CHAPTER 6

Birational contractions and two-dimensional log canonical singularities

THEOREM 6.0.6. *Let $(X/Z \ni o, D)$ be a log surface of local type, where $f\colon X \to Z \ni o$ is a contraction. Assume that $K_X + D$ is lc and $-(K_X + D)$ is f-nef and f-big. Then there exists an 1, 2, 3, 4, or 6-complement of $K_X + D$ which is not klt near $f^{-1}(o)$. Moreover, if there are no nonklt 1 or 2-complements, then $(X/Z \ni o, D)$ is exceptional. These complements $K_X + D^+$ can be taken so that $a(E, D) = -1$ implies $a(E, D^+) = -1$ for any divisor E of $\mathcal{K}(X)$.*

PROOF. Let H be an effective Cartier divisor on Z containing o and let $F := f^*H$. First we take the $c \in \mathbb{Q}$ such that $K_X + D + cF$ is maximally lc (see 5.3.3) and replace D with $D + cF$. This gives that $\mathrm{LCS}(X, D) \neq \varnothing$. Next we replace (X, D) with a log terminal modification. So we may assume that $K_X + D$ is dlt and $\lfloor D \rfloor \neq 0$. Then Proposition 4.4.3 and Theorem 4.1.10 give us that there exists a regular complement $K_X + D^+$ of $K_X + D$. By construction, $\lfloor D^+ \rfloor \geq \lfloor D \rfloor$. If $K_X + D$ is not exceptional, then there exists a $\mathbb{Q}$-complement $K_X + D'$ of $K_X + D$ and at least two divisors with discrepancy $a(\cdot, D') = -1$. Then we can replace D with D'. Taking a log terminal blowup, we obtain that $\lfloor D \rfloor$ is reducible. The rest follows by Theorem 4.1.10. □

COROLLARY 6.0.7. *Let (Z, Q) be a lc, but not klt two-dimensional singularity. Then the index of (Z, Q) is 1, 2, 3, 4, or 6.*

This fact has three-dimensional generalizations [**I**].

PROOF. Apply Theorem 6.0.6 to $f = \mathrm{id}$ and K_Z. We get an n-complement $K_Z + D$ with $n \in \{1, 2, 3, 4, 6\}$. Then $K_Z + D$ is lc and $n(K_Z + D) \sim 0$. But if $D \neq 0$, K_Z is klt (because $Q \in \mathrm{Supp} D$). □

COROLLARY 6.0.8. *Let $(X \ni P)$ be a normal surface germ. Let D be a boundary such that $D \in \Phi_{\mathbf{m}}$ and C a reduced Weil divisor on X. Assume that D and C have no common components. Then one of the following holds:*

(i) *$K_X + D + C$ is lc; or*
(ii) *$K_X + D + \alpha C$ is not lc for any $\alpha \geq 6/7$.*

Actually, we have more precise result 6.0.9. See [**Ko1**] for three-dimensional generalizations.

PROOF. Assume that $K_X+D+\alpha C$ is lc for some $\alpha \geq 6/7$. By Theorem 6.0.6 there is a regular complement $K_X+D^+ +\alpha^+ C$ near P. Since $D\in \Phi_{\mathbf{m}}$, $D^+\geq D$. By the definition of complements, $\alpha^+=1$. Hence K_X+D+C is lc. □

Let $(X\ni o)$ be a klt singularity and D an effective Weil divisor on X. Assume that D is $\mathbb{Q}$-Cartier. The *log canonical threshold* is defined as follows

$$c_o(X,D):=\sup\{c \mid K_X+cD \text{ is lc}\}.$$

COROLLARY 6.0.9 (cf. 10.3.7). *Let $(X\ni P)$ be a normal klt surface germ. Let D be a reduced Weil divisor on X. Assume that $c_P(X,D)\geq 2/3$, then*

$$c_P(X,D)\in \mathcal{S}:=\left\{\frac{2}{3},\frac{7}{10},\frac{3}{4},\frac{5}{6},1\right\}.$$

PROOF. Put $c:=c_P(X,D)$. Assume that $2/3<c<1$. Clearly, D is reduced. Let $f\colon (Y,C)\to X$ be an inductive blow up of (X,cD) (see 3.1.5). Then we can write $f^*(K_X+cD)=K_Y+C+cD_Y$, where D_Y is the proper transform of D. If K_Y+C+cD_Y is not plt, then by Theorem 6.0.6, K_X+cD is 1 or 2-complementary. Since $c\geq 2/3$, this gives us that $(cD)^+\geq D$, K_X+D is lc and $c=1$. Hence, we may assume that K_Y+C+cD_Y is plt. By 4.4.4, C intersects $\operatorname{Supp}D_Y$ transversally and

