possible. Moreover, X is unique up to isomorphism (see e.g., [**KeM**, 3.10]). On the other hand, $\mathbb{P}(1,2,3)$ is a Gorenstein del Pezzo of degree 6. □

REMARK. There is another way to treat the case $H^2 = 6$: since $\dim|H| = 6$, one can construct a 1-complement $K_X + C$ such that C has three components and then use Theorem 8.5.2.

PROOF OF THEOREM 10.2.1. Since $B \neq 0$ and $\rho(X) = 1$, $-K_X$ is ample. Hence X is rational. By Lemma 10.2.2 Then $C := C_1 + C_2$ has only normal crossings at smooth points of X, Supp$F$ does not pass $C_1 \cap C_2$ and $b_1 + b_2 < 13/7$ (by Lemma 9.1.8).

Write

$$C_i \equiv d_iH, \quad -K_X \equiv rH, \quad F \equiv qH.$$

We assume that $d_1 \leq d_2$. Since $-(K_X + B)$ is nef,

$$\frac{6}{7}(d_1 + d_2) \leq b_1d_1 + b_2d_2 + q \leq r. \tag{10.6}$$

Take b so that $K_X + C_1 + bC_2 + F \equiv 0$, i.e.

$$d_1 + bd_2 + q = r.$$

Then

$$b = \frac{r - q - d_1}{d_2} \geq \frac{b_1d_1 + b_2d_2 - d_1}{d_2} = b_2 - (1 - b_1)\frac{d_1}{d_2} \geq b_1 + b_2 - 1 \geq 5/7. \tag{10.7}$$

Since $K_X + C + F$ is ample, $b < 1$.

Recall that $K_X + C + F$ is analytically dlt except for the case (ii) of Lemma 10.2.2. In particular, X is smooth at points $C_1 \cap C_2$ and $C_1 \cap C_2 \cap \mathrm{Supp}F = \varnothing$. By Adjunction,

$$K_{C_1} + \mathrm{Diff}_{C_1}(bC_2 + F) \equiv 0. \tag{10.8}$$

If $p_a(C_1) > 0$, then $K_{C_1} = \mathrm{Diff}_{C_1}(bC_2 + F) = 0$. This is impossible because $C_1 \cap C_2 \neq \varnothing$. Therefore $C_1 \simeq \mathbb{P}^1$ and $\deg \mathrm{Diff}_{C_1}(bC_2 + F) = 2$.

10.2.6. Case: X is smooth. Then $X \simeq \mathbb{P}^2$ and $r = 3$. From (10.6) we obtain $(d_1, d_2) = (1, 2)$ or $(1, 1)$. On the other hand, $K_X + C + F$ is ample. This gives

$$q > 3 - d_1 - d_2.$$

If $(d_1, d_2) = (1, 2)$, then by (10.6), $0 < q \le 3 - \frac{18}{7} = \frac{3}{7} < \frac{1}{2}$, a contradiction. Therefore C_1, C_2 are lines on $X \simeq \mathbb{P}^2$. Then

$$(10.9) \quad \frac{1}{2}\sum \deg F_i \le q = \sum(1 - 1/m_i)\deg F_i \le 3 - 12/7 = 9/7, \qquad q > 1.$$

If $\deg F_1 \ge 2$, then $F = \frac{1}{2}F_1$, $\deg F_1 = 2$ and $q = 1$, a contradiction. Hence all the components of F are lines. From (10.9) we have only two possibilities: $F = \frac{1}{2}F_1 + \frac{2}{3}F_2$ and $F = \frac{1}{2}F_1 + \frac{3}{4}F_2$. These are cases (A_2^1) and $(\mathrm{A}_2^{1\prime})$.

From now on we assume that X is singular. Since $p_a(C) \le 1$, we have two possibilities: $\#C_1 \cap C_2 = 2$ and $\#C_1 \cap C_2 = 1$.

10.2.7. Case: $\#C_1 \cap C_2 = 2$. Let $C_1 \cap C_2 = \{P_1, P_2\}$. Then

$$2 = C_1 \cdot C_2 = d_1 d_2 H^2.$$

Equality (10.8) gives

$$\operatorname{Diff}_{C_1}(bC_2 + F) = bP_1 + bP_2 + \operatorname{Diff}_{C_1}(F).$$

Hence

$$\deg \operatorname{Diff}_{C_1}(F) = 2 - 2b \le 4/7.$$

By Inversion of Adjunction, $K_X + C_1 + F$ is plt near C_1. Assume that $\operatorname{Diff}_{C_1}(F) = 0$. Then $F = 0$ and $b = 1$, a contradiction with $b < 1$. Therefore $\operatorname{Diff}_{C_1}(F) \ne 0$.

Since $\operatorname{Diff}_{C_1}(F) \in \Phi_{\mathbf{sm}}$ (see Corollary 2.2.8), we have only one possibility: $\operatorname{Diff}_{C_1}(F) = \frac{1}{2}Q$, where $Q \in C_1$ is a single point $\ne P_1, P_2$. Moreover, $b = 3/4$ and $d_1 + \frac{3}{4}d_2 + q = r$.

