Part II: Complex Reflection Groups and Fake Degrees[1]

[1]Lectures at RIMS, Kyoto University (Japan) in 1997. Noted by Kenji Taniguchi. The author thanks Gunter Malle and Kenji Taniguchi for their keen observations, questions and comments, and Kenji Taniguchi for his commitment and skill in writing these notes.

1 Introduction

In this section, we introduce finite complex reflection groups, define the fake degrees, and explain the objectives of this paper.

Definition 1.1 Let V be an n-dimensional Hilbert space and let $U(V)$ be the group of unitary transformations on V. An element $r \in U(V)$ is called a *complex reflection* if the set $\mathrm{Ker}(r - \mathrm{Id}_V)$ of fixed points of r in V is a complex hyperplane H_r. A finite subgroup W of $U(V)$ is called *a finite complex (or also unitary) reflection group* if W is generated by complex reflections.

The following theorem is the fundamental fact characterizing such subgroups in $U(V)$.

Theorem 1.2 (Shephard-Todd [19]) *A subgroup W of $U(V)$ is a finite complex reflection group if and only if the subring P^W of the W-invariant elements in the ring P of polynomial functions on V is generated by n algebraically independent homogeneous elements $p_1, \ldots, p_n$.*

In this case, the homogeneous degrees $d_i = \deg(p_i)$ depend only on W, and are called the *primitive degrees* of W. The product of d_i is equal to $|W|$ and the sum $\sum_{i=1}^{n}(d_i - 1)$ is equal to the number of complex reflections in W.

We call W *irreducible* if W acts irreducibly on V. Shephard and Todd have given the complete classification of the irreducible complex reflection groups:

Theorem 1.3 (Shephard-Todd [19]) *If W is an irreducible complex reflection group, then W is isomorphic to one of the following list:*

(i) *The symmetric Group $\mathfrak{S}_n$, acting on $V = \{v \in \mathbb{C}^n \mid \sum v_i = 0\}$.*

(ii) *Let m, p, n be positive integers such that p divides m, $m \geq 2$, and $p = 1$ if $n = 1$. Let $\{e_i; i = 1, \ldots, n\}$ be an orthonormal basis of $\mathbb{C}^n$ and for each $i = 1, \ldots, n$, let ζ_i be an m-th root of unity, such that $(\prod_{i=1}^{n} \zeta_i)^{\frac{m}{p}} = 1$. We denote by $G(m, p, n)$ the group generated by*

$$e_i \mapsto \zeta_i e_{\sigma(i)} \quad (\sigma \in \mathfrak{S}_n).$$

This is a finite complex reflection group in $\mathbb{C}^n$. These groups are the imprimitive complex reflection groups.

(iii) *One of 34 exceptional cases (these cases contain of course the exceptional Coxeter groups).*

Remark 1.4 Many rank two cases are obtained as follows: Let $\Gamma \subset SL_2(\mathbb{C})$ be a finite subgroup corresponding to a platonic polyhedron. For suitable choice of $a \in \mathbb{N}$, $\Gamma \cdot \mu_a$ is a finite complex reflection group, where $\mu_a = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} ; \alpha^a = 1 \right\}$.

The action of W on P respects the grading by degree. Let P_+ be the graded ideal of polynomials vanishing at $0 \in V$ and let P_+^W be the ideal generated by the invariants in P_+. The coinvariant algebra P/PP_+^W is a representation space of W and we denote it by RW. This representation is isomorphic to the left regular representation on $\mathbb{C}[W]$. The space RW is graded by homogeneous degree and this grading is compatible with the action of W. Let RW_k be the homogeneous subspace of RW of degree k.

It is well known that the graded character of this representation is

$$\mathrm{tr}_{RW}(w) := \sum_{k\geq 0}(\mathrm{tr}(w)|_{RW_k})T^k = \frac{\prod_{i=1}^n(1-T^{d_i})}{\det_V(1-Tw^{-1})} \in \mathbb{C}[T]. \tag{1.1}$$

By this character, we have a map from the space $\mathrm{Class}(W)$ of class functions on W to $\mathbb{C}[T]$:

$$\mathrm{Class}(W) \ni \alpha \mapsto F_\alpha := \frac{1}{|W|}\sum_{w\in W}\alpha(w)\mathrm{tr}_{RW}(w^{-1}) \in \mathbb{C}[T].$$

Definition 1.5 For a representation τ of W we write $R_\tau = F_{\overline{\tau}} = F_{\overline{\chi_\tau}}$, where χ_τ denotes the character of τ. R_τ is called the *fake degree* of τ. Notice that $R_\tau \in \mathbb{Z}[T]$ with nonnegative coefficients.

Since $RW \simeq \mathbb{C}[W]$, $\dim \mathrm{Hom}_W(\tau, RW) = \deg\tau =: l$ for each irreducible representation $\tau \in \hat{W}$. Let $p_1^\tau \leq \cdots \leq p_l^\tau$ be the homogeneous embedding degrees of τ in the coinvariant algebra RW, i.e. the degrees of irreducible τ-components in RW.

Corollary 1.6 *For an irreducible representation $\tau \in \hat{W}$,*

$$F_\tau(T) = \frac{1}{|W|}\sum_{w\in W}\frac{\chi_\tau(w)}{\det_V(1-Tw)}\prod_{i=1}^n(1-T^{d_i}) \tag{1.2}$$

$$= \sum_{j=1}^l T^{p_j^\tau}. \tag{1.3}$$

Proof. By (1.1) and by the orthogonality of characters. □

The name *fake degree* was given by Lusztig. When W is a Coxeter group the notion of fake degree plays a role in the representation theory of finite Chevalley groups. By Lusztig's work, they can be regarded as approximations of the degrees of the principal series unipotent representations of a finite Chevalley group. In fact Lusztig has shown that every unipotent degree of a finite Chevalley group $G(\mathbb{F}_q)$ with Weyl group W, can be expressed as a rational linear combination of the fake degrees of W evaluated at $T = q$ (see [8], [14]).

The ideas and conjectures of Broué, Malle and Michel state, among many other things, that this role of W and its fake degrees for the study of unipotent representations can be extended naturally to the more general case where W is a complex reflection group that arises as the quotient $N(L)/L$ with $L \subset G$ a "d-cuspidal" Levi subgroup of G. We refer the reader to [5], [6] for an account of this subject. Also see [15], where the unipotent degrees and fake degrees were studied in the case of the general imprimitive group.

A central notion in these considerations of unipotent and fake degrees for a complex reflection group W is its *(cyclotomic) Hecke algebra*. It is a deformation of the group algebra of W, similar to the ordinary Hecke algebra of a Coxeter group. It was introduced by Broué and Malle in [5], and investigated in many subsequent papers (for example [15], [16], [7]). From the results of these papers it is clear that the cyclotomic Hecke algebra shares many of the properties which give the usual Hecke algebra its prominent role in representation theory. But also, there are still severe problems to give the cyclotomic Hecke algebra a transparent theoretical basis similar to the theory of ordinary Hecke algebras. Many results rely on classifications and computer aided computations. In fact there are many exceptional complex reflection groups for which it is still not known whether their Hecke algebras are free over the coefficient ring or not, or what the rank of the Hecke algebra is. The situation is better for the imprimitive cases. Here the theory is rather well understood, see [1] and [15].

In these lectures we approach the cyclotomic Hecke algebra from topology. It is the same approach as was used in the paper [7]. This way of thinking about the Hecke algebra is quite natural in the case of a Coxeter group W, given Brieskorn's description of the fundamental group of the regular orbit space as the braid group of W [4]. For Coxeter groups one can find results in this direction in [11], [12] (and many other papers). We use the monodromy representation of certain systems of differential equations to construct the "topological cyclotomic Hecke algebra" (terminology from [7]), which is known to be isomorphic to the cyclotomic Hecke algbra in many cases (and conjectured to be isomorphic in all cases) (see [7]).

There are two main results in these notes. The first is described in section 4. This is a transformation property of fake degrees of representations of W with respect to certain operations on representations of W. In view of the current difficulties with cyclotomic Hecke algebras we formulate and prove this result without the use of cyclotomic Hecke algebras. The price we have to pay is that the "operations" on representations of W are somewhat mysterious without the cyclotomic Hecke algebra, and not much can be said about their basic properties. In the second half of these notes we assume certain facts about the cyclotomic Hecke algebra, and with these assumptions we interpret the "operations" mentioned above as the natural action on $\mathrm{Irr}(W)$ of the geometric Galois group of the character field of the cyclotomic Hecke algebra. See section 7 for more details about this part of the story. The main result of this second part is Theorem 6.7, which implies that the geometric Galois group of the character field over the coefficient field is abelian (but the statement of Theorem 6.7 is more precise).

