Part I:
Lectures on Dunkl Operators

1 Prologue

These are the lecture notes of a series of 5 lectures held at RIMS in October/ November 1997, in one of the workshops of the research project "Harmonic analysis on homogeneous spaces and representation of Lie groups". In these lectures I have discussed Dunkl operators in the trigonometric, differential setting. This subject has been very dear to me for many years, and it was a great pleasure to have the opportunity to lecture on this subject in a stimulating environment. My warm thanks go out to those who made this possible: to prof. T. Oshima for inviting me to participate in the research project "Harmonic analysis on homogeneous spaces and representation of Lie groups" at RIMS; to prof. M. Kashiwara for being my host at the RIMS institute; and to the note takers T. Honda, H. Ochiai, N. Shimeno, and K. Taniguchi for their kindness to prepare these notes.

The choice of the subject is based on my personal experience and taste. In view of the recent developments concerning Dunkl operators, one may object that my choice represents a rather limited point of view. Indeed, in view of Cherednik's work, the trigonometric differential Dunkl operators seem to be only a degenerate limit of a theory of commuting difference operators. These difference operators arise from commutation formulae inside Cherednik's double affine Hecke algebra. This magnificent insight has changed the way in which we ought to think about Dunkl operators and their applications.

Nonetheless, I have restricted myself to discuss the differential case. There are various reasons for doing so. First of all, there are a number of recent expositions ([19], [25], [5]) of the new algebraic theory of Dunkl operators and the double affine Hecke algebra. Second, the trigonometric differential limit that we consider, is very rich and it has served as a guideline for developments in the general theory. Third, there are aspects in the differential theory that have resisted generalization to the general theory so far. Especially with respect to harmonic analysis, the differential theory has currently reached a higher level of maturity (although an exciting start of the harmonic analysis for the difference equations can be found in [6]). It is this analytic aspect of the theory of Dunkl operators I shall concentrate on. Finally, although we will only deal with the differential theory, on our way we shall meet with the (degenerated) double affine Hecke algebra several times. As has been mentioned before, Cherednik's approach has profoundly changed our perception of Dunkl operators, and of course this also manifests itself in the differential theory. In fact, I hope and even expect that for some readers, this modern treatment of Dunkl operators will be a motivation to look more closely at the double affine Hecke algebra.

Let me make some personal historical comments on the development of the theory we will be studying in these notes. Dunkl operators were conceived by Charles Dunkl in 1989 (see [8]). He found these operators in the so-called rational differential situation, which is the basic example. He proved the two fundamental properties, the W-equivariance (which is in fact immediate here) and the marvelous commutativity, and he used this to set up a theory analogous to the theory of spherical harmonics.

Almost at the same time, but unaware of Dunkl's fundamental results, Gerrit Heckman and I were seeking to generalize the theory of the spherical function of Harish-Chandra. Our goal was a theory of multi-variable hypergeometric functions associated with a root system. Inspired by Tom Koornwinder's work [21] in this direction (already in the early seventies) we set up such a theory in a series of papers [14], [9], [27], [28].

Soon afterwards I noticed [29] that this theory provided natural tools (shift operators) that could be successfully applied to a number of combinatorial and analytic problems that were related to root systems (most notably, Macdonald's constant term conjectures for root systems [24]). In spite of these applications, the hypergeometric theory itself was not in a very satisfactory state at the time. The main arguments were indirect and complicated, avoiding at all times to use explicit knowledge of the defining differential equations of our hypergeometric function. The obstacle, psychologically, was that it *seemed* hopeless to write down these defining differential equations explicitly, since this was already impossible (in general) for Harish-Chandra's spherical function itself.

These difficulties were resolved in a rather drastic way when Gerrit Heckman noticed [11] the connection with Dunkl's work. Dunkl's operators provided a very simple method for constructing the differential equations we needed, in the rational version of our theory. Heckman defined a trigonometric version of these operators as well [12]. There was however a remarkable difference with the rational case: the trigonometric operators that Heckman found were W-equivariant, but they did not commute. Nonetheless these "Dunkl-Heckman" operators were important and useful, because they were the building blocks for the desired commuting (higher order) differential operators (and shift operators) in the trigonometric case.

The next development was Ivan Cherednik's discovery of the connection between (degenerated) affine Hecke algebras on the one hand, and Dunkl and Dunkl-Heckman operators on the other hand [2], [3]. This discovery had some important consequences. From the structure theory of Hecke algebras it was now obvious that there also existed *commuting* Dunkl-type operators in the trigonometric case. It is an interesting fact that these commuting operators are not W-equivariant in the trigonometric case. The joint spectral theory of these commuting "Dunkl-Cherednik" operators will be the main subject of study in these notes. Non-compact spectral theory started with De Jeu's important paper [18] (the rational case), and was then further explored in the trigonometric case in [32], [33] and in Cherednik's paper [4].

Cherednik's discovery also created a natural way to discretize the theory (creating the difference operators alluded to in the second paragraph of this prologue), by using the affine Hecke algebra instead of the degenerated version. This led to the complete solution of the Macdonald conjectures (including the "q-version"), and many new results (see [5] for a very good account of these developments).

2 Dunkl operators in the trigonometric setting

The basic reference for this section is [32].

2.1 Notation

We assume that the reader is familiar with root systems and their basic properties. However, in order to fix notations and conventions we will review the definitions of these and related fundamental structures in this subsection.

Let $\mathfrak{a}$ be a Euclidean vector space of dimension n. For $\alpha \in \mathfrak{a}^*$ we denote by $X_\alpha \in \mathfrak{a}$ the element corresponding to α. When α is nonzero we introduce the covector $\alpha^\vee \in \mathfrak{a}$ of α by the formula

$$\alpha^\vee = \frac{2X_\alpha}{(X_\alpha, X_\alpha)}.$$

A nonzero α in $\mathfrak{a}^*$ determines the orthogonal reflection $r_\alpha \in O(\mathfrak{a})$ in the hyperplane $\ker(\alpha)$ of $\mathfrak{a}$. This reflection is given by the formula

$$r_\alpha(\xi) = \xi - \alpha(\xi)\alpha^\vee.$$

In many instances the orthogonal transformation r_α will act on spaces derived from $\mathfrak{a}$, such as the complexification of $\mathfrak{a}$, certain stable lattices in $\mathfrak{a}$, tori that are a quotient of $\mathfrak{a}$ by such a stable lattice, and also on the dual $\mathfrak{a}^*$. In all these situations we will simply use the same notation r_α, since there is no danger of confusion (in the last case, notice that $r_\alpha = r_{\alpha^\vee}$ when we identify $\mathfrak{a}$ and $\mathfrak{a}^*$.

A finite subset $R \subset \mathfrak{a}^* \backslash \{0\}$ is called a root system when it satisfies the following properties:

(R1) R spans $\mathfrak{a}^*$.

(R2) $^\forall\alpha \in R$, $r_\alpha(R) = R$.

(R3) $^\forall\alpha, \beta \in R$, $\alpha(\beta^\vee) \in \mathbf{Z}$.

The elements of R are called roots. We shall always assume that R is reduced[1], which means that $\mathbf{R}\alpha \cap R = \{\pm\alpha\}$ for every $\alpha \in R$.

Clearly the set $R^\vee = \{\alpha^\vee \mid \alpha \in R\} \subset \mathfrak{a}$ is also a root system, called the dual or coroot system.

The group generated by the reflections r_α is a finite reflection group, called the Weyl group and denoted by $W = W(R)$. We shall also assume throughout these lectures that W acts irreducibly on $\mathfrak{a}^*$, or equivalently, that R is *indecomposable.* Even though all the results, when formulated properly, also hold without this assumption, it would be a notational burden not to assume this. On the other hand,

[1]This assumption is not necessary. Actually, an important class of orthogonal polynomials (Koornwinder-polynomials) arises from the non-reduced root system of type BC_n. However, we employ this assumption for simplicity.

it would not add anything meaningful not to make this assumption in the theory of the Dunkl operator. The related structures for a general root system R simply decompose as direct products over the indecomposable components of R.

Because of (R3), $Q = Q(R) = \mathbf{Z}R$ and $Q^\vee = Q(R^\vee)$ are stable lattices for the action of W. These lattices are called the *root lattice* and the *coroot lattice*, respectively. The dual lattice $P = \mathrm{Hom}_{\mathbf{Z}}(Q^\vee, \mathbf{Z}) \subset \mathfrak{a}^*$ is called the weight lattice of R, and is of course also W stable.

We put $\mathfrak{h} = \mathfrak{a}_{\mathbf{C}}$ and $\mathfrak{t} = \sqrt{-1}\mathfrak{a}$, hence we have $\mathfrak{h} = \mathfrak{a} + \mathfrak{t}$. Let H be the complex torus $H = \mathrm{Hom}_{\mathbf{Z}}(P, \mathbf{C}^\times) = Q^\vee \otimes_{\mathbf{Z}} \mathbf{C}^\times$. The Weyl group W stabilizes P and $Q^\vee$, hence W also acts on H. We have $H = TA$, where T is a compact torus and A is the real split torus, corresponding to $\mathfrak{t}$ and $\mathfrak{a}$ in $\mathfrak{h}$ respectively.

Choose and fix a half-space in $\mathfrak{a}^*$ such that none of the roots of R are in the boundary of this half-space. The roots in this half-space are said to be positive, and the set of positive roots is called a positive subsystem $R_+ \subset R$. Let Q_+ be the $\mathbf{Z}_+$-span of R_+. It is well known that Q_+ is a simplicial cone over $\mathbf{Z}_+$, and is generated over $\mathbf{Z}_+$ by a basis of roots $\{\alpha_1, \ldots, \alpha_n\}$. Put $r_i = r_{\alpha_i}$, then $S = \{r_1, \ldots, r_n\}$ is a set of generators of W. In fact these generators give a presentation of W as a Coxeter group, with relations $r_i^2 = 1$ and $(r_i r_j)^{m_{ij}} = 1$.

The set Q_+ defines an important partial ordering $<$ in $\mathfrak{a}^*$ by $\lambda < \mu$ iff $\mu - \lambda \in Q_+$. This ordering is called the dominance ordering.

When $\lambda(\alpha_i^\vee) \geq 0 \ \forall i \in \{1, \ldots, n\}$ we call λ dominant (and we call λ strongly dominant when all the inequalities are strict). The set $\mathfrak{a}_+^*$ of all strongly dominant elements is called the Weyl chamber. It is well known that the closure of the Weyl chamber $\overline{\mathfrak{a}_+^*}$ is a fundamental domain for the action of W. Let $P_+ \subset P$ denote the set of dominant weights. It is generated over $\mathbf{Z}_+$ by the basis $\{\lambda_i\}$ dual to $\{\alpha_i^\vee\}$. The weights λ_i are called fundamental weights.

Let $\mathbf{C}[H]$ be the space of Laurent polynomials (finite linear combinations of algebraic characters e^λ with $\lambda \in P$). By restriction to T one may identify this space of functions with the space of Fourier polynomial on T.

2.2 Dunkl-Cherednik operator

Proposition 2.1 *The divided difference operator* $\dfrac{1}{1-e^{-\alpha}}(1 - r_\alpha)$ *maps* $\mathbf{C}[H]$ *into itself.*

Proof. This easily follows from the summation over geometric series. This operator sends

$$e^\lambda \mapsto \begin{cases} e^\lambda(1 + e^{-\alpha} + \cdots + e^{(1-\lambda(\alpha^\vee))\alpha}) & \text{if } \lambda(\alpha^\vee) > 0 \\ 0 & \text{if } \lambda(\alpha^\vee) = 0 \\ -e^{r_\alpha\lambda}(1 + e^{-\alpha} + \cdots + e^{(1+\lambda(\alpha^\vee))\alpha}) & \text{if } \lambda(\alpha^\vee) < 0 \end{cases} .$$

This proves the required property. □

Notice the asymmetry, the difference between the formulae for positive exponents and for negative exponents. Only the largest element of λ and $r_\alpha\lambda$, (in the dominance order) shows up in the support of the image of e^λ. This property plays an important role in the sequel.

Let us introduce the Weyl denominator

$$\Delta = \prod_{\alpha\in R_+} \left(e^{\alpha/2} - e^{-\alpha/2}\right) = e^\delta \prod_{\alpha\in R_+} (1 - e^{-\alpha}) \in \mathbf{C}[H],$$

where $\delta = \frac{1}{2}\sum_{\alpha\in R_+} \alpha \in P$.

Corollary 2.2 *Skew functions in $\mathbf{C}[H]$ are divisible by Δ. If we denote the set of W-skew Laurent polynomials by $\mathbf{C}[H]^{-W}$, then $\mathbf{C}[H]^{-W} = \Delta\mathbf{C}[H]^W$.*

Proof. Let $p \in \mathbf{C}[H]^{-W}$. The previous proposition says that $p \in (1 - e^{-\alpha})\mathbf{C}[H]$. Since the algebra $\mathbf{C}[H]$ has the unique factorization property, and $(1 - e^{-\alpha})$ are coprime, p can be divided by Δ. □

Corollary 2.3 *We put $\varepsilon(w) = \det_{\mathfrak{a}} w$. Then we have*

$$\Delta = \sum_{w\in W} \varepsilon(w) e^{w\delta}.$$

Proof. Since the right hand side is skew, we have

$$\frac{1}{\Delta}\sum_{w\in W} \varepsilon(w) e^{w\delta} \in \mathbf{C}[H]^W.$$

Moreover the leading term in the dominance ordering must be 1. □

Let $k_\alpha \in \mathbf{C}$ be W-invariant root labels, that is, $k_\alpha = k_\beta$ if α, β are in the same W-orbit. We call $k = (k_\alpha)_{\alpha\in R}$ a multiplicity function on R. In this lecture we mainly consider real multiplicity functions and often assume that $k_\alpha \geq 0$ for any $\alpha \in R$. We set

$$\rho(k) = \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha \alpha \in \mathfrak{h}^*.$$

The hero of our story is the *Dunkl-Cherednik operator*, given by the following formula:

Definition 2.4 (Dunkl-Cherednik operator) For $\xi \in \mathfrak{h}$ define

$$T_\xi(k) = \partial_\xi + \sum_{\alpha\in R_+} k_\alpha \alpha(\xi) \frac{1}{1 - e^{-\alpha}} (1 - r_\alpha) - \rho(k)(\xi).$$

Here ∂_ξ denote the invariant vector field on the torus H corresponding to $\xi \in \mathfrak{h}$.

Remark 2.5 By Proposition 2.1, $T_\xi(k)$ maps $\mathbf{C}[H]$ to itself. We may also think of $T_\xi(k)$ as an operator acting on other function spaces on $\mathfrak{h}$, for example, holomorphic functions, $C^\infty(A)$, or $C_c^\infty(A)$. The point is that in these spaces the ideal of functions vanishing at the hyperplane $e^\alpha = 1$ is generated by $e^\alpha - 1$.

2.3 Commutativity

Theorem 2.6 *For any $\xi, \eta \in \mathfrak{h}$, we have*

$$[T_\xi(k), T_\eta(k)] = 0.$$

Proof. There are basically three proofs. A direct computation as in Dunkl's original paper, Cherednik's approach from conformal field theory (KZ equation), and Heckman's proof using orthogonality. We give Heckman's proof here.

We introduce two important structures on $\mathbf{C}[H]$. In the rest of this section we assume $k_\alpha \geq 0$ for any $\alpha \in R$. First, we define the hermitian inner product

$$(f, g)_k = \int_T f\, \bar{g}\, \delta_k\, dt,$$

where the weight function is given by

$$\delta_k = \prod_{\alpha \in R_+} \left| e^{\alpha/2} - e^{-\alpha/2} \right|^{2k_\alpha} = \prod_{\alpha \in R} \left| 1 - e^\alpha \right|^{k_\alpha}.$$

Second, we introduce a partial ordering $\triangleleft$ on P as follows : $\lambda \triangleleft \mu$ if either $\lambda_+ < \mu_+$ in dominance ordering (with λ_+ the unique dominant weight in $W\lambda$), or if $\lambda_+ = \mu_+$ and $\lambda > \mu$. This last inequality is not a typographical error!

The following lemma explains the importance of the ordering and the inner product defined above:

Lemma 2.7 *The operator $T_\xi(k)$ is upper triangular with respect to $\triangleleft$, and $T_\xi(k)$ is symmetric with respect to $(\cdot, \cdot)_k$ if $\xi \in \mathfrak{a}$.*

Proof. Using Proposition 2.1, we check that $T_\xi(k)$ is upper triangular with respect to $\triangleleft$. The symmetry property is a simple direct computation left to the reader. □

Definition 2.8 Define a basis $\{E(\lambda, k)\,;\, \lambda \in P\}$ of $\mathbf{C}[H]$ by the following conditions.

(a) $E(\lambda, k) = e^\lambda + \sum_{\mu \triangleleft \lambda} c_{\lambda,\mu} e^\mu$.

(b) For any $\mu \triangleleft \lambda$, $(E(\lambda, k), e^\mu)_k = 0$.

