Chapter 1

Introduction

1.1 History

It is generally accepted that John Wallis first used the term 'hypergeometric' in his book "Arithmetica Infinitorum" (1655) to denote any series beyond the ordinary geometric series. In modern use the term applies to any series $\sum a_n x^n$ such that a_{n+1}/a_n is a rational function of n. The series which has become known as the ordinary **hypergeometric series** or the Gauss series is

$$(1) \quad 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots$$

Gauss called the series $F[a, b; c; x]$ in 1812. It is convenient to introduce Appell's notation

$$(a, n) = a(a+1)(a+2)\cdots(a+n-1).$$

Gauss already considered F as a function in four variables which may be real or complex and considered the problem of convergence of the series. If a or b is zero or a negative integer, the series reduces to a polynomial. If c is zero or a negative integer, the function is not defined. The ratio of two succesive terms of (1) is

$$\frac{(a+n)(b+n)}{(c+n)(1+n)}x = \frac{(1+a/n)(1+b/n)}{(1+c/n)(1+1/n)}x,$$

so that as $n \to \infty$, the absolute value of the ratio tends to $|x|$. Thus the series converges for $|x| < 1$ and diverges for $|x| > 1$. For $|x| = 1$, more delicate tests are required and convergence depends on a, b, c. The sum of the series inside its circle of convergence is called the **hypergeometric function**, and the same name is used for its analytic continuation outside the circle of convergence.

Barnes constructed more general hypergeometric series with p numerator parameters $(a) = (a_1, \ldots, a_p)$ and q denominator parameters $(c) = (c_1, \ldots, c_q)$:

$$ {}_pF_q[(a);(c);x] = \sum_{n=0}^{\infty} \frac{(a_1,n)\cdots(a_p,n)}{(c_1,n)\cdots(c_q,n)} \frac{x^n}{n!}. $$

In this terminology the original Gauss series is ${}_2F_1$. Many important special functions are represented by the series ${}_pF_q$. These include the trigonometric functions, the Legendre functions, the Bessell functions, and the Airy functions. For example:

$$
\begin{aligned}
e^x &= \sum \frac{x^n}{n!} = {}_0F_0\,[x] \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = {}_0F_1\,[1/2; -x^2/4] \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = x\; {}_0F_1\,[3/2; -x^2/4] \\
(1-x)^{-a} &= \sum (a,n)\frac{x^n}{n!} = {}_1F_0\,[a; x] \\
J_0(x) &= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} - \cdots = {}_0F_1\,[1; -x^2/4] \\
\int_0^x e^{t^2}\,dt &= \sum \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)} = x\, {}_1F_1\,[1/2; 3/2; -x^2] \\
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = x\, {}_2F_1\,[1,1;2;-x] \\
\arcsin(x) &= x + \frac{x^3}{2\cdot 3} + \frac{3x^5}{2\cdot 4\cdot 5} + \frac{3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots = x\, {}_2F_1\,[1/2,1/2;3/2;x^2].
\end{aligned}
$$

These and related special functions are important in probability theory, heat conduction, vibration, electromagnetic theory, fluid dynamics, boundary value problems of potential theory, and quantum mechanics. Since the generalized series are of peripheral importance to us, we shall use Gauss' original notation.

Gauss knew that the differential equation

$$ (2) \qquad x(1-x)\frac{d^2y}{dx^2} + [c - (1+a+b)x]\frac{dy}{dx} - aby = 0 $$

is satisfied by $F[a,b;c;x]$. This can be verified by using

$$ \frac{d}{dx}F[a,b;c;x] = \frac{ab}{c}F[a+1,b+1;c+1;x]. $$

If we assume the existence of solutions of (2) of the form $y = x^g \sum_{n=0}^{\infty} u_n x^n$, then direct calculation gives

$$
\begin{aligned}
g(g+c-1) &= 0, \\
(g+n+c)(g+n+1)u_{n+1} &= (g+n+a)(g+n+b)u_n.
\end{aligned}
$$

The root $g = 0$ leads to the solution $y_1 = F[a, b; c; x]$ provided c is not zero or a negative integer. The root $g = 1-c$ gives a second solution $y_2 = x^{1-c}F[1+a-c, 1+b-c; 2-c; x]$ provided c is not a positive integer ≥ 2. Hence one complete solution of the hypergeometric differential equation (2) is $y = Ay_1 + By_2$ for $|x| < 1$ and $c \notin \mathbb{Z}$. In 1836 Kummer listed in all twenty-four solutions which arise in different regions of the plane and for different values of the parameters. Since there are only two linearly independent solutions at any point, the validity of more solutions leads to interesting functional identities. One is Euler's identity

$$F[a, b; c; x] = (1-x)^{c-a-b}F[c-a, c-b; c; x].$$

An integral representation of $F[a, b; c; x]$ goes back to Euler in 1748. Recall the definition of the beta and gamma functions

$$B(x, y) = \int_0^1 u^{x-1}(1-u)^{y-1}du, \qquad \Gamma(x) = \int_0^\infty e^{-u}u^{x-1}du.$$

Let $|x| < 1$ and set

$$I = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du.$$

This integral exists and is convergent if the real part $\Re(a) > 0$ and $\Re(c-a) > 0$. Now $(1-xu)^{-b} = \sum_{n=0}^\infty (b, n)u^n x^n/n!$. Hence

$$\begin{aligned} I &= \sum_{n=0}^\infty \frac{(b,n)}{n!} x^n \int_0^1 u^{a+n-1}(1-u)^{c-a-1}du \\ &= \sum_{n=0}^\infty \frac{(b,n)}{n!} x^n \frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)} \\ &= \frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)} F[a, b; c; x]. \end{aligned}$$

Here we used the fact that $(a, n) = \Gamma(a+n)/\Gamma(a)$ and Euler's beta function identity:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3}$$

Thus we obtain the integral representation

$$\frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)} F[a, b; c; x] = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du \tag{4}$$

provided that $|x| < 1$, $\Re(a) > 0$, and $\Re(c-a) > 0$.