2 Minimal polynomial realization

In this section, we define the minimal τ-matrix and investigate the basic properties of it.

Let $\mathcal{C}$ be the set of W-orbits of reflection hyperplanes and let $\mathcal{A}$ be the full hyperplane arrangement $\cup_{C\in\mathcal{C}} C$. For a hyperplane H in an orbit $C \in \mathcal{C}$, the stabilizer

$$W_H := \{w \in W; wx = x \text{ for any } x \in H\}$$

is isomorphic to a cyclic group $\mathbb{Z}/e_C\mathbb{Z}$. Here, e_C is the order of W_H, which is determined by the orbit C. For a representation τ of W, we define a nonnegative integer $n^\tau_{C,j}$ by

$$\mathrm{Res}^W_{W_H}\tau \simeq \bigoplus_{j=0}^{e_C-1} n^\tau_{C,j} \det{}^{-j}. \tag{2.1}$$

Note that $\hat{W}_H = \{1 = \det^{-e_C}, \det^{-e_C+1}, \ldots, \det^{-1}\}$.

In this note, we denote $f(H) = f(C)$ for a function f on $\mathcal{C}$ if $H \in C \in \mathcal{C}$. For example, $e_H = e_C$ and $n^\tau_{H,j} = n^\tau_{C,j}$. Note that

$$\sum_{i=1}^{n}(d_i - 1) = |\{\text{complex reflections}\}| = \sum_{H\in\mathcal{A}}(e_H - 1), \tag{2.2}$$

$$\deg\tau = \sum_{j=0}^{e_H-1} n^\tau_{H,j} \quad (\text{for each } H \in \mathcal{A}). \tag{2.3}$$

Lemma 2.1 *For an irreducible representation τ,*

$$\sum_{j=1}^{l} p^\tau_j = \sum_{C\in\mathcal{C}} |C| \sum_{j=1}^{e_C-1} j n^\tau_{C,j}. \tag{2.4}$$

Proof. To show this, we use Corollary 1.6. We differentiate this equality by T and take the limit $T \to 1$. After we take the limit, (1.3) is equal to $\sum_{j=1}^{l} p^\tau_j$, and the only terms which survive in the sum $\sum_{w\in W}$ of right hand side of (1.2) are the terms where $w = e$ or where w is a complex reflection. The limit of the $w = e$ term equals

$$\lim_{T\to 1}\frac{\deg\tau}{|W|}\sum_{i=1}^{n}\left(\sum_{j_i=0}^{d_i-1} j_i T^{j_i-1}\right)\prod_{k\neq i}\left(\sum_{j_k=0}^{d_k-1} T^{j_k}\right)$$
$$=\frac{\deg\tau}{|W|}\sum_{i=1}^{n}\frac{1}{2}d_i(d_i-1)\prod_{k\neq i} d_k = \sum_{H\in\mathcal{A}}\sum_{j=0}^{e_H-1}\frac{e_H-1}{2}n^\tau_{H,j}.$$

Here, we used the facts (2.2), (2.3) and $\prod_{i=1}^{n} d_i = |W|$.

For $H \in \mathcal{A}$, we denote by ζ_H a primitive e_H-th root of unity. The limit of the sum of the terms with w a complex reflection equals

$$\begin{aligned}
&\lim_{T\to 1}\frac{d}{dT}\frac{1}{|W|}\sum_{H\in\mathcal{A}}\sum_{k=1}^{e_H-1}\sum_{j=0}^{e_H-1}\frac{n^{\tau}_{H,j}\zeta_H^{-jk}}{1-T\zeta_H^k}(1-T)\prod_{i=1}^{n}\left(\sum_{j_i=0}^{d_i-1}T^{j_i}\right)\\
&=-\sum_{H\in\mathcal{A}}\sum_{k=1}^{e_H-1}\sum_{j=0}^{e_H-1}\frac{n^{\tau}_{H,j}\zeta_H^{-jk}}{1-\zeta_H^k}\\
&=\dots\\
&=-\sum_{H\in\mathcal{A}}\sum_{j=0}^{e_H-1}n^{\tau}_{H,j}\frac{e_H-1-2j}{2}.
\end{aligned}$$

□

Definition 2.2 For a finite dimensional representation (τ, E) of W, we choose an explicit matrix realization $\tau : W \to GL(l, \mathbb{C})$ ($l = \deg\tau$) and fix it.

A matrix $M = (m_{ij}) \in Mat(l \times l, P)$ with $\det(M) \neq 0$ is called a *minimal τ-matrix* if it satisfies the following conditions:

(i) For each $w \in W$, $M^w := (m_{ij} \circ w^{-1}) = \tau(w)M$.

(ii) Let $\mathbb{E} := \sum_{i=1}^{n} x_i\partial/\partial x_i$ be the Euler vector field. The matrix M satisfies

$$\mathbb{E}M := (\mathbb{E}m_{ij}) = M \cdot C^M_{\mathbb{E}}$$

for some $C^M_{\mathbb{E}} \in Mat(l \times l, \mathbb{C})$.

(iii)

$$\operatorname{tr}(C^M_{\mathbb{E}}) = \sum_{C\in\mathcal{C}}|C|\sum_{j=1}^{e_C-1} j n^{\tau}_{C,j}.$$

Remark 2.3 (i) 2.2 Condition (iii) is equivalent to the following (iii)$'$:

$$\deg(\det(M)) = \sum_{C\in\mathcal{C}}|C|\sum_{j=1}^{e_C-1} j n^{\tau}_{C,j}.$$

This follows immediately by considering the action of the group $\mathbb{C}^{\times}$ (for which $\mathbb{E}$ is the infinitesimal generator) on M.

(ii) If M is a minimal τ-matrix and $g \in GL(l, \mathbb{C})$, then Mg is also a minimal τ-matrix.

Proposition 2.4 (i) *$C_{\mathbb{E}}^M$ is semisimple and its spectrum is contained in the set of non-negative integers.*

(ii) *For every finite dimensional representation τ of W, there exists a minimal τ-matrix M.*

(iii) *Let α_H be a linear function satisfying* $\mathrm{Ker}\alpha_H = H$. *We define $\pi_C = \prod_{H\in C}\alpha_H$ for $C \in \mathcal{C}$. Then*

$$\det(M) = const. \prod_{C\in\mathcal{C}} \pi_C^{\sum_{j=0}^{e_C-1} jn_{C,j}^\tau}. \tag{2.5}$$

(iv) *For any $N \in Mat(l\times l, P)$ satisfying Definition 2.2 (i), there exists $R \in Mat(l\times l, P^W)$ such that $N = MR$.*

(v) *Spectrum of $C_{\mathbb{E}}^M$ does not depend on M, only on τ.*

Proof. (i) Since $e^{2\pi\sqrt{-1}\mathbb{E}}$ acts on P by identity, we have $e^{2\pi\sqrt{-1}C_{\mathbb{E}}^M} = \mathrm{Id}$. It follows that $C_{\mathbb{E}}^M$ is semisimple and its eigenvalues are integers. The spectrum of $C_{\mathbb{E}}^M$ is contained in the set of non-negative integers since M has polynomial entries.

(ii) For the proof of (ii), we may assume τ to be irreducible. Let $\mathrm{Harm}(\tau)$ be the vector space spanned by harmonic polynomials associated with τ and let (τ^*, E^*) be the contragredient representation of (τ, E). Take bases ε_i of E and σ_j of $(\mathrm{Harm}(\tau)\otimes E^*)^W$ $(i,j = 1,\dots,l)$ and we define $M = (h_{ij}^\tau) := (\sigma_j(\varepsilon_i))$. By this construction, (i) and (ii) in Definition 2.2 are clear. We have to check Definition 2.2 (iii), but this is clear by $\mathrm{tr}(C_{\mathbb{E}}^M) = \sum_{j=1}^l \deg h_{jj}^\tau = \sum_{j=1}^l p_j^\tau$ (using Lemma 2.1).