Now we come back to the proof of Theorem 2.6. $T_\xi(k)E(\lambda,k)$ also satisfies (a) and (b), except that its expansion in (a) has leading term $(\tilde{\lambda}(\xi))e^\lambda$ for some $\tilde{\lambda}$. The uniqueness shows that

$$T_\xi(k)E(\lambda,k) = \tilde{\lambda}(\xi)E(\lambda,k). \tag{2.1}$$

Therefore $\{E(\lambda,k)\,;\,\lambda\in P\}$ diagonalize simultaneously the Dunkl-Cherednik operators $T_\xi(k)$, hence these operators must mutually commute. □

The eigenvalue $\tilde{\lambda}(\xi)$ can be calculated easily by Proposition 2.1:

Corollary 2.9 *Define* $\varepsilon : \mathbf{R} \to \{\pm 1\}$ *by*

$$\varepsilon(x) = \begin{cases} 1 & x > 0 \\ -1 & x \le 0 \end{cases}$$

Given $\lambda \in P$*, the eigenvalue in equation* (2.1) *is given by*

$$\tilde{\lambda} = \lambda + \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha \varepsilon(\lambda(\alpha^\vee))\alpha = \lambda + w_\lambda^*(\rho(k)),$$

where w_λ^* *is the longest element in* W *sending* λ_+ *to* λ.

Proof. By Proposition 2.1, the eigenvalue $\tilde{\lambda}$ is given by

$$\begin{aligned}\tilde{\lambda} &= \lambda - \rho(k) + \sum_{\alpha\in R_+,\lambda(\alpha^\vee)>0} k_\alpha \alpha \\ &= \lambda + \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha \varepsilon(\lambda(\alpha^\vee))\alpha \\ &= w_\lambda^*(\lambda_+ + \rho(k)) \\ &= \lambda + w_\lambda^*(\rho(k)).\end{aligned}$$

□

Notice that the function ε is not skew symmetric at $x = 0$. We can decompose $\mathfrak{a}^*$ in a non symmetric way in the disjoint "chambers"

$$C_w = \{\lambda \in \mathfrak{a}^* \mid \lambda(\alpha^\vee) > 0 \ \forall\alpha \in R_+ \cap w(R_+) \text{ and } \lambda(\alpha^\vee) \le 0 \ \forall\alpha \in R_+ \cap w(R_-)\}$$

(with w traversing W) which lie between $w(\mathfrak{a}_+^*)$ and $\overline{w(\mathfrak{a}_+^*)}$. The map $\lambda \mapsto \tilde{\lambda}$ restricted to C_w is a translation by the vector $w(\rho(k))$. So the chambers C_w are shifted apart from each other by this map, and the joint spectrum of the $T_\xi(k)$ operators on $\mathbf{C}[H]$ is obtained by applying this map to the lattice P.

Corollary 2.10 $\{E(\lambda,k)\,;\,\lambda\in P\}$ *is an orthogonal basis of* $\mathbf{C}[H]$ (*assuming still that* $k_\alpha \ge 0$ *for any* α).

Proof. The eigenvalues $\tilde{\lambda}$ are mutually distinct. □

"Macdonald theory" is concerned with these polynomials $E(\lambda, k)$ and their further properties, for example, the computation of their L^2 norm with respect to $(\cdot, \cdot)_k$, and their values at $e \in H$. To attack these problems effectively, we must investigate the algebraic structures attached to the $T_\xi(k)$. This is the main subject of the next three sections.

3 Degenerate double affine Hecke algebra

The results in this section are due to Ivan Cherednik, see [4], [5].

3.1 Affine Weyl group

The *affine Weyl group* W^a is the group acting on $\mathfrak{h}^*$, generated by the reflections r_a, $a = [\alpha^\vee, n] \in R^\vee + \mathbf{Z} \subset S(\mathfrak{a})$, defined by

$$r_a(\lambda) = r_{[\alpha^\vee,n]}(\lambda) = \lambda - (\lambda(\alpha^\vee) + n)\alpha.$$

We shall often write $a = \alpha^\vee + n$ as an element of $S(\mathfrak{a})$ instead of $[\alpha^\vee, n]$. In particular, this group contains all translations $t_\mathfrak{a} : \lambda \mapsto \lambda + \mathfrak{a}$ with $\mathfrak{a} \in Q$, since for any $\alpha \in R$,

$$r_{\alpha^\vee} r_{[\alpha^\vee,1]} = r_{[-\alpha^\vee,1]} r_{\alpha^\vee} = t_\alpha.$$

In fact, one has $W^a = W \ltimes Q$, the semi-direct product of Q by W. This is a Coxeter group of affine type, if we take the set of simple reflections for W^a equal to $\{r_0, r_1, \cdots, r_n\}$, with $r_i = r_{a_i}$, $a_0 = [-\theta^\vee, 1]$, and $a_i = \alpha_i^\vee$, $i > 0$. Here θ denotes the unique *highest short root.*

Remark 3.1 The usual definition of the affine Weyl group associated with R, especially in the theory of affine Lie algebras, is "dual" to our definition. In particular, the affine Dynkin diagram associated to our affine Weyl group W^a is $(R^\vee)^{(1)}$!

The affine positive roots are $R_+^a = R_+^\vee \cup (R^\vee + \mathbf{Z}_{>0})$, and the corresponding set of simple roots is denoted

$$S^a = \{a_0, a_1, \cdots, a_n\}.$$

The fundamental alcove C is

$$C = \{\lambda \in \mathfrak{a}^* \,;\, \lambda(a_i) > 0, \quad i = 0, 1, \cdots, n\}.$$

Then $\overline{C}$ is a fundamental domain for the action of W^a.

We shall work with $W^e = W \ltimes P$, the *extended affine Weyl group.* This is not a Coxeter group in general, but $W^a \lhd W^e$ and if

$$\Omega = \{\omega \in W^e \,;\, \omega(C) = C\},$$

then $\Omega \cong P/Q$, and

$$W^e = W^a \rtimes \Omega.$$

Clearly $\omega \in \Omega$ defines a permutation of the set S^a.

By *duality* the action of W^e on $\mathfrak{h}^*$ via affine transformations gives rise to a representation of W^e on the symmetric algebra $S(\mathfrak{h})$ of $\mathfrak{h}$ (viewed as polynomial functions on $\mathfrak{h}^*$). Notice that $S_n(\mathfrak{h})$ (the part of $S(\mathfrak{h})$ of degree $\leq n$) is stable under this action; for $n = 1$ this action gives the *reflection representation* of W^e on $\mathfrak{h} \oplus \mathbf{C}$, explicitly given by:

$$r_{[\alpha^\vee, n]}[\xi, u] = [\xi, u] - \alpha(\xi)[\alpha^\vee, n],$$

and

$$t_\lambda[\xi, u] = [\xi, u - \lambda(\xi)],$$

where $[\xi, u](\lambda) = \lambda(\xi) + u$. If $p \in S(\mathfrak{h})$, and $w \in W^e$, then write $p^w(\lambda) = p(w^{-1}\lambda)$.

Since we need to understand precisely the relation $\Omega \cong P/Q$ we introduce the following notion.

Definition 3.2 An element in $\overline{C} \cap P \setminus \{0\}$ is called a *minuscule weight.*

Proposition 3.3 *Let* $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ *denote the set of fundamental weights for the simple system* $\{r_1, r_2, \cdots, r_n\}$ *and* $\theta^\vee = \sum_{i=1}^n n_i a_i$ *the maximal coroot. Put* $O^* = \{i \in \{1, 2, \cdots, n\}\,;\ n_i = 1\}$. *Then* $\overline{C} \cap P \setminus \{0\} = \{\lambda_i\,;\ i \in O^*\}$.

Proof. Obviously $\overline{C} \cap P \setminus \{0\} \supset \{\lambda_i\,; i \in O^*\}$. In the other direction we argue as follows. If $\lambda \in \overline{C} \cap P \setminus \{0\}$ then $\lambda(\theta^\vee) = 1$. Write $\lambda = \sum_{i=1}^n m_i \lambda_i$, and notice that $m_i \in \mathbf{Z}_{\geq 0}$ and that $\lambda_i(\theta^\vee) \in \mathbf{Z}_{>0}$. Hence from

$$\lambda(\theta^\vee) = \sum_{i=1}^n m_i \lambda_i(\theta^\vee) = 1,$$

it follows that there exists an i such that $m_i = \lambda_i(\theta^\vee) = 1$ and $m_j = 0$ (for $i \neq j$). Thus $\lambda = \lambda_i$ and $i \in O^*$. □

For $r \in O^*$, let $\omega_r = t_{\lambda_r} w_{\lambda_r} w_0 \in W^e$, where w_{λ_r} is the longest element in the parabolic subgroup W_{λ_r} of W generated by $\{r_1, \cdots, r_{r-1}, r_{r+1}, \cdots, r_n\}$ (the stabilizer of λ_r) and w_0 is the longest element in W. The parabolic subsystem of roots that corresponds to W_{λ_i} is denoted by R_{λ_i}. Its basis of simple roots is $\{\alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots, \alpha_n\}$.

Proposition 3.4 $\Omega = \{\omega_r \in W^e\,; r \in O^*\} \cup \{id_{\mathfrak{a}^*}\}$. *In particular the set of all minuscule weights is a complete set of representatives of* $P/Q \setminus \{0\}$.

Proof. Let $\omega \in W^e$ such that $\omega(C) = C$. Then $\omega(S^a) = S^a$, where $S^a = \{a_0 = 1 - \theta^\vee, a_1, \cdots, a_n\}$. If $\omega(a_0) = a_0$, then $\omega(\{\alpha_1, \cdots, \alpha_n\}) = \{\alpha_1, \cdots, \alpha_n\}$, therefore $\omega = id_{\mathfrak{a}^*}$ by simple transitivity of the action on chambers of W. Hence we may and will label $\omega \in \Omega$ uniquely by the index $r \in \{0, \ldots, n\}$ such that $\omega_r(a_0) = a_r$. Now let $r \in \{1, 2, \cdots, n\}$, and write $\omega_r = t_{\mu_r} w_r$. Then $w_r(\theta^\vee) = -a_r$ and $\mu_r \in \overline{C} \cap P \setminus \{0\}$. Hence μ_r is a minuscule fundamental weight and $\mu_r(a_r) = 1$. In other words, it is the fundamental weight λ_r of a_r. Because $\omega_r^{-1} = w_r^{-1} t_{-\lambda_r}$

we have $w_0 w_r^{-1}(\lambda_r) \in \overline{C}$. Hence $w_0 w_r^{-1}(\lambda_r) = \lambda_r$. Moreover, for $i \neq r$ we have $w_0 w_r^{-1}(\alpha_i) = w_0(\alpha_j) \in R_-$ for some $j \in \{1, 2, \cdots, n\}$. Therefore we have $w_0 w_r^{-1} = w_{\lambda_r}$, hence $w_r = w_{\lambda_r} w_0$.

Vice versa, let λ_r be a minuscule fundamental weight. Since $w_0 \mu \in -C$ for $\mu \in C$ and $w_{\lambda_r}(a_i) \in R_{\lambda_r,-}$ $(i \neq 0, r)$, we have

$$\omega_r \mu(a_i) = \lambda_r(a_i) + w_0 \mu(w_{\lambda_r}(a_i)) = w_0 \mu(w_{\lambda_r}(a_i)) > 0.$$

Since $\theta^\vee \geq w_{\lambda_r}(a_r)$ and $w_0 \mu(\theta^\vee) > -1$, we have

$$\omega_r \mu(a_r) = \lambda_r(\alpha_r) + w_0 \mu(w_{\lambda_r}(a_r)) = 1 + w_0 \mu(w_{\lambda_r} a_r) > 1 + w_0 \mu(\theta^\vee) > 0.$$

On the other hand, $w_{\lambda_r}(\theta^\vee) \in R_+^\vee$ and λ_r is a minuscule weight, thus

$$\omega_r \mu(\theta^\vee) = 1 + w_0 \mu(w_{\lambda_r} \theta^\vee) < 1.$$

Thus we have $\omega_r C \subset C$, that is $\omega_r \in \Omega$. The map $O^* \ni r \mapsto \omega_r \in \Omega$ is injective since $\omega_r(0) = \lambda_r$. □

Corollary 3.5 *(of proof) If λ_r is a minuscule weight, then $\omega_r(1 - \theta^\vee) = a_r$.*

3.2 Hecke algebra

Definition 3.6 (Cherednik) The *degenerate extended double affine Hecke algebra* $\mathbf{H}^e(R_+, k)$ is the unique associative algebra over $\mathbf{C}$ such that

(1) $\mathbf{H}^e(R_+, k) \cong S(\mathfrak{h}) \otimes \mathbf{C}[W^e]$ as vector space over $\mathbf{C}$,

(2) $S(\mathfrak{h}) \ni p \mapsto p \otimes e \in \mathbf{H}^e(R_+, k)$, and $\mathbf{C}[W^e] \ni w \mapsto 1 \otimes w \in \mathbf{H}^e(R_+, k)$ are algebra homomorphisms,

(3) $(p \otimes e)(1 \otimes w) = p \otimes w$.
Write $p \cdot w$, or pw instead of $p \otimes w$ from now on.

(4) $r_i \cdot p - p^{r_i} \cdot r_i = -k_i(p - p^{r_i})/a_i$, $(i = 0, 1, \cdots, n)$, where $k_0 = k_\theta$.

(5) $\omega \cdot p = p^\omega \cdot \omega$ for all $\omega \in \Omega$.

Theorem 3.7 (Cherednik) *Let $\mathbf{A}$ denote a sub-algebra of* $\mathrm{End}(\mathbf{C}[H])$ *generated by e^λ $(\lambda \in P)$, $w \in W$, and $T_\xi(k)$ $(\xi \in \mathfrak{h})$. Then*

$$\pi : W^e \ni t_\lambda w \mapsto e^\lambda w \in \mathrm{End}(\mathbf{C}[H])$$

and

$$\pi : \mathfrak{h} \ni \xi \mapsto T_\xi(k) \in \mathrm{End}(\mathbf{C}[H])$$

extend to a representation of $\mathbf{H}^e(R_+, k)$ *on* $\mathbf{C}[H]$, *and* $\mathbf{H}^e(R_+, k)$ *is isomorphic to* $\mathbf{A}$ *via* π.

Proof. We need to check (4) and (5), the other points being obvious.

First notice that $\pi : W^e \to \mathrm{End}(\mathbf{C}[H])$, and $\pi : S(\mathfrak{h}) \to \mathrm{End}(\mathbf{C}[H])$ are well defined. We can check by simple direct computation that $T_\xi(k)$ and r_i $(i = 1, 2, \cdots, n)$ satisfy the relation (4). The case $r = 0$ requires a bit of special care: put

$$S_\xi(k) = \partial_\xi + \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha(\xi) \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} (1 - r_\alpha),$$

This operator is called the *Dunkl-Heckman operator.* Define $u_\xi(k)$ by $T_\xi(k) = S_\xi(k) - u_\xi(k)$, then

$$u_\xi(k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha(\xi) r_\alpha.$$

The operator $S_\xi(k)$ is independent of the choice of a positive system R_+ of R and $wS_\xi(k)w^{-1} = S_{w\xi}(k)$ for all $w \in W$, $\xi \in \mathfrak{h}$ (but $\{S_\xi \,;\, \xi \in \mathfrak{h}\}$ is not commutative). We leave it to the reader to verify by direct computation that

$$\pi(r_0)S_\xi(k)\pi(r_0) = S_{r_0(\xi)}(k) - \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha(r_0(\xi)) \left\{ \frac{(1 - e^{\theta(\alpha^\vee)\alpha})(1 + e^\alpha)}{1 - e^\alpha} \right\} r_\alpha,$$

and

$$\pi(r_0)u_\xi(k)\pi(r_0) = -\frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \varepsilon(\theta(\alpha^\vee))\alpha(r_0(\xi)) e^{\theta(\alpha^\vee)\alpha} r_\alpha.$$

Using that $\theta(\alpha^\vee) = 0$ or 1 we now check the desired relation $\pi(r_0)T_\xi(k)\pi(r_0) = T_{r_0(\xi)}(k) + k_0\theta(\xi)\pi(r_0)$.

Let's look at relation (5). For the minuscule fundamental weight λ_r of a simple root α_r, we put $\pi_r : \mathbf{C}[H] \to \mathbf{C}[H]$, $\pi_r = \pi(\omega_r) = e^{\lambda_r} w_r$. Straightforward computations show:

$$\pi_r S_\xi(k)\pi_r^{-1} = S_{\omega_r(\xi)}(k) + \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha(\xi)(r_{w_r\alpha} - r_{\omega_r\alpha})(-\lambda_r(w_r\alpha^\vee))$$

and

$$\pi_r u_\xi(k)\pi_r^{-1} = -\frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha(\xi) r_{\omega_r\alpha},$$

hence

$$\begin{aligned} \pi_r T_\xi(k)\pi_r^{-1} &= S_{\omega_r(\xi)}(k) + \frac{1}{2} \sum_{\alpha \in R_+} \varepsilon(-\lambda_r(w_r\alpha^\vee)) k_\alpha \alpha(\xi) r_{w_r\alpha} \\ &= T_{\omega_r(\xi)}(k). \end{aligned}$$

Finally we show that π is an isomorphism. Obviously π is surjective. Suppose that $\sum_{w\in W} p_w(T(k))w = 0$ in $\mathbf{A}$. If we write $\sum_{w\in W} p_w(T(k))w = \sum_{w\in W} D_w w$, then $D_w = 0$ for all $w \in W$. On the other hand, let w' be such that the degree of $p_{w'}$ is maximal and let q denote its highest degree part. Then the highest order part of $D_{w'}$ equals ∂_q, hence $q = 0$. Consequently, $p_w = 0$ for all $w \in W$. □

We can give a more intrinsic definition of the model representation:

Definition 3.8 (Drinfeld[7], Lusztig[22]) $\mathbf{H}(R_+, k) \cong S(\mathfrak{h}) \otimes \mathbf{C}[W] \subset \mathbf{H}^e(R_+, k)$ is called the *degenerate affine Hecke algebra* or *graded affine Hecke algebra.*

Definition 3.9 We can define a one-dimensional representation of $\mathbf{H}(R_+, k)$ by

$$\begin{cases} \xi \cdot 1 = -\rho(k)(\xi)1 & (\xi \in \mathfrak{h}) \\ w \cdot 1 = 1 & (w \in W). \end{cases}$$

This representation is called the *trivial representation* of $\mathbf{H}(R_+, k)$, which we denote by triv.

