Part I

Chapter 2

Hypergeometric Integrals

2.1 Local System Cohomology

Definition 2.1.1. *A sheaf $\mathcal{L}$ on M is* **locally constant** *if there is a cover $\{U_i\}$ of M so that each restriction $\mathcal{L}_{U_i}$ is a constant sheaf.*

One of our central objects is a locally constant sheaf $\mathcal{L}_\lambda$ on M which we define next. Recall the holomorphic 1-forms $\omega_H = d\alpha_H/\alpha_H$. Given the complex weight system $\lambda = \{\lambda_H \mid H \in \mathcal{A}\}$, define

$$\omega_\lambda = \sum_{H\in\mathcal{A}} \lambda_H\,\omega_H, \qquad \nabla_\lambda = d + \omega_\lambda \wedge$$

where d is the ordinary exterior differential. Define a presheaf on M as follows. For $U \subset M$, let $\mathcal{L}_\lambda(U) = \{f : U \to \mathbb{C} \mid \nabla_\lambda(f) = 0\}$ where f is a holomorphic function and the restriction maps are ordinary restrictions.

Proposition 2.1.2. *Define a multivalued holomorphic function on M by $\Phi_\lambda = \prod_{H\in\mathcal{A}} \alpha_H^{\lambda_H}$. The presheaf $\mathcal{L}_\lambda$ is a locally constant sheaf whose local sections are isomorphic to constant multiples of Φ_λ^{-1}.*

Proof. Since $\omega_\lambda = d(\log\Phi_\lambda) = d\Phi_\lambda/\Phi_\lambda$, we get

$$\nabla_\lambda(\Phi_\lambda^{-1}) = -\Phi_\lambda^{-2}d\Phi_\lambda + (d\Phi_\lambda/\Phi_\lambda)\Phi_\lambda^{-1} = 0.$$

If $\nabla_\lambda(f) = 0$, then $df = -f\omega_\lambda = -fd\Phi_\lambda/\Phi_\lambda$. Thus $d(f\Phi_\lambda) = df\Phi_\lambda + fd\Phi_\lambda = 0$ so $f \in \mathbb{C}\Phi_\lambda^{-1}$. Cover M with contractible open sets to see that the sheaf $\mathcal{L}_\lambda$ is locally constant. □

We are interested in $\mathcal{L}_\lambda$ because the function Φ_λ occurs as the integrand of multidimensional hypergeometric integrals and hypergeometric functions. Euler's Beta function identity, Introduction (3), is related to the arrangement $\{0, 1\} \subset \mathbb{C}$

and $\lambda = (x-1, y-1)$. Similar considerations apply to the Appell, Dirichlet and Selberg integrals. These groups are called local system cohomology groups in the literature. This name is justified as follows. We use Spanier [Sp] as a general reference for local systems and sheaf cohomology. Local systems are defined in [Sp, Ch.1, Ex.F]. Let $m \in M$ be a base point. Let Γ be a local system on M with $\Gamma(m) = \mathbb{C}$. Since M is path connected, it follows from this exercise that a local system is determined by a homomorphism $\rho : \pi_1(M, m) \longrightarrow Aut(\mathbb{C})$. We call the corresponding local system Γ_ρ. Homology and cohomology groups with coefficients in a local system are defined in [Sp, Ch.5, Ex.I and J]. Local systems and sheaves are discussed in [Sp, Ch.6, Ex.F]. To each local system Γ a locally constant (pre)sheaf $\bar{\Gamma}$ is associated. Since M is locally path connected and semilocally 1-connected, the exercise implies that this association is a bijection between the respective equivalence classes. It follows from the exercise that in our case $H^p(M, \Gamma) \simeq H^p(M, \bar{\Gamma})$ where the left side is local system cohomology and the right is sheaf cohomology. Since a local system is determined (up to its equivalence class) by a representation $\rho : \pi_1(M, m) \longrightarrow Aut(\mathbb{C})$, it is natural to ask what representation is induced by $\mathcal{L}_\lambda$. The group $\pi_1(M, m)$ has a presentation with one generator for each hyperplane [OT1, 5.3]. Let $\gamma_H \in \pi_1(M, m)$ be the standard generator around the hyperplane H. Since the homology classes $[\gamma_H]$ freely generate $H_1(M, \mathbb{Z})$, it suffices to determine $\rho(\gamma_H)$.