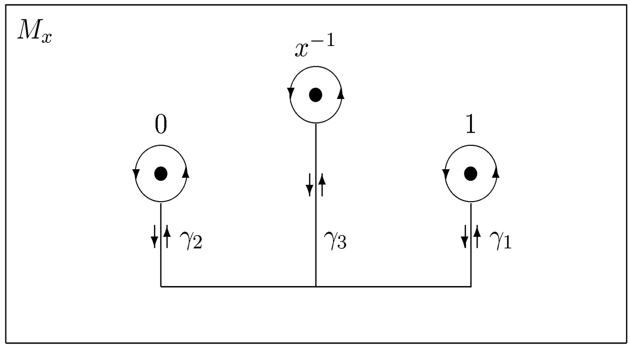

Figure 1.1: Generators

1.2 The Classical Example

Since all ingredients of the generalized theory are present in the classical example above, we describe it first in modern terms that allow generalization. For fixed x, not 0,1, there are three distinct points 1, 0, x^{-1}, in the complex line. This is an example of an arrangement, defined below. We view this as the *static* setup because x is fixed. The three points are the zeros of the linear functions $1-u$, u, and $1-xu$. We have a set of parameters λ and a multivalued holomorphic function defined in the complement of the points. Under suitable conditions on the parameters λ, there exist certain paths in the complement of the points so that the corresponding line integrals are defined and a basis for the space of integrals is obtained by choosing a basic set of paths. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be complex parameters. Let $N_x = \{1, 0, x^{-1}\}$ and let $M_x = \mathbb{C} - N_x$. Then

$$\Phi(u; \lambda; x) = (1-u)^{\lambda_1} u^{\lambda_2} (1-xu)^{\lambda_3} \tag{5}$$

defines a multivalued holomorphic function on M_x. In order to write down suitable integrals, we must introduce twisted versions of homology and cohomology. The twisting comes from the change in the value of Φ as we prolong it by analytic continuation while moving around a point of N_x. Choose a base point $p \in M_x$ and simple loops around the points of N_x representing standard generators of $\pi_1(M_x, p)$ as in Figure 1.1. A rank one local system $\mathcal{L}$ over M_x is given by the representation $\rho : \pi_1(M_x, p) \to \mathrm{Aut}(\mathbb{C})$, where $\rho(\gamma_j) = \exp(-2\pi i \lambda_j)$ for $j = 1, 2, 3$.

The holomorphic 1-form

$$\omega_\lambda = d(\log \Phi) = \frac{d\Phi}{\Phi} = -\lambda_1 \frac{du}{1-u} + \lambda_2 \frac{du}{u} - \lambda_3 \frac{xdu}{1-xu}$$

is single valued. Define $\nabla : \mathcal{O}_{M_x} \longrightarrow \Omega^1_{M_x}$ by $\nabla(f) = df + f\omega_\lambda$ Note that $\nabla(\Phi^{-1}) = -\Phi^{-2}d\Phi + \Phi^{-1}(d\Phi/\Phi) = 0$. If $0 = \nabla(f) = df + f(d\Phi/\Phi)$, then $df/f = -d\Phi/\Phi$, so f is locally a constant multiple of Φ^{-1}. This identifies $\ker \nabla$ and $\mathcal{L}$. Let $\mathcal{L}^\vee$

denote the dual local system. The theory shows that integration is a nondegenerate pairing. In our case this pairing

$$H^1(M_x, \mathcal{L}) \times H_1(M_x, \mathcal{L}^\vee) \longrightarrow \mathbb{C} \tag{6}$$

is defined by $\int_\gamma \Phi\eta$ where $\gamma \in H_1(M_x, \mathcal{L}^\vee)$ and $\eta \in H^1(M_x, \mathcal{L})$. Identifying a basis of twisted cycles is easy in our example. Moreover, for suitably general λ,

$$H^1(M_x, \mathcal{L}) \simeq (\mathbb{C}\frac{d(1-u)}{1-u} + \mathbb{C}\frac{du}{u} + \mathbb{C}\frac{d(1-xu)}{1-xu})/\nabla(1).$$

For generic values of λ we may choose the basis $\{\varphi_1 = \lambda_2 du/u,\ \varphi_2 = \lambda_3 x du/(xu - 1)\}$ for $H^1(M_x, \mathcal{L})$.

If we allow x to vary in $\mathbb{C} - \{0,1\}$, then we obtain a family of arrangements of three points in the plane parametrized by x. We view this as the *dynamic* setup because x is allowed to vary. In this setup, we may derive the Gauss hypergeometric differential equation (2) as follows. Let $\gamma \in H_1(M_x, \mathcal{L}^\vee)$ and for $i = 1, 2$ define $\hat{\varphi}_i = \int_\gamma \Phi\varphi_i$. Working in $H^1(M_x, \mathcal{L})$, a direct calculation in Chapter 8 shows that $\hat{\varphi}_1$ and $\hat{\varphi}_2$ satisfy the following system of first-order differential equations:

$$\begin{aligned}\frac{d}{dx}\begin{pmatrix}\hat{\varphi}_1\\ \hat{\varphi}_2\end{pmatrix} dx = \begin{pmatrix}0 & 0\\ \lambda_3 & \lambda_1+\lambda_3\end{pmatrix}\begin{pmatrix}\hat{\varphi}_1\\ \hat{\varphi}_2\end{pmatrix}\frac{dx}{x-1}\\ + \begin{pmatrix}-\lambda_3 & \lambda_2\\ 0 & -\lambda_1-\lambda_2-\lambda_3\end{pmatrix}\begin{pmatrix}\hat{\varphi}_1\\ \hat{\varphi}_2\end{pmatrix}\frac{dx}{x}.\end{aligned} \tag{7}$$

Define a flat connection ∇_Ω on $\mathcal{O}^2_{\mathbb{C}\setminus\{1,0\}}$ by

$$\nabla_\Omega\begin{pmatrix}f_1\\ f_2\end{pmatrix} = \frac{d}{dx}\begin{pmatrix}f_1\\ f_2\end{pmatrix}dx - \Omega\begin{pmatrix}f_1\\ f_2\end{pmatrix}$$

where

$$\Omega = \begin{pmatrix}0 & 0\\ \lambda_3 & \lambda_1+\lambda_3\end{pmatrix}\frac{dx}{x-1} + \begin{pmatrix}0 & \lambda_2\\ 0 & -\lambda_1-\lambda_2\end{pmatrix}\frac{dx}{x}.$$

Define $F_i = x^{\lambda_3}\hat{\varphi}_i$ for $i = 1, 2$. Equation (7) shows that these functions provide a flat section of the connection ∇_Ω:

$$\nabla_\Omega\begin{pmatrix}F_1\\ F_2\end{pmatrix} = 0.$$

An easy calculation shows that F_1 satisfies the second-order differential equation

$$x(1-x)\frac{d^2y}{dx^2} + [(\lambda_1+\lambda_2+1) - (\lambda_2-\lambda_3+1)x]\frac{dy}{dx} + \lambda_2\lambda_3 y = 0.$$

Setting

$$\lambda_1 = c-a-1,\ \lambda_2 = a,\ \lambda_3 = -b,$$

we recover the Gauss hypergeometric differential equation (2).