If $Q \in X$ is smooth, then $F = \frac{1}{2}F_1$, where F_1 is irreducible, $F_1 \cap C_1 = \{Q\}$ and $F_1 \cdot C_1 = 1$. Thus C_1 is Cartier (see 2.2.4), $d_1 \in \mathbb{N}$ and $r = d_1 + \frac{3}{4}d_2 + q > \frac{7}{4}$. By Lemma 10.2.4 X is a cone over a rational normal curve of degree $d \ge 2$. In this case $r = (d+2)/d > 7/4$ and $d = 2$. Therefore $X \subset \mathbb{P}^3$ is a quadratic cone. Further, $d_1 = d_2 = 1$, so C_1, C_2 are hyperplane sections (and they do not pass through the vertex of the cone). Finally, from $F_1 \cdot C_1 = 1$ we see that F_1 is a generator of the cone. This is case (I_2^1).

Therefore $Q \in X$ is singular. Then it must be DuVal of type A_1. Moreover, $F = 0$ and $2C_1$ is Cartier (but C_1 is not, because C_1 is smooth at Q). Hence $d_1 \in \frac{1}{2}\mathbb{N}$. Further, $d_1 + \frac{3}{4}d_2 = r$.

If $d_1 \ge 1$, then $d_2 \ge 1$ and $r \ge 7/4$. By Lemma 10.2.4 and our assumption that X is singular, $r = 2$ and X is a quadratic cone. But then $d_2 = 4/3$, a contradiction. Hence $d_1 = 1/2$, $d_2 \ge 1/2$. Put $k := C_1 \cdot H \in \mathbb{N}$. Then $H^2 = 2k$,

$2 = C_1 \cdot C_2 = \frac{1}{2}d_2H^2$, so $d_2 = 2/k \geq 1/2$, $k \leq 4$. This gives $r = \frac{1}{2} + \frac{3}{4}d_2 = \frac{1}{2} + \frac{3}{2k}$. On the other hand, by Lemma 10.2.4, $r = \frac{l}{k} - 1$, where $l \in \mathbb{N}$. Therefore $3k+3 = 2l$ and $k \in \{1, 3\}$. If $k = 1$, then $l = 3$, $r = 2$, $d_2 = 2$. But this contradicts $\frac{6}{7}(d_1 + d_2) \leq r$. We obtain $k = 3$, $l = 6$, $r = 1$, $d_2 = 2/3$, $H^2 = 6$. By Lemma 10.2.5, $X \simeq \mathbb{P}(1,2,3)$. We may assume that $C_1 \in |\mathcal{O}_{\mathbb{P}}(3)|$ and $C_2 \in |\mathcal{O}_{\mathbb{P}}(4)|$. Then $C_1 = \{x_3 = 0\}$ and $C_2 = \{x_2^2 = \alpha_1 x_1^4 + \alpha_2 x_1^2 x_2 + \alpha_3 x_1 x_3\}$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. But $\alpha_3 \neq 0$ (otherwise C_2 is singular at $(0,0,1)$). Moreover, $(\alpha_1, \alpha_2) \neq (0,0)$, because $C_1 \cap C_2$ consists of two points. This is case (I_2^2).

10.2.8. Case: $p_a(C_2) = 1$. By 10.2.7 we may assume that $C_1 \cap C_2$ is a single point, say P. As in (10.7) take b' so that $K_X + b'C_1 + C_2 + F \equiv 0$, i.e.

$$b'd_1 + d_2 + q = r.$$

Since $K_{C_2} + \mathrm{Diff}_{C_2}(b'C_1 + F) \equiv 0$, we have $\deg(K_{C_2} + \mathrm{Diff}_{C_2}(b'C_1)) \leq 0$ and $K_{C_2} = 0$, $b' \leq 0$. This yields

$$b' = \frac{r - q - d_2}{d_1} = b_1 - (1 - b_2)\frac{d_2}{d_1} \leq 0,$$

$$\text{(10.10)} \qquad \frac{6}{7}d_1 \leq b_1 d_1 \leq (1 - b_2)d_2 \leq \frac{1}{7}d_2, \quad 6d_1 \leq d_2.$$

Assume that $r \leq 1$. Then

$$\text{(10.11)} \qquad 1 \geq r \geq b_1 d_1 + b_2 d_2 + q \geq (b_1 + 6b_2)d_1 + q \geq 6d_1.$$

On the other hand, by (10.8),

$$\deg \mathrm{Diff}_{C_1}(F) = 2 - b,$$

where

$$\text{(10.12)} \qquad 1 > b \geq b_2 - (1 - b_1)\frac{d_1}{d_2} \geq b_2 + \frac{1}{6}b_1 - \frac{1}{6} \geq \frac{5}{6}.$$

(see (10.7) and (10.10)). Hence

$$1 < \deg \mathrm{Diff}_{C_1}(F) \leq 7/6.$$

Since $\mathrm{Diff}_{C_1}(F) \in \mathbf{\Phi_{sm}}$, we have only one possibility $\mathrm{Diff}_{C_1}(F) = \frac{1}{2}Q_1 + \frac{2}{3}Q_2$ and $b = 5/6$. In particular, $6C_1$ is Cartier (see Theorem 2.2.4), so $d_1 \geq 1/6$. On the other hand, $d_1 \leq 1/6$ (see (10.11)). Hence $d_1 = 1/6$ and kC_1 is not Cartier for $1 \leq k \leq 5$. This gives us that $F = 0$. Moreover, in (10.12) equalities hold, so $1 = 6d_1 = d_2$ and $b_1 = b_2 = 6/7$. From (10.11) we have $r \geq 6d_1 = 1$. Hence $r = 1$. Further, $C_1 \cdot C_2 = \frac{1}{6}H^2 = 1$, gives $H^2 = K_X^2 = 6$. By Lemma 10.2.5, $X \simeq \mathbb{P}(1,2,3)$. We get case ($\mathrm{A}_2^6$).