Proposition 2.1.3. *The locally constant sheaf $\mathcal{L}_\lambda$ corresponds to a local system Γ_ρ induced by the representation $\rho : \pi_1(M, m) \to Aut(\mathbb{C})$ with $\rho(\gamma_H) = \exp(-2\pi i \lambda_H)$. An equivalent local system arises if we replace λ_H by $\lambda_H + k_H$ for $k_H \in \mathbb{Z}$.*

Proof. We may consider the image of γ_H represented by an embedded circle around H. Cover it by contractible open sets U_i, $1 \le i \le r$ so that for each i the sets $U_{i-1} \cap U_i$ and $U_i \cap U_{i+1}$ are nonempty and contractible and all other intersections are empty. (Here $U_{r+1} = U_1$.) Choose a nonzero holomorphic function $f_i : U_i \to \mathbb{C}$ so that $\mathcal{L}_\lambda(U_i) \simeq \mathbb{C} f_i$. Then there are nonzero constants a_i so that $f_i = a_{i+1} f_{i+1}$ on $U_i \cap U_{i+1}$. (Here $f_r = a_1 f_1$.) It follows that $\rho(\gamma_j) = a_1 a_2 \cdots a_r$.

Now let g_1 be a branch of Φ_λ^{-1} on U_1. Let g_2 be the analytic continuation so that g_1 and g_2 agree on $U_1 \cap U_2$. It follows from Proposition 2.1.2 that if we continue in this fashion, we get a branch g_r on U_r with the property that on $U_1 \cap U_r$ we have $g_r = \exp(-2\pi i \lambda_j) g_1$. Write $g_i = b_i f_i = b_i a_{i+1} f_{i+1}$. Since $g_i = g_{i+1}$ for $1 \le i < r$, we get $b_{i+1} = a_{i+1} b_i$ in this range. Finally, $g_r = b_r f_r = b_r a_1 f_1 = \exp(-2\pi i \lambda_H) g_1 = \exp(-2\pi i \lambda_H) b_1 f_1$. Thus $\rho(\gamma_H) = a_1 a_2 \cdots a_r = \exp(-2\pi i \lambda_H)$. $\square$

In particular, $H^p(M, \mathcal{L}_\lambda) \simeq H^p(M, \Gamma_\rho)$. This justifies reference to $H^p(M, \mathcal{L}_\lambda)$ as local system cohomology groups. Until further notice we fix the arrangement $\mathcal{A}$ and the complex weights λ and no longer indicate dependence on them. For example, we may write $\Phi = \Phi_\lambda$, $\nabla = \nabla_\lambda$, $\mathcal{L} = \mathcal{L}_\lambda$.

2.2 Homology with Local Coefficients

The local system $\mathcal{L}^\vee$ on M dual to $\mathcal{L}$ is the sheaf whose sections over $U \subset M$ form the abelian group $\mathcal{L}^\vee(U) = \mathbb{C}\Phi_U$.

Definition 2.2.1. *The p-th chain group $C_p(M, \mathcal{L}^\vee)$ with coefficients in $\mathcal{L}^\vee$ is the complex vector space with basis $\sigma \otimes \Phi_\sigma$ where σ is a singular p-simplex of M and Φ_σ is a branch of Φ on the image of σ. Let σ^j denote the j-th face of σ and let Φ_{σ^j} denote the restriction. Define $\partial : C_p(M, \mathcal{L}^\vee) \to C_{p-1}(M, \mathcal{L}^\vee)$ by*

$$\partial(\sigma \otimes \Phi_\sigma) = \sum_{j=0}^{p} (-1)^j \sigma^j \otimes \Phi_{\sigma^j}.$$

The homology groups of this complex are denoted $H_p(M, \mathcal{L}^\vee)$.

Example 2.2.2. *In $V = \mathbb{C}$ let $\mathcal{A} = \{0, 1\}$ with $\alpha_0 = u$ and $\alpha_1 = u - 1$. Fix $\lambda_0, \lambda_1 \in \mathbb{C} - \mathbb{Z}$.*

Let $\epsilon << 1$ and let Δ be a small simple closed curve with center 0 oriented counterclockwise, see Figure 2.1. Write $c_j = \exp(2\pi i \lambda_j)$ for $j = 0, 1$. Fix a branch of Φ on $(0, 1)$. Then

$$\partial(\Delta \otimes \Phi) = [\epsilon] \otimes c_0 \Phi_\epsilon - [\epsilon] \otimes \Phi_\epsilon = ([\epsilon] \otimes \Phi_\epsilon)(c_0 - 1).$$

Let Δ^* be a small simple closed curve with center 1 oriented counterclockwise. A similar calculation gives

$$\partial(\Delta^* \otimes \Phi) = ([1 - \epsilon] \otimes \Phi_{1-\epsilon})(c_1 - 1).$$

Since $\partial([\epsilon, 1 - \epsilon] \otimes \Phi) = [1 - \epsilon] \otimes \Phi_{1-\epsilon} - [\epsilon] \otimes \Phi_\epsilon$, we conclude that

$$(c_0 - 1)^{-1}(\Delta \otimes \Phi) + [\epsilon, 1 - \epsilon] \otimes \Phi - (c_1 - 1)^{-1}(\Delta^* \otimes \Phi)$$

is a twisted cycle. It is called a **regularization** of the open interval $(0, 1)$.