$$\text{(6.1)}\quad \operatorname{Diff}_C(cD)=\sum_{i=1}^{s}\frac{n_i-1+c}{n_i}P_i+\sum_{j=1}^{q}\frac{r_j-1}{r_j}Q_j,$$

$$\text{where}\quad \{P_1,\dots,P_s\}=C\cap \operatorname{Supp}D,\quad n_i,r_j\in\mathbb{N},$$

$$\{Q_1,\dots,Q_q\}=\operatorname{Sing}Y\setminus \operatorname{Supp}D.$$

Since $C\simeq \mathbb{P}^1$, $\deg \operatorname{Diff}_C(cD)=2$. If $s\geq 3$, then $s=3$ and in (6.1), $\frac{n_i-1+c}{n_i}=\frac{2}{3}$ for $i=1,2,3$, a contradiction with $c>2/3$. Assume that $s=2$. Then in (6.1) we have $2>\sum\frac{n_i-1+c}{n_i}>\frac{4}{3}$ and $0<\sum\frac{r_j-1}{r_j}<2/3$. Hence $\sum\frac{r_j-1}{r_j}=\frac{1}{2}$ and $\sum_{i=1}^{2}\frac{n_i-1+c}{n_i}=\frac{3}{2}$. This yields

$$\frac{1}{2}=\frac{1-c}{n_1}+\frac{1-c}{n_2}<\frac{1}{3n_1}+\frac{1}{3n_2}.$$

Therefore $n_1=n_2=1$ and $c=3/4$. Finally, assume that $s=1$. Similarly, in (6.1) we have $1<\sum_{j=1}^{q}\frac{r_j-1}{r_j}<4/3$. From this $q=2$ and up to permutations (r_1,r_2) is one of the following: $(2,3)$, $(2,4)$, $(2,5)$. Thus $\frac{n_1-1+c}{n_1}=\frac{5}{6},\frac{3}{4}$, or $\frac{7}{10}$. In all cases $n_1=1$, so $c\in\mathcal{S}$. □

6.1. Classification of two-dimensional log canonical singularities

Two-dimensional log terminal singularities (=quotient singularities) were classified for the first time by Brieskorn [**Br**] (see also [**Il**], [**Ut**, ch. 3]). We reprove this classification in terms of plt blowups. It is expected that this method has higher-dimensional generalizations, cf. [**Sh3**], [**P1**]. Recall that two-dimensional log terminal singularities are exactly quotient singularities (see [**K**]).

Let (Z, Q) be a two-dimensional klt singularity. If K_Z is 1-complementary, then by 2.1.3 (Z, Q) is analytically isomorphic to a cyclic quotient singularity. Assume further that K_Z is not 1-complementary. By Lemma 3.1.4 there exists a plt blowup $f\colon (X, C) \to Z$ (where C is the exceptional divisor of f). Further, we classify these blowups and propose the method to construct the minimal resolution. This method also allows us to describe klt singularities as quotients (see Proposition 6.2.6).

LEMMA 6.1.1. *Let $f\colon X \to Z \ni o$ be a plt blowup of a surface klt singularity and C the (irreducible) exceptional divisor. Then*

(i) *X has at most three singular points on C;*
(ii) *near each singular point the pair $C \subset X$ is analytically isomorphic to $(\{x = 0\} \subset \mathbb{C}^2)/\mathbb{Z}_{m_i}(1, a_i)$, where $\gcd(a_i, m_i) = 1$;*
(iii) *if X has one or two singular points on C, then $K_X + C$ is 1-complementary;*
(iv) *if X has three singular points on C, then $(m_1, m_2, m_3) = (2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$ and $K_X + C$ is respectively 2, 3, 4, or 6-complementary in these cases.*

PROOF. By Proposition 2.1.2 we get that all singular points $P_1, \dots, P_r \in X$ are cyclic quotients:

$$(X \supset C \ni P_i) \simeq (\mathbb{C}^2 \supset \{x = 0\} \ni 0)/\mathbb{Z}_{m_i}(1, a_i), \quad \gcd(a_i, m_i) = 1,$$

where the action of $\mathbb{Z}_{m_i}$ on $\mathbb{C}^2$ is free outside of 0. Therefore $C \simeq \mathbb{P}^1$, $\operatorname{Diff}_C(0) = \sum(1 - 1/m_i)P_i$, where $K_C + \operatorname{Diff}_C(0) = (K_X + C)|_C$ is negative on C. From this it is easy to see that for $(m_1, \dots, m_r)$ there are only the possibilities (m), (m_1, m_2), $(2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. Since $-(K_X + C)$ is f-ample, n-complements for $K_C + \operatorname{Diff}_C(0)$ can be extended to n-complements of $K_X + C$. By 4.1.10 we have the desired n-complements. This proves (iii) and (iv). □