1.3 Outline

We generalize these ideas to several variables. The notation introduced here will be used throughout these notes. Let V be a complex affine space of dimension ℓ and let $\mathcal{A}$ be an arrangement of affine hyperplanes in V. Let $N = N(\mathcal{A}) = \cup_{H\in\mathcal{A}} H$ be the divisor of $\mathcal{A}$ and let $M = M(\mathcal{A}) = V - N(\mathcal{A})$ be the complement of $\mathcal{A}$. Choose coordinates $u = \{u_1, \ldots, u_\ell\}$ in V and for each $H \in \mathcal{A}$ a degree one polynomial α_H with $H = \ker \alpha_H$. Let $\lambda_H \in \mathbb{C}$ be complex weights. Define

$$\Phi(u;\lambda) = \prod_{H\in\mathcal{A}} \alpha_H^{\lambda_H}.$$

A generalized hypergeometric integral is of the form

$$\int_\sigma \Phi\eta \tag{8}$$

where σ is a suitable domain of integration and η is a holomorphic form on M. Appell defined four series in 1880 which generalize (1) to two variables, see [E, p. 222]. They have integral representations. Corresponding to $F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$ is the integral

$$\int\int u^{\beta-1} v^{\beta'-1} (1-u-v)^{-\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'}\, du\, dv$$

where the domain of integration is $u \geq 0$, $v \geq 0$, $u+v \leq 1$ and we assume $\Re\beta > 0$, $\Re\beta' > 0$, $\Re(\gamma - \beta - \beta') > 0$. The corresponding arrangement consists of the lines $u = 0$, $v = 0$, $u+v = 1$, $u = 1/x$, $v = 1/y$.
There are several generalizations of the beta function identity (3). Here are two. Dirichlet's integral is

$$\int\cdots\int u_1^{x_1-1} u_2^{x_2-1} \cdots u_\ell^{x_\ell-1} (1-u_1-\ldots-u_\ell)^{x_{\ell+1}-1}\, du_1 \ldots du_\ell$$

where the domain of integration is $u_i > 0$, $u_1 + \cdots + u_\ell < 1$ and we assume that $\Re x_i > 0$. The corresponding arrangement consists of the coordinate hyperplanes and $u_1 + \cdots + u_\ell = 1$. Selberg's integral [Se] is

$$\int_0^1 \cdots \int_0^1 (u_1 \cdots u_\ell)^{x-1} [(1-u_1)\cdots(1-u_\ell)]^{y-1} |\Delta(u)|^{2z}\, du_1 \ldots du_\ell$$

where $\Delta(u) = \prod_{i<j}(u_j - u_i)$, $\Re x > 0$, $\Re y > 0$, $\Re z > -\min\{1/\ell, \Re x/(\ell-1), \Re y/(\ell-1)\}$. The corresponding arrangement consists of the coordinate hyperplanes, their parallels $u_i = 1$, and the diagonals $u_i = u_j$ for $i < j$. We call this the Selberg arrangement and use it to illustrate constructions.

In Part I we work in the static setup. The constructions involve a fixed arrangement in analogy with fixed x in the classical example. The purpose of Chapter 2 is to interpret hypergeometric integrals as the result of the hypergeometric pairing. This already appears in the book by Aomoto and Kita [AK]. Singular homology and cohomology with real coefficients are algebraic duals. This provides a perfect pairing between them. Ordinary integrals on manifolds are interpreted by de Rham theory using this pairing. Smooth triangulation represents homology classes and de Rham's theorem represents cohomology classes as globally defined differential forms. The hypergeometric pairing has an analogous interpretation. A rank one local system $\mathcal{L}$ on M is defined with monodromy $\exp(-2\pi i\lambda_H)$ around the hyperplane H. It defines the cohomology groups $H^p(M,\mathcal{L})$. The dual local system $\mathcal{L}^\vee$ has monodromy $\exp(2\pi i\lambda_H)$ around the hyperplane H. It defines homology groups $H_p(M,\mathcal{L}^\vee)$. These groups are algebraic duals. This provides a perfect pairing between them:

$$H^p(M,\mathcal{L}) \times H_p(M,\mathcal{L}^\vee) \longrightarrow \mathbb{C}. \tag{9}$$

Smooth triangulation of M represents twisted homology classes. In order to interpret cohomology, let $\mathcal{O} = \mathcal{O}_M$ denote the sheaf of germs of holomorphic functions on M and let $\Omega^\cdot = \Omega^\cdot_M$ be the de Rham complex of germs of holomorphic differentials on M, where $\Omega^0 = \mathcal{O}$. Let $\omega_H = d\alpha_H/\alpha_H$ and

$$\omega_\lambda = d(\log\Phi) = \sum_{H\in\mathcal{A}} \lambda_H\omega_H, \qquad \nabla = d + \omega_\lambda\wedge$$

as in the one-variable case. We see that $\nabla : \Omega^0 \to \Omega^1$ is a flat connection whose kernel is $\mathcal{L}$. Extend ∇ to a derivation of degree one. The sequence

$$0 \to \mathcal{L} \to \Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Omega^\ell \to 0$$

is exact. Cartan's Theorem B implies that $H^n(M,\Omega^p) = 0$ for $n > 0$ and all p since M is a Stein manifold. Thus the exact sequence above is an acyclic resolution of $\mathcal{L}$. We obtain the holomorphic de Rham theorem

$$H^p(M,\mathcal{L}) \simeq H^p(\Gamma(M,\Omega^\cdot),\nabla)$$

where Γ denotes global sections. A twisted version of Stokes theorem shows that the bilinear pairing (9) is given by the integral (8).