Theorem 3.10 *The representation π is isomorphic to the induced representation* $\mathrm{Ind}_{\mathbf{H}(R_+,k)}^{\mathbf{H}^e(R_+,k)}(\mathrm{triv})$.

Proof. For $1 \in \mathbf{C}[H]$, $T_\xi(k) \cdot 1 = -\rho(k)$ and $w \cdot 1 = 1$. Hence there exist a unique epimorphism $\varphi : \mathrm{Ind}_{\mathbf{H}(R_+,k)}^{\mathbf{H}^e(R_+,k)}(\mathrm{triv}) \to \pi$ such that $\varphi(1) = 1$. On the other hand, as a $\mathbf{C}[H]$-module, $\mathrm{Ind}_{\mathbf{H}(R_+,k)}^{\mathbf{H}^e(R_+,k)}(\mathrm{triv})$ is isomorphic to the left regular representation of $\mathbf{C}[H]$. Hence, as a $\mathbf{C}[H]$ module, $\mathrm{Ind}_{\mathbf{H}(R_+,k)}^{\mathbf{H}^e(R_+,k)}(\mathrm{triv}) \cong \pi$. Therefore, as a $\mathbf{H}^e(R_+, k)$ module, $\mathrm{Ind}_{\mathbf{H}(R_+,k)}^{\mathbf{H}^e(R_+,k)}(\mathrm{triv})$ is isomorphic to π via φ. □

4 Intertwiners

The intertwining operators between minimal principal series representations of (graded) affine Hecke algebras are built from certain intertwining elements of these algebras. This is a main topic of study in the representation theory of Hecke algebras. In this section we will extend this construction to the double affine situation, and discuss the basic applications to Macdonald theory. The ideas in this section are mainly due to Ivan Cherednik.

4.1 Intertwining elements in the degenerate double affine Hecke algebra

In the degenerate graded Hecke algebra there exist elements I_w for $w \in W^e$ with the property that the conjugate inside $\mathbf{H}^e(R_+, k)$ of an element $p \in S(\mathfrak{h})$ by I_w is equal to p^w. These elements are called "intertwiners", because they give rise to intertwining maps between minimal principal series modules. In our context this means that we find operators $\pi(I_w)$ which map solutions of (2.1) to solutions of (2.1) with spectral parameter $w\lambda$.

Definition 4.1

$$I_i \doteq r_i a_i + k_i \ \in \mathbf{H}^e(R_+, k) \quad (i = 0, 1, \cdots, n)$$

Theorem 4.2 *(a)* $I_i^2 = k_i^2 - a_i^2$.
(b) $I_i p = p^{r_i} I_i \ \forall p \in S(\mathfrak{h})$.
(c) $I_i I_j I_i \cdots = I_j I_i I_j \cdots$
with m_{ij} factors on both sides. Here m_{ij} denotes the order of the element $r_i r_j \in W^a$.
(d) Assume that $k_\alpha \geq 0$ for all $\alpha \in R$. Then we have $(I_i f, g)_k = -(f, I_i g)_k$ for all $i = 0, 1, \cdots, n$.

Proof. (a) and (b) are trivial reformulation of (4) in Definition 3.6, and (d) follows directly from the symmetry of $T_\xi(k)$. Statement (c) is equivalent with the following; if we have two reduced expressions $r_{i_1} r_{i_2} \cdots r_{i_n} = r_{i'_1} r_{i'_2} \cdots r_{i'_n}$ for w, then $I_{i_1} I_{i_2} \cdots I_{i_n} = I_{i'_1} I_{i'_2} \cdots I_{i'_n}$. For a reduced expression $r_{i_1} r_{i_2} \cdots r_{i_n}$, we put $I_w = I_{i_1} I_{i_2} \cdots I_{i_n}$. Notice that we can write

$$I_w = w \prod_{a \in R_+^a, w(a) \in R_-^a} a + \sum_{w' < w} p_{w,w'} w',$$

where $p_{w,w'} \in S(\mathfrak{h})$, thus, if we allow rational coefficients, we also have

$$I_w = w \prod_{a \in R_+^a, w(a) \in R_-^a} a + \sum_{w' < w} r_{w,w'} I_{w'}.$$

The top coefficient is independent of the reduced expression for w; so if I_w and I'_w are different, then the difference $I''_w = I_w - I'_w$ is of the form $\sum_{w'<w} r'_{w,w'} I_{w'}$ and also have intertwining property $I''_w p = p^w I''_w$ ($p \in S(\mathfrak{h})$). Thus we have $I''_w = 0$. □

By the above theorem, we can define I_w for $w \in W^a$ as follows; if $w = r_{i_1} r_{i_2} \cdots r_{i_n}$ is a reduced expression for w, then we put

$$I_w = I_{i_1} I_{i_2} \cdots I_{i_n}.$$

Obviously, we also have $\omega I_i = I_j \omega$ if $\omega \in \Omega$ and $\omega r_i = r_j \omega$. Hence we may also use Ω to build intertwiners for arbitrary elements of W^e:

Definition 4.3 For a reduced expression $w = \omega r_{i_1} r_{i_2} \cdots r_{i_n}$ for $w \in W^e$, we define the *general intertwiner* $I_w \in \mathbf{H}^e(R_+, k)$ for w by

$$I_w = \omega I_{i_1} I_{i_2} \cdots I_{i_n}.$$

Remark 4.4 The equality $I_w = \omega I_{i_1} I_{i_2} \cdots I_{i_n}$ is true only if the expression $w = \omega r_{i_1} r_{i_2} \cdots r_{i_n}$ is reduced. Denote by I_w^λ the right evaluation of I_w at λ. In other words, I_w^λ is the element of $\mathbf{C}[W^e]$ defined by

$$I_w^\lambda = \omega I_{i_1}^{r_{i_2} \cdots r_{i_n} \lambda} I_{i_2}^{r_{i_3} \cdots r_{i_n} \lambda} \cdots I_{i_n}^\lambda$$

with $I_i^\lambda = \lambda(a_i) r_i + k_i$. If we normalize these elements of $\mathbf{C}[W]$ as follows:

$$\tilde{I}_w^\lambda = \frac{I_w^\lambda}{\prod_{a \in R_+^a \cap w^{-1}(R_-^a)} (\lambda(a) + k_a)}$$

then the $\tilde{I}_w^\lambda$ behave as a W^e cocycle:

$$\tilde{I}_{ww'}^\lambda = \tilde{I}_w^{w'\lambda} \tilde{I}_{w'}^\lambda,$$

for all w and w'.

4.2 Applications of affine intertwiners

Intertwiners generate recurrent relations between the eigenfunctions of the Dunkl-Cherednik operators whose spectral parameters differ by an element of the weight lattice P. For example, the following result follows in a straightforward way from Definition 4.3:

Corollary 4.5 *For $w \in W^e$ we have*

$$I_w(1) = d(w,k) E(w(0), k),$$

where

$$d(w,k) = \prod_{a \in R_+^a \cap w^{-1} R_-^a} a(-\rho(k)).$$

Of course, Corollary 4.5 makes it possible to reduce the problem of computing the norm of the polynomials $E(\lambda, k)$, and also the computation of their value at the identity element of H, to the same problems for the simplest polynomial $E(0, k) = 1$. For the complete computation of the norm of the polynomials this is not sufficient, since the computation of the norm of 1 is still nontrivial (in fact, this was the original Macdonald constant term conjecture!). To solve this part of the story we need another technique, coming from the so-called shift principle. This will be discussed in the next section.

4.2.1 Evaluation at the identity element

Now we shall formulate the result of the evaluation at the identity element. We need to introduce some important functions, the non-symmetric generalizations of Harish-Chandra's c-function.

Definition 4.6 For $w \in W$ we put

$$\delta_w(\alpha) = \begin{cases} 0 & \text{if } \alpha \in w^{-1}R_+ \\ 1 & \text{if } \alpha \in w^{-1}R_- \end{cases} .$$

We define meromorphic functions c_w^* and $\tilde{c}_w$ in λ and k by

$$c_w^*(\lambda, k) = \prod_{\alpha \in R_+} \frac{\Gamma(-\lambda(\alpha^\vee) - k_\alpha + \delta_w(\alpha))}{\Gamma(-\lambda(\alpha^\vee) + \delta_w(\alpha))}, \tag{4.1}$$

$$\tilde{c}_w(\lambda, k) = \prod_{\alpha \in R_+} \frac{\Gamma(\lambda(\alpha^\vee) + \delta_w(\alpha))}{\Gamma(\lambda(\alpha^\vee) + k_\alpha + \delta_w(\alpha))}. \tag{4.2}$$

In particular we put $\tilde{c} = \tilde{c}_e$.

For $\lambda \in P_+$, we put $W_\lambda = \{w\,;\, w\lambda = \lambda\}$ and $W^\lambda = \{w\,;\, l(ww') \geq l(w) \forall w' \in W_\lambda\}$. It is well known that in each right W_λ-coset $wW_\lambda \subset W$ there exists a unique element of minimal length. Hence W^λ can be characterized as the system of representatives of W/W_λ which are of minimal length in their cosets. Let w_λ denote the longest element in W_λ.

Theorem 4.7 *For $\lambda \in P_+$ and $w \in W^\lambda$, we have*

$$E(w\lambda, k)(e) = \frac{\tilde{c}_{w_0}(\rho(k), k)}{\tilde{c}_{ww_\lambda}(\lambda + \rho(k), k)}.$$

Proof. Use Corollary 4.5. □

Corollary 4.5 can easily be generalized to non-polynomial eigenfunctions of the Dunkl-Cherednik operators (see Section 6), and this will play an important role in one possible approach to the inversion formula of the harmonic analysis on A, which will be discussed in Section 8.

4.2.2 Jack Polynomials

Another important application of the affine intertwiners was given by Knop and Sahi, in the case of the root system A_n. In this case, Knop and Sahi used the intertwiners to verify the integrality and positivity conjecture for Jack polynomials (also in the non-symmetric case).

Theorem 4.8 (F.Knop and S.Sahi [20]) *For a partition λ of n let $m_i(\lambda)$ be the number of parts which are equal to i and let $u_\lambda = \prod_{i\geq 1} m_i(\lambda)!$. If the Jack polynomial $J_\lambda(x;\alpha)$ has an expansion*

$$J_\lambda(x;\alpha) = \sum_{\nu\geq 0} v_{\lambda,\nu}(\alpha) m_\nu(x)$$

by monomial symmetric functions m_ν (ν : partition of n), then all functions $\tilde{v}_{\lambda,\nu} = u_\lambda^{-1} v_{\lambda,\nu}(\alpha)$ are polynomials in α with positive integral coefficients.

Here, in terms of our notations, $\alpha = \frac{1}{k}$ is the reciprocal of the multiplicity k, and

$$J_\lambda(x;\alpha) = \prod_{b\in\lambda} c_\lambda(b) \frac{1}{|W_\lambda|} \sum_{w\in W} E^w(\lambda, k),$$

where, for λ and $b = (i,j) \in \lambda$; a box in λ, $c_\lambda(b) = \alpha(\lambda_i - j) + (\mathrm{leg}(b) + 1)$.

Remark 4.9 In fact Knop and Sahi proved a stronger result, namely a combinatorial formula for the Jack polynomial.

5 The shift principle

In the previous section we introduced operators that act on the spectral parameter λ of (2.1). In this section we will study operations on the multiplicity parameter k. There exist so-called shift operators that induce translations in a certain lattice in the parameter space K. The most fundamental example of this kind of operator is already sufficient to prove Macdonald's constant term and evaluation conjectures, and therefore we will restrict ourselves to the discussion of this simplest example of a shift operator.

It is remarkable that these shift operators act naturally on the W symmetrizations of solutions of (2.1), rather than on the solutions themselves. However, on the solution space of (2.1), symmetrization for the action of W is invertible by a differential operator. This will become clear in the section on the KZ equation (see Remark 7.4).

The W symmetrizations of solutions of (2.1) are eigenfunctions of an important system of commuting differential operators that will play the leading part in the next section. This system is called the hypergeometric system of differential equations. In the section on the KZ equations we shall see that this system is generically equivalent to (2.1) (Matsuo's theorem), but it represents a different point of view (somewhat like spherical representations versus principal series representations).

When considering these hypergeometric differential operators, yet another symmetry in the parameter space K arises naturally. This is the reflection symmetry $k'_\alpha = 1 - k_\alpha$, and this will also be discussed in this section.

5.1 Translation symmetry in the multiplicity parameter

In this section we use the notation $\mathbf{H} = \mathbf{H}(R_+, k)$ for the degenerate affine Hecke algebra. Here k is a multiplicity such that $k_\alpha \geq 0$ for all $\alpha \in R$.

Lemma 5.1 $Z(\mathbf{H}) = S(\mathfrak{h})^W$.

Proof. The following formula can be checked by induction on the length of w:

$$w \cdot \xi \cdot w^{-1} = w(\xi) + \sum_{\alpha \in R_+ \cap w R_-} k_\alpha \alpha(w\xi) r_\alpha. \tag{5.1}$$

From this formula one deduces easily that $Z(\mathbf{H}) \subset S(\mathfrak{h})$. Then one may use Definition 3.6 (4) to prove the result. □

Definition 5.2 Let us define a subspace $M(\lambda, k)$ of $\mathbf{C}[H]$ by

$$M(\lambda, k) = \{f \in \mathbf{C}[H]\,;\, p(T_\xi(k))f = p(\lambda)f, p \in S(\mathfrak{h})^W\}.$$

Corollary 5.3 *$M(\lambda, k)$ is a module over* $\mathbf{H}$.

Proof. This follows from Lemma 5.1. □

Proposition 5.4 *For all* $\lambda \in \mathfrak{h}^*$ *we have*

$$M(\lambda, k) = \begin{cases} \mathrm{Span}\{E(\nu, k)\}_{\nu \in W\bar{\lambda}} & \text{if } \exists \bar{\lambda} \in P_+ \text{ s.t. } \lambda \in W(\bar{\lambda} + \rho(k)), \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. This follows from Corollary 2.9. □

Proposition 5.5 *As a module for* $\mathbf{C}[W]$*,* $M(\tilde{\lambda}, k)$ *is independent of* k*, and isomorphic to* $\cong \mathbf{C}[W/W_\lambda] = \mathbf{C}[W^\lambda]$.

Proof. For $k = 0$ this is obvious from Proposition 5.4 and the fact that $E(\lambda, 0) = e^\lambda$. It remains to prove that the structure as a W-module is independent of k. By Corollary 4.5 we see that the polynomials $E(\lambda, k)$ have rational coefficients in k with respect to the monomials e^λ. It follows that the values of the character of W in $M(\tilde{\lambda}, k)$ are rational functions in k. On the other hand, the character values are integral over $\mathbf{Z}$. The conclusion is that the character is constant. □

In particular, there is a *unique* W-invariant element up to a scalar multiple.

Definition 5.6 For $\lambda \in P_+$, the *Jacobi polynomial* $P(\lambda, k) \in M(\tilde{\lambda}, k)$ is defined by

$$P(\lambda, k) = \sum_{w \in W^\lambda} E^w(\lambda, k),$$

where E^w denote the function on T defined by $E^w(t) = E(w^{-1}t)$. Then it is of the form

$$P(\lambda, k) = \sum_{\nu \in P_+, \nu \leq \lambda} c_{\lambda,\nu}(k) m_\nu, \quad c_{\lambda,\lambda}(k) = 1.$$

The common characterization of the Jacobi polynomials is either by orthogonality relations (using the inner product $(\cdot, \cdot)_k$, but restricted to the W-invariant Laurent polynomials), or as eigenfunctions of differential operators. The above definition of the Jacobi polynomials is not the usual one, but by the orthogonality properties of the $E(\lambda, k)$ it is immediately clear that the above definition is equivalent to the usual one.

The Jacobi-polynomials for the root system A_n are the Jack polynomials $J_\lambda(x, \alpha)$ mentioned in Section 4.2.2, whith $\alpha = 1/k$.

For later use we observe the following consequence of Definition 5.6.