(iii) By (2.1) and Definition 2.2 (i), there exists $D \in GL(l,\mathbb{C})$ such that $D\tau(s)D^{-1}$ $(s\in W_H)$ is diagonal and

$$DM = M_H M',$$

where M' is a W_H-invariant matrix with polynomial entries and

$$M_H = \mathrm{diag}(I_{n_{H,0}^\tau}, \alpha I_{n_{H,1}^\tau}, \dots, \alpha_H^{e_H-1} I_{n_{H,e_H-1}^\tau}). \tag{2.6}$$

(I_p is the p-th unit matrix.) Note that M_H is a minimal $\tau|_{W_H}$-matrix and $\det(M') \neq 0$.

By the above discussion, $\det(M)$ is divisible by $\alpha_H^{\sum_{j=0}^{e_H-1} jn_{H,j}^\tau}$ for each $H \in \mathcal{A}$ and also by the right hand side of (2.5). Both hand sides of (2.5) have the same degree since $\deg(\det(M)) = \mathrm{tr}(C_{\mathbb{E}}^M) = \sum_{j=1}^l p_j^\tau = \sum_{C\in\mathcal{C}} |C| \sum_{j=1}^{e_C-1} jn_{C,j}^\tau = \deg\left(\prod_{C\in\mathcal{C}} \pi_C^{\sum_{j=0}^{e_C-1} jn_{C,j}^\tau}\right)$. This proves (2.5).

(iv) By the same discussion as above, we can show that, for any $N \in Mat(l\times l, P)$ satisfying (1), there exists a W_H-invariant matrix $N' \in Mat(l\times l, P)$ such that $DN = M_H N'$. Hence, for each reflection hyperplane H, $M^{-1}N = (M')^{-1}N'$

is W_H-invariant and regular at H. It follows that $M^{-1}N$ is W-invariant and the entries of $M^{-1}N$ are polynomials because of (2.5).

(v) By Remark 2.3 and (i) of this proposition, we may assume that $C_{\mathbb{E}}^{M_\nu}$ ($\nu = 1,2$) are diagonal matrices.

Let $n_1^\nu, \ldots, n_l^\nu$ be the spectrum of $C_{\mathbb{E}}^{M_\nu}$ ($\nu = 1,2$). Since

$$a^{C_{\mathbb{E}}^{M_\nu}} = \operatorname{diag}(a^{n_1^\nu}, \ldots, a^{n_l^\nu}),$$

the matrix $R = M_1^{-1}M_2$ satisfies $R(ax)_{ij} = a^{n_j^2 - n_i^1} R(x)_{ij}$, and it follows that $R(x)_{ij} = 0$ if $n_j^2 < n_i^1$. But we know $\sum_{i=1}^l n_i^1 = \sum_{i=1}^l n_i^2$ by Definition 2.2 (iii). If the sequences $(n_1^\nu, \ldots, n_l^\nu)$ ($\nu = 1,2$) do not coincide, then, for every $\sigma \in \mathfrak{S}_l$, there exists i such that $n_{\sigma(i)}^2 < n_i^1$. This implies $\det R = 0$, which is a contradiction. □

Remark 2.5 This generalizes a construction of Stanley [20], who proved that for every one dimensional representation $\tau \in \hat{W}$ the pseudo-invariants of type τ in P form a rank one free module over P^W, with generator

$$\prod_{C \in \mathcal{C}} \pi_C^{\sum_{j=0}^{e_C - 1} j n_{C,j}^\tau}$$

which is the minimal τ-matrix in this situation. So Stanley's result is Proposition 2.4 (iv) in this special case.

Corollary 2.6 *Let M be a minimal τ-matrix and let $n_1 \le \cdots \le n_l$ be the spectrum of $C_{\mathbb{E}}^M$. Then*

$$F_\tau(T) = \sum_{i=1}^{l} T^{n_i}.$$

3 Knizhnik-Zamolodchikov equations

We are going to construct deformed minimal τ-matrices using certain differential equations. This is a generalization of the construction of a minimal τ-matrix using harmonic polynomials.

Let us choose labels $k = (k_{C,j})_{C \in \mathcal{C}, j=0,\ldots,e_C-1}$ with $k_{C,j} \in \mathbb{C}$. Put $q_{C,j} = \exp(-2\pi\sqrt{-1}k_{C,j})$. We shall sometimes use the notation q for the vector

$$(q_{C,j})_{C \in \mathcal{C}, j=0,\ldots,e_C-1}.$$

Let $\varepsilon_j(H)$ be the idempotent element $\frac{1}{e_H}\sum_{w \in W_H} \det^j(w) w$ in $\mathbb{C}[W_H]$.

Definition 3.1 We define

$$\omega = \sum_{H \in \mathcal{A}} a_H \omega_H,$$

where $a_H = \sum_{j=0}^{e_H - 1} e_H k_{H,j} \varepsilon_j(H)$ and $\omega_H = d(\log \alpha_H) = \frac{d\alpha_H}{\alpha_H}$.

Let

$$V^{\mathrm{reg}} = \{v \in V; v \notin H \text{ for any } H \in \mathcal{A}\}$$

and let us denote by $\mathcal{O}[W]$ (resp. $\Omega^1[W]$) the sheaf of germs of $\mathbb{C}[W]$-valued holomorphic functions (resp. holomorphic 1-forms) on V^{reg}. Then ω is an element of $\Omega^1(V^{\mathrm{reg}})[W]$ satisfying $w \cdot (\omega \circ w^{-1}) \cdot w^{-1} = \omega$.

Theorem 3.2 (Kohno, Broué-Malle-Rouquier, Opdam) (i) *ω is integrable one form, i.e. $\omega \wedge \omega = 0$. This means that the connection*

$$\nabla(k) : \mathcal{O}[W] \ni \Phi \mapsto d\Phi + \omega\Phi \in \Omega^1[W]$$

is completely integrable, i.e. the $\nabla(k)$-flat local sections $\Phi \in \mathcal{O}[W]$ form a vector space of dim $|W|$.

(ii) *The connection $\nabla(k)$ commutes with right W-multiplication.*

(iii) *We define an action $\Phi \mapsto \Phi^w$ of W on $\mathcal{O}[W] \simeq \mathcal{O} \otimes \mathbb{C}[W]$ and $\Omega^1[W] \simeq \Omega^1 \otimes \mathbb{C}[W]$ by $w \otimes$* (left multiplication of w). *Then $\nabla(k)$ commutes with this action.*

Proof. Basically, the proof of Kohno [13] is still valid. □

Let (τ, E) be a representation of W, and let e_τ be an idempotent of the ring $\mathbb{C}[W]^N$ such that $\mathbb{C}[W]^N \cdot e_\tau$ is isomorphic to (τ, E) (we choose N large enough). By Theorem 3.2 (ii), $\nabla(k)^N$ descends to a connection on the bundle $V^{\mathrm{reg}} \times E \to V^{\mathrm{reg}}$. Moreover, by Theorem 3.2 (iii), $\nabla(k)^N$ descends to $\mathcal{E} := \mathcal{O}(X^{\mathrm{reg}}) \otimes_{\mathbb{C}} \mathcal{L}(E)$, where $X = W \backslash V$, $X^{\mathrm{reg}} = W \backslash V^{\mathrm{reg}}$ and $\mathcal{L}(E)$ is the local system $V^{\mathrm{reg}} \times_W E \to X^{\mathrm{reg}}$. The resulting connection on $\mathcal{E}$, called *Knizhnik-Zamolodchikov (KZ) connection*, is denoted by $\nabla_\tau(k)$.

Definition 3.3 We define $\mathcal{E}^{\nabla_\tau(k)} \to X^{\mathrm{reg}}$ to be the local system of $\nabla_\tau(k)$-flat sections in $\mathcal{E}$.

Let x_0 be a base point in X^{reg}, and v_0 a lift of x_0 in V^{reg}. For every $H \in \mathcal{A}$, we choose a path $l_H : v_0 \to s_H v_0$ in V^{reg}, where s_H is a generator for W_H such that $\det(s_H) = \zeta_H = e^{2\pi\sqrt{-1}/e_H}$. We denote $\pi_1(V^{\mathrm{reg}}, v_0)$ and $\pi_1(X^{\mathrm{reg}}, x_0)$ by P and B, and we call them the *pure braid group* and the *braid group*, respectively.

The following theorem is due to Broué, Malle and Rouquier:

Theorem 3.4 (see [7]) (i) *B is generated by $\{l_H\}_{H\in\mathcal{A}}$.*

(ii) *P is generated by $\{l_H^{e_H}\}_{H\in\mathcal{A}}$.*

(iii) *We have the following short exact sequence:*

$$1 \to P \to B \to W \to 1,$$

where the map $B \to W$ is given by $l_H \mapsto s_H$.