Now assume that $r > 1$. Then X is a cone. From $2 \geq r \geq b_1 d_1 + b_2 d_2 + q \geq (b_1 + 6b_2)d_1 + q \geq 6d_1$ we see that $d_1 \leq 1/3$ and C_1 is not Cartier. Hence C_1 contains the vertex and C_2 does not. Thus C_2 is Cartier. Finally, $C_1 \cdot C_2 = 1$. Therefore C_1 is a generator of the cone and C_2 is a smooth hyperplane section. But then C_2 is rational, a contradiction.

10.2.9. Case $C_1 \cap C_2 = \{P\}$ and $p_a(C_1) = p_a(C_1) = 0$. Then $C_1 \cdot C_2 = 1$. By (10.7), $1 > b \geq 5/7$. Hence $1 < \deg(\mathrm{Diff}_{C_1}(F)) = 2 - b \leq 9/7$. Using $\mathrm{Diff}_{C_1}(F) \in \Phi_{\mathbf{sm}}$ we get the following cases:

$$\mathrm{Diff}_{C_1}(F) = \frac{1}{2}Q_1 + \frac{2}{3}Q_2, \quad \frac{1}{2}Q_1 + \frac{3}{4}Q_2. \tag{10.13}$$

By Inversion of Adjunction, $K_X + C_1 + F$ is plt near C_1. In particular, either $4C_1$ or $6C_1$ is Cartier (see 2.2.4) and F has at most two components. Thus $4d_1$ or $6d_1 \in \mathbb{N}$. Note that

$$d_1 = \frac{1}{H \cdot C_2} \leq 1, \qquad d_2 = \frac{1}{H \cdot C_1} \leq 1.$$

10.2.9.1. *Subcase* $d_2 = 1$. It is easy to see $H \cdot C_1 = d_1 H^2 = 1$, so $d_1 = 1/H^2$. We claim that $r > 1$. Indeed, if $r \leq 1$, then

$$1 \geq r \geq \frac{6}{7}(1 + d_1) \tag{10.14}$$

and $d_1 \leq 1/6$. Thus mC_1 is not Cartier for $m < 6$. By (10.13) we have that $6C_1$ is Cartier, $\mathrm{Diff}_{C_1}(F) = \frac{1}{2}Q_1 + \frac{2}{3}Q_2$ and $d_1 \geq 1/6$. Therefore $d_1 = 1/6$ and in 10.14 the equality holds. In particular, $r = 1$, $K_X^2 = H^2 = 6C_1 \cdot C_2 = 6$. By Lemma 10.2.5, $X \simeq \mathbb{P}(1,2,3)$ and $\mathrm{Weil}_{\mathrm{lin}}(X) \simeq \mathbb{Z}$. But then $C_2 \sim -K_X \sim H$ is Cartier and $p_a(C_2) = 1$, a contradiction.

Thus $r > 1$ and $X \subset \mathbb{P}^{d+1}$ is a cone of degree $d := H^2$ (see 10.2.4). Hence C_2 is a smooth hyperplane section and C_1 is a generator of the cone (i.e., $d_2 = 1$, $d_1 = 1/d$). Write $F_i \equiv \frac{q_i}{d}H$. (Note that $q_i \in \mathbb{N}$ and $F_i \sim q_i C_1$ because $\mathrm{Weil}_{\mathrm{lin}}(X) \simeq \mathbb{Z} \cdot C_1$ in our case). We have

$$\begin{aligned} 1 + \frac{1}{d} + \sum \left(1 - \frac{1}{m_i}\right)\frac{q_i}{d} > r = \\ \frac{d+2}{d} \geq b_2 + \frac{1}{d}b_1 + \sum \left(1 - \frac{1}{m_i}\right)\frac{q_i}{d}, \\ q_i \in \mathbb{N}, \quad m_i \in \{0, 2, 3, 4, 5, 6\} \end{aligned} \tag{10.15}$$

Assume that F has a component F_1 which does not pass through the vertex. Then $q_1 \geq d$, so

$$1 + \frac{2}{d} \geq b_2 + \frac{1}{d}b_1 + 1 - \frac{1}{m_1} \geq \frac{6}{7}\left(1 + \frac{1}{d}\right) + 1 - \frac{1}{m_1},$$

$$8 \geq d\left(6 - \frac{7}{m_1}\right) \geq \frac{5}{2}d.$$

This gives $d = 2$ or $d = 3$. If $d = 3$, then $m_1 = 2$. From (10.15) we get $F = \frac{1}{2}F_1$, i.e., case (A_2^3). If $d = 2$, then $m_1 = 2$ or $m_1 = 3$. In both cases by (10.15) we have $F = \left(1 - \frac{1}{m_1}\right) F_1$. For $m_1 = 2$ we derive a contradiction with the left side of (10.15). We obtain case (A_2^2).