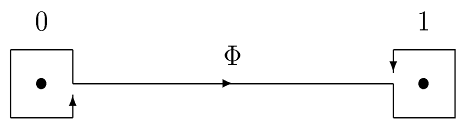

Figure 2.1: A Twisted Cycle

The groups $H^p(M, \mathcal{L})$ and $H_p(M, \mathcal{L}^\vee)$ are algebraic duals so they provide a perfect pairing

$$(1) \qquad \langle\, , \,\rangle : H^p(M, \mathcal{L}) \times H_p(M, \mathcal{L}^\vee) \to \mathbb{C}$$

induced from the evaluation of an $\mathcal{L}$-valued cochain $f = \sum_\sigma \ell_\sigma f_\sigma$ on an $\mathcal{L}^\vee$-valued (finite) chain $c = \sum_\tau m_\tau \tau$. Here $\ell_\sigma = a_\sigma \Phi_\sigma^{-1}$ with $a_\sigma \in \mathbb{C}$, f_σ is a characteristic function, and $m_\tau = b_\tau \Phi_\tau$ with $b_\tau \in \mathbb{C}$. Thus $\langle f, c\rangle = \sum_\tau a_\tau b_\tau$ is a finite sum. Let $C_p^{lf}(M, \mathcal{L}^\vee)$ denote the p-th locally finite chain group with coefficients in $\mathcal{L}^\vee$ and let $H_p^{lf}(M, \mathcal{L}^\vee)$ denote the corresponding locally finite homology group. Let $H_c^p(M, \mathcal{L})$ denote compactly supported cohomology. Since M is an open 2ℓ-manifold, Poincaré duality gives isomorphisms

$$H^p(M, \mathcal{L}) \simeq H_{2\ell-p}^{lf}(M, \mathcal{L}), \quad H_p(M, \mathcal{L}) \simeq H_c^{2\ell-p}(M, \mathcal{L}). \tag{2}$$

This provides another perfect pairing

$$\langle\, , \rangle : H_c^p(M, \mathcal{L}) \times H_p^{lf}(M, \mathcal{L}^\vee) \to \mathbb{C}. \tag{3}$$

These dualities and the fact that M is a Stein manifold of the homotopy type of a finite cell complex of dimension $\leq \ell$ yield the following:

Proposition 2.2.3. *(1) $H^p(M, \mathcal{L}) = H_{2\ell-p}^{lf}(M, \mathcal{L}) = 0$ for $p > \ell$,*
(2) $H_p(M, \mathcal{L}) = H_c^{2\ell-p}(M, \mathcal{L}) = 0$ for $p > \ell$,

Our goal is to interpret hypergeometric integrals as values of either pairing. Clearly, we must give M a locally finite smooth triangulation and represent each twisted cycle by a locally smooth cycle (finite or locally finite). If our model is the hypergeometric integral of Gauss, formula (3) in the Introduction, then *neither* of these two pairings will suffice. In that integral we integrate over the open interval $(0, 1)$, which is a cycle in locally finite homology, while the integrand is a globally defined holomorphic form which has noncompact support. We will show in Theorem 7.1.1 that under suitable conditions locally finite cycles are represented by finite cycles. Representation of local system cohomology classes by global holomorphic differential forms requires the holomorphic de Rham theorem of the next section.

2.3 Hypergeometric Pairing

Let $\mathcal{O} = \mathcal{O}_M$ denote the sheaf of germs of holomorphic functions on M and let $\Omega^\cdot = \Omega^\cdot_M$ be the de Rham complex of germs of holomorphic differentials on M, where $\Omega^0 = \mathcal{O}$.

Theorem 2.3.1. *The operator $\nabla : \Omega^0 \to \Omega^1$ is a flat connection whose kernel is $\mathcal{L}$. The sequence*

$$0 \to \mathcal{L} \to \Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Omega^\ell \to 0$$

is exact. We call $(\Omega^\cdot, \nabla)$ the twisted de Rham complex.