The next four chapters are devoted to the explicit calculation of the groups $H^p(M,\mathcal{L})$. If all $\lambda_H \in \mathbb{Z}$, then the local system is trivial. Calculation of the cohomology groups $H^*(M,\mathbb{C})$ was initiated by Arnold [Ar], who found a beautiful formula for the Poincaré polynomial of the of the cohomology of the complement of the braid arrangement $\{u_i = u_j\}, i \neq j$. Arnold conjectured and Brieskorn [Bri] showed that the algebra $\mathsf{B}(\mathcal{A})$ generated by 1 and the holomorphic 1-forms ω_H, $H \in \mathcal{A}$ is isomorphic to $H^*(M,\mathbb{C})$. When the local system is nontrivial, the groups $H^p(M,\mathcal{L})$ are not known in general. Let $\Omega^p(*\mathcal{A})$ denote the group of globally defined rational p-forms on V with poles on N. These forms are holomorphic on

M so $\Omega^p(*\mathcal{A}) \to \Gamma(M, \Omega^{\cdot})$ is an inclusion. Note that $(\Omega^{\cdot}(*\mathcal{A}), \nabla)$ is a complex because $\omega_\lambda \in \Omega^1(*\mathcal{A})$. It follows from the algebraic de Rham theorem of Deligne and Grothendieck that the inclusion is a quasiisomorphism of complexes and hence

$$H^p(M, \mathcal{L}) \simeq H^p(\Omega^{\cdot}(*\mathcal{A}), \nabla).$$

This reduces the original analytic problem to the algebraic problem of computing cohomology of rational forms on V with poles on N, but it is still very difficult. Deligne's work [D1] may be used to reduce the problem from poles of arbitrary order on N to computing in a complex of forms with poles of order one. In order to apply the results of [D1] we must compactify M with a normal crossing divisor.

In Chapter 3 we embed $V \subset \mathbb{CP}^\ell$ by adding the infinite hyperlane, H_∞. Define the projective closure of $\mathcal{A}$, as $\mathcal{A}_\infty = \mathcal{A} \cup H_\infty$. The divisor $N(\mathcal{A}_\infty) = N \cup H_\infty$ may have non-normal crossings. To get a normal crossing divisor we need to determine where these non-normal crossings occur and blow up their singularities. We review basic notions of arrangements from [OT1]. A nonempty intersection of hyperplanes is an edge. An edge is called dense if the subarrangement of hyperplanes containing it is irreducible: the hyperplanes cannot be partitioned into nonempty sets so that after a change of coordinates hyperplanes in different sets are in different coordinates. The divisor $N(\mathcal{A}_\infty)$ does not have normal crossings along a dense edge. In higher dimensions it is difficult to use the definition directly to decide which edges are dense. We find a combinatorial condition. Let $L(\mathcal{A})$ denote the set of edges. Define a partial order on $L(\mathcal{A})$ by reverse inclusion $X \leq Y \iff Y \subseteq X$. Thus V is the unique minimal element of L. Let $\mu : L \to \mathbb{Z}$ be the Möbius function of L defined by $\mu(V) = 1$, and for $X \neq V$ by the recursion $\sum_{Z \leq X} \mu(Z) = 0$. The characteristic polynomial of $\mathcal{A}$ is defined as $\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(X) t^{\dim X}$. The rank of $\mathcal{A}$, $r = r(\mathcal{A})$, is the maximal number of linearly independent hyperplanes in $\mathcal{A}$. Call $\mathcal{A}$ essential if $r(\mathcal{A}) = \ell$. The β-invariant of $\mathcal{A}$ is $\beta(\mathcal{A}) = (-1)^r \chi(\mathcal{A}, 1)$. The product $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ is a defining polynomial for $\mathcal{A}$. It is unique up to a constant. An arrangement is called central if the intersection of all its hyperplanes is nonempty. This intersection is the center. Given a central $(\ell+1)$-arrangement $\mathcal{C}$, we obtain a projective ℓ-arrangement $\mathbb{P}\mathcal{C}$ by viewing the defining homogeneous polynomial $Q(\mathcal{C})$ as a polynomial in projective coordinates. Given a central $(\ell+1)$-arrangement $\mathcal{C}$ and a hyperplane $H \in \mathcal{C}$, we define an affine ℓ-arrangement $\mathbf{d}_H\mathcal{C}$, called the decone of $\mathcal{C}$ with respect to H. We construct the projective quotient $\mathbb{P}\mathcal{C}$ and choose coordinates so that $\mathbb{P}H = \ker u_0$ is the hyperplane at infinity. By removing it, we obtain the affine arrangement $\mathbf{d}_H\mathcal{C} = \mathbb{P}\mathcal{C} - \mathbb{P}H$. It is easy to see that $\beta(\mathbf{d}_H\mathcal{C})$ is independent of H, so we may omit H in the notation. Given $X \in \mathcal{A}$, define $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}$. It is a central arrangement with center X.

Theorem 1.3.1. *The edge $X \in L(\mathcal{A}_\infty)$ is dense if and only if $\beta(\mathbf{d}(\mathcal{A}_\infty)_X) > 0$.*

In Chapter 4 we show that there is a "minimal" resolution $\tau : \bar{X} \to \mathbb{CP}^\ell$ so that the proper transform $Y = \tau^{-1}(N(\mathcal{A}_\infty))$ has normal crossings. This resolution is obtained by blowing up successively with centers the proper transforms of dense edges of dimensions $0, 1, \dots, \ell-2$. We apply Deligne's results [D1] to this resolution and find that under certain conditions on λ, the local system cohomology groups may be computed in terms of a complex of forms that are holomorphic on M and have logarithmic poles on Y. Let $\lambda_\infty = -\sum_{H \in \mathcal{A}} \lambda_H$ be the weight of H_∞. For $X \in L(\mathcal{A}_\infty)$, define $\lambda_X \in \mathbb{C}$ by

$$\lambda_X = \sum_{X \subset H} \lambda_H, \quad H \in \mathcal{A}_\infty.$$

Since $\omega_\lambda \wedge \omega_\lambda = 0$, wedge product with ω_λ provides a finite dimensional subcomplex $(\mathsf{B}^{\cdot}, \omega_\lambda \wedge)$ of $(\Omega^{\cdot}(*\mathcal{A}), \nabla)$:

$$0 \to \mathsf{B}^0 \xrightarrow{\omega_\lambda \wedge} \mathsf{B}^1 \xrightarrow{\omega_\lambda \wedge} \dots \xrightarrow{\omega_\lambda \wedge} \mathsf{B}^\ell \to 0.$$

The next result was proved by Esnault, Schechtman, and Viehweg [ESV] and improved by Schechtman, Terao, and Varchenko [STV]:

Theorem 1.3.2. *Assume that $\lambda_X \notin \mathbb{Z}_{>0}$ for every dense edge $X \in L(\mathcal{A}_\infty)$. Then*

$$H^p(M, \mathcal{L}) \simeq H^p(\mathsf{B}^{\cdot}(\mathcal{A}), \omega_\lambda \wedge). \tag{10}$$

The conditions on λ appear when Deligne's theorem is applied to the normal crossing divisor $Y = \tau^{-1}(N(\mathcal{A}_\infty))$.