Definition 3.5 Let $\tau(k)$ be the monodromy action of B on $\mathcal{E}_{x_0}^{\nabla_\tau(k)}$.

Theorem 3.6 (Broué-Malle-Rouquier, Opdam)

$$\prod_{j=0}^{e_H-1}(\tau(k)(l_H) - q_{H,j}\zeta_H^j) = 0.$$

(Note that $q_{H,j}\zeta_H^j = \exp\left(2\pi\sqrt{-1}(j - e_H k_{H,j})/e_H\right)$.) Moreover, with respect to a fixed basis of $\mathcal{E}_{x_0} = E$ the matrix of τ will have coefficients in S, the ring of entire functions in the labels $(k_{C,j})_{C\in\mathcal{C},j=0,\ldots,e_C-1}$.

Proof. The proof we present here differs a little bit from the one in [7], and will give important additional information about the local behaviour of flat sections near the reflection hyperplanes.

We first address the last assertion of Theorem 3.6. This is a basic fact, which we prove anyway, for want of a good reference. The connection matrix of ∇_τ depends polynomially on the labels $(k_{C,j})_{C\in\mathcal{C},j=0,\ldots,e_C-1}$. By definition of analytic continuation along a path, it therefore suffices to prove the following local fact (the monodromy matrix of a loop is a composition of finitely many local steps like these): Suppose we have a holomorphic, first order, linear system of differential equations on the bundle $\mathbb{C}^l \times D$ over the unit disc D, and the coefficient matrix depends polynomially on a parameter $\kappa \in \mathbb{C}$. Given $v \in \mathbb{C}^l$, the unique solution $\nu(z,\kappa)$ ($z \in D$) such that $\nu(0,\kappa) = v$ is an entire function of κ. For this it suffices to check that the power series expansion of ν on D converges locally uniformly in κ, and this is an elementary exercise left to the reader.

Choose $H_0 \in \mathcal{A}$ and fix it. For notational convenience, we abbreviate α_{H_0} as α_0, s_{H_0} as s_0 and so on. Let $x_0 \in H_0$ be a regular point and let (x,α_0) be a coordinate in a tubular neighborhood $U \times I$ of x_0, where $U \subset H_0$ and $I \subset \mathbb{C}\alpha_0$. For every $\varepsilon_0 \in E$ with $s_0\varepsilon_0 = \zeta_0^{-j}\varepsilon_0$, we shall construct a flat section $\varepsilon(x,\alpha_0)$ in $U \times I$.

Contracting the equation $d\varepsilon + \omega\varepsilon = 0$ with vector field α_0^*, we have

$$\frac{\partial\varepsilon}{\partial\alpha_0} + \frac{1}{\alpha_0}a_0\varepsilon + A(x,\alpha_0)\varepsilon = 0, \tag{3.1}$$

where $A(x, \alpha_0) = \sum_{H \in \mathcal{A}, H \neq H_0} \frac{(\alpha_0^*, \alpha_H)}{\alpha_H} a_H$. Notice that

$$\begin{aligned} s_0 A(x \circ s_0^{-1}, \alpha_0 \circ s_0^{-1}) s_0^{-1} &= \sum_{H \neq H_0} \frac{(\alpha_0^*, \alpha_H)}{\alpha_H \circ s_0^{-1}} s_0 a_H s_0^{-1} \\ &= \sum_{H \neq H_0} \frac{(s_0(\alpha_0^*), \alpha_{s_0(H)})}{\alpha_{s_0(H)}} a_{s_0(H)} \\ &= \zeta_0 A(x, \alpha_0). \end{aligned}$$

Hence,

$$B(x, \alpha_0) := \alpha_0 A(x, \alpha_0)$$

is W_0-invariant, i.e. $wB(x \circ w^{-1}, \alpha_0 \circ w^{-1}) w^{-1} = B(x, \alpha_0)$ for any $w \in W_0$, and satisfies $B(x, 0) = 0$. It follows that $B(x, \alpha_0)$ can be expressed as

$$B(x, \alpha_0) = \sum_{n=1}^{\infty} B_n(x) \alpha_0^n. \tag{3.2}$$

By definition, $s_0 B_n(x) s_0^{-1} = \det^n(s_0) B_n(x)$.

The equation (3.1) is equivalent to

$$\alpha_0 \frac{\partial \varepsilon}{\partial \alpha_0} + a_0 \varepsilon + B(x, \alpha_0) \varepsilon = 0. \tag{3.3}$$

Let

$$\varepsilon(x, \alpha_0) = \alpha_0^c \sum_{n=0}^{\infty} \varepsilon_n(x) \alpha_0^n \tag{3.4}$$

be a solution of (3.3) satisfying $\varepsilon_0(x) = \varepsilon_0$. By (3.2), (3.3) and (3.4), we have

$$\sum_{n=0}^{\infty} \left(\left(c + n + \sum_{l=0}^{e_0 - 1} k_{H_0, l} \sum_{w \in W_0} \det{}^l(w) w \right) \varepsilon_n(x) + \sum_{l=1}^{n} B_l(x) \varepsilon_{n-l}(x) \right) \alpha_0^{n+c} = 0. \tag{3.5}$$

By this equation, we have $c = -e_0 k_{H_0, j}$ since $w\varepsilon_0 = \det^{-j}(w) \varepsilon_0$ $(w \in W_0)$.

For a moment, assume that $e_0 k_{H_0, l} \not\equiv e_0 k_{H_0, l'}$ (mod $\mathbb{Z}$) for $l \not\equiv l'$ (mod $e_0 \mathbb{Z}$). Then $n - e_0 k_{H_0, j} + \sum_{l=0}^{e_0 - 1} k_{H_0, l} \sum_{w \in W_0} \det^l(w) w$ is invertible since the eigenvalues of w are $\det^{-l'}(w)$ $(l' = 0, \dots, e_0 - 1)$. It follows that equation (3.5) is uniquely solved and we have $w\varepsilon_n(x) = \det^{n-j}(w) \varepsilon_n(x)$ for $w \in W_0$ by induction. Hence

$$\varepsilon(x, \alpha_0) = \alpha_0^{j - e_0 k_{H_0, j}} \sum_{n=0}^{\infty} \varepsilon_n(x) \alpha_0^{n-j}. \tag{3.6}$$

Since the series $\sum_{n=0}^{\infty} \varepsilon_n(x)\alpha_0^{n-j}$ is W_0-invariant, we have proved Theorem 3.6 for generic k. By the continuity of $\varepsilon(x, \alpha_0)$ with respect to k, Theorem 3.6 is proved. □

Definition 3.7 Let $\{\varepsilon_i; i = 1, \ldots, l = \deg\tau\}$ be a local basis of $\mathcal{E}^{\nabla_\tau(k)}$ around $x_0 \in X^{\mathrm{reg}}$ and let $\{\sigma_j; j = 1, \ldots, l\}$ be a basis of $(\mathrm{Harm}(\tau) \otimes E^*)^W$, as in the proof of Proposition 2.4. For $k = (k_{C,j})_{C,j}$, we define a matrix

$$M(k) = (\sigma_j(\varepsilon_i))_{i,j=1,\ldots,l}.$$

Corollary 3.8 (i) *The matrix $M_0 := M(k = 0)$ is a minimal τ-matrix.*

(ii) *For all k, $M(k)$ is a nonsingular matrix. For any element b of B, we define a matrix $\tau(k)(b)$ by*

$$\mu(b)M(k) = \tau(k)(b)M(k),$$

where $\mu(b)M(k)$ is analytic continuation along b. Then $\tau(k)$ is the monodromy representation of $\mathcal{E}^{\nabla_\tau(k)}$ (cf. Theorem 3.6), written as matrix with respect to the local basis $\{\varepsilon_i; i = 1, \ldots, l = \deg\tau\}$. In particular, $\tau(k)(b)$ is an element of $GL(l, S)$ where S denotes the ring of entire functions in k.