Corollary 5.7 *Suppose that* R *is of type* B_n, C_n, F_4 *or* G_2*. Suppose that the subsystem* $R' := \{\mathfrak{a} \in R \mid k_\mathfrak{a} \neq 0\} \subset R$ *consists of a single* W*-orbit. Choose positive* *roots $R'_+ \subset R_+$ and denote by P' the weight lattice of R'. Denote by $k' \in \mathbf{R}_+$ the value of the multiplicity function k on R'. Let W' be the Weyl group of R'. If $\lambda \in P_+$ we denote by $\{\lambda'_i\}_{i=1}^l \subset P'$ the set of positive representatives for the W'-orbits in $W\lambda$. With these notation we have:*

$$P(R, \lambda, k) = \sum_{i=1}^{l} P(R', \lambda'_i, k')$$

Proof. By the characterization of $E(\lambda, k)$ as eigenfunction of the Dunkl-Cherednik operators we see that $E(R, \lambda, k) = E(R', \lambda, k')$. Now use Definition 5.6. □

If λ is *regular* in P_+, $M(\tilde{\lambda}, k)$ also contains a one-dimensional skew-invariant subspace, and we can define a skew-invariant function

$$P^-(\lambda, k) = \sum_{w \in W} \varepsilon(w) E^w(\lambda, k).$$

The next theorem is the heart of the "shift principle". It is a direct generalization of Weyl's character formula.

Theorem 5.8 (Generalized Weyl character formula) *Denote by* 1 *the multiplicity function which takes the value* $1 \in \mathbf{R}$ *on each root* $\alpha \in R$.

$$P^-(\lambda + \delta, k) = \Delta P(\lambda, k + 1)$$

or

$$P(\lambda, k + 1) = \frac{P^-(\lambda + \delta, k)}{\Delta} = \frac{P^-(\lambda + \delta, k)}{P^-(\delta, k)}.$$

Proof. The assertion follows directly from the divisibility (Corollary 2.2) of skew polynomials by Δ and the definition of the $E(\lambda, k)$ using orthogonality. □

It is not difficult to show that $M(\tilde{\lambda}, k)$ is *irreducible* as $\mathbf{H}$-module. Consequently, the shift principle is effective to understand properties of $M(\tilde{\lambda}, k)$ if $k_\alpha \in \mathbf{Z}_{>0}$ for all $\alpha \in R$, because it reduces everything to the trivial situation of $M(\lambda + \tilde{\rho}(k), 0)$, via induction on k. For example we can compute the norms of the Jacobi polynomials and the polynomials $E(\lambda, k)$ in this way (see prove 5.16 below).

Definition 5.9 If $q \in S(\mathfrak{h})$ we denote by $D_q^{\pm}(k)$ the differential operator that coincides with $q(T_\xi(k))$ on $\mathbf{C}[H]^{\pm W}$.

Lemma 5.10 *We put*

$$\pi^{\pm}(k) = \prod_{\alpha \in R_+} (\alpha^\vee \pm k_\alpha) \in S(\mathfrak{h}) \subset \mathbf{H},$$

and denote by $\varepsilon^{\pm}$ the idempotents in $\mathbf{C}[W]$ corresponding to the trivial representation (ε^+) and the sign representation (ε^-), respectively. Then

(a) $\varepsilon^{\mp} \cdot \pi^{\pm}(k) \cdot \varepsilon^{\pm} = \pi^{\pm}(k) \cdot \varepsilon^{\pm}$.

(b) $\varepsilon^{\pm} \cdot \mathbf{H}(k) \cdot \varepsilon^{\pm} = Z(\mathbf{H}(k)) \cdot \varepsilon^{\pm}$. *The map* $Z(\mathbf{H}(k)) \to Z(\mathbf{H}(k)) \cdot \varepsilon^{\pm}$, $z \mapsto z \cdot \varepsilon^{\pm}$ *is an isomorphism of commutative algebras, and the map* $Rad^{\pm} : \mathbf{H}(k) \to Z(\mathbf{H}(k))$ *defined by* $\varepsilon^{\pm} \cdot h \cdot \varepsilon^{\pm} = Rad^{\pm}(h) \cdot \varepsilon^{\pm}$ *respects the filtering by degree.*

(c) $\varepsilon^{\mp} \cdot \mathbf{H}(k) \cdot \varepsilon^{\pm} = Z(\mathbf{H}(k))\pi^{\pm}(k) \cdot \varepsilon^{\pm}$ *The map* $Z(\mathbf{H}(k)) \to Z(\mathbf{H}(k))\pi^{\pm}(k) \cdot \varepsilon^{\pm}$, $z \mapsto z\pi^{\pm}(k) \cdot \varepsilon^{\pm}$ *is a linear isomorphism, and the map* ${}^{\pm}Rad : \mathbf{H}(k) \to Z(\mathbf{H}(k))\pi^{\pm}(k)$ *defined by* $\varepsilon^{\mp} \cdot h \cdot \varepsilon^{\pm} = {}^{\pm}Rad(h) \cdot \varepsilon^{\pm}$ *respects the filtering by degree.*

Proof. To prove (a) it is enough to show that for all simple reflections r_i,

$$(r_i \cdot \pi^{\pm}(k) + \pi^{\pm}(k) \cdot r_i) \cdot \varepsilon^{\pm} = 0.$$

This follows from Definition 3.6 (4). As to (b), first observe that it is enough to show that for all $p \in S(\mathfrak{h})$, $\varepsilon^{\pm} \cdot p \cdot \varepsilon^{\pm} \in Z(\mathbf{H}(k)) \cdot \varepsilon^{\pm}$. Using formula (5.1) and Lemma 5.1 this is clear, by induction on the degree of p. The remaining statements follow trivially from this proof. Essentially the same arguments, combined with (a), proves (c). □

Definition 5.11 The *fundamental shift operators* $G_{\pm}(k)$ are defined by

$$G_{+}(k) = \Delta^{-1} D^{+}_{\pi^{+}(k)}(k),$$

and

$$G_{-}(k+1) = D^{-}_{\pi^{-}(k)}(k)\Delta.$$

The shift principle is equivalent with the following action of the shift operators on Jacobi polynomials:

Theorem 5.12 *We have the following shift relations* ($\lambda \in P_{+}$):

$$G_{+}(k)P(\lambda, k) = \prod_{\alpha \in R_{+}} (k_{\alpha} - (\lambda + \rho(k))(\alpha^{\vee}))P(\lambda - \delta, k+1)$$

and

$$G_{-}(k+1)P(\lambda, k+1) = \prod_{\alpha \in R_{+}} (k_{\alpha} + (\lambda + \delta + \rho(k)(\alpha^{\vee}))P(\lambda + \delta, k)$$

Proof. Both relations are proved in the same manner. Let us do the first one. By Lemma 5.10 it is clear that

$$D^{+}_{\pi^{+}(k)}(k)P(\lambda, k) = c \cdot P^{-}(\lambda, k)$$

for some constant c. To compute this constant one has to recall that the Dunkl operators are triangular with respect to the ordering $\triangleleft$. With respect to this ordering, the highest order term in the expansion of $P(\lambda, k)$ is $e^{w_0\lambda}$, and the highest order term of $P^-(\lambda, k)$ is $\varepsilon(w_0)e^{w_0\lambda}$. Using Corollary 2.9 and the shift principle it is now straightforward to verify the asserted relation. □

We collect some basic properties of the shift operators in the following theorem.

Theorem 5.13 (a) *$G_\pm(k)$ transforms $\mathbf{C}[H]^W$ to $\mathbf{C}[H]^W$*

(b) *For all $f, g \in \mathbf{C}[H]$, $(G_+(k)f, g)_{k+1} = (f, G_-(k+1)g)_k$*

(c) *For all $p \in S(\mathfrak{h})^W$, $D_p(k \pm 1)G_\pm(k) = G_\pm(k)D_p(k)$*

(d) *For any W-invariant holomorphic germ f at $x = e$, we have*

$$(G_-(k+1)f)(e) = \frac{\tilde{c}(\rho(k), k)}{\tilde{c}(\rho(k+1), k+1)} f(e).$$

Proof. (a) In the case of $G_+(k)$ this is immediate from Remark 2.5, and in the case of $G_-(k)$ we use Lemma 5.10 and the divisibility of W-skew Laurent polynomials by Δ.

(b) From the definitions and the symmetry of the Dunkl-Cherednik operators with respect to the inner product $(\cdot,\cdot)_k$, we see that one has to verify (in the terminology of Lemma 5.10 (c)) that ${}^-Rad(\pi^+(k)) = {}^-Rad(\pi^-(k))$. This is true because Lemma 5.10 (c) implies that ${}^-Rad$ kills polynomials with degree lower than $|R_+|$.

(c) This is an immediate consequence of Theorem 5.12.

(d) By power series expansion at e it can be proved that

$$(G_-(k+1)f)(e) = c \cdot f(e) \tag{5.2}$$

for a some constant c. When we apply this to the function $f = 1 = P(0, k+1)$ and use Theorem 5.12 we find that

$$c = \prod_{\alpha \in R_+} (k_\alpha + (\delta + \rho(k))(\alpha^\vee))P(\delta, k, e) \tag{5.3}$$

Taking $f = P(\lambda, k+1)$ in (5.2) we now obtain

$$P(\lambda, k+1, e)P(\delta, k, e) \prod_{\alpha \in R_+} (k_\alpha + (\delta + \rho(k))(\alpha^\vee)) = \tag{5.4}$$

$$P(\lambda + \delta, k, e) \prod_{\alpha \in R_+} (k_\alpha + (\lambda + \delta + \rho(k))(\alpha^\vee)). \tag{5.5}$$

This is a recurrent formula for $P(\lambda, k, e)$. Using this recurrent relation we will prove that

$$P(\lambda, k, e) = \frac{\tilde{c}(\rho(k), k)}{\tilde{c}(\lambda + \rho(k), k)}. \tag{5.6}$$

Let us remark beforehand that the right hand side of this formula has to be interpreted in such a way that it is a continuous function of $k_\alpha \in \mathbf{R}_{\geq 0}$. The value on the boundary can be computed by using Macdonald's results [23] on the product formula for the Poincaré polynomial of a Coxeter group. Using identity 2.8 of that paper, we find that

$$\lim_{k \downarrow 0} c(\lambda + \rho(k), k) = \lim_{k \downarrow 0} \prod_{\alpha \in R_+} \frac{(\lambda + \rho)(\alpha^\vee) + k_\alpha}{(\lambda + \rho)(\alpha^\vee)} = |W_\lambda|. \tag{5.7}$$

By this formula we can also compute the limiting value of the c-function when we are in the situation of an indecomposable root system with two distinct root lengths when we send only one of the two values of the multiplicity function to 0. In addition to the notation of Corollary 5.7 we write k'' for the value of the multiplicity function k on the roots which are *not* in R'. Write δ' for half the sum of the positive roots in R'. We obtain:

$$\lim_{k'' \downarrow 0} c(R, \lambda + \rho(k), k) = |W_{\lambda + \delta'}| c(R', \lambda + k'\delta', k'). \tag{5.8}$$

After these preliminary remarks, let us prove equation 5.6. First we check that the right hand side indeed solves the recurrent relation 5.4. This is a simple computation left to the reader. Next assume that we are in the special case that the multiplicity function is constant on R. In this case everything reduces by 5.4 to the verification that

$$P(\lambda, 0, e) = |W^\lambda| = \lim_{k \downarrow 0} \frac{\tilde{c}(\rho(k), k)}{\tilde{c}(\lambda + \rho(k), k)}.$$

This is true by 5.7. Now consider the general case where we assume that the multiplicity function has two distinct values. By 5.4 we reduce to a situation where one of the values of the multiplicity function is 0 and the other value is a positive integer. By 5.8 and the above special case, the right hand side of 5.6 now reduces to

$$|W_{\delta'}/W_{\lambda+\delta'}| P(R', \lambda, k', e).$$

In view of Corollary 5.7 this is correct since the set of positive representatives of the W'-orbits in $W\lambda$ is equal to $W_{\delta'}\lambda$.

Now the constant c of equation (5.2) finally follows from (5.3) and (5.6). □

Remark 5.14 Of course, the evaluation formula (5.6) is equivalent to Theorem 4.7. So we have derived this formula in two different ways, first using the affine intertwiners in the previous section, and now using the shift operators.

Remark 5.15 When there are two distinct, integral root labels we can reduce to the trivial situation where both the root labels are zero, using the shift operator G_-. This is a two step process: first we reduce to a situation where one of the root labels is zero, then we change to the root subsystem of the roots whose labels are nonzero. The type of argument used in the proof of Theorem 5.13(d) is typical.

As a corollary of Theorem 5.13, we may now finally compute the norms of the polynomials $P(\lambda, k)$ and $E(\lambda, k)$. Again we have to argue as indicated in Remark 5.15.

Theorem 5.16 *Assume that* $k_\alpha \geq 0$ *for all* $\alpha \in R$, *and let* $\lambda \in P_+$ *and* $w \in W^\lambda$. *Then:*

(a)

$$||E(w\lambda, k)||_k^2 = \frac{c^*_{ww_\lambda}(-(\lambda + \rho(k)), k)}{\tilde{c}_{ww_\lambda}(\lambda + \rho(k), k)}.$$

(b)

$$||P(\lambda, k)||_k^2 = |W| \frac{c^*(-(\lambda + \rho(k)), k)}{\tilde{c}(\lambda + \rho(k), k)}.$$

Proof. We can compute the square norms of the Jacobi polynomial $P(\lambda, k)$ with respect to $(\cdot, \cdot)_k$ by the recursion relation that follows from Theorem 5.12 and Theorem 5.13 (b). This can be translated in terms of the $E(\lambda, k)$ as well, using the structure of the module introduced in Proposition 5.4. Details are left to the reader (see [29]). □

5.2 Another reflection symmetry and application

The operators $D_p(k)$ have another symmetry in the parameter k that gives a direct relation between the two shift operators G_- and G_+. This has an important application because it gives a proof of the conjecture by Yano and Sekiguchi concerning the explicit form of the b-function for the discriminant of a crystallographic reflection group.

Theorem 5.17 (see Proposition 2.2 of [14]) *Let* $1-k \in K$ *be defined by* $(1-k)_\alpha = 1 - k_\alpha$. *Then we have:*

$$D_p(1-k) = \delta_{k-1/2} \circ D_p(k) \circ \delta_{1/2-k}.$$

Proof. (Sketch) When $p_2 = \sum \xi_i^2$ this is a direct computation using the explicit formula in Example 6.2 for $D_{p_2}(k) = L(k) + (\rho(k), \rho(k))$. It is not difficult and standard to see that an operator D that commutes with $D_{p_2}(1-k)$, and that has an asymptotic expansion as in 6.2, is determined by its image $p = \gamma(D)$ under the Harish-Chandra homomorphism (see also (6.8)). Therefore the conjugation formula holds for all $p \in S(\mathfrak{h})^W$. □

By a similar argument one proves the following consequence:

Corollary 5.18

$$G_+(-1/2-k) \circ \delta_{k+1} = \delta_k \circ G_-(3/2+k)$$

Now apply this identity to the constant function 1, and take the lowest homogeneous part of the identity thus obtained. Use 5.13(d). This gives:

Corollary 5.19 *Take $k_\alpha = k \ \forall \alpha \in R$. Let D be the lowest homogeneous part of $G_+(-1/2-k)$ at the unit element of H. Let*

$$\pi = \prod_{\alpha \in R_+} \alpha^2 \tag{5.9}$$

be the discriminant of the reflection group W. Then

$$D\pi^{k+1} = |W| \prod_{i=1}^{n} \prod_{j=1}^{d_i-1} (d_i(k+1/2)+j)\pi^k. \tag{5.10}$$

where $d_1, \ldots, d_n$ are the primitive degrees of W.

As an application of these considerations we will compute the so-called b-function or Bernstein polynomial of the discriminant of W. Let us first define the b-function in general.

Let p be a polynomial on $\mathbf{C}^n$. According to a famous result of Bernstein, there exist nonzero polynomials $B \in \mathbf{C}[s]$ and differential operators

$$D \in \mathbf{C}[s]\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

such that the relation

$$Dp^{s+1} = B(s)p^s \tag{5.11}$$

holds. The collection of polynomials B for which such a relation exists forms an ideal in $\mathbf{C}[s]$. The b-function is by definition the monic generator of this ideal.

The existence proof due to Bernstein for nontrivial relations of the form 5.11 is based on algebraic considerations concerning rings of differential operators and their modules. This proof gives no information on the explicit form of b in specific examples, and finding such explicit formulas is difficult in general. It is known that the zeroes of b are negative rational numbers, and that these numbers are related to the singularities of the locus $p = 0$.

The following result was conjectured by Yano and Sekiguchi in [35]. It is an easy application of 5.10 now.

Theorem 5.20 ([29], Theorem 7.1) *Denote by $p_1, \ldots, p_n$ a set of fundamental polynomial invariants for the action of W on $\mathfrak{h}$. Let $\pi \in \mathbf{C}[p_1, \ldots, p_n]$ be the discriminant of W, i.e. the W-invariant polynomial 5.9, considered as an element of the polynomial ring generated by the fundamental invariants $p_1, \ldots, p_n$. The b-function of the discriminant π is given by:*

$$b(s) = \prod_{i=1}^{n} \prod_{j=1}^{d_i - 1} (s + 1/2 + \frac{j}{d_i}).$$

Remark 5.21 We have introduced two shift operators $G_\pm$ in this section, associated to the sign character of W. In fact one can associate a raising and a lowering operator to each linear character of W. For the purpose of this section we did not need this construction so we have skipped it. The interested reader is advised to consult [29] and [16] for the properties of these shift operators.

