CHAPTER 0

Introduction

These notes grew out of lectures I gave at Moscow University. Basically we follow works of V. V. Shokurov [**Sh2**], [**Sh3**]. These notes can help the reader to understand the main ideas of the theory. In particular, they can be considered as an introduction to [**Sh3**].

One of the basic problems in the modern birational minimal model program is the problem of constructing a divisor with rather "good" singularities in the anticanonical or multiple anticanonical linear system. Probably this question arises for the first time in the classification of Fano 3-folds (see e.g., [**Is**]) and was solved by V. V. Shokurov [**Sh**]. Later this technique was improved by Kawamata and others (see e.g., [**R**]). However, this method only applies to linear systems of Cartier divisors.

Another approach to constructing "good" divisors in the anticanonical linear system was proposed by Mori [**Mo**] in the proof of existence of three-dimensional flips. Unfortunately, now this method was applied only in analytic situation and in dimension 3 (see [**KoM**], [**P**] and also [**Ka**]).

The concept of complement was introduced by V. V. Shokurov in his proof of the existence of good divisors near nontrivial fibers of small contractions [**Sh2**]. Roughly speaking, an n-complement of the canonical divisor K_X is an element of the multiple anticanonical linear system $D \in |-nK_X|$ such that the pair $\left(X, \frac{1}{n}D\right)$ has only log canonical singularities (for precise definitions we refer to 4.1.3). Thus the question of the existence of complements can be posed for Fano or Calabi-Yau varieties (i.e., with numerically effective anticanonical divisor), and also for varieties with a fiber space structure on varieties of such types. For example, if on a smooth (or even with log canonical singularities) variety X the anticanonical divisor $-K_X$ is ample, then by Bertini's theorem there is an n-complement for some $n \in \mathbb{N}$. In case of Calabi-Yau varieties of the property of the canonical divisor K_X to be n-complementary is equivalent to triviality of nK_X. For example, it is known that for smooth surfaces such n can be taken in $\{1, 2, 3, 4, 6\}$.

These notes aim to do two things. The first is to give an introduction to the theory of complements, with rigorous proofs and motivating examples. For the first time the reader can be confused by rather tricky definition of n-complements (see (iii) of 4.1.3). However this definition is justified because of their useful properties – both birational and inductive (see 4.3 and 4.4). This shows that the property of the existence of complements is very flexible: we can change the birational model and then extend complements from the reduced part of the boundary.

The second is to prove several concrete results on the existence of complements. Our results, presented mainly in dimension two, illustrate main ideas and difficulties for higher-dimensional generalizations (cf. [**Sh3**, Sect.7], [**P1**], [**P2**], [**P3**]).

The first type of results concerns with the inductive implication (local) $\Longrightarrow$ (lower dimensional global). Although we prove this result in dimension two only, it can be generalized for the case of arbitrary dimension without difficulties (see [**PSh**]). The next type of results are inductive implications (global) $\Longrightarrow$ (lower dimensional global). Roughly speaking these results assert that if a log pair has singularities worse than Kawamata log terminal, then complements can be induced from lower dimensions. This result is not difficult for varieties of log Calabi-Yau and log Fano type (see [**I1**], [**PSh**, Sect. 5]) and becomes very hard for varieties of intermediate numerical anticanonical dimension. We expect that this result can be generalized in arbitrary dimension modulo log MMP, groups of automorphisms of Calabi-Yau and Kodaira-type formula. Our proof (which is a reworked version of [**Sh3**]) is based on detailed analysis of the reduced part of a (non-klt) boundary.

The last group of discussed problems is the boundedness of exceptional log del Pezzo surfaces (refer to 4.5 for the precise definition). The ideal result (which is expected to be valid in any dimension) is that these surfaces have bounded complements and they belong to a finite number of "families". The proofs uses heavily Alexeev-Nikulin's theorems about boundedness of log del Pezzo surfaces with ϵ-log terminal singularities and inductive results (see above). In recent work [**KeM**] similar techniques were used to study families of rational curves on nonprojective surfaces. It is very important problem here is to classify exceptional log del Pezzos. Some results in this direction are obtained in [**KeM**], [**Sh3**], [**Ab**] but in general the question is still open. We present, following [**Sh3**], the classification of exceptional log Del Pezzos with "large" boundary. This classification is interesting for applications to three-dimensional birational geometry. For example, there is a conjecture that any Mori contraction is nonexceptional (see 4.1.6, cf. [**RY**, 6.4]). Adopting this conjecture one can by using the inductive procedure construct an 1, 2, 3, 4, or 6-complements which leads to the classification of all Mori contractions (see [**Sh3**, Sect. 7], [**P2**], cf. [**KoM**]).

As applications of the developed techniques, we reprove several well known facts such as classifications of quotient singularities and singular fibers of elliptic fibrations. This outlines ways for higher-dimensional generalizations. Note, however, that these expected generalizations cannot be straightforward. For example the classification of all three-dimensional log-canonical (and even canonical) singularities is not a reasonable problem (cf. [**IP**]).

The following diagram exhibit the logical structure of the results (proved in dimension 2 and conjectured in higher dimensions). We omit some technical details and conditions (for example restrictions on the coefficients of the boundaries).

$(X/Z \ni o, D)$: a d-dimensional log pair of local type (i.e. a contraction with $\dim Z > 0$) such that $-(K_X + D)$ is nef and big

EXCEPTIONAL ?

YES	NO
$\Downarrow$	$\Downarrow$
complements are induced from $d-1$-dimensional log Fano pairs (possibly of global type)	complements are induced from $d-2$-dimensional log Fano pairs

(X, D): a d-dimensional log pair of global type (i.e. X is compact) with nef and big $-(K_X + D)$

EXCEPTIONAL ?

YES	NO
$\Downarrow$	$\Downarrow$
complements are bounded by a constant $C(d)$	complements are induced from $d-1$-dimensional log Fano pairs

Our notes are organized as follows. We start with three introductory chapters: Chapter 1 gives precise definitions of singularities of pairs and gives some known constructions. Chapter 2 contains some important tools: Inversion of Adjunction and Connectedness Lemma. In Chapter 3 we collect facts about log terminal modifications (these are consequences of the Minimal Model Program, and therefore restricted to dimension two or three). In Chapter 4 we define complements and present some properties. The following three Chapters deal with applications of complements to surface geometry. The central part of these notes is chapters 8 and 9. Roughly speaking, all log surfaces can be divided into two classes: exceptional and nonexceptional. If the log surface is nonexceptional, then under some additional assumptions there is a regular (i.e., 1, 2, 3, 4 or 6-complement). This is done in Chapter 8. The main result there is the Inductive Theorem 8.3.1. On the contrary, exceptional log surfaces can have no regular complements. However, we show in Theorems 9.1.7 and 9.1.11 that they are bounded (again under additional assumptions). In Chapter 10 we discuss the problem of classification of exceptional complements.

Our presentation of the subject is rather elementary. The notes contain many examples.

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