Now we assume that all components of F pass through the vertex v of the cone (in particular, $F \neq 0$). Since $K_X + C + F$ is plt at v (see Lemma 10.2.2), there is at most one such a component and $F = (1 - \frac{1}{m_1})F_1$. We claim that either $q_1 = 1$ or $q_1 \geq d + 1$. Indeed, assume that $1 < q_1 \leq d$. Then

$$F_1 \cdot C_1 = \frac{q_1}{d^2} H^2 = \frac{q_1}{d} \leq 1.$$

Since X is smooth outside of v, $F_1 \cap C_1 = \{v\}$. By Adjunction, $\lfloor \mathrm{Diff}_{C_1}(F) \rfloor = 0$ at v. On the other hand, by 2.2.8, the coefficient of $\mathrm{Diff}_{C_1}(F)$ at v is

$$1 - \frac{1}{d} + \left(1 - \frac{1}{m_1}\right)(F_1 \cdot C_1) = 1 - \frac{1}{d} + \left(1 - \frac{1}{m_1}\right) \frac{q_1}{d}.$$

We obtain

$$\frac{1}{d} - \left(1 - \frac{1}{m_1}\right) \frac{q_1}{d} > 0, \quad 1 > \left(1 - \frac{1}{m_1}\right) q_1 \quad \text{and} \quad q_1 < \frac{m_1}{m_1 - 1} \leq 2,$$

a contradiction. Therefore $q_1 = 1$ or $q_1 \geq d + 1$. But the second case is impossible by the right side of (10.15). Hence $q_1 = 1$. But this contradicts to the left side of (10.15).

From now on we assume that $d_1 \leq d_2 < 1$.

REMARK 10.2.10. If $r > 1$, then X is a cone and contains exactly one singular point, say P, and $P \notin C_1 \cap C_2$. Hence we may assume that $P \notin C_1$ and C_1 is Cartier. Thus we may assume that $r \leq 1$ and C_1, C_2 are not Cartier.

10.2.10.1. *Subcase* $d_1 = 1/2$. Then we have

$$1 = C_1 \cdot C_2 = d_1 H \cdot C_2, \quad H \cdot C_2 = 2, \quad d_2 H^2 = 2.$$

Since $1 > d_2 = \frac{2}{H^2} \geq d_1 = \frac{1}{2}$, $H^2 = 3$ or $H^2 = 4$. On the other hand, $H \cdot C_1 = \frac{1}{2}H^2 \in \mathbb{N}$. Hence $H^2 = 4$, $d_2 = 1/2$ and $\mathbb{N} \ni -K_X \cdot H = rH^2 = 4r$. By symmetry, taking into account $d_1 = d_2 = 1/2$, one can see that (10.13) holds also for C_2:

$$\mathrm{Diff}_{C_2}(F) = \frac{1}{2}Q_1' + \frac{2}{3}Q_2', \quad \text{or} \quad \frac{1}{2}Q_1' + \frac{3}{4}Q_2'.$$

From $r \geq \frac{6}{7}(d_1 + d_2) = \frac{6}{7}$ we get $r \geq 1$. Thus $r = 1$ and X is Gorenstein by 10.2.10 and Lemma 10.2.5. By Theorem 5.2.3 all singular points are contained in C. Since $K_X + C$ is dlt (see Lemma 10.2.2), we obtain that X has only DuVal points of types A_{n_i}, $i = 1, \dots, s$. Since $\rho(X) = 1$, $\sum_{i=1}^{s} n_i = 10 - 4 - \rho(X) = 5$. By (10.13), $n_i \leq 3$ and $(n_1, \dots, n_s) \neq (1, 1, 1, 1, 1)$. Now we can use the classification of Gorenstein del Pezzo surfaces with $\rho = 1$ (see e.g., [**Fu**]). The configuration of singular points on X is $\{2A_1A_3\}$. We may assume that C_1 contains the point of type A_3. Hence $\mathrm{Diff}_{C_1}(F) = \frac{1}{2}Q_1 + \frac{3}{4}Q_2$ (see (10.13)). At least one of points Q_1, Q_2, Q_1', Q_2' is smooth. Hence $F \neq 0$ and $\mathrm{Supp} F \cap C_1 = Q_1$. Thus $F = \frac{1}{2}F_1$, where $F_1 \cap C_1 = Q_1$ and $F_1 \cdot C_1 = 1$. This implies $F_1 \equiv C_2 \equiv \frac{1}{2}H$. But then $1 = r < \frac{6}{7}(d_1 + d_2) + q = \frac{6}{7} + \frac{1}{4}$, a contradiction.