In Chapter 5 we introduce combinatorial tools to study the groups $H^p(\mathsf{B}^{\cdot}, \omega_\lambda \wedge)$. The transformation of the original analytic problem into a problem in combinatorics is completed by the Orlik-Solomon algebra, A. It is a finite dimensional $\mathbb{C}$-algebra defined combinatorially using the intersection poset $L(\mathcal{A})$. There is a graded algebra isomorphism [OT1, 5.90]:

$$\mathsf{A}^{\cdot}(\mathcal{A}) \simeq \mathsf{B}^{\cdot}(\mathcal{A}). \tag{11}$$

Let a_H be the image of ω_H and let a_λ be the image of ω_λ under this isomorphism. Thus we must compute the cohomology of the combinatorial complex $(\mathsf{A}^{\cdot}, a_\lambda \wedge)$.

We define the simplicial complex $\mathsf{NBC}(\mathcal{A})$. Write $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ and introduce a linear order in $\mathcal{A}$ by $H_p \prec H_q$ if $p < q$. Different linear orders may result in different complexes, but it follows from our results that the number of simplexes in each dimension and the homotopy type of the complex is independent of the linear order in $\mathcal{A}$. Thus we may ignore dependence on the linear order. Let $S = \{H_{i_1}, \dots H_{i_q}\}$ be a set of hyperplanes and write $\cap S = H_{i_1} \cap \dots \cap H_{i_q}$ and $|S| = q$. We say that S is independent if $\cap S \neq \emptyset$ and $\operatorname{codim}(\cap S) = |S|$. A maximal independent set is called a frame. Every frame has cardinality r. We say that S is dependent if $\cap S \neq \emptyset$ and $\operatorname{codim}(\cap S) < |S|$. An inclusion-minimal dependent set is called a circuit. A broken circuit is a set S for which there exists $H \prec \min(S)$ such that $\{H\} \cup S$ is a circuit. The collection of subsets of $\mathcal{A}$ which have nonempty intersection and contain no broken circuits is called the **nbc** set of $\mathcal{A}$. This set has the properties of a simplicial complex which we denote $\mathsf{NBC}(\mathcal{A})$. We agree to include the empty set in **nbc** and the empty simplex of dimension -1 in $\mathsf{NBC}(\mathcal{A})$. This results in reduced homology and cohomology. A maximal element (simplex) of **nbc** is called an nbc frame. It follows from the general theory [OT1, 3.55] that the elements of **nbc** provide a $\mathbb{C}$-basis for the algebra $\mathsf{A}^\cdot$. Next we compute the cohomology groups of NBC:

$$H^p(\mathsf{NBC}(\mathcal{A})) = \begin{cases} 0 & \text{if } p \neq r(\mathcal{A}) - 1, \\ \text{free of rank } \beta(\mathcal{A}) & \text{if } p = r(\mathcal{A}) - 1. \end{cases} \tag{12}$$

We conclude Chapter 5 with construction of a basis for $H^{r-1}(\mathsf{NBC}(\mathcal{A}))$. A frame B is called a βnbc frame if B is an nbc frame and for every $H \in B$ there exists $H' \prec H$ in $\mathcal{A}$ such that $(B \setminus \{H\}) \cup \{H'\}$ is a frame. Let β**nbc** be the set of all βnbc frames. Direct calculation shows that $|\beta\mathbf{nbc}| = \beta(\mathcal{A})$ and the β**nbc** set provides a cohomology basis.

We compute the cohomology of the combinatorial complex $(\mathsf{A}^\cdot, a_\lambda\wedge)$ in Chapter 6. Let $\mathbf{y} = \{\mathbf{y}_H \mid H \in \mathcal{A}\}$ be a system of indeterminates in one-to-one correspondence with the hyperplanes of $\mathcal{A}$. Let $\mathbb{C}[\mathbf{y}]$ be the polynomial ring in $\mathbf{y}$. Let

$$\mathsf{A}^\cdot_{\mathbf{y}} = \mathsf{A}^\cdot_{\mathbf{y}}(\mathcal{A}) = \mathbb{C}[\mathbf{y}] \otimes_{\mathbb{C}} \mathsf{A}^\cdot(\mathcal{A})$$

and let $a_{\mathbf{y}} = \sum_{H \in \mathcal{A}} \mathbf{y}_H \otimes a_H \in \mathsf{A}^1_{\mathbf{y}}$. Then $(\mathsf{A}^\cdot_{\mathbf{y}}(\mathcal{A}), a_{\mathbf{y}}\wedge)$ is called the Aomoto complex. We prove that over a suitable ring of quotients of $\mathbb{C}[\mathbf{y}]$, the Aomoto complex is isomorphic to the cochain complex of $\mathsf{NBC}(\mathcal{A})$. The collection of invertible elements is determined by the set of dense edges of $\mathcal{A}$. We show that if $\lambda_X \neq 0$ for every dense edge X, then

$$H^p(\mathsf{A}^\cdot(\mathcal{A}), a_\lambda\wedge) \simeq H^{p-1}(\mathsf{NBC}(\mathcal{A}), \mathbb{C}). \tag{13}$$

This reduces the original problem of computing the cohomology groups of a complex which depends on λ to computing the cohomology groups of the simplicial complex $\mathsf{NBC}(\mathcal{A})$ with constant coefficients. We combine (10), (11), and (13) to obtain:

Theorem 1.3.3. *If $\lambda_X \notin \mathbb{Z}_{\geq 0}$ for all dense edges X, there are isomorphisms*

$$H^p(M, \mathcal{L}) \simeq H^p(\mathsf{B}^\cdot(\mathcal{A}), \omega_\lambda\wedge) \simeq H^p(\mathsf{A}^\cdot(\mathcal{A}), a_\lambda\wedge) \simeq H^{p-1}(\mathsf{NBC}(\mathcal{A}), \mathbb{C}).$$

It follows from (12) that there is only one nontrivial group, $H^r(M, \mathcal{L})$. Falk and Terao [FT] constructed a basis for $H^r(\mathsf{B}^\cdot, \omega_\lambda\wedge)$ which is in natural bijection with the set β**nbc**. For $X \in L(\mathcal{A})$ define $\omega_\lambda(V) = 1$ and

$$\omega_\lambda(X) = \sum_{H \in \mathcal{A}_X} \lambda_H \omega_H \in \mathsf{B}^1(\mathcal{A}).$$