6 Away from polynomials

This section is a review of the hypergeometric function for root systems, which is a k-deformation of the elementary spherical function on symmetric spaces. This function was introduced and studied by Heckman and Opdam in [14] and a series of subsequent papers. An introduction to the hypergeometric system and the hypergeometric function is [16, Part I], where one can find further references.

In the previous section, we have introduced the differential operator $D_p(k) = D_p^+(k)$ for $p \in S(\mathfrak{h})^W$, which maps $\mathbf{C}[H]^W$ to itself. By Chevalley's theorem $\mathbf{C}[H]^W \cong \mathbf{C}[z_1, z_2, \ldots, z_n]$ with $z_i = \sum_{\mu \in W\mu_i} e^\mu$, so we have a system of commuting partial differential operators on the affine space $W\backslash H$. We want to study the general eigenvalue problem for these operators. We have seen that when we want polynomial eigenfunctions $\varphi \in \mathbf{C}[H]^W$, we are forced to take the eigenvalue $\lambda \in \mathfrak{h}^*$ in the system

$$D_p(k)\varphi = p(\lambda)\varphi, \qquad {}^\forall p \in S(\mathfrak{h})^W$$

equal to $\mu + \rho(k)$ for some $\mu \in P_+$. This means that the eigenvalue has to satisfy a certain integrality condition in this situation. However, for values of λ that are not integral in this sense, we can still find germs of holomorphic solutions at any point $h \in H$. The most elementary case is the case where h is regular for the action of W. We will see in the next subsection that in this case the space of germs of holomorphic solutions has dimension $|W|$. For generic parameters we can give a basis of series solutions which are convergent in an open neighborhood of A_+, and which behave asymptotically free (the Harish-Chandra series).

The important conclusion at this point is that the sheaf of germs of holomorphic solutions of these equations (6.1) is a local system of rank $|W|$ on the regular W orbit space of H. A further understanding of the equations (6.1) is obtained from the investigation of the monodromy of the local system, in subsection 6.2.

6.1 Harish-Chandra series

We denote the set of regular elements by

$$H^{\mathrm{reg}} = \{h \in H\,;\, \Delta^2(h) \neq 0\}.$$

We choose a base point $z \in W\backslash H^{\mathrm{reg}}$ with a representative $h \in H^{\mathrm{reg}}$. By definition, the germ $\mathcal{O}_z$ of holomorphic functions at z is the germ $\mathcal{O}_{Wh}^W$ of W-invariant holomorphic functions on Wh. Remark that $\mathcal{O}_{Wh} = \oplus_{w \in W} \mathcal{O}_{wh}$.

Definition 6.1 The *hypergeometric system* of differential equations at $z \in W\backslash H^{\mathrm{reg}}$ with a spectral parameter $\lambda \in \mathfrak{h}^*$ is the system of differential equations

$$D_p(k)\varphi = p(\lambda)\varphi, \qquad p \in S(\mathfrak{h})^W \tag{6.1}$$

for an unknown function $\varphi \in \mathcal{O}_z \cong \mathcal{O}_{Wh}^W$.

We denote the set of solutions for this system by

$$S(\lambda,k)^W = \{\varphi \in \mathcal{O}^W_{Wh}\,;\, D_p(k)\varphi = p(\lambda)\varphi,\ p \in S(\mathfrak{h})^W\}.$$

Example 6.2 Let $\xi_1, \ldots, \xi_n$ be an orthonormal basis of $\mathfrak{a}$. Then $p = \sum \xi_i^2$ is a W-invariant quadratic, and the corresponding differential operator is

$$D_p(k) = L(k) + (\rho(k), \rho(k)),$$

where

$$L(k) = \sum_{i=1}^{n} \partial^2_{\xi_i} + \sum_{\alpha \in R_+} \frac{1}{2} k_\alpha \frac{1+e^{-\alpha}}{1-e^{-\alpha}} (\alpha,\alpha)\partial_{\alpha^\vee}.$$

Let $\mathfrak{g}$ be a real semi-simple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace, and $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ the restricted root system with root labels $m_\alpha = \dim(\mathfrak{g}^\alpha)$. Then the radial part of the Laplace-Beltrami operator on the corresponding symmetric space G/K with respect to left action of K equals $L(k)$, if we identify R with 2Σ and $k_{2\alpha} = \frac{1}{2} m_\alpha$. So (6.1) becomes the system of differential equations for the elementary spherical function φ_λ restricted to A.

Example 6.3 Let us consider the rank 1 case, and in order to be even more convincing, we do the non-reduced case BC_1, $R = \{\pm\alpha, \pm 2\alpha\}$. Let us introduce notation. $H = \mathbf{C}^\times$, $\mathbf{C}[H] = \mathbf{C}[y, y^{-1}]$, with $y = e^\alpha$; If $\xi = (2\alpha)^\vee$, then $Q^\vee = P^\vee$ is generated by ξ, and $\partial_\xi = \theta = y\frac{d}{dy}$. Normalize $|\xi| = 1$. We set $\lambda = \lambda(\xi)$, $k_1 = k_\alpha$, $k_2 = k_{2\alpha}$. Now (6.1) becomes

$$\left\{\theta^2 + \left(k_1 \frac{1+y^{-1}}{1-y^{-1}} + 2k_2 \frac{1+y^{-2}}{1-y^{-2}}\right)\theta + \left((\frac{1}{2}k_1 + k_2)^2 - \lambda^2\right)\right\}\varphi = 0.$$

Let $z = \frac{1}{2} - \frac{1}{4}(y + y^{-1})$ be a coordinate on $W\backslash H$, then this becomes

$$\left\{z(1-z)\frac{d^2}{dz^2} + (c - (1+a+b)z)\frac{d}{dz} - ab\right\}\varphi = 0$$

with $a = \lambda + \frac{1}{2}k_1 + k_2$, $b = -\lambda + \frac{1}{2}k_1 + k_2$, $c = \frac{1}{2} + k_1 + k_2$.

To understand system (6.1), we first consider the easiest examples of solutions, the asymptotically free solutions on A_+ (also called the Harish-Chandra series).

The crucial point is the observation that the equations themselves have an asymptotic expansion as follows.

Lemma 6.4 *For any $p \in S(\mathfrak{h})^W = \mathbf{C}[\mathfrak{h}^*]^W$ one has an asymptotic expansion of the following kind on A_+:*

$$D_p(k) = \partial(p(\cdot + \rho(k))) + \sum_{\kappa \in Q_- \backslash \{0\}} e^\kappa \partial(p_\kappa) \tag{6.2}$$

where $p_\kappa \in \mathbf{C}[\mathfrak{h}^*]$ has lower degree than p. More generally, for any p in $S(\mathfrak{h}) = \mathbf{C}[\mathfrak{h}^*]$ and $w_0 \in W$ the longest element of W, we have the following asymptotic expansion on A_+ (compare with [32], Lemma 6.4):

$$w_0 D_{p^{w_0}}(k) w_0 = \partial(p(\cdot + \rho(k))) + \sum_{\kappa \in Q_- \setminus \{0\}} e^\kappa \partial(p_\kappa)$$

Proof. We prove the second asymptotic formula, by induction on the degree of p. Let p be of the form $p = \xi q$ with $\xi \in \mathfrak{h}$ and let $w \in W$. Then

$$\begin{aligned} w^{-1} D_{(\xi q)^w}(k) w = (\partial_\xi - w^{-1}\rho(k)(\xi)) w^{-1} D_{q^w}(k) w + \\ \sum_{\alpha \in R_+} k_\alpha \alpha(\xi) \frac{1}{1 - e^{-w^{-1}\alpha}} (w^{-1} D_{q^w}(k) w - w^{-1} r_\alpha D_{q^w}(k) r_\alpha w) \end{aligned} \tag{6.3}$$

(just check that the right-hand side is a differential operator that restricts to $w^{-1}(p^w(T))w$ on W-invariant functions). From (6.3) it follows by induction that $w^{-1} D_{p^w} w$ has an asymptotic expansion on A_+ of the form:

$$\sum_{\kappa \in Q_-} e^\kappa \partial(p_\kappa) \tag{6.4}$$

with $\deg(p_\kappa) \leq \deg(p)$, with equality if and only if $\kappa = 0$. In the special case where $w = w_0$ we want to prove that $p_0(\lambda) = p(\lambda + \rho(k))$. Observe that in this special case none of the terms of the second line of (6.3) contribute to the leading term (using (6.4)). Hence the result follows from (6.3) by induction on the degree. □

Substitute a formal series

$$\varphi = \sum_{\nu \leq \mu} c_\nu e^\nu, \qquad c_\mu = 1$$

into (6.1). By Lemma 6.4 we obtain the following indicial equation for the leading exponent:

$$p(\mu + \rho(k)) = p(\lambda), \qquad p \in S(\mathfrak{h})^W. \tag{6.5}$$

This means that

$$\lambda \in W(\mu + \rho(k)).$$

We put $\lambda = \mu + \rho(k)$, and put $c_\nu = \Gamma_\kappa(\lambda, k)$ if $\kappa = \nu - \mu \in Q_-$. Just using the explicit second order operator $L(k)$ we arrive at the following recurrence relations.

$$-(2\lambda + \kappa, \kappa)\Gamma_\kappa(\lambda, k) = 2 \sum_{\alpha > 0} k_\alpha \sum_{j \geq 1} (\lambda - \rho(k) + \kappa + j\alpha, \alpha) \Gamma_{\kappa + j\alpha}(\lambda, k) \tag{6.6}$$

These have a unique solution if we fix $\Gamma_0(\lambda, k) = 1$, and then the coefficients $\Gamma_\kappa(\lambda, k)$ are rational, with poles possibly at the hyperplanes $H_{\kappa'}$ for some $\kappa' < 0$, where

$$H_\kappa = \{\lambda \in \mathfrak{h}\,;\, (2\lambda + \kappa, \kappa) = 0\}. \tag{6.7}$$

Next we want to show that the eigenfunctions of the second order equation which we have just constructed, are in fact solutions of all the equations (6.1). The following well known and beautiful argument is due to Harish-Chandra. The uniqueness of the asymptotic solution, combined with the Lemma 6.4 and the commutativity of the operators $\{D_p\,;\, p \in S(\mathfrak{h})^W\}$ implies that

$$\Phi(\lambda, k) = \sum_{\kappa \in Q_-} \Gamma_\kappa(\lambda, k) e^{\lambda - \rho(k) + \kappa}, \quad \Gamma_0(\lambda, k) = 1$$

is a joint eigenfunction of the commuting family of differential operators $\{D_p\,;\, p \in S(\mathfrak{h})^W\}$. It is easy to find the eigenvalues by considering the leading exponents, taking Lemma 6.4 into consideration. We find that

$$D_p(k)\Phi(\lambda, k) = p(\lambda)\Phi(\lambda, k).$$

In other words, we have indeed constructed formal series solutions of (6.1). In this context one traditionally writes

$$p(\lambda) = \gamma(D_p(k))(\lambda), \tag{6.8}$$

and then one calls γ the "Harish-Chandra homomorphism".

The series $\Phi(\lambda, k)$ converges on

$$A_+ = \{a \in A\,;\, a^\alpha = e^\alpha(a) > 1, \quad {}^\forall \alpha > 0\}.$$

as one easily verifies using the defining recurrence relations.

As we have seen in the descriptions above, there are possibly singularities in the parameter space $\mathfrak{h}^* \times K$ of our series solutions $\Phi(\lambda, k)$. These are simple poles along the hyperplanes H_κ as defined in (6.7). However, the actual set of poles of $\Phi(\lambda, k)$ turns out to be a much smaller subset of hyperplanes:

Lemma 6.5 *The (apparent) simple pole of $\Phi(\lambda, k)$ (as a function of λ!) along H_κ is removable unless $\kappa = n\alpha$ for some $n \in \mathbf{Z}_-$ and $\alpha \in R_+$. If $\kappa = n\alpha$ then the residue of $\Phi(\lambda, k)$ at H_κ is a constant multiple of $\Phi(r_\alpha(\lambda), k)$.*

Proof. From the recurrence relations it is easy to see that the residue of $\Phi(\lambda, k)$ at H_κ is a constant multiple of $\Phi(\lambda + \kappa, k)$. Suppose it is nonzero. Then by the indicial equation (6.5), the leading exponent $\lambda + \kappa$ of the residue must be of the form $w\lambda$ for some $w \in W$, and this must hold for all $\lambda \in H_\kappa$. Hence $w = r_\alpha$ for some $\alpha \in R$, and $\kappa = n\alpha$ for some $n \in \mathbf{Z}$. It is obvious that κ has to be negative in the dominance ordering. □

The equation that defines $H_{n\alpha}$ can be rewritten as

$$\lambda(\alpha^\vee) + n = 0.$$

We now change the notation for this hyperplane to $H_{n,\alpha}$, so as to also include the case $n = 0$ of the hyperplane perpendicular to the root α. We will call λ generic if

$$\lambda \notin \bigcup_{n \in \mathbf{Z}, \alpha \in R} H_{n,\alpha}. \tag{6.9}$$

Remark 6.6 Notice that the set of generic parameters is precisely the set of regular points for the action of the affine Weyl group introduced in Section 3. There is a natural action of the affine Weyl group on the space of non-symmetric eigenfunctions of the Dunkl operators T_ξ, via the intertwiners of Section 4. The relation between such non-symmetric eigenfunctions and our space of solutions of (6.1) is the subject of the next section.

If λ is generic then, by Lemma 6.5, the dimension of the solution space for the eigenfunction equations (6.1) on A_+ is at least equal to $|W|$. The next theorem tells us that this is in fact an equality which holds for any λ, and moreover that this is the dimension of the solution space of these equations in the space of holomorphic germs at any regular point of H.

Theorem 6.7 *System* (6.1) *is holonomic of rank* $|W|$. *If* $\lambda \in \mathfrak{h}^*$ *is generic then* $\{\Phi(w\lambda, k; \cdot)\,;\, w \in W\}$ *forms a basis of the solution space.*

Proof. For any homogeneous $p \in S(\mathfrak{h})^W$,

$$D_p(k) = \partial(p) + (\text{lower order terms}).$$

Then in the left ideal generated by $D_p(k) - p(\lambda)$, we have operators of the form

$$\partial(q) + (\text{lower order terms}), \qquad {}^\forall q \in S(\mathfrak{h})S(\mathfrak{h})^W_+,$$

where $S(\mathfrak{h})^W_+$ denotes the space of the elements of $S(\mathfrak{h})^W$ without constant term. Hence the left $\mathcal{O}_z$-module

$$\mathcal{D}_z / \sum_{p \in S(\mathfrak{h})^W} \mathcal{D}_z(D_p(k) - p(\lambda))$$

is generated by the operators

$$\partial(q), \qquad \text{with } q \in S(\mathfrak{h}),\ W\text{-harmonic polynomials.}$$

Then the holonomic rank at the base point z is less than or equal to $|W|$. Conversely, we found, generically, the linearly independent asymptotically free solutions $\Phi(w\lambda, k; \cdot)$. Combining these, we conclude that the holonomic rank equals $|W|$ generically.

A more precise version of this argument shows that $(\partial(q))$ ($q \in S(\mathfrak{h})$: harmonic) always gives an $\mathcal{O}_z$-basis for the $\mathcal{D}_z$-module, independent of the parameter choice (see [14] or [16]). This point will also become quite clear in Section 7, when we study the relation between (6.1) and the KZ connection. □

6.2 Monodromy

We need to understand the *monodromy action* of $\pi_1(W\backslash H^{\text{reg}}, z_0)$ on the solution space of (6.1). Take a base point $x_0 \in A_+ \subset A^{\text{reg}}$ such that $z_0 = \overline{x_0}$. For each simple reflection r_i we consider an element l_i in $\pi_1(W\backslash H^{\text{reg}}, z_0)$ defined as follows: l_i can be represented by a path from x_0 to $r_i(x_0)$ which we can take arbitrarily close to the "straight" line segment between these two end points, but near the wall $a^{\alpha_i} = 1$ we replace a subsegment that intersects the wall by a half circle going around the wall in positive direction.

For each $v \in Q^\vee$ we define the closed loop l_v by

$$l_v(t) = x_0 \exp(2\pi\sqrt{-1}tv) \quad (t \in [0,1]).$$

Given φ, a local solution at x_0 of (6.1), we denote $T_i\varphi$ for the solution obtained by continuing φ analytically along the path l_i, and composing the result with r_i, and we denote $T_v\varphi$ for the continuation of φ along the loop l_v.

System (6.1) has regular singularities at infinity and and also along the walls. Moreover the structure of the fundamental group $\pi_1(W\backslash H^{\text{reg}}, z_0)$ allows the method of rank one reduction, which enables us to compute the connection formula for $\{\Phi(w\lambda, k; \cdot)\,;\, w \in W\}$ explicitly in terms of the c-function:

Theorem 6.8 (Looijenga, v.d.Lek (part(a)), Heckman-Opdam (other parts)) *Assume that $\lambda \in \mathfrak{h}^*$ satisfies condition* (6.9).