(iii) *Let τ be irreducible. The Euler vector field $\mathbb{E}$ acts on $M(k)$ in the following way:*

$$\mathbb{E}M(k) = M(k)C_{\mathbb{E}}^{M}(k) \tag{3.7}$$

with

$$C_{\mathbb{E}}^{M}(k) = C_{\mathbb{E}}^{M_0} - s(\tau, k)\mathrm{Id}.$$

and where the scalar $s(\tau, k)$ is given by:

$$s(\tau, k) := \frac{1}{\deg\tau} \sum_{C \in \mathcal{C}} e_C |C| \sum_{j=0}^{e_C - 1} k_{C,j} n_{C,j}^{\tau}.$$

(iv) *We have*

$$\det(M(k)) = \mathit{const.} \prod_{C \in \mathcal{C}} \pi_C^{\epsilon_C(k)},$$

where

$$\epsilon_C(k) = \sum_{j=0}^{e_C - 1} j n_{C,j}^{\tau} - e_C \sum_{j=0}^{e_C - 1} k_{C,j} n_{C,j}^{\tau}.$$

(v) *If $k_{C,j}$ is an integer for every C and j, then $\tau(k)(b)$ only depends on the image of b in W (cf. Theorem 3.4(iii)). This induces a map from $\mathbb{Z}^{\sum_{C\in\mathcal{C}} e_C}$ to the space* Funct(Rep(W), Rep(W)) *of functors from* Rep(W) *to* Rep(W) *by*

$$\begin{aligned}\mathbb{Z}^{\sum_{C\in\mathcal{C}} e_C} \ni k = (k_{C,j}) \\ \mapsto (\gamma(k) : \tau \mapsto \tau(-k)) \in \mathrm{Funct}(\mathrm{Rep}(W), \mathrm{Rep}(W)).\end{aligned}$$

Moreover, the matrix elements of $M(k)$ are rational functions on V in this situation, with their poles contained in $\mathcal{A}$.

(vi) *If k is integral, then the local data (cf.* (2.1)*) $n^{\tau}_{C,j}$ and $n^{\tau(k)}_{C,j}$ coincide for any integral k.*

Proof. (i) This is a direct consequence of Definition 3.7.

(ii) The regularity of $M(k)$ is a direct consequence of its definition. The remaining statement follows from Theorem 3.6.

(iii) The Euler vector field $\mathbb{E}$ acts on $\mathcal{E}^{\nabla_\tau(k)}$. By definition, $\nabla_\tau(k)_{\mathbb{E}} = \mathbb{E} + \langle \mathbb{E}, \omega\rangle$ is 0 on $\mathcal{E}^{\nabla_\tau(k)}$. On the other hand, $\langle \mathbb{E}, d\log\alpha_H\rangle = 1$ for any $H \in \mathcal{A}$, and $\langle \mathbb{E}, \omega\rangle = \sum_{H\in\mathcal{A}} a_H$ is an element of the center of $\mathbb{C}[W]$, hence it acts on E by scalar multiplication since we assume τ to be irreducible in this item. This scalar is equal to $s(k,\tau)$, since

$$\begin{aligned}\mathrm{tr}\sum_{H\in\mathcal{A}} a_H &= \sum_{H\in\mathcal{A}}\sum_{j=0}^{e_C-1} k_{H,j}\sum_{w\in W_H}\det{}^j(w)\mathrm{tr}_E w \\ &= \sum_{H\in\mathcal{A}}\sum_{j=0}^{e_C-1} k_{H,j}\sum_{w\in W_H}\sum_{j'=0}^{e_C-1} n^{\tau}_{C,j'}\det{}^{j-j'}(w) \\ &= \sum_{C\in\mathcal{C}} e_C|C|\sum_{j=0}^{e_C-1} k_{H,j} n^{\tau}_{C,j}.\end{aligned}$$

Therefore the Euler vector field $\mathbb{E}$ acts on $\mathcal{E}^{\nabla_\tau(k)}$ by the scalar $-s(\tau,k)$, and $\mathbb{E}M(k) = M(k)C^M_{\mathbb{E}}(k)$, as was claimed.

(iv) We may and will assume that τ is irreducible here. Since the series $\sum_{n=0}^{\infty} \varepsilon_n(x)\alpha_{H_0}^{n-j}$ in (3.6) is W_{H_0}-invariant, $\sigma(\varepsilon_n(x)\alpha_{H_0}^{n-j})$ is a W_{H_0}-invariant function for any $\sigma \in (\mathrm{Harm}(\tau)\otimes E^*)^W$. It follows that this function is regular at H_0 because the pole of it is of order less than e_{H_0}.

As in the proof of Proposition 2.4 (iii), locally at $H \in C$, there exists a matrix $D \in GL(l,\mathbb{C})$ such that

$$DM(k) = M_H(k)M'(k),$$

where $M'(k)$ is a holomorphic W_H-invariant matrix and

$$M_H(k) = \mathrm{diag}(\alpha_H^{-k_{H,0}e_H} I_{n^\tau_{H,0}}, \alpha_H^{1-k_{H,1}e_H} I_{n^\tau_{H,1}}, \ldots, \alpha_H^{e_H-1-k_{H,e_H-1}e_H} I_{n^\tau_{H,e_H-1}}). \tag{3.8}$$

It follows that

$$\det M(k) = \prod_{C \in \mathcal{C}} \pi_C^{\epsilon_C(k)} \cdot r,$$

where, r is a W-invariant, entire function. Using (iii) it follows that $\mathbb{E}r = 0$ and (iv) is proved.

(v) If $k_{C,j}$ is an integer for every C and j, then $\mu(l_H)^{e_C} = \mu(l_H^{e_C}) = 1$ by Theorem 3.6. Hence $\tau(k)(b)$ only depends on the image of b in W by Theorem 3.4. This means that, with respect to the chosen basis of E, all the local sections of $\mathcal{E}^{\nabla_\tau(k)}$ extend to global holomorphic sections on V^{reg}. Let us prove the rationality of $M(k)$ now. The connection ∇_τ has simple poles at the reflection hyperplanes, and also at ∞. It follows that the pole orders at the reflection hyperplanes and at infinity are bounded by the eigenvalues of the residues of ∇_τ. Multiply a global flat section σ with a suitable power of $\prod_{C\in\mathcal{C}} \pi_C$ so that the resulting product σ' is entire on V. Then the restriction of σ' to any complex line $L \subset V$ is a polynomial of degree $\leq N$ for some suitable $N \in \mathbb{N}$. Therefore, σ' is killed by all homogeneous constant coefficient differential operators of order $> N$, which implies that σ' is a polynomial. We conclude that $M(k)$ is a matrix of rational functions on V, whose poles are possibly at $\mathcal{A}$, and that $\tau(k)$ descends to a representation of W.

We have

$$\begin{aligned}\mathbb{Z}^{\sum_{C\in\mathcal{C}} e_C} \ni k &= (k_{C,j}) \\ &\mapsto (\gamma(k) : \tau \mapsto \tau(-k)) \in \mathrm{Funct}(\mathrm{Rep}(W), \mathrm{Rep}(W)),\end{aligned}$$

since it is easy to verify the functoriality.

(vi) This follows from (3.8). □

Remark 3.9 (i) The functor $\gamma(k)$ does not respect $\otimes$, Hom and $*$ (contragredient). The reason for this is simply that the tensor product of two KZ connections is not equal to the KZ connection of the tensor product of the two representations of W involved (and similarly for the other constructions from linear algebra that are mentioned).

(ii) If W is a simply laced Coxeter group, the functor $\gamma(0,1)$ corresponds to the involution $i : \hat{W} \longrightarrow \hat{W}$ that was introduced by Lusztig, with the property that $i(\cdot) \otimes \det$ maps special representations to special representations. Similarly, for a Coxeter group that is not simply laced, the involution i defined by Lusztig is equal to $\gamma((0,1),(0,1))$. This involution i plays a role in the study of cells and special representations of W.

Conjecture 3.10 *For k and k' in $\mathbb{Z}^{\sum_{C\in\mathcal{C}} e_C}$, $\gamma(k)\gamma(k')$ equals $\gamma(k+k')$.*

Remark 3.11 (i) Conjecture 3.10 is clearly true if the following question, raised by Deligne-Mostow [10], has an affirmative answer: For a given irreducible representation τ of W, is it *uniquely* possible to deform τ to a family of representations τ_q of B such that the eigenvalues of $\tau_q(l_H)$ satisfy the relation

$$\prod_{j=0}^{e_C-1}(\tau_q(l_H)-\zeta_C^j q_{C,j})=0?$$

In turn, this question has an affirmative answer (locally in a neighbourhood of $q_{C,j}=1$ at least) if the so-called cyclotomic Hecke algebra of W, introduced by Broué and Malle [5], can be generated by $|W|$ elements over its coefficient ring. This was conjectured by Broué, Malle and Rouquier [7], and checked in many cases. We will discuss these matters extensively in Section 6 and Section 7.