By our assumptions, K_Z is not 1-complementary and by Lemma 6.1.1 we have exactly three singular points on C. Consider now the minimal resolution $g\colon Y \to X$ and put $h := f \circ g\colon Y \to Z$. Then on this resolution C corresponds to some curve, say C', and the singular points P_i, $i = 1, 2, 3$ correspond to "tails" meeting C' and consisting of smooth rational curves (see Fig. 6.1)

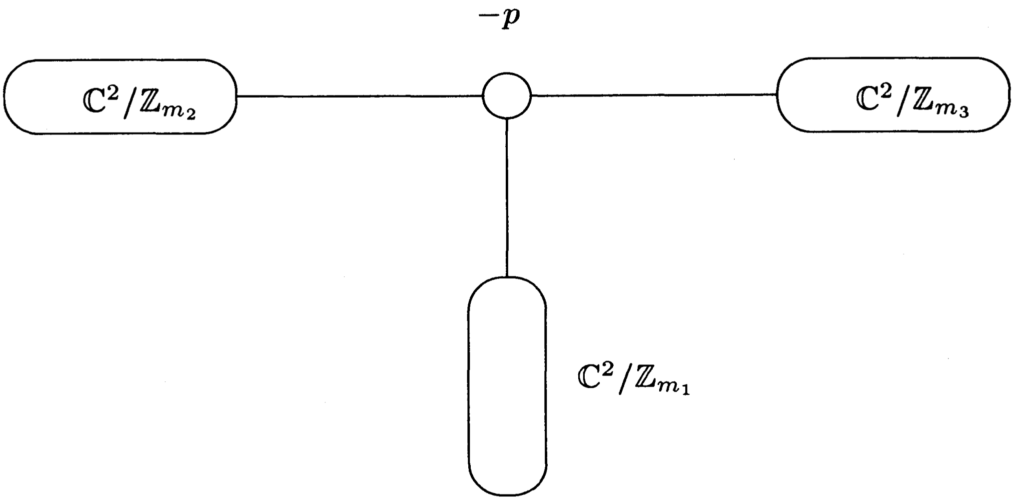


FIGURE 6.1

Here each oval is a chain for the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_m$ (see 2.1.1). Thus for $m \leq 5$ it is one of the following:

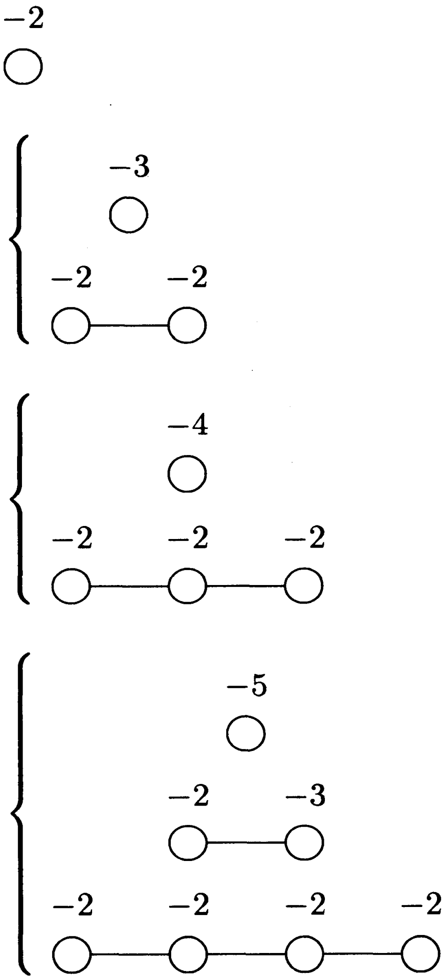
(6.2)

$m = 2$

$m = 3$

$m = 4$

$m = 5$



From Corollary 4.1.11 we obtain cases for (m_1, m_2, m_3) in figures 6.2-6.5. Since the intersection matrix of exceptional divisors is negative definite, $p \geq 2$. Then taking into account (6.2) it is easy to get the complete list of klt singularities (see [**Br**], [**Il**], [**Ut**, ch. 3]).