10.2.10.2. *Subcase* $d_1 = 1/3$. Since $4C_1$ is not Cartier, $\mathrm{Diff}_{C_1}(F) = \frac{1}{2}Q_1 + \frac{2}{3}Q_2$ and $Q_2 \in X$ is singular (of type A_2 or $\frac{1}{3}(1,1)$). Moreover, no components of F pass through Q_2. Further,

$$1 = C_1 \cdot C_2 = d_1 H \cdot C_2, \quad H \cdot C_2 = 3, \quad d_2 H^2 = 3.$$

Since $1 > d_2 = \frac{3}{H^2} \geq d_1 = \frac{1}{3}$, $9 \geq H^2 \geq 4$. On the other hand, $H \cdot C_1 = \frac{1}{3}H^2 \in \mathbb{N}$. Thus $H^2 = 6$ or 9. Further, by Lemma 10.2.4, $r = \frac{2l}{H^2} - 1$, where $l \in \mathbb{N}$ and $l \leq H^2 + 1$.

If $H^2 = 6$, then $d_2 = 1/2$ and

$$1 \geq r = \frac{l}{3} - 1 \geq \frac{6}{7}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{7}.$$

This gives $l = 6$ and $r = 1$. By Lemma 10.2.5, $X \simeq \mathbb{P}(1,2,3)$. In particular, $\mathrm{Weil}_{\mathrm{lin}}(X) \simeq \mathbb{Z}$. Since $-(K_X + C) \equiv (1 - 1/3 - 1/2)H$ is ample, $F \neq 0$. Therefore $Q_1 = \mathrm{Supp}F \cap C_1$ and moreover $Q_1 \in X$ is smooth, $F = \frac{1}{2}F_1$ and the intersection of F_1 and C_1 is transverse. Thus $1 = F_1 \cdot C_1 = \frac{1}{3}F_1 \cdot H$ and $F_1 \equiv \frac{1}{2}H$. We may assume that $C_1 = \{x_2 = 0\}$, $C_2 = \{x_3 = 0\}$, and $F_1 = \{x_3 = \alpha_1 x_1^3 + \alpha_2 x_1 x_2\}$, $\alpha_1, \alpha_2 \in \mathbb{C}$. But if $F_1 = \{x_3 = x_1^3\}$, then $K_X + C + F$ is not lc at $(0,1,0)$. On the other hand, if $F_1 = \{x_3 = x_1 x_2\}$, then F_1 passes through the point $C_1 \cap C_2$, a contradiction. Therefore $\alpha_1, \alpha_2 \neq 0$ and we may put $F_1 = \{x_3 = x_1^3 + x_1 x_2\}$. This is case ($\mathrm{A}_2^4$).

If $H^2 = 9$, then $d_2 = 1/3$ and

$$1 \geq r = \frac{2l}{9} - 1 \geq \frac{6}{7}\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{4}{7}, \quad l \in \mathbb{Z}.$$

This gives $l = 9$ or $l = 8$. But in the first case $r = 1$ which is a contradiction with $H^2 = 9$ (see 10.2.5). Hence $l = 8$ and $r = 7/9$. Since $d_1 = d_2$, similar to 10.13 we have $\mathrm{Diff}_{C_2}(F) = \frac{1}{2}Q_1' + \frac{2}{3}Q_2'$. In particular, this means that C contains no points of index > 3. But $X \setminus (C)$ contains such a point (because $r = 7/9$), a contradiction with 5.2.3.

10.2.10.3. *Subcase* $d_1 = 1/4$. Since mC_1 is not Cartier for $m < 4$, $\mathrm{Diff}_{C_1}(F) = \frac{1}{2}Q_1 + \frac{3}{4}Q_2 \geq \mathrm{Diff}_{C_1}(0)$ and $Q_2 \in X$ is a singular point of type A_3 or $\frac{1}{4}(1,1)$. By Theorem 5.2.3, $Q_1 \in X$ is smooth. Thus $F = \frac{1}{2}F_1$, where $F_1 \cap C_1 = Q_1$ and $C_1 \cdot F_1 = 1$. Put $k := H \cdot C_1$. Then $H^2 = 4k$, $d_2 = 1/k$. Since $d_2 \geq d_1$, $k \leq 4$. If $F_1 \equiv q_1 H$, then $1 = C_1 \cdot F_1 = \frac{1}{4}q_1 H^2 = q_1 k$. Hence $F_1 \equiv \frac{1}{k}H$. Further, by Lemma 10.2.4,

$$r = \frac{l}{2k} - 1 \geq \frac{6}{7}(d_1 + d_2) + \frac{1}{2}q_1 = \frac{6}{7}\left(\frac{1}{4} + \frac{1}{k}\right) + \frac{1}{2k}, \quad l - 2k - 2 \geq \frac{3k+5}{7}.$$

On the other hand, $K_X + C + \frac{1}{2}F_1$ is ample, so $0 < -r + d_1 + d_2 + \frac{1}{2}q_1$. This gives

$$0 < -\frac{l}{2k} + 1 + \frac{1}{4} + \frac{1}{k} + \frac{1}{2k} = \frac{-l + 2k + k/2 + 2 + 1}{2k}, \quad l - 2k - 2 < k/2 + 1.$$

We get the following case:

$$k = 3, \quad l = 10, \quad r = 2/3, \quad d_2 = 1/3, \quad F_1 \equiv \frac{1}{3}H, \quad H^2 = 12.$$