(ii) If Conjecture 3.10 is true, then $\gamma(k)$ maps $\hat{W}$ to itself. This follows from the fact that $\gamma(k)$ respects direct sums and dimensions. If 3.10 is true, $\gamma(k)$ has inverse $\gamma(-k)$, but the $\gamma(-k)$ pre-image of an irreducible representation must be irreducible by the above. In this situation we have an action $k\to\gamma(k)$ of the lattice $\mathbb{Z}^{\sum_{C\in\mathcal{C}} e_C}$ on $\hat{W}$. Let I be the kernel of this action, and denote by G_W the finite abelian group $\mathbb{Z}^{\sum_{C\in\mathcal{C}} e_C}/I$. We will identify this group with its image in $\mathrm{Per}(\hat{W})$. In Section 6 we will show that this group is isomorphic with the geometric Galois group of the character field of the cyclotomic Hecke algebra, in the situation where this algebra is free over its coefficient ring. In particular we prove in this case that the character field is an *abelian* extension of the coefficient field of the cyclotomic Hecke algebra, assuming that the coefficient field contains $\mathbb{C}$.

4 Symmetries of fake degrees

In this section, we prove the "fake degree symmetry", which is the main theorem of this note.

The key result is the following lemma, which shows how one may construct minimal matrices for W using the KZ connection. We use the notations introduced in Definition 3.7 and Corollary 3.8.

Lemma 4.1 *Choose $b=(b_C)_{C\in\mathcal{C}}$, where $b_C\in\{0,1,\dots,e_{C-1}\}$ for each $C\in\mathcal{C}$. Define $k_b\in\mathbb{Z}^{\sum_{C\in\mathcal{C}} e_C}$ by putting $(k_b)_{C,j}=1$ if $j\geq b_C$, and $(k_b)_{C,j}=0$ if $j<b_C$. Let χ_b be the one dimensional representation of W associated with $\pi_b=\prod_{C\in\mathcal{C}}\pi_C^{e_C-b_C}$. Then $\pi_b\cdot M(k_b)$ is a minimal matrix of type $\tau(k_b)\otimes\chi_b$ $(=\gamma(k_b)(\tau)\otimes\chi_b)$.*

Proof. By Corollary 3.8 (v), $M(k_b)$ is a rational matrix with poles in $\mathcal{A}$, and by (3.8) one sees that $\pi_b \cdot M(k_b)$ has polynomial entries. It is a nonsingular matrix of type $\tau(k_b) \otimes \chi_b$ by Corollary 3.8 (ii). Definition 2.2 (ii) is satisfied because of Corollary 3.8 (iii). It remains to prove the minimality. We will appeal to Remark 2.3 (i) for the proof of minimality. By that remark it suffices to compute the degree of the determinant of $\pi_b \cdot M(k_b)$. From Corollary 3.8 (iv) and (vi) we find that the degree equals $\sum_C |C|\epsilon'_C(k_b)$ with

$$\begin{aligned}
\epsilon'_C(k_b) &= \epsilon_C(k_b) + (e_C - b_C)\dim(\tau) \\
&= \sum_{j=0}^{e_C-1} jn^{\tau}_{C,j} - e_C \sum_{j=b_C}^{e_C-1} n^{\tau}_{C,j} + (e_C - b_C)\dim(\tau) \\
&= \sum_{j=0}^{b_C-1} (j + e_C - b_C)n^{\tau}_{C,j} + \sum_{j=b_C}^{e_C-1} (j - b_C)n^{\tau}_{C,j} \\
&= \sum_{j=0}^{b_C-1} (j + e_C - b_C)n^{\tau(k_b)}_{C,j} + \sum_{j=b_C}^{e_C-1} (j - b_C)n^{\tau(k_b)}_{C,j} \\
&= \sum_{j=0}^{e_C-1} jn^{\tau(k_b)\otimes\chi_b}_{C,j}
\end{aligned}$$

This shows the minimality of $\pi_b \cdot M(k_b)$ by Remark 2.3 (i). □

Theorem 4.2 (Fake degree symmetry) *Let τ be irreducible. With notations as in Lemma 4.1,*

$$F_{\chi_b \otimes \tau(k_b)}(T) = T^{N(\tau,b)} F_\tau(T).$$

where

$$N(\tau, b) = \sum_{C \in \mathcal{C}} |C| \sum_{j=0}^{b_C-1} \left(\frac{e_C n^{\tau}_{C,j}}{\deg \tau} - 1\right)$$

Proof. This is a direct consequence of (3.7), in view of Lemma 4.1 and Corollary 2.6. □

5 Coxeter-like presentation of W

So far we used only one straightforward topological fact, the relation between the braid group B and the pure braid group P, Theorem 3.4. We also avoided completely the use of the so called cyclotomic Hecke algebra, although we already mentioned in Remark 3.11 that this algebra plays an important role when one tries to say more about the meaning and properties of the functor $\gamma(k)$. In the next sections we will discuss this in some detail. Unfortunately, one is forced to make the serious assumption that the cyclotomic Hecke algebra is a free algebra over its coefficient ring. This was conjectured by Broué, Malle and Rouquier in [7], but at present no general proof is known. It has been proved for the infinite series by Ariki (see [1]) and by Broué and Malle (see [5]) in about half of the exceptional cases.

Let us now start the discussion of the cyclotomic Hecke algebra and its relation to the previous sections. The results and ideas in this and the next section are entirely due to Broué, Malle and Rouquier. A good reference for this material is [7].

To give the relation between the braid group and the cyclotomic Hecke algebra one first needs *Coxeter-like* presentations of W, that are well behaved with respect to the braid group B.

Definition 5.1 A Coxeter-like presentation of W is a presentation of W given by a minimal set of generators $\mathcal{S} \subset W$ such that

(i) The set $\mathcal{S}$ consists of reflections of W, and all relations are generated by homogeneous *braid relations* and the *order relations* (relations of the form $s^d = e$, for $s \in \mathcal{S}$).

(ii) There exists a choice of l_H (for any H with $s_H \in \mathcal{S}$) such that $\{l_H\}_{H, s_H \in \mathcal{S}}$ with just the homogeneous relations from (i) form a presentation of B.

Remark 5.2 (i) It is well known that Coxeter groups have such presentations (see [4], [9]).

(ii) Broué-Malle-Rouquier have shown that there exists such a presentation of W for all but $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}$ and G_{34}. For these groups, the existence of such a presentation is conjectural.

6 Cyclotomic Hecke algebras and monodromy

In this section, we define the Hecke algebra $H_u(W)$ of finite complex reflection groups which have a Coxeter-like presentation.

Let S be the ring of entire functions in $(k_{C,j})_{C \in \mathcal{C}, j=0,\ldots,e_C-1}$ and let K be its field of fractions. We also define $R = \mathbb{Z}[u_{C,j}^{\pm}; C \in \mathcal{C}, j = 0, \ldots, e_C - 1]$ ($u_{C,j}$ are indeterminant) and let Q be its field of fractions. The following definition is from [7].

Definition 6.1 The (topological) Hecke algebra $H_u(W)$ is the R-algebra $R[B]/J$, where J is the ideal generated by the elements $\prod_{j=0}^{e_H-1}(l_H - u_{H,j})$.

Corollary 6.2 *If W has Coxeter-like presentation, then the image $\{T_s \mid s \in \mathcal{S}\}$ of $\{l_H \mid H \in \mathcal{A}$ such that $s_H = s \in \mathcal{S}\}$ generates $H_u(W)$ and is subject to the relations $\prod_{j=0}^{e_C-1}(T_s - u_{s,j}) = 0$. Together with the braid relations (which already hold in B) this is a presentation of $H_u(W)$. (Here $u_{s,j} = u_{H,j}$ if $s = s_H$.)*

With this definition, the next theorem follows directly from Theorem 3.6, except for the last assertion (iii).