(a) *Put $T_0 = T_{\theta^\vee} T_{i_1} \cdots T_{i_k}$ with $r_{i_1} \cdots r_{i_k}$ a reduced expression for $r_{\theta^\vee}$. This is independent of the reduced expression, and T_0, T_1, ... , T_n satisfy the braid relations of W^a. These operators generate all monodromy on $W\backslash H^{\text{reg}}$ (in other words, the corresponding elements of $\pi_1(W\backslash H^{reg}, z_0)$ form a set of generators).*

(b) $(T_i - 1)(T_i + q_i) = 0$ *for all* $i = 0, 1, \ldots, n$, *with* $q_i = e^{-2\pi\sqrt{-1}k_i}$.

(c) $T_v\Phi(\lambda, k) = e^{2\pi\sqrt{-1}(\lambda - \rho(k))(v)}\Phi(\lambda, k)$.

(d) $\tilde{c}(\lambda, k)\Phi(\lambda, k) + \tilde{c}(r_i\lambda, k)\Phi(r_i\lambda, k)$ *is fixed for* T_i $(i = 1, \ldots, n)$.

(e) $\tilde{c}(-r_i\lambda, 1-k)\Phi(\lambda, k) + \tilde{c}(-\lambda, 1-k)\Phi(r_i\lambda, k)$ *has eigenvalue* $-q_i$ *with respect to* T_i $(i = 1, \ldots, n)$.

Proof. As indicated, these results come from various sources; we refer to [16, Part I, Lecture 4] for more details and references.

(a) is known from the work of Looijenga and v.d.Lek on the fundamental group $\pi_1(W\backslash H^{\text{reg}}, x_0)$, and is a nontrivial result.

(b) follows from (d) and (e).

(c) is trivial.

(d) and (e) form the heart of the matter. The proof is not difficult, and reduces to the rank one case. Let us sketch the idea of the proof. From the braid relations (a) it follows that if $v \in Q^\vee$ such that $\alpha_i(v) = 0$, then T_i and T_v commute (already in the fundamental group). Hence by (c) we see that, for generic λ, span$(\Phi(\lambda, k), \Phi(r_i\lambda, k))$ is closed for T_i. Now one takes limiting values of

$$e^{-\lambda+\rho(k)}\Phi(\lambda, k, b\cdot \exp(t\alpha_i^\vee))$$

when $b \to \infty$ in the wall $b^{\alpha_i} = 1$. The resulting limits are formal series solutions (asymptotically free at ∞) of Example 6.3, and here the monodromy of such series is explicitly known. For the precise argument, see [14, Theorem 6.7], [9, Theorem 1.1], and [16, Part I, Lecture 1, Section 4.3].

□

Motivated by these facts, we define the affine Hecke algebra $\mathbf{H}^{\text{aff}}(R_+, q_i)$ generated by T_i's and T_v's with the relations (a) and (b) in Theorem 6.8. This algebra contains two important sub-algebras; the finite dimensional Hecke algebra $\mathbf{H}^{fin}(R_+, q_i) = \langle T_i\rangle_{i=1}^n$ (describing the monodromy locally at the unit element of H), and the group algebra $\mathbf{C}[Q^\vee] = \langle\theta_v\rangle_{v\in Q^\vee}$, where θ_v is defined by $\theta_v = e^{2\pi\sqrt{-1}\rho(k)(v)}T_v$ (describing the monodromy "at infinity" in A_+). As a vector space, the algebra $\mathbf{H}^{\text{aff}}(R_+, q_i)$ is naturally isomorphic to the tensor product of these two algebras:

$$\mathbf{H}^{\text{aff}}(R_+, q_i) \cong \mathbf{H}^{fin}(R_+, q_i) \otimes \mathbf{C}[Q^\vee].$$

The relations between the T_i and the θ_v are given by Lusztig's formula:

$$T_i\theta_v - \theta_{r_iv}T_i = (q_i - 1)\left(\frac{\theta_v - \theta_{r_iv}}{1-\theta_{-\alpha_i^\vee}}\right) \tag{6.10}$$

Corollary 6.9 *The monodromy is, for generic parameters, equal to the representation*

$$\text{Ind}_{\mathbf{C}[Q^\vee]}^{\mathbf{H}^{\text{aff}}(R_+,q_i)} e^{2\pi\sqrt{-1}(\lambda)}$$

Here we consider $e^{2\pi\sqrt{-1}(\lambda)}$ as a character of $\mathbf{C}[Q^\vee]$.

Remark 6.10 At this point it is natural to invoke the result that the holonomic system of differential equations (6.1) has regular singularities, both at the "hyperplanes" $e^\alpha = 1$ in H and "at infinity" when we consider the torus H as a quasi-projective variety (for instance via an embedding in a projective toric variety). These facts have simple proofs which will be given in Section 7, when we study the equivalence of (6.1) and the KZ connection. The point is that the KZ connection visibly meets these regularity requirements.

Corollary 6.11 *Let λ be generic. The linear combination*

$$\tilde{c}(\lambda, k)\Phi(\lambda, k) + \tilde{c}(r_i\lambda, k)\Phi(r_i\lambda, k)$$

as mentioned in Theorem 6.8(d) extends holomorphically in a neighborhood of $int(\overline{A_+ \cup r_i(A_+)})$, and is r_i invariant. Hence for generic λ, the function (for $\tilde{c}$, see (4.2)*)*

$$\tilde{F}(\lambda, k; a) = \sum_{w \in W} \tilde{c}(w\lambda, k)\Phi(w\lambda, k; a)$$

extends holomorphically from A_+ to a tubular neighborhood of A in H, and is W-invariant there.

Proof. The linear combination of Harish-Chandra series under consideration has no monodromy with respect to l_i by Theorem 6.8, which means that it extends to an r_i-invariant holomorphic function on an open set of the form

$$U \cdot int(\overline{A_+ \cup r_i(A_+)}) \backslash \{e^{\alpha_i} = 1\}$$

where $e \in U$, $U \subset T$ open and connected. By Remark 6.10 this function has moderate growth towards $\{e^{\alpha_i} = 1\}$, hence it will extend meromorphically to $int(\overline{A_+ \cup r_i(A_+)})$. Let us denote its pole order along $\{e^{\alpha_i} = 1\}$ by $d \in \{0, 2, 4, \dots\}$. But now consider the operator $L(k)$ of Example 6.2, and take $\xi_1 = \frac{1}{2}\alpha_i^\vee |\alpha_i|$. It follows directly from the explicit Definition 2.4 for T_{ξ_1} that such a meromorphic function can be an eigenfunction of $L(k)$ only if

$$d(d + 1 - 2k_i) = 0 \tag{6.11}$$

(In other words, the operator $L(k)$ has exponents 0 and $(1 - 2k_i)/2$ (in the sense of Oshima [34])) along the wall $\{e^{\alpha_i} = 1\}$ (considered in the orbit space $W\backslash H^{reg}$). Hence for generic k it is clear that we must have $d = 0$. But an irreducible component of the set of singularities of a meromorphic function cannot have codimension > 1, hence the result is true for arbitrary k. □

Remark 6.12 The first part of Corollary 6.11 is remarkable, and it is not so easy to prove directly for Harish-Chandra series without the deformation theory in k. The reason is that in the situation of a symmetric space, the two exponents of

$L(k)$ along a wall are 0 and a non-positive integer (by 6.11). In this case there possibly exist true meromorphic solutions, but by the deformation in k it is clear that this possibility does not occur for the linear combination of Harish-Chandra series considered in the Corollary.

6.3 The hypergeometric function

The function $\tilde{F}$ is more beautiful and well behaved than Φ. When normalized at $e \in H$ this function will be denoted $F(\lambda, k; h)$, and this function will be called the hypergeometric function for the root system R. It is the natural generalization of the elementary spherical function on a symmetric space with restricted root system R (compare with Example 6.2).

Theorem 6.13 ([28], Theorem 2.8) *$\tilde{F}$ extends to an entire function of λ, k and h (in a tubular neighborhood of A).*

Proof. From Lemma 6.5 and the explicit formula for the c-function it is clear that $\tilde{F}$ may have first order poles along hyperplanes of the form $(\lambda, \alpha^\vee) = n$. First consider the case $n = 0$. In this case the first order pole has to be removable since $\tilde{F}$ is W invariant in λ. Next if $n \neq 0$ we may assume that $\alpha = \alpha_i$ is simple and $n > 0$ by W invariance. Take the residue $\mathrm{Res}_{n,i}$ of $\tilde{F}$ at the hyperplane H_{n,α_i}. Clearly $\mathrm{Res}_{n,i}$ is also a solution of (6.1), defined on a tubular neighborhood of A in H and W invariant there. Let W_i be the rank one parabolic subgroup $W_i = \{1, r_i\}$ and let W^i denote the set of elements w in W such that $l(wr_i) > l(w)$. By Lemma 6.5, there exists an asymptotic expansion on A_+ of the form ($\lambda \in H_{n,\alpha_i}$):

$$\mathrm{Res}_{n,i}(a) = \sum_{w \in W} d_w(\lambda, k)\Phi(w\lambda, k, a)$$

with $d_w = 0$ if $w \in W^i$ (and in particular, $d_e = 0$). The remaining leading exponents have, for generic $\lambda \in H_{n,\alpha_i}$, no mutual differences in P. Hence we may, for any $j \in \{1, \ldots, n\}$, separate $\mathrm{Res}_{n,i}$ into sub-sums

$$\Sigma_{w,j}(a) = \sum_{x \in W_j} d_{xw}(\lambda, k)\Phi(xw\lambda, k, a)$$

using the monodromy action of the θ_v (see text preceding Corollary 6.9) such that $r_j v = v$. By Lusztig's formula (6.10) we have $[T_j, \theta_v] = 1$ for such v. Hence these sub-sums $\Sigma_{w,j}$ are still T_j invariant. Therefore, the boundary value of $\Sigma_{w,j}$ along the wall $e^{\alpha_j} = 1$ is a multiple of an ordinary hypergeometric function. From the theory of asymptotic expansion of the ordinary hypergeometric function we obtain that $d_w = d_{r_j w} = 0$ if either $d_w = 0$ or $d_{r_j w} = 0$. This, combined with the prior remark that $d_e = 0$, implies that $d_w = 0 \ \forall w \in W$, by a simple inductive argument on the length of w. Hence the pole at H_{n,α_i} was removable. □

Theorem 6.14 (Gauss summation formula [30]) *The function $\tilde{F}$ can be evaluated explicitly at the unit element of H: $\tilde{F}(\lambda, k; e) = \tilde{c}(\rho(k), k)$. This evaluation is equivalent to the following limit formulae: When $k_\alpha \leq 0$ for all α, then*

$$\lim_{a \in A_+, a \downarrow e} \Phi(\lambda, k; a) = \tilde{c}(-\lambda, 1 - k).$$

Proof. (Sketch) We normalize

$$F(\lambda, k; a) = \frac{1}{\tilde{c}(\rho(k), k)} \tilde{F}(\lambda, k; a)$$

and consider the value at the identity $f(\lambda, k) := F(\lambda, k; e)$. It follows from Theorem 5.13(d) that, since

$$G_-(k+1)\tilde{F}(\lambda, k+1) = \tilde{F}(\lambda, k),$$

one has in any case the property that $f(\lambda, k)$ is entire and periodic in k. One can show $f(\lambda, k)$ is non-vanishing. We also see that $f(\lambda, k) \in \mathbf{R}$ if λ, k are real. Finally one can show that $k \mapsto f(\lambda, k)$ is entire with growth order ≤ 1. (This is technical, but essentially based on the recurrence relations (6.6) for $\Gamma_\kappa(\lambda, k)$.) By Hadamard's factorization theorem for entire functions one concludes that a function with these properties must be constant in k, and therefore $f(\lambda, k) = f(\lambda, 0) = 1$ for all λ and k. For the formulation in terms of the limits of Harish-Chandra series: consult [30]. □

Definition 6.15 $F(\lambda, k; a)$ is called the *hypergeometric function* for the root system R.

7 The KZ connection

The goal of this section is to understand properly the analogue of the polynomials $E(\lambda, k)$ for arbitrary $\lambda \in \mathfrak{h}^*$. We call this analogue *non-symmetric hypergeometric functions.* The construction of non-symmetric local solutions of the T_ξ on a W-orbit leads naturally to the study of the so called Knizhnik-Zamolodchikov connection. We will gain a lot of insight in the equations (6.1) by doing this exercise. Most importantly perhaps, it will become plain that the system has regular singularities. Also, it will naturally bring into play the action of the affine Weyl group by virtue of the affine intertwiners of Cherednik as discussed in Section 4.

Basic references for this section are [32] and [13].

7.1 Non-symmetric hypergeometric functions

For each element $h \in H^{\text{reg}}$, we define

$$S_{Wh}(\lambda, k) = \left\{\varphi \in \mathcal{O}_{Wh} \,;\, p(T_\xi(k))\varphi = p(\lambda)\varphi \text{ for any } p \in S(\mathfrak{h})^W\right\}.$$

Proposition 7.1 *The space $S_{Wh}(\lambda, k)$ is an $\mathbf{H}(R_+, k)$-module and the dimension of the subspace $S_{Wh}(\lambda, k)^W$ of W-invariant elements is $|W|$.*

Proof. Recall that $\mathbf{H}(R_+, k)$ is realized as the algebra generated by W and $\{T_\xi(k)\,;\,\xi \in \mathfrak{h}\}$ and also that the center of $\mathbf{H}(R_+, k)$ is $\{p(T_\xi(k))\,;\, p \in S(\mathfrak{h})^W\}$ (Lemma 4.1). Hence, $S_{Wh}(\lambda, k)$ is a module for $\mathbf{H}(R_+, k)$. By definition of D_p (Definition 5.9), $S_{Wh}(\lambda, k)^W$ is the space of solutions of the hypergeometric system (6.1). Then, by Theorem 6.7, $\dim S_{Wh}(\lambda, k)^W = |W|$. □

We now want to understand the weight subspace

$$S_{Wh}(\lambda, k)^\lambda = \{\varphi \in \mathcal{O}_{Wh} \,;\, T_\xi(k)\varphi = \lambda(\xi)\varphi \text{ for any } \xi \in \mathfrak{h}\}.$$

We have a map from $S_{Wh}(\lambda, k)^\lambda$ to $S_{Wh}(\lambda, k)^W$ given by $\varphi \mapsto \sum_{w \in W} \varphi^w$. (As in Section 3, we use the notation $\varphi^w = \varphi(w^{-1}\cdot)$ for a function φ). The following simple algebraic lemmata serve to prove that this is an isomorphism if λ satisfies certain conditions.

Lemma 7.2 *Let $\mathbf{C}_\lambda$ denote the one dimensional $S(\mathfrak{h})$-module in which $f \in S(\mathfrak{h})$ act by multiplication with $f(\lambda)$. The $\mathbf{H}$-module $\mathcal{I}_\lambda = \operatorname{Ind}_{S(\mathfrak{h})}^{\mathbf{H}}(\mathbf{C}_\lambda)$ is called the minimal principal series module induced from the character λ. It is isomorphic to the regular representation as $\mathbf{C}[W]$-module. Suppose that λ satisfies $\lambda(\alpha^\vee) \neq 0, \pm k_\alpha$ for all $\alpha \in R_+$. Then $\mathcal{I}_\lambda$ is the direct sum of its one dimensional weight spaces $\mathcal{I}_\lambda^\mu$ with $\mu \in W\lambda$. Moreover, $\mathcal{I}_\lambda$ is irreducible and the map*

$$p : \mathcal{I}_\lambda^\mu \ni v \mapsto \sum_{w \in W} wv \in \mathcal{I}_\lambda^W$$

is an isomorphism for any $\mu \in W\lambda$. *Recall that the center* $Z(\mathbf{H})$ *of* $\mathbf{H}$ *equals* $S(\mathfrak{h})^W$. *We say that a module* M *of* $\mathbf{H}$ *has central character* λ *if the center of* $\mathbf{H}$ *acts in* M *by* $z \cdot m = z(\lambda)m$ *(for* $z \in Z(\mathbf{H})$ *and* $m \in M$*). Every module over* $\mathbf{H}$ *with central character* λ *and dimension* $\leq |W|$ *is isomorphic to* $\mathcal{I}_\lambda$.

Proof. Recall the intertwiners I_w of Subsection 4.1. Observe that $I_w \in \mathbf{H}(R_+, k)$ if $w \in W$, and that the defining property of the intertwiner implies that

$$I_w(I_\lambda^\mu) \subset I_\lambda^{w(\mu)}.$$

Under the assumption on λ we see that the kernel of the intertwiners I_w cannot have a nontrivial intersection with the weight space $\mathcal{I}_\lambda^\lambda$. Hence all weight spaces of the form $\mathcal{I}_\lambda^\mu$ with $\mu \in W\lambda$ are at least one dimensional. Thus by a dimension count every weight space $\mathcal{I}_\lambda^\mu$ is one dimensional, and the intertwiners I_w act as isomorphisms. The irreducibility of $\mathcal{I}_\lambda$ follows from the remark that any nonzero submodule has to contain at least one (nonzero) weight vector, but we have seen that all weight vectors are cyclic. Suppose that $0 \neq v \in \mathcal{I}_\lambda^\mu$ and that $p(v) = 0$. Then $\mathbf{H}v = \mathbf{C}[W]v$ has dimension less than $|W|$, contradicting the irreducibility. If M is a module with central character λ and dimension $\leq |W|$, then we argue as before that all its weight spaces with weight $\mu \in W\lambda$ have dimension 1. In particular, there is a nonzero weight vector of weight λ, which gives rise to an isomorphism with $\mathcal{I}_\lambda$. □

Lemma 7.3 *Let* M *be any* $\mathbf{H}(R_+, k)$*-module with central character* λ. *Denote by* M^λ *the weight space with weight* λ *and by* M^W *the subspace of* W*-invariant elements. If* $\lambda(\alpha^\vee) \neq 0, \pm k_\alpha$ *for all* $\alpha \in R_+$*, then* M *is semi-simple and isotypic of type* $\mathcal{I}_\lambda$. *The map*

$$p : M^\lambda \ni v \mapsto \sum_{w \in W} wv \in M^W$$

is an isomorphism. If M^W *is finite dimensional then* M *itself is finite dimensional with* $\dim(M) = |W|\dim(M^W)$.