We claim that $K_X + C$ is 1-complementary. Note that $-(K_X + C) \equiv (\frac{2}{3} - \frac{1}{4} - \frac{1}{3})H$ is ample. By Theorem 5.2.3 and because $r = 2/3$, C_2 contains exactly one singular point of X, say Q'. Therefore $\operatorname{Diff}_C(0)$ is supported at two points Q' and Q_2. It is easy to verify that $K_C + \dot{Q}' + Q_2$ is an 1-complement. By Proposition 4.4.3 this complement gives an 1-complement $K_X + C + \Theta$, where Θ is reduced and $\Theta \cap C = \{Q', Q_2\}$. By Theorem 8.5.2, $(X, C + \Theta)$ is a toric pair. Such X is defined by a fan Δ in $\mathbb{R}^2 = \mathbb{Z}^2 \otimes \mathbb{R}$. Let v_1, v_2, v_3 be generators of one-dimensional cones in Δ. Since $X \setminus C$ is smooth, we may assume that v_1 and v_2 generate $\mathbb{Z}^2$. Thus we can put $v_1 = (1,0,0)$ and $v_2 = (0,1,0)$. Therefore X is a weighted projective space $\mathbb{P}(1, a_2, a_3)$, $C_1 \sim \mathcal{O}_{\mathbb{P}}(a_2)$, $C_2 \sim \mathcal{O}_{\mathbb{P}}(a_3)$ and $-K_X \sim \mathcal{O}_{\mathbb{P}}(1 + a_2 + a_3)$. Since $X \ni Q_2$ is singular of type $\frac{1}{4}(1, s)$, where $s = 1$ or 3 and $Q_2 \in C_1$, we can take $a_3 = 4$. Finally, from

$$K_X^2 = \left(\frac{2}{3}H\right)^2 = \frac{16}{3}, \quad K_X^2 = \frac{(a_1 + a_2 + a_3)^2}{a_1 a_2 a_3} = \frac{(5 + a_2)^2}{4a_2}$$

we obtain $a_2 = 3$. This is case (A_2^5).

10.2.10.4. *Subcase* $d_1 = 1/6$. Since mC_1 is not Cartier for $m < 6$, $\operatorname{Diff}_{C_1}(F) = \frac{1}{2}Q_1 + \frac{2}{3}Q_2 \geq \operatorname{Diff}_{C_1}(0)$ and $\operatorname{Diff}_{C_1}(0) = \operatorname{Diff}_{C_1}(F) = \frac{1}{2}Q_1 + \frac{2}{3}Q_2$. Hence $F = 0$ and points $Q_1, Q_2 \in X$ are singular. This contradicts to Theorem 5.2.3.

Theorem 10.2.1 is proved. □

Theorem 10.2.1 completes the classification of log pairs with $\delta(X, B) = 2$. The case $\delta(X, B) = 1$ was studied by Abe [**Ab**]. In particular, he completely described so called "elliptic curve case", i.e., the case $p_a(C) = 1$. A different approach to the classification of exceptional complements was given in [**KeM**].

10.3. Examples

EXAMPLE 10.3.1. Let $X = \mathbb{P}^2$ and $B = \sum d_i B_i$, where all B_i are lines on $\mathbb{P}^2$ such that no three of them pass through one point, and $d_i = 1 - 1/m_i$. Assume that $-(K_X + B)$ is ample. By definition, $K_X + B$ is n-complementary if and only if $\deg(-nK_X - \lfloor (n+1)B \rfloor) \geq 0$ (i.e., $\sum \lfloor (n+1)(1 - 1/m_i) \rfloor \leq 3n$). We give the list of all possibilities for $(m_1, \dots, m_r)$ (with $m_1 \leq \dots \leq m_r$). These were found by means of a computer program. Here $n = \operatorname{compl}(X, B)$.

NONEXCEPTIONAL PAIRS

$n = 1$: (m), (m_1, m_2), (m_1, m_2, m_3) (type $\mathbb{A}1_0^3$, see 5.3.7);
$n = 2$: $(2, 2, m_1, m_2)$, $(2, 2, 2, 2, m)$ (types $\mathbb{D}2_0^2$ and $\mathbb{E}2_0^1$, respectively);
$n = 3$: $(2, 3, 3, m)$, $(3, 3, 3, m)$ (type $\mathbb{E}3_0^1$);
$n = 4$: $(2, 3, 4, m)$, $(2, 4, 4, m)$ (type $\mathbb{E}4_0^1$);
$n = 5$: $(2, 3, 5, 5)$ (there is also a regular 6-complement of type $\mathbb{E}6_0^1$);
$n = 6$: $(2, 3, 5, m)$, $m \geq 6$, $(2, 3, 6, m)$ (type $\mathbb{E}6_0^1$).