Theorem 6.3 *We take Knizhnik-Zamolodchikov connection with values in $E = \mathbb{C}[W]$, and let $\mu(k)$ be its monodromy representation.*

(i) *For $b \in B$, $\mu(k)(b)$ coincides with the left multiplication by an element $\mu(k)_b \in (\mathbb{C}[W])^\times$.*

(ii) *Let us embed R in S by means of the substitution*

$$u_{H,j} = \det(s_H)^j \exp(-2\pi\sqrt{-1}k_{C,j}).$$

Then the representation $\mu(k)$ factorizes as follows:

$$\begin{array}{ccc} S[B] & \xrightarrow{\mu(k)} & S[W] \\ & \searrow \quad\quad \nearrow & \\ & S \otimes_R H_u(W) & \end{array}.$$

A similar factorization holds for the matrix representations $\tau(k)$ defined in Corollary 3.8(ii). These representations can be obtained by restriction of $\mu(k)$ to a minimal left ideal of $\mathbb{C}[W]$ of type τ.

(iii) *(Broué-Malle-Rouquier [7]) If W has a Coxeter-like presentation and $H_u(W)$ can be generated by $|W|$ elements as R-module, then $\mu(k)$ induces an K-algebra isomorphism*

$$K \otimes_R H_u(W) \xrightarrow{\sim} K[W]. \tag{6.1}$$

Moreover, in this situation $H_u(W)$ is free over R of rank $|W|$.

Remark 6.4 The assumption of Theorem 6.3 (iii) is conjectured to be true in all cases and is known for the basic series by the work of Ariki and Koike [2] and Ariki [1], and the work of Broué and Malle [5] in many of the exceptional cases.

Hypothesis 6.5 From now on we will assume that W has a Coxeter-like presentation and that $H_u(W)$ can be generated by $|W|$ elements, as required in Theorem 6.3 (iii).

Corollary 6.6 *The algebra $Q \otimes H_u(W)$ is a semisimple algebra of rank $|W|$, which splits over $K \supset Q$.*

Let F denote the character field of $Q \otimes H_u(W)$, which we define as the subfield of K generated by the values of the irreducible characters of $K \otimes H_u(W)$ on $Q \otimes H_u(W)$. It is a subfield of K containing Q. Note that the characters take values in S on an R-basis of $H_u(W)$, by Theorem 3.6 and Theorem 6.3(ii), (iii). In fact, the values of these characters on $H_u(W)$ are even in the integral closure of R in S, by a well known argument due to Steinberg (for example, see [8], Proposition 10.11.4). In particular, F is a Galois extension of Q. Let G_F denote its Galois group. This Galois group acts naturally on the irreducible characters of $K \otimes H_u(W)$. The evaluation homomorphism $f \to f(0)$ of S, applied to the characters, defines a bijection between the irreducible characters of $K \otimes H_u(W)$ and of W (by Tits' specialization theorem). Hence we arrive at a certain action of G_F on $\hat{W}$. The Galois group of $\mathbb{C} \cdot F$ over $\mathbb{C} \cdot Q$ is called the geometric Galois group, which we will denote by G_F^{geom}. By restriction to $F \subset \mathbb{C} \cdot F$ we have a canonical homomorphism $G_F^{geom} \to G_F$, which is obviously injective. We identify G_F^{geom} with its image in G_F. Via this embedding of G_F^{geom} in G_F we have now defined an action of G_F^{geom} on $\hat{W}$, which we will denote by α. Note that α is an injective homomorphism $G_F^{geom} \to \mathrm{Per}(\hat{W})$.

Theorem 6.7 *If Hypothesis 6.5 holds, then the field $\mathbb{C} \cdot F$ is contained in a field that is obtained from $\mathbb{C} \cdot Q$ by adjoining radicals of the form $(u_{C,j})^{1/N}$, for a suitable N. In particular, $\mathbb{C} \cdot F$ is an abelian extension of $\mathbb{C} \cdot Q$.*

Proof. As was explained above, the characters of $K \otimes H_u(W)$ take values in the integral closure of R in S on $H_u(W)$. The lattice $L = \mathbb{Z}^{\sum_{C \in \mathcal{C}} e_C}$ (as in Corollary 3.8 (v)) acts on S by means of the action $(\beta(l)f)(k) = f(k-l)$. Notice that Q is fixed for this action! By Theorem 6.3(ii), for every $\tau \in \hat{W}$ we have a representation $\tau(k) : S \otimes H_u(W) \to \mathrm{Mat}(d_\tau \times d_\tau, S)$ (where d_τ is the degree of τ) and these representations exhaust the equivalence classes of irreducible representations of $K \otimes H_u(W)$ (by Theorem 6.3(iii)). We define $\beta(l)$ ($l \in L$) in the obvious way on $S \otimes H_u(W)$ and on $\mathrm{Mat}(d_\tau \times d_\tau, S)$ (i.e. coefficient-wise) and then define $\beta(l)(\tau) = \beta(l) \circ \tau \circ \beta(-l)$. Obviously, $\beta(l)(\tau)$ is S-linear, and it is also a representation of $S \otimes H_u(W)$ because $\beta(l)$ is a ring homomorphism of $S \otimes H_u(W)$. Using Tits' specialization theorem, applied to specializations of S at lattice points, we find that the new representation so obtained by translation, is irreducible. In this way, we have an action β of the lattice L on the finite set $\mathrm{Irr}(K \otimes H_u(W))$. Hence there exists a $N \in \mathbb{N}$ such that $N \cdot L$ fixes every representation. This means that a character value $\chi_\tau(b)$ is periodic with respect to the lattice $N \cdot L$. Therefore it can be viewed as a univalued function of the coordinates $z_{C,j} = (u_{C,j})^{1/N}$, which is as such still a holomorphic function outside the coordinate hyperplanes $z_{C,j} = 0$, and integral over Q, hence certainly integral over the larger ring of polynomials in $z_{C,j}$. But this polynomial ring is integrally closed in the field of meromorphic functions in the $z_{C,j}$, because a solution of a monic equation is bounded in norm by the sum of the norms of the coefficients of the equation (including the top coefficient 1). Our conclusion is that $\chi_\tau(b)$ is a polynomial in the coordinates $z_{C,j} = (u_{C,j})^{1/N}$.

□

Corollary 6.8 (i) *Hypothesis 6.5 implies that Conjecture 3.10 holds. In particular, as was mentioned in Remark 3.11 (ii), the functors $\gamma(l)$ of Corollary 3.8 (v) define an action of L on $\hat{W}$.*

(ii) *The action α of G_F^{geom} on $\hat{W}$ defines an isomorphism of G_F^{geom} with the group G_W described in Remark 3.11 (ii).*

Proof. In fact, everything is clear by the remark that the action β of L defined in the proof of Theorem 6.7 is equal to the action γ when we identify $\mathrm{Irr}(K \otimes H_u(W))$ and $\hat{W}$ via the specialization $f \to f(0)$ of S, as always. From Theorem 6.7 is is clear that G_F^{geom} is the quotient of $L/(N \cdot L)$ by the subgroup that fixes all the irreducible characters via the action β, and this is isomorphic to G_W by the definition of G_W. □

7 Applications

The phenomenology in the field of cyclotomic Hecke algebras is much further developed than the theory, and the present paper does not change this situation very much! Recently, Gunter Malle made a thorough study of the character and splitting fields of representations of cyclotomic Hecke algebras in [16]. His study is based on the theory developed in [1] for the case of the cyclotomic Hecke algebra of the groups $G(m,p,n)$, and on a case by case analysis of the primitive groups (always assuming Hypothesis 6.5 of course). There are many intriguing observations in Malle's paper [16], and some of these are closely related to the questions we have discussed here. *One remark has to be made beforehand. In the comparison of results of [16] and the results of the present paper one must realize that we are discussing the topological Hecke algebra here, with its presentation as described in Corollary 6.2. However, Malle uses the abstract Hecke algebra, defined with a similar presentation but where the topological braid group B is replaced by the abstract braid group associated with a certain choice of a presentation of W (by removing the order relations from the presentation of W). In other words, only when the presentation of W that Malle uses is a Coxeter-like presentation it is clear that we are discussing the same algebra. Recall that there are still some groups for which the existence of a Coxeter-like presentation is not known. See section 5.* With this understood, let me list the main facts revealed in Malle's paper, and comment on these from the point of view of the theory in this paper.

(i) The character field F is a regular, abelian extension of $k(u_{C,j})$, where k denotes the splitting field of the group algebra $\mathbb{Q}[W]$. (It is known that k is the character field of the reflection representation of W, by a result of [3]). This means that $G_F^{geom} = \mathrm{Gal}(F/k(u_{C,j}))$, and that this group is abelian.

The equality of these two Galois groups is based on the (empirical!) fact that k contains all the roots of unity of order d, whenever d is the order of an element of G_F^{geom} (*loc. cit.* Cor. 4.8).