Given an element $B = (H_{i_1}, \ldots, H_{i_r})$ of β**nbc** with $i_1 < \ldots < i_r$, associate to it the flag $\xi(B) = (X_0 > \cdots > X_r)$, where $X_r = V$, $X_{p-1} = \bigcap_{k=p}^r H_{i_k}$ for $1 \leq p \leq r$, and the element of B^r

$$\zeta(B) = \wedge_{p=0}^r \omega_\lambda(X_p), \quad X_p \in \xi(B).$$

Theorem 1.3.4. *Let $\mathcal{A}$ be an affine arrangement with projective closure $\mathcal{A}_\infty$. Assume that $\lambda_X \notin \mathbb{Z}_{\geq 0}$ for every dense edge $X \in L(\mathcal{A}_\infty)$. Then the set*

$$\{\zeta(B) \in H^r(M, \mathcal{L}) \mid B \in \beta\mathbf{nbc}\}$$

is a basis for the only nonzero local system cohomology group, $H^r(M, \mathcal{L})$.

We may also ask for a basis of the twisted homology group $H_r(M, \mathcal{L}^\vee)$. Morse theoretic arguments are used in [OSi] to construct a $\beta\mathbf{nbc}$ local system homology basis for arbitrary arrangements. Here we present a special case used in Chapter 7. We say that $\mathcal{A}$ is a complexified real arrangement if the polynomials α_H have real coefficients. In this case let $V_\mathbb{R} = \mathbb{R}^\ell$ be the real part of V and let $M_\mathbb{R} = M \cap V_\mathbb{R}$ be the real complement. It is a disjoint union of open convex subsets called chambers. Let $\mathsf{ch}(\mathcal{A})$ denote the set of chambers in $M_\mathbb{R}$. If we assume that $\mathcal{A}$ is essential so $r = \ell$, some chambers may be bounded. Let $\mathsf{bch}(\mathcal{A})$ denote the set of bounded chambers in $M_\mathbb{R}$. Zaslavsky [Za] proved that $|\mathsf{ch}(\mathcal{A})| = (-1)^\ell \chi(\mathcal{A}, -1)$ and $|\mathsf{bch}(\mathcal{A})| = \beta(\mathcal{A})$. Let $\rho = (X_0 > X_1 > \cdots > X_\ell)$ be a simplex of edges $X_i \in L(\mathcal{A})$ with $\dim X_i = i$ $(i = 0, \ldots, \ell)$. Let $\Delta \in \mathsf{bch}(\mathcal{A})$ and $\bar{\Delta}$ be its closure in $\mathbb{R}^\ell$. We say that ρ is adjacent to Δ if $\dim(X_i \cap \bar{\Delta}) = i$ for $i = 0, \ldots, \ell$. There exists a unique bijection

$$\tau : \mathsf{bch}(\mathcal{A}) \longrightarrow \beta\mathbf{nbc}(\mathcal{A})$$

with the property that $\xi(\tau(\Delta))$ is adjacent to Δ. Let $C_p^{lf}(M, \mathcal{L}^\vee)$ denote the p-th locally finite chain group with coefficients in $\mathcal{L}^\vee$ and let $H_p^{lf}(M, \mathcal{L}^\vee)$ denote the corresponding locally finite homology group. There is a natural inclusion $i : C_p(M, \mathcal{L}^\vee) \to C_p^{lf}(M, \mathcal{L}^\vee)$ which induces a map in homology. If Δ is a bounded chamber in $M_\mathbb{R}$, then $\Delta \in C_\ell^{lf}(M, \mathcal{L}^\vee)$ is a cycle. Let $[\Delta]$ denote its locally finite homology class. The following holds for nonresonant weights λ:

(1) $H_p(M(\mathcal{A}), \mathcal{L}^\vee) = H_p^{lf}(M(\mathcal{A}), \mathcal{L}^\vee) = 0$ for $p \neq \ell$.

(2) The natural map $i_h : H_\ell(M(\mathcal{A}), \mathcal{L}^\vee) \longrightarrow H_\ell^{lf}(M(\mathcal{A}), \mathcal{L}^\vee)$ is an isomorphism.

(3) $\{[\Delta] \mid \Delta \in \mathsf{bch}(\mathcal{A})\}$ forms a basis for $H_\ell^{lf}(M(\mathcal{A}), \mathcal{L}^\vee)$.

Theorem 1.3.5. *Let $\mathcal{A}$ be an essential complexified real arrangement with projective closure $\mathcal{A}_\infty$. Assume that $\lambda_X \notin \mathbb{Z}_{\geq 0}$ for every dense edge $X \in L(\mathcal{A}_\infty)$. Then the set*

$$\{\tau^{-1}(B) \in H_\ell^{lf}(M, \mathcal{L}^\vee) \mid B \in \beta\mathbf{nbc}(\mathcal{A})\}$$

is a basis for the only nonzero local coefficient homology group, $H_\ell^{lf}(M, \mathcal{L}^\vee)$.

In Chapter 7 we assume that $\mathcal{A}$ is an essential complexified real arrangement. Introduce a linear order in $\beta\mathbf{nbc}(\mathcal{A})$ using the lexicographic order on the hyperplanes read from right to left. Write the ordered set $\beta\mathbf{nbc}(\mathcal{A}) = \{B_j\}_{j=1}^\beta$. Write $\psi_j = \zeta(B_j)$ to get the associated linearly ordered basis of global holomorphic forms for $H^\ell(M, \mathcal{L})$, $\Psi(\mathcal{A}) = \{\psi_j\}_{j=1}^\beta$. Write $\Delta_j = \tau^{-1}(B_j)$ to get the associated linearly ordered basis of bounded chambers for $H_\ell^{lf}(M,\mathcal{L}^\vee)$, $\mathsf{bch}(\mathcal{A}) = \{\Delta_j\}_{j=1}^{\beta}$. Each bounded chamber has an intrinsic orientation. Define the hypergeometric period matrix $\mathsf{PM}(\mathcal{A},\lambda)$ by

$$\mathsf{PM}(\mathcal{A},\lambda)_{i,j} = \int_{\Delta_j} \Phi_\lambda \psi_i.$$