EXCEPTIONAL PAIRS

$n = 4$: (3,3,4,4), (3,4,4,4), (2,2,2,3,3), (2,2,2,3,4);
$n = 5$: (2,4,5,5), (2,5,5,5) (in these cases there are also regular 6-complements);
$n = 6$: (2,4,5,6), (2,4,6,6), (2,5,5,6), (2,5,6,6), (3,3,4,5), (3,3,5,5), (3,3,5,6), (3,3,4,6), (2,2,2,3,5);
$n = 7$: (2,3,7,7);
$n = 8$: (2,3,7,8), (2,3,8,8), (2,4,5,7), (2,4,5,8), (2,4,6,7), (2,4,6,8), (2,4,7,7), (2,4,7,8);
$n = 9$: (2,3,7,9), (2,3,8,9), (2,3,9,9), (3,3,4,7), (3,3,4,8), (3,3,4,9);
$n = 10$: (2,3,7,10), (2,3,8,10), (2,3,9,10), (2,3,10,10), (2,4,5,9), (2,4,5,10), (2,5,5,7), (2,5,5,8), (2,5,5,9);
$n = 12$: (2,3,7,11), (2,3,7,12), (2,3,8,11), (2,3,8,12), (2,3,9,11), (2,3,9,12), (2,3,10,11), (2,3,10,12), (2,3,11,11), (2,3,11,12), (2,4,5,11), (2,4,5,12), (2,4,6,9), (2,4,6,10), (2,4,6,11), (3,3,4,10), (3,3,4,11), (3,4,4,5);
$n = 14$: (2,3,7,13), (2,3,7,14);
$n = 15$: (3,3,5,7), (2,3,7,15);
$n = 16$: (2,3,7,16), (2,3,8,13), (2,3,8,14), (2,3,8,15), (2,3,8,16), (2,4,5,13), (2,4,5,14), (2,4,5,15), (2,4,5,16);
$n = 18$: (2,3,7,17), (2,3,7,18), (2,3,8,17), (2,3,8,18), (2,3,9,13), (2,3,9,14), (2,3,9,15), (2,3,9,16), (2,3,9,17);
$n = 20$: (2,4,5,17), (2,4,5,18), (2,4,5,19);
$n = 21$: (2,3,7,19), (2,3,7,20), (2,3,7,21);
$n = 22$: (2,3,7,22);
$n = 24$: (2,3,7,23), (2,3,7,24), (2,3,8,19), (2,3,8,20), (2,3,8,21), (2,3,8,22), (2,3,8,23);
$n = 28$: (2,3,7,25), (2,3,7,26), (2,3,7,27), (2,3,7,28), (2,4,7,9);
$n = 30$: (2,3,7,29), (2,3,7,30), (2,3,10,13), (2,3,10,14), (2,5,6,7);
$n = 36$: (2,3,7,31), (2,3,7,32), (2,3,7,33), (2,3,7,34), (2,3,7,35), (2,3,7,36);
$n = 42$: (2,3,7,37), (2,3,7,38), (2,3,7,39), (2,3,7,40), (2,3,7,41);
$n = 66$: (2,3,11,13).

Thus the set of all compl(X, B) in this case is

$$\{1,2,3,4,5,6,7,8,9,10,12,14,15,16,18,20,21,\\ 22,24,28,30,36,42,66\}.$$

It is easy to see that this set is contained in $\{n \in \mathbb{N} \mid \varphi(n) \leq 20, n \neq 60\}$, which is related to automorphisms of $K3$ surfaces [**I**] (see also [**Ts**, Sect. 2]).

EXAMPLE 10.3.2. Replace the condition of the ampleness of $-(K_X + B)$ in Example 10.3.1 with numerical triviality. We obtain only exceptional cases:

$n = 2$: (2,2,2,2,2,2);

$n = 4$: $(4,4,4,4)$, $(2,2,2,4,4)$;
$n = 6$: $(2,6,6,6)$, $(3,3,6,6)$, $(2,2,2,3,6)$, $(2,2,3,3,3)$;
$n = 8$: $(2,4,8,8)$;
$n = 10$: $(2,5,5,10)$;
$n = 12$: $(2,3,12,12)$, $(2,4,6,12)$, $(3,3,4,12)$, $(3,4,4,6)$;
$n = 18$: $(2,3,9,18)$;
$n = 20$: $(2,4,5,20)$;
$n = 24$: $(2,3,8,24)$;
$n = 30$: $(2,3,10,15)$;
$n = 42$: $(2,3,7,42)$

In these cases (X, B) is a log Enriques surface and $n(K_X+B) \sim 0$. Construction 1.3 gives a ramified cyclic cover $\varphi\colon X' \to \mathbb{P}^2$ such that $K_{X'} = \varphi^*(K_X + B)$. Then $K_{X'} \sim 0$ and is plt, so X' is a surface with Du Val singularities and $K_{X'} \sim 0$. Note that if we replace the condition $B \in \mathbf{\Phi_{sm}}$ with $B \in \Phi_m$, we can get bigger values of $\operatorname{compl}(X, B)$. For example, take $B = \frac{1}{2}B_1 + \frac{2}{3}B_2 + \frac{18}{19}B_3 + \frac{101}{114}B_4$, where, as above, $B_i \subset \mathbb{P}^2$ are lines such that no three of them pass through one point. Then $\operatorname{compl}(X, B) = 78$.

EXAMPLE 10.3.3. Let $G \subset \mathrm{PGL}_3(\mathbb{C})$ be a finite subgroup, $X := \mathbb{P}^2/G$, and $f\colon \mathbb{P}^2 \to X$ the quotient morphism. Define the boundary B on X by $K_{\mathbb{P}^2} = f^*(K_X + B)$ (see (1.4) and (1.5)). Then (X, B) is exceptional if and only if G has no semiinvariants of degree ≤ 3 (see [**MP**]). There are only four types of such groups up to conjugation in $\mathrm{PGL}_3(\mathbb{C})$.