(ii) The order of G_F^{geom} can be arbitrarily large (but of course subject to the condition imposed by (i)), even if all the reflections in W have order 2 (cf *loc. cit.* Ex. 4.5). The orders of elements of G_F^{geom} do not necessarily divide the order of the center $Z(W)$ (but in the "well-generated" case they do, see *loc. cit.* Prop. 7.2).

(iii) A Beynon-Lusztig type of "semi-palindromicity" for fake degrees of rational representations of the 1-parameter cyclotomic Hecke algebra (*loc. cit.* Thm. 6.5).

(iv) The group W can be generated by n complex reflections if and only if the fake degree of the reflection representation is semi-palindromic (*loc. cit.* Prop. 6.12).

We have not much to say about (i), since the methods used here do not seem fit for the study of the full Galois group of F. The only thing we have proved is the fact that G_F^{geom} is abelian (Theorem 6.7).

It is implied by (i) that the order of every element of G_F^{geom} is a divisor of the order of the group of roots of unity in k. Looking at the explicit results of Malle and of Broué and Malle [5] in the primitive cases, and at (ii) mentioned above, this seems to be the only general rule. From the point of view of the present paper this fact is rather mysterious, except in some cases where the order of all nontrivial elements in G_F^{geom} is two. This includes all the Coxeter cases. The case of Coxeter groups was dealt with in the paper [17], but since the argument given there is incomplete, we will fill in the details here.

We start with some elementary relations:

Proposition 7.1 *Let $g_{C,j} \in G_W = G_F^{geom}$ be the image under γ of the basis element $b_{C,j}$ of the lattice L, defined by $(b_{C,j})_{C',j'} = -\delta_{C,C'}\delta_{j,j'}$. Let m_C denote the permutation of irreducible representations of W given by: $m_C(\tau) = \tau \otimes \chi_C$, where χ_C is the linear character associated to the pseudo invariant π_C. We have the following relations:*

(i) *The $g_{C,j}$ mutually commute.*

(ii) $g_{C,0} \cdots g_{C,e_C-1} = \mathrm{id}$.

(iii) $m_C^{e_C} = \mathrm{id}$.

(iv) $g_{C,j}m_{C'} = m_{C'}g_{C,j}$ *($C \neq C'$).*

(v) $g_{C,j}m_C = m_C g_{C,j+1}$ *(cyclic).*

In particular, we have $(1 \le j \le e_C)$: $(m_C g_{C,1} \cdots g_{C,e_C-1})^j = m_C^j g_{C,j} \cdots g_{C,e_C-1}$. *Thus the order of* $m_C g_{C,1} \cdots g_{C,e_C-1}$ *divides* e_C. *(Note that the powers of*

$$m_C g_{C,1} \cdots g_{C,e_C-1}$$

are the operations on the representation τ *that occur in the fake degree symmetry formula Theorem 4.2.)*

Proof. In the notation of Definition 3.3, assertion (ii) follows from the remark (verified by means of direct computation) that the flat sections of $\nabla_\tau(b_{C,0} + \cdots + b_{C,e_C-1})$ are of the form $\pi_C^{e_C} \cdot e$ with $e \in E$. This has type τ since $\pi_C^{e_C}$ is W-invariant. Likewise, (iv) and (v) follow from a direct computation of the tensor product of the KZ-connection $\nabla_\tau(k)$ and the (trivial) KZ-connection $\nabla_{\chi_C}(0)$. It is left to the reader. The remaining statements are trivial. □

Remark 7.2 It is informative to have a look at Malle's table 7.1, displaying all the irrationalities for the primitive groups, at this point. For example, one can check Proposition 7.1 (ii) directly from this table: when we specialize $u_{C,j} \to u_C$ everything becomes rational in the u_C.

Now we may reprove the following well known results:

Proposition 7.3 *Let* W *be a Coxeter group, or of type* $G(4,2,n)$ $(n > 2)$, *or* G_{31}, *in which case we need hypothesis 6.5, as always. Then the elements* $g_{C,j}$ $(j = 0, 1)$ *have order 2, and* $g_{C,0} = g_{C,1}$. *(In other words,* $\mathbb{C} \cdot F$ *is a subfield of the field obtained from* $\mathbb{C} \cdot Q$ *by adjunction of the square roots* $\sqrt{q_{C,0}q_{C,1}}$.*)*

Proof. The point is that there exists a Kazhdan-Lusztig involution j of the (cyclotomic) Hecke algebra, defined by $j(T_i) = -q_{C,0}q_{C,1}T_{i'}^{-1}$ where s_i and $s_{i'}$ belong to hyperplanes in C. In fact, if W is Coxeter we may take $i = i'$, but for the complex cases we have to flip elements according to the symmetry axes in the diagram, in order to preserve the circular relations. This induces an involution on the irreducible representations, also denoted by j, defined by $j(\tau) = \tau \circ j$. It is simple to see that $j(\tau) = \tau \otimes \det = \prod_C m_C(\tau)$ in the Coxeter case. In the complex cases we mentioned this is also true because the aforementioned flip of generators is an *inner* automorphism. (For G_{31} this follows from a calculation in [5], Bemerkung 6.5, and for the infinite series $G(4,2,n)$ it is similar.)

On the other hand, j obviously commutes with the elements $g_{C,i}$ since it is rational in the $q_{C,i}$. Combining this with Proposition 7.1 we obtain the result. □

There is a bewildering number of groups acting on $\hat{W}$: $Gal(F/Q)$, the group of linear characters (via tensoring), complex conjugation, diagram automorphisms, and the symmetric groups $\mathfrak{S}_C$ of permutations of $\{u_{C,j} \mid 0 \le j \le e_C - 1\}$. There are some easy observations about the groups that some of these actions generate. For example, the actions of symmetric group $\mathfrak{S}_C$ and the lattice L_C (i.e. all the parameters are 0 for hyperplanes not in C) combine to give an action of the affine Weyl group $\mathfrak{S}_C^{\mathrm{aff}}$. Or, as remarked in [16], complex conjugation and $g_{C,0}$ generate an action of the infinite dihedral group. Another example is the cyclic group of order e_C described in Proposition 7.1. Maybe it is important to investigate this systematically, but I did not see anything useful other than Proposition 7.3

The next issue (iii) from Malle's paper, the Beynon-Lusztig type of "semi-palindromicity" of the fake degrees of rational representations, can be fully understood in terms of our results. The Galois operation δ he introduces is easily identified in our notations as $\delta = (\prod_C g_{C,0})^{-1}$ (recall how R is embedded in S, see Theorem 6.3). With this notation, Malle observes the following:

Proposition 7.4 *([16], Theorem 6.5) Let $\mathcal{R}$ denote the set of reflections in W. In the notations of Definition 1.5, we have*

$$R_\tau(T) = T^c R_{\delta(\overline{\tau})}(T^{-1}) \tag{7.1}$$

where $c = \#\mathcal{R} - \sum_{r\in\mathcal{R}} \chi_\tau(r)/\chi_\tau(1)$.

Proof. Replace T by T^{-1} in (7.1), multiply both sides by $T^{\#\mathcal{R}}$ and rewrite the left hand side using the well known and elementary formula:

$$T^{\#\mathcal{R}} R_\tau(T^{-1}) = F_{\tau\otimes\det}(T).$$

On the right hand side we use Definition 1.5 and the observation $\overline{\delta(\overline{\tau})} = \delta^{-1}(\tau)$, to obtain

$$F_{\tau\otimes\det}(T) = T^{N-c} F_{\delta^{-1}(\tau)}(T)$$

Finally replace τ by $\delta(\tau)$ and use Proposition 7.1(ii) to see that (in the notation of Theorem 4.2) $\delta(\tau)\otimes\det = \tau(k_b)\otimes\chi_b$ where the vector b is defined by $b_C = 1\forall C$. The result now follows from Theorem 4.2. □

Finally, we have quoted assertion (iv) of Malle's paper because it is such a nice result related to fake degees. It goes back to Orlik and Solomon, who proved it by inspection. Malle suggests an alternative approach to this result, based on two empirical facts. First, the reflection character of the "1-parameter specialisation" of the Hecke algebra is rational if and only if W is generated by n reflections. Next, the reflection character of the "1-parameter specialisation" of the Hecke algebra is rational if and only if its fake degree is semi-palindromic (see [16], Corollary 4.9 and Proposition 6.12). Hence 7.4 gives an a priori proof of one out of the four implications in these two statements.

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