Proof. For a given $v \in M$ let us consider the submodule $\mathbf{H}v$. This is a quotient of the module $Q_\lambda = \mathbf{H}/J_\lambda$ with J_λ the ideal generated by the central elements $p - p(\lambda)$ with $p \in S(\mathfrak{h})^W$. It is clear that Q_λ can be represented by $\mathfrak{H} \otimes \mathbf{C}[W]$ with $\mathfrak{H}$ the harmonic elements in $S(\mathfrak{h})$. Hence Q_λ^W has dimension $|W|$, and for every $q \in Q_\lambda^W$, Hq is isomorphic to $\mathcal{I}_\lambda$ by the previous lemma. Thus Q_λ is a direct sum of $|W|$ copies of $\mathcal{I}_\lambda$. Now everything claimed follows from the previous lemma. □

Remark 7.4 The inverse of

$$p : M^\lambda \ni v \mapsto \sum_{w \in W} wv \in M^W$$

is given by the application of the element $q \in S(\mathfrak{h})$ given by

$$q = \prod_{\alpha \in R_+} \left(1 - \frac{k_\alpha}{\lambda(\alpha^\vee)}\right)^{-1} \prod_{w \in W, w \neq e} \frac{\xi - w\lambda(\xi)}{\lambda(\xi) - w\lambda(\xi)},$$

where ξ is any element in $\mathfrak{h}$ satisfying $\lambda(\xi) \neq w\lambda(\xi)$ for all $w \neq e$

Proof. It is sufficient to prove this for $M = \mathcal{I}_\lambda$. Consider the following identity in $\mathbf{C}[W]$:

$$|W|\epsilon^+ = \sum_w c_w(\lambda) \tilde{I}_w^\lambda$$

(notations as in Remark 4.4 and Lemma 5.10). We compute the coefficients c_w easily by the following remarks. First of all, one verifies directly that

$$c_{w_0}(\lambda) = \prod_{\alpha \in R_+} \frac{\lambda(\alpha^\vee) + k_\alpha}{\lambda(\alpha^\vee)}.$$

Using the cocycle relation of Remark 4.4 and the observation $\epsilon^+ \cdot \tilde{I}_w^\lambda = \epsilon^+$ it follows that $c_w(\lambda) = c_{w_0}(w_0 w \lambda)$, hence

$$c_w(\lambda) = \prod_{\alpha \in R_+} \frac{w\lambda(\alpha^\vee) - k_\alpha}{w\lambda(\alpha^\vee)}$$

Apply this decomposition of $p = |W|\epsilon^+$ to $v = 1 \in \mathcal{I}_\lambda^\lambda$ and we see that $q \circ p(1) = 1$, as desired. □

Corollary 7.5 *Retain the assumptions of Lemma 7.3. The dimension of $S_{Wh}(\lambda, k)$ is $|W|^2$, and this defines a local system $S(\lambda, k)$ of $\mathbf{H} = \mathbf{H}(R_+, k)$-modules with central character λ on the regular orbit space. The monodromy of this local system centralizes the $\mathbf{H}$-module structure, and gives $S_{Wh}(\lambda, k)$ the structure of a $\mathbf{H}^{\mathrm{aff}}(R_+, q)$-module. More precisely, $S_{Wh}(\lambda, k)$ is the direct sum of $|W|$ copies of the monodromy of the equations (6.1).*

Proof. We leave to the reader the easy verification that monodromy of $S(\lambda, k)$ commutes with the actions on $S(\lambda, k)$ by W and by Dunkl operators. By the previous lemmata, $S(\lambda, k)$ is the direct sum of weight spaces $S(\lambda, k)^\mu$ all of which are isomorphic to $S(\lambda, k)^W$ via the intertwiner p for the monodromy. (And of course, $S(\lambda, k)^W$ is nothing but the local system of solutions of (6.1).) □

Corollary 7.6 *If $\operatorname{Re} k_\alpha \geq 0$ for any $\alpha \in R_+$, then there exists a unique holomorphic function $G(\lambda, k; \cdot)$ in a tubular neighborhood of A such that*

$$T_\xi(k) G(\lambda, k; \cdot) = \lambda(\xi) G(\lambda, k; \cdot), \qquad (1)$$

$$G(\lambda, k; e) = 1. \qquad (2)$$

Proof. For λ satisfying $\lambda(\alpha^\vee) \neq 0, \pm k_\alpha$ for any $\alpha \in R_+$, we define

$$G(\lambda, k; \cdot) = |W| D_q F(\lambda, k; \cdot),$$

where q is as defined in Remark 7.4. By Remark 7.4 it is clear that (1) holds.

Since this function satisfies (again by Remark 7.4):

$$F(\lambda, k; \cdot) = \frac{1}{|W|} \sum_{w \in W} G^w(\lambda, k; \cdot),$$

(2) follows from Theorem 6.14. The apparent poles in λ are removable because of the next lemma, from which the uniqueness also follows.

Lemma 7.7 *Let $\varphi \in S(\lambda, k)^\lambda$ be a holomorphic function in a neighborhood of $e \in A$. If $\operatorname{Re} k_\alpha \geq 0$ for any $\alpha \in R_+$, then $\varphi(e) = 0$ implies $\varphi = 0$.*

Proof. Let $\{\xi_i\}$ be an orthonormal basis of $\mathfrak{a}$ and let $\{\xi_i^*\}$ be the dual basis. The lowest homogeneous part of the operator

$$\sum_{i=1}^n \xi_i^* T_{\xi_i}(k) = \sum_{i=1}^n \xi_i^* \partial_{\xi_i} + \sum_{\alpha \in R_+} \frac{k_\alpha \alpha}{1 - e^{-\alpha}} (1 - r_\alpha)$$

at the origin is equal to

$$E(k) = \sum_{i=1}^n \xi_i^* \partial_{\xi_i} + \sum_{\alpha \in R_+} k_\alpha (1 - r_\alpha).$$

Assume that $\varphi \neq 0$ and let f be the lowest homogeneous part of φ with degree $m \geq 0$. By the equation $\sum_{i=1}^n \xi_i^* T_{\xi_i}(k) \varphi = \lambda \varphi$, we have

$$E(k) f = \left(m + \sum_{\alpha \in R_+} k_\alpha (1 - r_\alpha) \right) f = 0.$$

Since $\mathbf{C}[W] f$ is a $\mathbf{C}[W]$-module, we can express f as a sum $\sum_{\delta \in \hat{W}} f_\delta$ of δ-equivariant parts f_δ for each $\delta \in \hat{W}$. The element $\sum_{\alpha \in R_+} k_\alpha (1 - r_\alpha)$ is central in $\mathbf{C}[W]$, hence acts on an irreducible $\mathbf{C}[W]$-module δ by a scalar. It is easy to see that this scalar is equal to

$$\epsilon_\delta(k) = \sum_{\alpha \in R_+} k_\alpha (1 - \chi_\delta(r_\alpha) / \chi_\delta(e)),$$

where χ_δ is the character of δ, and we have the following equation:

$$(m + \epsilon_\delta(k)) f_\delta = 0 \quad \text{for each } \delta \in \hat{W}.$$

On the other hand, since $\mathrm{Re}\,\epsilon_\delta(k)$ is not less than zero for each $\delta \in \hat{W}$ by assumption, we have $f_\delta = 0$ unless $m = 0$. Contradiction. □

We shall prove the removability of poles of $G(\lambda, k)$. Assume that $G(\lambda, k)$ has a singularity. Since $F(\lambda, k)$ is an entire function of (λ, k) and by the expression $G(\lambda, k) = |W| D_q F(\lambda, k)$, $G(\lambda, k)$ is meromorphic in (λ, k) and its singular set is the zero set of a function that depends only on (λ, k). Let (λ_0, k_0) be a regular point and let φ be an irreducible holomorphic function in a neighborhood V of (λ_0, k_0) such that the zero set of φ is equal to the singular set in V. Let $l \in \mathbf{N}$ be the smallest integer such that $\tilde{G} = \varphi^l G$ extends holomorphically to V. By continuity and the property (2), $\tilde{G}(\lambda, k, e) = 0$ for any singular point (λ, k) in V and, by Lemma 7.7, $\tilde{G}(\lambda, k) \equiv 0$ for these points. This is a contradiction. □

Example 7.8 Let us consider the BC_1 case, i.e. $R = \{\pm\alpha, \pm 2\alpha\}$. We use the notation in Example 6.3. The functions F and G are expressed as follows:

$$\begin{cases} F(\lambda, k; x) = {}_2F_1(a, b, c; z), \\ G(\lambda, k; z) = {}_2F_1(a, b, c; z) + \frac{1}{4b}(y - y^{-1})_2F_1'(a, b, c; z), \end{cases}$$

where ${}_2F_1(a, b, c; z)$ is Gauss' hypergeometric function.

Remark 7.9 We have seen that $p : S^\lambda \to S^W$ is an isomorphism if $\lambda(\alpha^\vee) \neq 0, \pm k_\alpha$ for all $\alpha \in R$, and that this map is an intertwiner for the monodromy representation of $\mathbf{H}^{\mathrm{aff}}(R_+, q_i)$. In fact, for sufficiently generic parameters, we have two isomorphisms:

$$\begin{aligned} S(\lambda, k) &\simeq \mathcal{I}_\lambda^{|W|} \quad \text{(as } \mathbf{H}\text{-module)}, \\ &\simeq \left(\mathrm{Ind}_{\mathbf{C}[Q^\vee]}^{\mathbf{H}^{\mathrm{aff}}} e^{2\pi\sqrt{-1}(\lambda)}\right)^{|W|} \quad \text{(as } \mathbf{H}^{\mathrm{aff}}\text{-module)}. \end{aligned}$$

These two actions commute with each other. Notice that also the shift operators $G_\pm(k) : S(\lambda, k)^W \to S(\lambda, k \pm 1)^W$ and the intertwiners $I_w : S(\lambda, k)^\lambda \to S(w\lambda, k)^{w\lambda}$ $(w \in W^e)$ commute with the $\mathbf{H}^{\mathrm{aff}}$-action.

Remark 7.10 Since $T_\xi(k)$ is *not* W-equivariant, $G(w\lambda, k; a)$ and $G^w(\lambda, k, a)$ do not coincide. The correct relationship between them is given by affine intertwiners:

$$I_w G(\lambda, k) = \left(\prod_{a \in R_+^a \cap w^{-1} R_+^a} (\lambda(a) + k_a) \right) G(w\lambda, k) \quad \text{for} \quad w \in W^e.$$

7.2 The role of the Knizhnik-Zamolodchikov connection

Let Ω^l be the sheaf of holomorphic l-forms on $\mathfrak{h}^{\mathrm{reg}}$. We use the notation Ω^l_h and Ω^l_{Wh} analogously to $\mathcal{O}_h$ and $\mathcal{O}_{Wh}$.

Define an operator $d(\lambda,k):\Omega^l_{Wh}\to\Omega^{l+1}_{Wh}$ by

$$d(\lambda,k)=d-d(\lambda+\rho(k))+\sum_{\alpha\in R_+}k_\alpha(1-e^{-\alpha})^{-1}d\alpha\otimes(1-r_\alpha).$$

As in the proof of Lemma 7.7, let $\{\xi_i\}$ be an orthonormal basis of $\mathfrak{a}$ and let $\{\xi_i^*\}$ be its dual basis of $\mathfrak{a}^*$. Since the action of $d(\lambda,k)$ is expressed as

$$\begin{aligned} d(\lambda,k)(\varphi\otimes dx_1\wedge\cdots\wedge dx_l) &= \sum_{i=1}^{n}\left(\partial_{\xi_i}-(\lambda+\rho(k))(\xi_i)+\sum_{\alpha\in R_+}\frac{k_\alpha\alpha(\xi_i)}{1-e^{-\alpha}}(1-r_\alpha)\right)\varphi \\ &\qquad\otimes d\xi_i^*\wedge dx_1\wedge\cdots\wedge dx_l \\ &= \sum_{i=1}^{n}(T_{\xi_i}(k)-\lambda(\xi_i))\,\varphi\otimes d\xi_i^*\wedge dx_1\wedge\cdots\wedge dx_l, \end{aligned}$$

we have $d(\lambda,k)^2=0$, and

$$0\longrightarrow S^\lambda\overset{\text{inj.}}{\longrightarrow}\mathcal{O}_{Wh}\overset{d(\lambda,k)}{\longrightarrow}\Omega^1_{Wh}\overset{d(\lambda,k)}{\longrightarrow}\Omega^2_{Wh}\longrightarrow\cdots$$

is a cochain complex.

Note that Ω^l_{Wh} is isomorphic to $(\Omega^l_{Wh}\otimes\mathbf{C}[W])^W$ by

$$\Omega^l_{Wh}\ni\varphi\overset{\sim}{\longmapsto}\sum_{w\in W}\varphi^w\otimes w\in(\Omega^l_{Wh}\otimes\mathbf{C}[W])^W.$$

On the other hand, $(\Omega^l_{Wh}\otimes\mathbf{C}[W])^W$ is also isomorphic to $\Omega^l_h\otimes\mathbf{C}[W]$ by

$$\Omega^l_h\otimes\mathbf{C}[W]\ni\varphi\otimes v\overset{\sim}{\longmapsto}\sum_{w\in W}\varphi^w\otimes wv\in(\Omega^l_{Wh}\otimes\mathbf{C}[W])^W.$$

Via these isomorphisms, we have a new cochain complex:

$$0\longrightarrow\mathcal{L}^\lambda\overset{\text{inj.}}{\longrightarrow}\mathcal{O}_h\otimes\mathbf{C}[W]\overset{\nabla(\lambda,k)}{\longrightarrow}\Omega^1_h\otimes\mathbf{C}[W]\overset{\nabla(\lambda,k)}{\longrightarrow}\Omega^2_h\otimes\mathbf{C}[W]\longrightarrow\cdots$$

Since the isomorphism $\Omega^l_{Wh}\overset{\sim}{\longrightarrow}\Omega^l_h\otimes\mathbf{C}[W]$ is given by

$$(\varphi_w)_{w\in W}\mapsto\sum_{w\in W}\varphi^w_{w^{-1}}\otimes w\quad(\varphi_w\in\Omega^l_{w\cdot h})$$

and the inverse is given by

$$\sum_{w\in W}\psi_w\otimes w\mapsto(\psi^w_{w^{-1}})_{w\in W},$$

the operator $\nabla(\lambda, k)$ is expressed as follows:

$$\nabla(\lambda,k)(\psi\otimes w\otimes dx_1\wedge\cdots\wedge dx_l) = \sum_{i=1}^{n}\nabla_{\xi_i}(\lambda,k)(\psi\otimes w)\otimes d\xi_i^*\wedge dx_1\wedge\cdots\wedge dx_l,$$

with

$$\begin{aligned}\nabla_\xi(\lambda,k) =& w\left(T_{w^{-1}\xi}(k)-w\lambda(\xi)\right)w^{-1} \quad \text{(multiplication in } \mathbf{H}(R_+,k))\\ =& \partial_\xi+\frac{1}{2}\sum_{\alpha\in R_+}k_\alpha\left(\alpha(\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\otimes(1-r_\alpha)+\alpha(\xi)\otimes r_\alpha\epsilon_\alpha\right)-w\lambda(\xi),\end{aligned}$$

and $\epsilon_\alpha(w) = -\mathrm{sgn}(w^{-1}\alpha)w$. The last expression is a consequence of (5.1), and the reflections in $\nabla_\xi(\lambda,k)$ act on $\mathbf{C}[W]$ by left multiplication.

Definition 7.11 We call the connection $\nabla(\lambda,k)$ the (trigonometric) *Knizhnik-Zamolodchikov connection* (KZ-connection in the sequel).

Corollary 7.12 (Matsuo [26]) *The KZ connection is integrable and the map*

$$\sum_{w\in W}\psi_w\otimes w\mapsto\sum_{w\in W}\psi_w$$

gives an isomorphism from $\mathcal{L}_\lambda$ *to* S^W *if* $\lambda(\alpha^\vee)\neq 0,\pm k_\alpha$ *for any* $\alpha\in R$.

The isomorphism in Corollary 7.12 is called the Matsuo isomorphism.

Remark 7.13 We can easily extend this isomorphism to the weaker condition "$\lambda(\alpha^\vee)\neq k_\alpha$ for any $\alpha\in R_+$".