In many cases the individual entries of this matrix are impossible to express in closed form. However, Varchenko [V1] proved that for certain arrangements the determinant of this matrix has a beautiful expression and he conjectured a formula for this determinant for all arrangements. The conjecture was proved by Douai and Terao [DT]. For $X \in L(\mathcal{A}_\infty)$, define

$$\rho(X) = |e(M(\mathbb{P}(\mathcal{A}_\infty)_X))\, e(M((\mathcal{A}_\infty)^X))|.$$

Here $e(M)$ is euler characteristic, $(\mathcal{A}_\infty)^X = \{H\cap X \mid H \in \mathcal{A}_\infty \setminus (\mathcal{A}_\infty)_X,\ H\cap X \neq \emptyset\}$, and $\mathbb{P}(\mathcal{A}_\infty)_X$ is the projective quotient of the central arrangement $(\mathcal{A}_\infty)_X$. There is a disjoint union $L(\mathcal{A}_\infty) = L_+(\mathcal{A}_\infty)\cup L_-(\mathcal{A}_\infty)$ where $L_+(\mathcal{A}_\infty) = L(\mathcal{A})$ consists of edges not in H_∞ and $L_-(\mathcal{A}_\infty) = L({\mathcal{A}_\infty}^{H_\infty})$ consists of edges in H_∞. The beta function of $\mathcal{A}$ is the following product of gamma functions:

$$B(\mathcal{A},\lambda) = \prod_{X\in L_+(\mathcal{A}_\infty)} \Gamma(\lambda_X+1)^{\rho(X)} \prod_{X\in L_-(\mathcal{A}_\infty)} \Gamma(-\lambda_X+1)^{-\rho(X)}.$$

Fix a branch of $\alpha_p^{\lambda_p}$ on each Δ_j. Choose $x_{p,j}\in\bar\Delta_j$ so that $|\alpha_p^{\lambda_p}(x_{p,j})| \geq |\alpha_p^{\lambda_p}(y)|$ for all $y\in\bar\Delta_j$. Define the complex number

$$R(\mathcal{A},\lambda) = \prod_{p=1}^{n}\prod_{j=1}^{\beta} \alpha_p^{\lambda_p}(x_{p,j}).$$

Theorem 1.3.6. *Suppose $\Re\lambda_p > 0$ for all p and $\lambda_X \notin \mathbb{Z}$ for every dense edge $X \in L(\mathcal{A}_\infty)$. Then*

$$\det \mathsf{PM}(\mathcal{A},\lambda) = R(\mathcal{A},\lambda)B(\mathcal{A},\lambda).$$

In Part II we work in the dynamic setup. The constructions involve families of arrangements with a constant combinatorial structure in analogy with the family of arrangements of three points in the complex line parametrized by x in the classical example. There are two fundamental problems. The first problem is to find an adequate description of the moduli space of combinatorially equivalent arrangements of a fixed type. Given such a moduli space, assume that the weights are nonresonant. Assign to each point of the moduli space the only nonzero local system homology group. The Gauss-Manin connection describes the flat sections of this bundle. The second problem is to calculate the Gauss-Manin connection.

In Chapter 8 we describe the case of arrangements of n points in $\mathbb{C}$. Define the arrangement $\mathcal{C}$ in $\mathbb{C}^{n+1}$ by

$$Q(\mathcal{C}) = \prod_{j=1}^{n}(u-t_j)\prod_{1\leq j<k\leq n}(t_j-t_k),$$

the arrangement $\mathcal{B}$ in $\mathbb{C}^n$ by

$$Q(\mathcal{B}) = \prod_{1 \leq j < k \leq n} (t_j - t_k),$$

and the arrangement $\mathcal{A}_{\mathbf{t}}$ in $\mathbb{C}$ by

$$Q(\mathcal{A}_{\mathbf{t}}) = \prod_{j=1}^{n} (u - t_j).$$

Here $\mathcal{B}$ is the braid arrangement and $\mathcal{C}$ is a discriminantal arrangement. Let $\mathsf{M} = M(\mathcal{C})$ and $\mathsf{B} = M(\mathcal{B})$, the pure braid space. Then the natural projection induces a fibration $\pi : \mathsf{M} \to \mathsf{B}$ whose fiber at $\mathbf{t} = (t_1, \ldots, t_n) \in \mathsf{B}$ is $\mathsf{M}_{\mathbf{t}} = M(\mathcal{A}_{\mathbf{t}})$. This justifies calling B the parameter space of all (combinatorially equivalent) arrangements of n distinct points in $\mathbb{C}$. In this setup , let $\lambda_1, \ldots, \lambda_n$ be nonresonant weights of $\mathcal{A}_{\mathbf{t}}$ and let $\mathcal{L}$ be the corresponding local system. Then $\mathcal{H}^1 = \bigcup_{\mathbf{t} \in \mathsf{B}} H^1(\mathsf{M}_{\mathbf{t}}, \mathcal{L})$ is a local system of rank $n-1$. We showed in Chapter 6 that there exists a global frame

$$\varphi_i = \frac{\lambda_{i+1}\, du}{u - t_{i+1}} \quad (i = 1, \ldots, n-1).$$

Since π is locally trivial, we have the bundle of duals $\mathcal{H}_1 = \bigcup_{\mathbf{t} \in \mathsf{B}} H_1(\mathsf{M}_{\mathbf{t}}, \mathcal{L}^\vee)$. We want to describe the sections of $\mathcal{H}_1$ by finding differential equations for them. An isomorphism $\mathcal{H}_1 \otimes \mathcal{O}_{\mathsf{B}} \simeq \mathcal{O}_{\mathsf{B}}^{n-1}$ is given by $\gamma \mapsto (\hat{\varphi}_1, \ldots, \hat{\varphi}_{n-1})^T$, where $\hat{\varphi}_j = \int_\gamma \Phi \varphi_j$. Let d' be the exterior differential in B. Then we have the following

Theorem 1.3.7. *The vector $(\hat{\varphi}_1, \ldots, \hat{\varphi}_{n-1})^T$ satisfies the system of first order differential equations*

$$d' \begin{pmatrix} \hat{\varphi}_1 \\ \vdots \\ \hat{\varphi}_{n-1} \end{pmatrix} = \Omega \wedge \begin{pmatrix} \hat{\varphi}_1 \\ \vdots \\ \hat{\varphi}_{n-1} \end{pmatrix},$$

where Ω is the $(n-1) \times (n-1)$ matrix of the form

$$\Omega = \sum_{i<j} \Omega_{i,j} \frac{d'(t_i - t_j)}{t_i - t_j}.$$

This is a Knizhnik-Zamolodchikov equation, which arose originally in conformal field theory. The explicit form of $\Omega_{i,j}$ is given in Theorem 8.2.1. The connection $\nabla' = d' - \Omega \wedge$ on $\mathcal{O}_{\mathsf{B}}^{n-1}$ is called the Gauss-Manin connection and the differential equations above are the equations for flat sections with respect to the connection ∇'. Since the connection is flat and $d'\Omega = 0$, the matrix Ω satisfies the equation $\Omega \wedge \Omega = 0$. This is equivalent to the infinitesimal pure braid relations among the $\Omega_{i,j}(1 \leq i < k \leq n)$ introduced by Kohno [Ko4]. The fundamental group of B is the pure braid group. Its action on the local system $\mathcal{H}^1$ turns out to be the Gassner representation.