Proof. We defined $\mathcal{L}$ as the kernel of ∇. Since $d\omega_\lambda = 0 = \omega_\lambda \wedge \omega_\lambda$, we have

$$\begin{aligned}
\nabla\nabla(\eta) &= \nabla(d\eta + \omega_\lambda \wedge \eta) \\
&= d(d\eta + \omega_\lambda \wedge \eta) + \omega_\lambda \wedge (d\eta + \omega_\lambda \wedge \eta) \\
&= dd\eta + d\omega_\lambda \wedge \eta - \omega_\lambda \wedge d\eta + \omega_\lambda \wedge d\eta + \omega_\lambda \wedge \omega_\lambda \wedge \eta \\
&= 0
\end{aligned}$$

Thus the connection is flat. To show exactness, consider the diagram of stalks at $x \in M$

$$\begin{tikzcd}
\Omega_x^{p-1} \arrow[r, "\nabla"] \arrow[d, "\Phi"'] & \Omega_x^p \arrow[d, "\Phi"] \\
\Omega_x^{p-1} \arrow[r, "d"'] & \Omega_x^p
\end{tikzcd}$$

where the bottom row is the ordinary de Rham complex. The diagram commutes because

$$d(\Phi\eta) = d\Phi \wedge \eta + \Phi d\eta = \Phi\left(\frac{d\Phi}{\Phi} \wedge \eta + d\eta\right) = \Phi(\nabla\eta).$$

Exactness of the twisted complex follows from the Poincaré Lemma for the ordinary de Rham complex. □

Since M is a Stein manifold, Cartan's Theorem B implies that $H^n(M, \Omega^p) = 0$ for $n > 0$ and all p. Thus the exact sequence of Theorem 2.3.1 is an acyclic resolution of $\mathcal{L}$. We obtain the **holomorphic de Rham theorem**

$$H^p(M, \mathcal{L}) \simeq H^p(\Gamma(M, \Omega^\cdot), \nabla)$$

where Γ denotes global sections. In order to define the hypergeometric pairing, we need a twisted version of Stokes theorem.

Proposition 2.3.2. *Let $\eta \in \Gamma(M, \Omega^p)$ and $\sigma \otimes \Phi_\sigma \in C_p(M, \mathcal{L}^\vee)$. Define $\langle \eta, \sigma \otimes \Phi_\sigma \rangle = \int_\sigma \Phi_\sigma \eta$ and extend this to a bilinear pairing. Then*

$$\langle \eta, \partial c \rangle = \langle \nabla\eta, c \rangle.$$

Proof. We may assume that $c = \sigma \otimes \Phi_\sigma$. Then

$$\begin{aligned}
\langle \eta, \partial c \rangle &= \langle \eta, \partial(\sigma \otimes \Phi_\sigma) \rangle = \langle \eta, \sum_{j=0}^{p} (-1)^j \sigma^j \otimes \Phi_{\sigma^j} \rangle \\
&= \sum_{j=0}^{p} (-1)^j \langle \eta, \sigma^j \otimes \Phi_{\sigma^j} \rangle = \sum_{j=0}^{p} (-1)^j \int_{\sigma^j} \Phi_{\sigma^j} \eta = \int_{\partial\sigma} \Phi_\sigma \eta \\
&= \int_\sigma d(\Phi_\sigma \eta) = \int_\sigma \Phi_\sigma(\nabla\eta) = \langle \nabla\eta, \sigma \otimes \Phi_\sigma \rangle = \langle \nabla\eta, c \rangle.
\end{aligned}$$

Equality at the second line break follows from the usual Stokes theorem. □

Definition 2.3.3. *The* **hypergeometric pairing**

$$\langle\ ,\ \rangle : H^p(M,\mathcal{L}) \times H_p(M,\mathcal{L}^\vee) \to \mathbb{C}$$

is defined as follows. Let $[\eta] \in H^p(M,\mathcal{L})$ *be represented by a global holomorphic form* η *and let* $[c] \in H_p(M,\mathcal{L}^\vee)$ *be represented by a locally smooth* $\mathcal{L}^\vee$*-valued finite chain* $c = \sum_\tau m_\tau \tau$ *where* $m_\tau = b_\tau \Phi_\tau$ *with* $b_\tau \in \mathbb{C}$*. Then*

$$\langle [\eta], [c] \rangle = \sum_\tau b_\tau \int_\tau \Phi_\tau \eta.$$

Let $\Omega^p(*\mathcal{A})$ denote the group of globally defined rational p-forms on V with poles on N. These forms are holomorphic on M so $\Omega^p(*\mathcal{A}) \to \Gamma(M, \Omega^\cdot)$ is an inclusion. Note that $(\Omega^\cdot(*\mathcal{A}), \nabla)$ is a complex because $\omega_\lambda \in \Omega^1(*\mathcal{A})$. It follows from the algebraic de Rham theorem of Deligne and Grothendieck that the inclusion is a quasiisomorphism of complexes and hence

$$H^p(M,\mathcal{L}) \simeq H^p(\Omega^\cdot(*\mathcal{A}), \nabla).$$

This reduces the original analytic problem to the algebraic problem of computing cohomology of rational forms on V, but it is still very difficult. Deligne's work [D1] may be used to reduce the problem to computing in a complex of forms with logarithmic poles, but in order to apply the results of [D1] we must compactify M with a normal crossing divisor. Thus we must resolve the singularities of N.