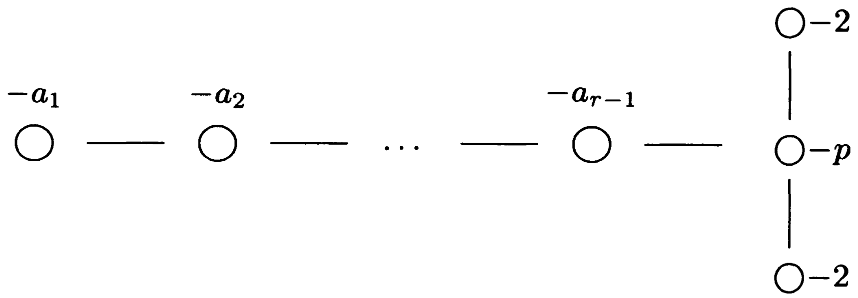


FIGURE 6.2. Case $(2, 2, m)$

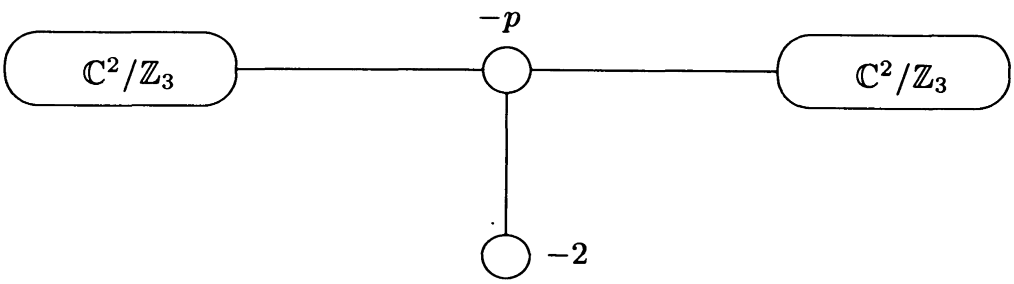


FIGURE 6.3. Case $(2, 3, 3)$

THEOREM 6.1.2. *Let (Z, Q) be a two-dimensional log terminal singularity. Then one of the following holds:*

(i) *(Z, Q) is nonexceptional and then it is either cyclic quotient (case $\mathbb{A}_n$ see 2.1.1) or the dual graph of its minimal resolution is as in Fig. 6.2 (case $\mathbb{D}_n$);*

(ii) *(Z, Q) is exceptional and then the dual graph of its minimal resolution is as in Fig. 6.3-6.5 (cases $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$).*

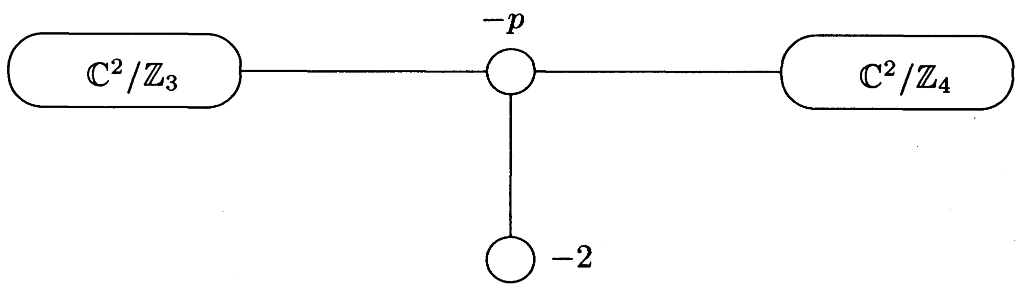


FIGURE 6.4. Case $(2, 3, 4)$

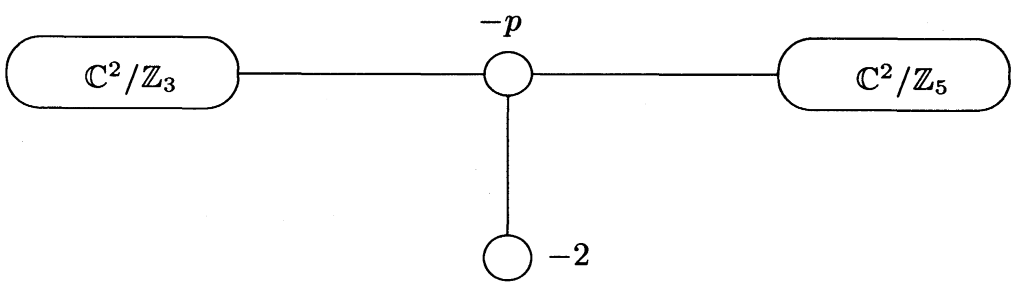


FIGURE 6.5. Case $(2, 3, 5)$

REMARK 6.1.3. (i) Note that our classification uses only the *numerical* definition of log terminal singularities (by using numerical pull backs [**S1**], see [**K**]).

(ii) In all cases of noncyclic quotient singularities (i.e. Fig. 6.2-6.5) the plt blowup is unique.