EXAMPLE 10.3.4 ([**Ab**]). Let $X := \mathbb{P}(1,2,3)$. Take a general member $E \in |-K_X|$ (a smooth elliptic curve) and let L be a line on X (with respect to $-K_X$). Then $E \sim 6L$. Since (X, L) is toric, $K_X + L$ is plt. Hence $(X, \alpha E + \beta L)$ is a log del Pezzo if and only if $6\alpha + \beta < 6$, $\alpha \le 1$, $\beta \le 1$. Moreover, if $\alpha \ge 6/7$ and $\beta \in \mathbf{\Phi_m}$, then $(X, \alpha E + \beta L)$ is exceptional. Indeed, by Corollary 8.4.2 it is sufficient to show that there are no regular nonklt complements. If $K_X + B^+$ is such a complement, then $B^+ \ge E + \beta L$, a contradiction. This gives the following exceptional cases with $\delta = 1$:

$\beta = 1/2$	$6/7 \le \alpha < 11/12$
$\beta = 2/3$	$6/7 \le \alpha < 8/9$
$\beta = 3/4$	$6/7 \le \alpha < 7/8$
$\beta = 4/5$	$6/7 \le \alpha < 13/15$
$\beta = 5/6$	$6/7 \le \alpha < 31/36$.

EXAMPLE 10.3.5 ([**Ab**]). Let $X \subset \mathbb{P}^3$ be a quadratic cone, $E \in |-K_X|$ a smooth elliptic curve, and L a generator of the cone. Then $(X, \frac{6}{7}E + \frac{1}{2}L)$ is an exceptional log del Pezzo with $\delta = 1$ and $K_X + \frac{6}{7}E + \frac{4}{7}L$ is a 7-complement.

EXERCISE 10.3.6. Let $C \subset \mathbb{P}^2$ be a smooth curve of degree d. Assume that $-(K_X + (1 - 1/m)C)$ is nef. Prove that $K_X + (1 - 1/m)C$ is exceptional only if and only if $(d, m) \in \{(4,3), (4,4), (5,2), (6,2)\}$. For $(d, m) = (4,3)$, $(5,2)$ such log Del

Pezzos can appear as exceptional divisors of plt blowups of canonical singularities (see [**P1**]). *Hint.* The nontrivial part is to prove that $K_X + (1 - 1/m)C$ is exceptional in these cases. Assuming the opposite we have a regular nonklt complement $K_X + B$. Then we can use the following simple fact: if $\sum d_i B_i$ is a boundary on $\mathbb{C}^2$ such that all the B_i are smooth curves and $\sum d_i \leq 1$, then $(\mathbb{C}^2, B)$ is canonical.

EXAMPLE 10.3.7. Let $(X \ni o)$ be a three-dimensional klt singularity and D an effective reduced Weil divisor on X. Assume that D is $\mathbb{Q}$-Cartier. Let $c_o(X, D)$ be the log canonical threshold. Assume that $1 > c := c_o(X, D) > 6/7$. Let $f\colon Y \to X$ be a plt blowup of (X, D). Write $K_Y + S + cB = f^*(K_X + cD)$, where B is the proper transform of D. Then $(S, \mathrm{Diff}_S(cB))$ is a log Enriques surface with $\delta \geq 1$. We claim that $K_S + \mathrm{Diff}_S(cB)$ is klt. Indeed, if $K_S + \mathrm{Diff}_S(cB)$ is not klt, then by the Inductive Theorem 8.3.1 there is a regular complement $K_S + \mathrm{Diff}_S(cB)^+$. Since $-(K_Y + S + (c - \varepsilon)B)$ is f-ample for $\varepsilon > 0$, by Proposition 4.4.1 we have a regular complement $K_Y + S + (c - \varepsilon)B$. This gives a regular complement $K_X + A$ of $K_X + (c - \varepsilon)D$. We can take ε so that $c - \varepsilon > 6/7$. Then A is reduced and $A = D$. Hence $c = 1$, a contradiction. This method can help to describe the set of all lc thresholds in the interval $[6/7, 1]$ (cf. [**Ku**]). For example, take $X = \mathbb{C}^3$ and $D = \{\psi(x, y, z) = 0\}$, where $\psi(x, y, z) = x^3 + yz^2 + x^2y^2 + x^5 z$ (see [**Ku**]). Then $c_o(\mathbb{C}^3, D) = 11/12$ and $f\colon Y \to \mathbb{C}^3$ is the weighted blowup with weights $(4, 2, 5)$. So $S = \mathbb{P}(4, 2, 5)$. It is easy to compute that $\mathrm{Diff}_S(cD) = \frac{11}{12}C + \frac{1}{2}L$, where $C := \{x^3 + yz^2 + x^2y^2 = 0\}$ and $L := \{z = 0\}$. Both C and L are smooth rational curves which intersect each other twice at smooth points of S. Such complements were studied in [**Ab**, Sect. 2] and called there "sesqui rational curve" complements.