In Chapter 9 we consider the general case of arbitrary ℓ-arrangements with n hyperplanes. This is clearly a difficult problem, since the natural moduli space of combinatorially equivalent arrangements is much more intricate than the pure braid space (for $\ell = 1$). We use a naive approach to construct the moduli space of the universal family of arrangements of a given combinatorial type. Let the ith hyperplane be defined by the equation $t_i^{(0)} + \sum_{j=1}^{\ell} t_i^{(j)} u_j = 0$. Consider the $(\ell+1) \times (n+1)$ matrix of coefficients, which includes the infinite hyperplane in the last column

$$\mathsf{T} = \begin{pmatrix} t_1^{(0)} & \cdots & t_n^{(0)} & 1 \\ t_1^{(1)} & \cdots & t_n^{(1)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ t_1^{(\ell)} & \cdots & t_n^{(\ell)} & 0 \end{pmatrix}.$$

The matrix T determines a point of $(\mathbb{CP}^\ell)^n$ by regarding each column, except the last one, as the homogeneous coordinates of $\mathbb{CP}^\ell$. If we only consider essential arrangements, the combinatorial type of an affine arrangement is completely described by the vanishing of certain $(\ell+1)$-minors in T and the nonvanishing of the other $(\ell+1)$-minors in T. Geometrically speaking, the moduli space of combinatorially equivalent essential arrangements is a locally closed subset of $(\mathbb{CP}^\ell)^n$ parametrized by subsets of $\left(\binom{[n+1]}{\ell+1}\right) = \{(i_1, \ldots, i_{\ell+1}) \mid 1 \le i_1 < \cdots < i_{\ell+1} \le n+1\}$. We introduce notation to describe the moduli space. For $\mathcal{S} \subseteq \left(\binom{[n+1]}{\ell+1}\right)$, let $\mathsf{B}_\mathcal{S}$ be the moduli space corresponding to $\mathcal{S}$ so that B_ϕ is the largest moduli space consisting of arrangements in general position, which is dense in $(\mathbb{CP}^\ell)^n$. It is shown in Proposition 9.1.7 that $\bar{\mathsf{B}}_\mathcal{S} \setminus \mathsf{B}_\mathcal{S}$ is a divisor in $\bar{\mathsf{B}}_\mathcal{S}$. In Proposition 9.3.3 we describe explicitly the irreducible components of the divisor and the way its irreducible components intersect each other, provided that the codimension of $\bar{\mathsf{B}}_\mathcal{S}$ in $(\mathbb{CP}^\ell)^n$ does not exceed one.

In Chapter 10 we describe the Gauss-Manin connection in general and find formulas to describe the logarithmic poles of the connection. Fix an essential affine ℓ-arrangement $\mathcal{A}$ with n hyperplanes. Let $\pi : \mathsf{M} \longrightarrow \mathsf{B}$ be the complete family of affine arrangements in $\mathbb{C}^\ell$ with n hyperplanes which are combinatorially equivalent to $\mathcal{A}$. As we saw in Chapter 9, B is a locally closed subset of $(\mathbb{CP}^\ell)^n$ and its "boundary" $\mathsf{D} = \bar{\mathsf{B}} \setminus \mathsf{B}$ is a divisor. Let $\mathsf{D} = \bigcup_{s=1}^{t} \mathsf{D}_s$ be its irreducible decomposition. As in Chapter 8, let $\lambda_1, \cdots, \lambda_n$ be nonresonant weights of $\mathcal{A}_\mathsf{t}$ and $\mathcal{L}$ be the corresponding rank one local system. Then $\mathcal{H}^\ell = \bigcup_{\mathsf{t} \in \mathsf{B}} H^\ell(\mathsf{M}_\mathsf{t}, \mathcal{L})$ is a local system of rank $\beta(\mathcal{A}) = |\chi(\mathsf{M}_\mathsf{t})|$. We know from Section 6.3 that there exists a global β**nbc** frame $[\Xi_1], [\Xi_2], \ldots, [\Xi_\beta]$. Since π is locally trivial, we have the bundle of duals $\mathcal{H}_\ell = \bigcup_{\mathsf{t} \in \mathsf{B}} H_\ell(\mathsf{M}_\mathsf{t}, \mathcal{L}^\vee)$. Let σ be a local section of $\mathcal{H}_\ell$. Let d'

be the exterior differential in B. Then we have the following system of first order differential equations

$$d' \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ \cdot\cdot \\ \cdot\cdot \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix} = \Omega \wedge \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ \cdot\cdot \\ \cdot\cdot \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix}$$

where Ω is the $\beta \times \beta$ matrix of differential 1-forms. The connection $\nabla' = d' - \Omega\wedge$ on $\mathcal{O}_{\mathsf{B}}^{\beta}$ is the Gauss-Manin connection and the differential equations above are the equations of flat sections. This equation generalizes the Knizhnik-Zamolodchikov equations. As we see in Theorems 10.2.1 and 10.2.2, the connection matrix Ω turns out to have logarithmic poles along the boundary divisor $\mathsf{D} = \bar{\mathsf{B}} \setminus \mathsf{B}$. In Sections 10.3 and 10.4 we obtain explicit formulas for the connection matrix in the case of codimension ≤ 1. We use the β**nbc** basis of Chapter 6 together with the geometric results on D from Chapter 9. We follow Aomoto and Kita [AK] in the general position case (= the codimension zero case) and [T] to describe the codimension one case.

It follows from the general theory, that the entries of the connection matrix are polynomials with rational coefficients in the variables λ_H, where $H \in \mathcal{A}$ and λ_X^{-1}, where λ_X runs over the set of dense edges. Remarkably, in all our calculations the matrix entries are integer linear combinations of the weights. We conjecture that this is the case in general.