COROLLARY 6.1.4. *Fix $\varepsilon > 0$. There is only a finite number of two-dimensional exceptional ε-lt singularities (up to analytic isomorphisms).*

PROOF. Let E_0 be the "central" exceptional divisor of the minimal resolution and E_1, E_2, E_3 exceptional divisors adjacent to E_0. Write $K_Y = h^*K_Z + \sum a_i E_i$. Intersecting both sides with E_0, we obtain

$$p - 2 = -pa_0 + a_1 + a_2 + a_3.$$

This yields

$$p\varepsilon < p(1 + a_0) \leq 2, \qquad p < 2/\varepsilon.$$

□

EXERCISE 6.1.5. Classify two-dimensional singularities (Z, Q) with klt $K_Z + D$, where

$$D = (1 - 1/m_1)D_1 + (1 - 1/m_2)D_2 + (1 - 1/m_3)D_3, \quad Q \in D_i.$$

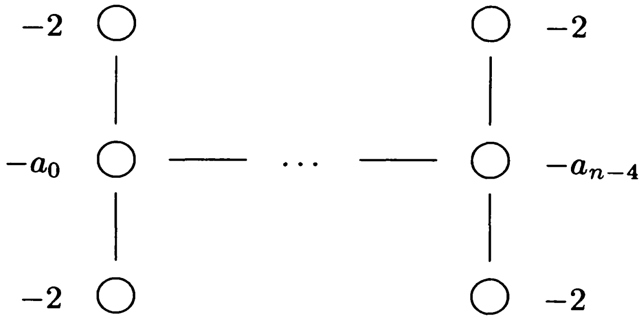
THEOREM 6.1.6 ([**S2**], [**K**], [**Ut**, Ch. 3]). *Let (Z, Q) be a two-dimensional lc, but not klt singularity, and let $f: X \to Z$ be the minimal resolution. Write $K_X + D = f^*K_Z$ and put $C := \lfloor D \rfloor$, $B := \{D\}$. Then one of the following holds:*

$Ell - \widetilde{A}_n$: $B = 0$, $p_a(C) = 1$ and C is either

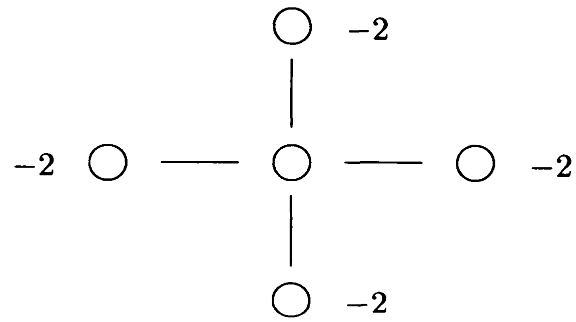
*Ell: a smooth elliptic curve (*simple elliptic *singularity),*

*$\widetilde{A}_n$: a rational curve with a node, or a wheel of smooth rational curves (*cusp *singularity);*

$\widetilde{D}_n$: the dual graph of $f^{-1}(Q)$ is given by



or (when $n = 4$)



Exc: the dual graph of $f^{-1}(Q)$ is as in Fig. 6.1, where (m_1, m_2, m_3) is one of the following: $(3, 3, 3)$ (case $\widetilde{E}_6$), $(2, 4, 4)$ (case $\widetilde{E}_7$), or $(2, 3, 6)$ (case $\widetilde{E}_8$), cf. 4.1.12.

SKETCH OF PROOF. Similar to the proof of Theorem 6.1.2. Instead of plt blowup we can use a minimal log terminal modification $f: X \to Z$. Let $K_X + C = f^*K_Z$ be the crepant pull back. Then C is a reduced divisor and $K_X + C$ is dlt. If C is reducible, we can use Lemmas 6.1.7 and 6.1.9 below. If C is irreducible, then either C is a smooth elliptic curve (and X is also smooth) or $C \simeq \mathbb{P}^1$. In the second case as in the proof of we Theorem 6.1.2 have cases according to 4.1.12. We need to check only that $p \geq 2$ in Fig. 6.1. This follows by the fact that the intersection matrix is negative definite. $\square$

LEMMA 6.1.7. *Let $(X/Z, D)$ be a log surface such that $K_X + D$ is dlt and $-(K_X + D)$ is nef over Z, $C := \lfloor D \rfloor \neq \varnothing$ and $B := \{D\}$. Assume that C is compact, connected and not a tree of smooth rational curves. Then X is smooth along C, $C \cap \operatorname{Supp} B = \varnothing$, $p_a(C) = 1$ and C is either a smooth elliptic curve or a wheel of smooth rational curves.*

PROOF. One has

$$0 \le 2p_a(C) - 2 + \deg \operatorname{Diff}_C(B) = (K_X + C + B) \cdot C \le 0.$$

This yields $p_a(C) = 0$ and $\operatorname{Diff}_S(B) = 0$. In particular, C is contained in the smooth locus of X and $C \cap \operatorname{Supp} B = \varnothing$. Moreover, it follows from $(K_X + C + B) \cdot C = 0$ that $(K_X + C + B) \cdot C_i = 0$ for any component $C_i \subset C$. Similarly we can write

$$0 \le 2p_a(C_i) - 2 + \deg \operatorname{Diff}_{C_i}(C - C_i) = (K_X + C) \cdot C_i = 0.$$

If $C = C_i$ is irreducible, then $p_a(C) = 1$ and C is a smooth elliptic curve (because $K_X + C$ is dlt). If $C_i \subsetneq C$, then $p_a(C_i) = 0$, $C_i \simeq \mathbb{P}^1$ and $\deg \operatorname{Diff}_{C_i}(C - C_i) = 2$. Since $K_X + C$ is dlt, C_i intersects $C - C_i$ transversally at two points. The only possibility is when C is a wheel of smooth rational curves. □

REMARK 6.1.8. Assuming that $K_X + D$ is only analytically dlt, we have additionally the case when C is a rational curve with a node.

Similar to 6.1.7 one can prove the following

LEMMA 6.1.9. *Let $(X/Z, D)$ be a log surface such that $K_X + D$ is dlt and numerically trivial over Z, $C := \lfloor D \rfloor \neq \varnothing$ and $B := \{D\}$. Assume that $B \in \Phi_{\mathbf{m}}$, C is compact and it is a (connected and reducible) tree of smooth rational curves. Then C is a chain. Further, write $C = \sum_{i=1}^r C_i$ where C_1, C_r are ends. Then*

(i) $\operatorname{Diff}_C(B) = \frac{1}{2}P_1^1 + \frac{1}{2}P_2^1 + \frac{1}{2}P_1^r + \frac{1}{2}P_2^r$, *where $P_1^1, P_2^1 \in C_1$, $P_1^r, P_2^r \in C_r$ are smooth points of C;*
(ii) $C \cap (\operatorname{Sing} X \cup \operatorname{Supp} B) \subset \{P_1^1, P_2^1, P_1^r, P_2^r\}$;
(iii) *for each P_j^i, $(i,j) \in \{(1,1), (1,2), (r,1), (r,2)\}$ we have one of the following:*
   (a) *X is smooth at P_j^i and there is exactly one component B_k of B passing through P_j^i, in this case C_i intersects B_k transversally and the coefficient of B_k is equal to $1/2$;*
   (b) *X has at P_j^i Du Val point of type A_1 and no components of B pass through P_j^i.*

In particular, if $B = 0$, then (X, C) looks like that on Fig. 6.6, X is singular only at $P_1^1, P_2^1 \in C_1$, $P_1^r, P_2^r \in C_r$ and these singularities are Du Val of type A_1.

REMARK 6.1.10. (i) We have $nK_Z \sim 0$, where $n = 1, 2, 3, 4, 6$ in cases $Ell - \widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$, respectively (see Corollary 6.0.7). This gives that any two-dimensional lc but not klt singularity is a quotient of a singularity of type $Ell - \widetilde{A}_n$ by a cyclic group of order 1, 2, 3, 4, or 6.
(ii) The singularity (Z, Q) is exceptional exactly in cases Ell, $\widetilde{D}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$.

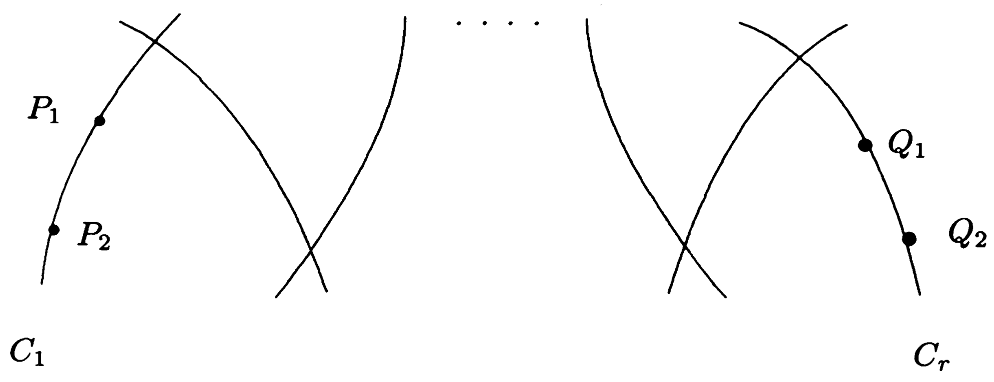


FIGURE 6.6

Recall that a normal surface singularity $Z \ni Q$ is said to be *rational* [**Ar**] (resp. *elliptic*) if $R^1 f_* \mathcal{O}_X = 0$ (resp. $R^1 f_* \mathcal{O}_X$ is one-dimensional) for any resolution $f\colon X \to Z$.

COROLLARY 6.1.11 ([**K**]). *Let* $(Z \ni Q)$ *be a two-dimensional lc singularity and* $f\colon X \to Z$ *its minimal resolution. Write* $K_X + D = f^*K_Z$. *Then one of the following holds:*

(i) $\{D\} \neq 0$ *and* $Z \ni Q$ *is a rational singularity;*
(ii) $\{D\} = 0$ *and* D *is either a smooth elliptic curve (type Ell), a rational curve with a node or a wheel of smooth rational curves (type* $\widetilde{A}_n$*). In this case,* $(Z \ni Q)$ *is a Gorenstein elliptic singularity.*

Note that exceptional log canonical singularities are rational except for the case *Ell*.

EXERCISE 6.1.12. Prove that the following hypersurface singularities are lc but not klt:

$$x^3 + y^3 + z^3 + axyz = 0, \quad a^3 + 27 \neq 0;$$
$$x^2 + y^4 + z^4 + ay^2z^2 = 0, \quad a^2 \neq 4;$$
$$x^2 + y^3 + z^6 + ay^2z^2 = 0, \quad 4a^3 + 27 \neq 0.$$

6.2. Two-dimensional log terminal singularities as quotients

Now we discuss the relation between two-dimensional log terminal singularities and quotient singularities. We use the following standard notation:

$\mathfrak{S}_n$	symmetric group;
$\mathfrak{A}_n$	alternating group;
$\mathfrak{D}_n = \langle \alpha, \beta \mid \alpha^n = \beta^2 = 1, \beta\alpha\beta = \alpha^{-1} \rangle$	dihedral group of order $2n$.

PROPOSITION 6.2.1. *Notation as in Lemma 6.1.1. Then we have*

(i) *if X has three singular points on C, $f\colon X \to Z$ is the quotient of the minimal resolution of the cyclic quotient singularity $\mathbb{C}^2/\mathbb{Z}_r(1,1)$ by the group $\mathfrak{D}_m$, $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$, in cases $(m_1, m_2, m_3) = (2,2,m)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$, respectively;*

(ii) *if moreover $-K_X$ is f-ample, then X has at most two singular points on C (and $K_X + C$ is 1-complementary).*

PROOF. Consider X as a small analytic neighborhood of $C \simeq \mathbb{P}^1$. We calculate the fundamental group of $X \setminus \{P_1, P_2, P_3\}$. Denote by $\Gamma(m_1, m_2, m_3)$ the group generated by $\alpha_1, \alpha_2, \alpha_3$ with relations

$$\alpha_1^{m_1} = \alpha_2^{m_2} = \alpha_3^{m_3} = \alpha_1\alpha_2\alpha_3 = 1.$$

LEMMA 6.2.2 (cf. [**Mo**, 0.4.13.3]).

$$\pi_1(X \setminus \{P_1, P_2, P_3\}) \simeq \Gamma(m_1, m_2, m_3).$$

PROOF. Let $U_i \subset X$ be a small neighborhood of P_i and $U_i^o := U_i \setminus \{P_i\}$. From Theorem 2.1.2 we have $\pi_1(U_i^o) \simeq \mathbb{Z}_{m_i}$. Denote by α_i the generators of these groups. The set $X \setminus \{P_1, P_2, P_3\}$ is homotopically equivalent to $\mathbb{P}^1 \setminus \{P_1, P_2, P_3\}$ glued along $\alpha_1, \alpha_2, \alpha_3$ with sets U_1^0, U_2^0, U_3^0. Denote loops around P_i (with the appropriate orientation) also by α_i. Then $\pi_1(\mathbb{P}^1 \setminus \{P_1, P_2, P_3\}) \simeq \langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_1\alpha_2\alpha_3 = 1 \rangle$. From the description of points 2.1.2 it follows also that the map

$$\pi_1(C \cap U_i^o) \simeq \mathbb{Z} \to \pi_1(U_i^o) \simeq \mathbb{Z}_{m_i}$$

is surjective. Now the lemma follows by Van Kampen's theorem. □

Now for (i) we notice that the groups $\Gamma(2,2,m)$, $\Gamma(2,3,3)$, $\Gamma(2,3,4)$ and $\Gamma(2,3,5)$ have finite quotient groups isomorphic to $\mathfrak{D}_m$, $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$, respectively, such that the images of the elements α_i have orders m_i. This follows from the fact that there exist actions of $\mathfrak{D}_m$, $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ on $\mathbb{P}^1$ with ramification points of orders (m_1, m_2, m_3). Then this finite group determines a finite cover $\widehat{X} \to X$ unramified outside of P_1, P_2, P_3, where $\widehat{X}$ is smooth. The Stein factorization gives a contraction $\widehat{X} \to \widehat{Z}$ of an irreducible curve $\mathbb{P}^1 \simeq \widehat{C} \subset \widehat{X}$. If $\widehat{C}^2 = -r$, then this contraction is the minimal resolution of the singularity $\mathbb{C}^2/\mathbb{Z}_r(1,1)$. Finally, if $-K_X$ is ample, then so is $-K_{\widehat{X}}$. Thus $r = 1$, i.e., $Z \ni o$ is a smooth point. But the groups $\mathfrak{D}_m$, $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ cannot act on $(Z \ni o) \simeq (\mathbb{C}^2, 0)$ freely in codimension one. This proves (ii). □

COROLLARY 6.2.3 ([**KMM**, 0-2-17]). *Any two-dimensional klt singularity is a quotient singularity.*

COROLLARY 6.2.4 ([**Br**]). *Let (Z, Q) be a two-dimensional klt singularity. Then $\pi_1(Z \setminus \{Q\})$ is finite.*

EXAMPLE 6.2.5. Let $a, b, m \in \mathbb{N}$, $\gcd(a,b) = 1$. Consider a cyclic quotient singularity $o \ni Z = \mathbb{C}^2/\mathbb{Z}_m(a,b)$ (the case $m = 1$ is not excluded). Any weighted blowup $f\colon X \to Z$ with weights (a,b) is an extremal contraction with exceptional divisor $C \simeq \mathbb{P}^1$. By Lemma 3.2.1, $K_X = f^*K_Z + ((a+b)/m - 1)C$. Hence for $a + b > m$ the divisor $-K_X$ is f-ample.

PROPOSITION 6.2.6 (cf. Conjecture 2.2.18). *Let $f\colon X \to Z$ be a birational contraction of normal surfaces. Assume that f contracts an irreducible curve C (i.e. $\rho(X/Z) = 1$) and $K_X + C$ is a plt and f-antiample (i.e., f is a plt blowup; see 3.1.4). Assume also that X has at most two singular points on C. Then f is a weighted blowup.*

REMARK 6.2.7. The condition of the antiampleness of $K_X + C$ is equivalent to the klt property of $f(C) \in Z$. The condition that X has ≤ 2 singular points is equivalent to that $Z \ni f(C)$ is a cyclic quotient singularity (or smooth).

PROOF. By Proposition 6.2.1, $K_X + C$ and K_Z are 1-complementary. Therefore there are two curves C_1, C_2 such that $K_X + C + C_1 + C_2$ is lc and linearly trivial over Z. Moreover, by Theorem 2.1.3 up to analytic isomorphisms we may assume that $(Z, f(C_1) + f(C_2))$ is a toric pair. For example, assume that X has exactly two singular points. Consider the minimal resolution $\mu\colon X' \to X$ and $f'\colon X' \to Z$ the composition. It is sufficient to show that the morphism f' is toric. By 2.1.3, in a fiber over $o \in Z$ we have the following configuration of curves:

$$\ominus \;—\; \bigcirc \;\cdots\; \bigcirc \;—\; \bullet \;—\; \bigcirc \;\cdots\; \bigcirc \;—\; \ominus,$$

where the black vertex corresponds to a fiber (and has self-intersection number $a \leq -1$), white vertices correspond to exceptional divisors and have self-intersection numbers $b_i \leq -2$, and the vertices $\ominus$ correspond to the curves C_1, C_2. If $a < -1$, f is the minimal resolution of a cyclic quotient singularity $o \in Z$ and in this case the morphism f' is toric. If $a = -1$, then $f'\colon X' \to Z$ factors through the minimal resolution $g\colon Y \to Z$ of the singularity $o \in Z$ (which is a toric morphism) and $X' \to Y$ is a composition of blowups with centers at points of intersections of curves. Such blowups preserve the action of the two-dimensional torus, hence f' is a toric morphism. □

EXAMPLE 6.2.8 ([**Mor**]). Let $f\colon X \to Z \ni o$ be a K_X-negative extremal birational contraction of surfaces. Assume that X has only Du Val singularities. Then

(i) Z is smooth;
(ii) f is a weighted blowup (see 3.2) with weights $(1,q)$ (and then X contains only one singular point, which is of type A_{q-1}).

EXERCISE 6.2.9 (cf. 2.2.18). Let $f\colon X \to Z \ni o$ be a birational two-dimensional contraction and D a boundary on X such that $K_X + D$ is lc and $-(K_X + D)$ is nef over Z. Prove that

$$\rho_{\mathrm{num}}(X/Z) + 2 \geq \sum d_i,$$

where $\rho_{\text{num}}(X/Z)$ is the rank of the quotient of Weil(X) modulo numerical equivalence. Moreover, the equality holds only if $(X/Z \ni o, \lfloor D \rfloor)$ is a toric pair.