Remark 7.14 By Corollary 7.6, $G(\lambda,k)\in S(\lambda,k)^\lambda$. Then, by the above discussion, the vector $\sum_{w\in W}G^w(\lambda,k)\otimes w$ is an element of

$$\mathcal{L}_\lambda=\{\psi\in\mathcal{O}_h\otimes\mathbf{C}[W]\,;\,\nabla(\lambda,k)\psi=0\}.$$

8 Harmonic Analysis on A

In this section we study the eigenfunction transform $\mathcal{F}$ for the algebra of Dunkl operators acting on $C_c^\infty(A)$. We shall prove a Paley-Wiener theorem and an explicit inversion formula for $\mathcal{F}$, when $k_\alpha \geq 0$ for all $\alpha \in R$. The transform $\mathcal{F}$ was called the Cherednik transform in [32] and the Opdam transform in [4]. We will simply use the generic name "Fourier transform" here.

8.1 Paley-Wiener theorem

For f, $g \in C_c^\infty(A)$, define

$$(f,g)_k = \int_A f(a)\overline{g(a)}\delta_k(a)da,$$

where

$$\delta_k(a) = \prod_{\alpha\in R_+} \left|a^{\alpha/2} - a^{-\alpha/2}\right|^{2k_\alpha}$$

and da is the Lebesgue measure on A normalized by $\mathrm{vol}(A/\exp(Q^\vee)) = 1$. In this section we assume that $k_\alpha \geq 0$ for all $\alpha \in R$. In this and the next section we shall only give complete proofs when there is something new to add to the ideas in the literature. Otherwise we shall content ourselves with references.

The following lemma is an easy computation.

Lemma 8.1 ([32, Lemma 7.8])

$$(T_\xi f, g)_k = (f, (-w_0 T_{w_0(\bar{\xi})} w_0)g)_k.$$

Here w_0 is the longest element in W.

Definition 8.2 For $f \in C_c^\infty(A)$ and $\lambda \in \mathfrak{h}^*$, define

$$\mathcal{F}(f)(\lambda) = \int_A f(a)G(-w_0\lambda, k; w_0 a)\delta_k(a)da.$$

And for φ a "nice function" on $\mathfrak{h}^*$, define

$$\mathcal{J}(\varphi)(a) = \int_{\sqrt{-1}\mathfrak{a}^*} \varphi(\lambda)G(\lambda, k; a)\sigma(\lambda)d\mu(\lambda),$$

where

$$\sigma(\lambda) = \prod_{\alpha\in R_+} \frac{\Gamma(\lambda(\alpha^\vee) + k_\alpha)\Gamma(-\lambda(\alpha^\vee) + k_\alpha + 1)}{\Gamma(\lambda(\alpha^\vee))\Gamma(-\lambda(\alpha^\vee) + 1)},$$

and $d\mu(\lambda)$ is the translation invariant holomorphic n-form such that the volume of $\sqrt{-1}\mathfrak{a}/2\pi\sqrt{-1}P$ equals 1.

First we need to show that $C_c^\infty(A)$ is mapped by $\mathcal{F}$ in a space of nice functions, so that the composition $\mathcal{J}\mathcal{F}(f)$ makes sense. Given $a \in A$, let C_a denote the convex hull of Wa and let H_a denote the support function given by

$$H_a(\lambda) = \sup\{\lambda(\log b)\,;\, b \in C_a\}.$$

An entire function φ on $\mathfrak{h}^*$ is said to be of *Paley-Wiener type* a if

$$^\forall N \in \mathbf{N},\ ^\exists C > 0 \,:\, |\varphi(\lambda)| \leq C(1+|\lambda|)^{-N} \exp(H_a(-\mathrm{Re}(\lambda))) \quad (\lambda \in \mathfrak{h}^*).$$

Let $PW(a)$ be the space of entire functions of the Paley-Wiener type a and $PW = \bigcup_{a\in A} PW(a)$.

Theorem 8.3 ([32, Proposition 6.1, Corollary 6.2]) *For all $k \in K^{reg}$ (here regular means: $\tilde{c}(\rho(k), k) \neq 0$) and all compact subset D of A, and all $p \in S(\mathfrak{h})$, there exists $C > 0$ and $N \in \mathbf{N}$ such that*

$$\sup_{a\in D} |\partial(p)G(\lambda, k; a)| \leq C(1 + |\lambda|^N) \exp(\max_w\{\mathrm{Re}(w\lambda(\log a))\}).$$

Proof. (Sketch) If a and ξ are regular elements in the same Weyl chamber, we can see that

$$\partial_\xi(a^{-2\mu} \sum_w |G(\lambda, k, w^{-1}a)|^2) \leq 0$$

from KZ connection, where $\mu \in W\mathrm{Re}\lambda$ such that $\mu(\xi) = \max_w\{\mathrm{Re}(w\lambda(\xi))\}$. This proves the theorem for $p \equiv 1$. The statement for general $p \in S(\mathfrak{h})$ follows from Cauchy's formula. □

Theorem 8.4 (Paley-Wiener theorem [32, Theorem 8.6])

(a) $\mathcal{F} \,:\, C_c^\infty(C_a) \to PW(a)$

(b) $\mathcal{J} \,:\, PW(a) \to C_c^\infty(C_a)$

Proof. (a) follows directly from Theorem 8.3. Using asymptotic expansion (b) can be proved in the same way as Helgason's proof of the Paley-Wiener theorem for Riemannian symmetric spaces [17]. □

8.2 Inversion and Plancherel formula

Theorem 8.5 (see [32]) *$\mathcal{F}\mathcal{J}$ and $\mathcal{J}\mathcal{F}$ are identical on PW and $C_c^\infty(A)$ respectively.*

Proof. The theorem was first proved by Opdam[32]. Here we will give an outline of Cherednik's proof of Theorem 8.5 ([4]). It is a very nice proof, based on the action of the affine intertwiners. The non-symmetric theory is essential now.

One checks by direct computation that

$$\begin{aligned} \mathcal{F}(I_i f)(\lambda) &= -(\lambda(a_i) + k_i)\mathcal{F}(f)(r_i\lambda), &(8.1)\\ \mathcal{F}(T_\xi f)(\lambda) &= \lambda(\xi)\mathcal{F}(f)(\lambda), &(8.2)\end{aligned}$$

Combined these formulae show that

$$\mathcal{F}(f^{r_i}) = \mathcal{F}(f)^{r_i} - k_i \frac{\mathcal{F}(f) - \mathcal{F}(f)^{r_i}}{a_i} = Q_i(\mathcal{F}(f)). \tag{8.3}$$

Here Q_i is the Lusztig operator, which is the action of r_i in the module

$$\mathrm{Ind}_{\mathbf{C}[W]}^{PW\otimes_{S(\mathfrak{h})}\mathbf{H}(R_+,k)}(\mathrm{triv}).$$

Next one checks that

$$\mathcal{J}(Q_i(\varphi)) = \mathcal{J}(\varphi)^{r_i} \quad i = 0, 1, \ldots, n. \tag{8.4}$$

This is delicate if $i = 0$, since we need a contour shift here (the proof for $i \neq 0$ is the same, but without the shift). If $i = 0$ it is only true for $k_\alpha \geq 0$ ($\alpha \in R$). For the proof we need

$$\left(1 + \frac{k_i}{\lambda(a_i)}\right)\sigma(\lambda) = \left(1 - \frac{k_i}{\lambda(a_i)}\right)\sigma(r_i\lambda), \tag{8.5}$$

which follows easily from the definition of σ.

We have

$$\begin{aligned}\mathcal{J}(Q_i(\varphi)) =& \int_{\sqrt{-1}\mathfrak{a}^*} Q_i(\varphi)(\lambda)G(\lambda,k;a)\sigma(\lambda)d\mu(\lambda)\\
=& \int_{\sqrt{-1}\mathfrak{a}^*} \left(\varphi^{r_i} - k_i\frac{\varphi - \varphi^{r_i}}{\lambda(a_i)}\right) G(\lambda,k;a)\sigma(\lambda)d\mu(\lambda) \quad \text{(by (8.3))}\\
=& \int_{\sqrt{-1}\mathfrak{a}^*} \varphi(r_i\lambda)\left(1 + \frac{k_i}{\lambda(a_i)}\right) G(\lambda,k;a)\sigma(\lambda)d\mu(\lambda)\\
&- k_i \int_{\sqrt{-1}\mathfrak{a}^*} \varphi(\lambda)\frac{1}{\lambda(a_i)}G(\lambda,k;a)\sigma(\lambda)d\mu(\lambda)\\
=& \int_{r_i(\sqrt{-1}\mathfrak{a}^*)} \varphi(\lambda)\left(1 + \frac{k_i}{\lambda(a_i)}\right) G(r_i\lambda,k;a)\sigma(\lambda)d\mu(\lambda)\\
&- k_i \int_{\sqrt{-1}\mathfrak{a}^*} \varphi(\lambda)\frac{1}{\lambda(a_i)}G(\lambda,k;a)\sigma(\lambda)d\mu(\lambda) \quad \text{(by (8.5))}\\
=& \int_{\sqrt{-1}\mathfrak{a}^*} \varphi(\lambda)\left(\left(1 + \frac{k_i}{\lambda(a_i)}\right) G(r_i\lambda,k;a) - \frac{k_i}{\lambda(a_i)}G(\lambda,k;a)\right)\sigma(\lambda)d\mu(\lambda)\\
=&\mathcal{J}(\varphi)^{r_i}.\end{aligned}$$

In last steps we use shift of contour for $i = 0$ and a formula for G^{r_i} based on the formula for I_iG (cf. Remark 7.10):

$$G^{r_i}(\lambda, k; a) = \left(1 + \frac{k_i}{\lambda(a_i)}\right) G(r_i\lambda, k; a) - \frac{k_i}{\lambda(a_i)} G(\lambda, k; a).$$

Observe that the necessary shift of contour when $i = 0$ is allowed when $k_\alpha > 0$, since the only pole of σ that possibly needs to be reckoned with is canceled by the factor

$$1 + \frac{k_0}{\lambda(a_0)} = \frac{1 - \lambda(\theta^\vee) + k_\theta}{1 - \lambda(\theta^\vee)}.$$

However, when $k_\alpha < 0$ the poles at $\lambda(a_i) + k_i$ enter into the positive chamber, and these destroy the symmetry for $i = 0$.

By (8.3) and (8.4), $\mathcal{J} \circ \mathcal{F}$ commutes with action of W^e on $C_c^\infty(A)$. In particular, $\mathcal{J} \circ \mathcal{F}$ commutes with multiplications by e^λ ($\lambda \in P$). It is easy to see that the ideal i_{x_0} of functions in $C_c^\infty(A)$ that vanish at some point $x_0 \in A$ can be written as $j_{x_0} C_c^\infty(A)$, where j_{x_0} denotes the maximal ideal at x_0 in $\mathbf{C}[P]$. Hence $\mathcal{J} \circ \mathcal{F}$ maps i_{x_0} into itself, for all x_0. Therefore it has to be multiplication by a $f \in C^\infty(A)$. Since $\mathcal{J} \circ \mathcal{F}$ is also W equivariant, f must be W invariant. Finally, by (8.2), it has to also commute with T_ξ-action on $C_c^\infty(A)$. Thus we have

$$T_\xi f = \partial_\xi f = 0 \quad \text{for all } \xi,$$

and f must be a constant. One can prove that the constant is 1 by considering the asymptotics.

Conversely $\mathcal{F} \circ \mathcal{J}$ commutes with multiplications by polynomials $p \in S(\mathfrak{h})$. As before, $\mathcal{F} \circ \mathcal{J}$ has to be multiplication by some function g. Computing $\mathcal{J}\mathcal{F}\mathcal{J}(\varphi)$ in two ways, we have

$$\mathcal{J}(\varphi) = \mathcal{J}(g\varphi).$$

At $e \in A$ we have

$$\int_{\sqrt{-1}\mathfrak{a}^*} \varphi(\lambda)\sigma(\lambda) d\mu(\lambda) = \int_{\sqrt{-1}\mathfrak{a}^*} g(\lambda)\varphi(\lambda)\sigma(\lambda) d\mu(\lambda),$$

hence $g \equiv 1$. □

The inversion formula we have derived now is NOT the inversion formula of the spectral decomposition of $\mathbf{C}_c^\infty(A)$ for the action of the commutative algebra of Dunkl-Cherednik operators (this algebra of operators is not even closed with respect to the $*$ operator!). Accordingly, the function σ is not positive (not even real), we have no Plancherel formula and no extension of $\mathcal{F}$ to an L_2 space. One can fix this by considering the decomposition of $\mathbf{C}_c^\infty(A)$ with respect to its structure as a pre-unitary module of the action of the non-commutative $*$ algebra $\mathbf{H}$, and this point of view was used in [32]. A simpler way out of this is the reduction of the transform to the $|W|$-symmetric situation. If $f \in C_c^\infty(A)$ is W-invariant, then

$$\mathcal{F}(f)(\lambda) = \int_A f(a)F(-\lambda, k; a)\delta_k(a)da, \tag{8.6}$$

which coincides with the Harish-Chandra transform for spherical functions if the parameter k corresponds to the root multiplicities of a Riemannian symmetric space.

The W-invariance of f results in the W-invariance of $\mathcal{F}(f)$. We may thus change from G to F in the transform $\mathcal{J}$ using the following formula (we leave it to the reader to prove this easy formula):

$$F(\lambda, k; a) := |W|^{-1} \sum_{w \in W} \sum_{\alpha \in R_+} \left(1 - \frac{k_\alpha}{w\lambda(\alpha^\vee)}\right) G(w\lambda, k; a).$$

We obtain

$$f(a) = \int_{\sqrt{-1}\mathfrak{a}^*} \mathcal{F}(f)(\lambda)F(\lambda, k; a)\sigma'(\lambda)d\mu(\lambda), \tag{8.7}$$

where

$$\sigma'(\lambda) = \prod_{\alpha \in R_+} \frac{\Gamma(\lambda(\alpha^\vee) + k_\alpha)\Gamma(-\lambda(\alpha^\vee) + k_\alpha)}{\Gamma(\lambda(\alpha^\vee))\Gamma(-\lambda(\alpha^\vee))}.$$

Notice that

$$\sigma'(\lambda) = \frac{1}{c(\lambda, k)c(-\lambda, k)} = \frac{1}{|c(\lambda, k)|^2} > 0,$$

where

$$c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)}.$$

Formula (8.7) is a k-deformation of Harish-Chandra's inversion formula for spherical transform. For arbitrary k ($k_\alpha \geq 0$, $\alpha \in R$) it had been conjectured by Heckman and Opdam and was proved by Opdam[32]. For group case, see [17, Ch IV].

9 The attractive case (Residue Calculus)

In the previous section we gave the inversion formula for $\mathcal{F}$ for the repulsive case, $k_\alpha \geq 0$ for all $\alpha \in R$. In this section we consider the attractive case, $k_\alpha < 0$ for all $\alpha \in R$ (cf. [33]). The spectral decomposition involves lower dimensional spectra.

9.1 Paley-Wiener theorem and Plancherel theorem

The formula

$$(f, g)_k = \int_A f(a)\overline{g(a)}\delta_k(a)da,$$

gives an inner product only as long as $\delta_k(a)$ is locally integrable.

Theorem 9.1 ([13, Proposition 5.1], [33, Proposition 1.1]) *$\delta_k(a)$ is locally integrable if and only if k is in the connected component of $\{k\,;\, \tilde{c}(\rho(k), k) > 0\}$ containing $k_\alpha \geq 0$ for all $\alpha \in R$. In particular this is satisfied in the following two situations:*

(a) $k_\alpha \geq 0$ *for all* $\alpha \in R$.

(b) $k_\alpha < 0$ *for all* $\alpha \in R$, *and* $\rho(k)(\theta^\vee) + k_\theta + 1 > 0$.

Here, as always, θ is the highest short root. In case (a), $\delta_k(a)$ is locally integrable and in case (b), $\delta_k(a)$ is even integrable.

Remark 9.2 If R is simply laced, the condition for k in the theorem means that $k > -1/d_n$, where d_n is the Coxeter number.

Remark 9.3 If $\delta_k(a)$ is integrable, then $G(-\rho(k), k, \cdot) = 1$ is square integrable with respect to $\delta_k(a)da$. On the other hand, in the sense of the previous section its Fourier transform is zero. Clearly the inversion formula with purely continuous spectrum as in the previous section now fails!

From now on we assume that we are in the situation of Theorem 9.1(b) (the so-called attractive case). And we will restrict ourselves to the W-symmetric case, in view of the remarks made in the last part of the previous section.

We define $\mathcal{F}$ as before, but we define $\mathcal{J}$ by

$$(\mathcal{J}\varphi)(a) = \int_{\gamma+\sqrt{-1}\mathfrak{a}^*} \varphi(\lambda)\Phi(\lambda, k; a)\frac{d\mu}{c(-\lambda, k)}, \quad \varphi \in PW, \tag{9.1}$$

where $\gamma \in \mathfrak{a}^*_- = \{\lambda \in \mathfrak{a}^*\,;\, \lambda(\alpha^\vee) < 0\,{}^\forall\alpha > 0\}$ such that $\gamma(\alpha^\vee) < -k_\alpha$ and $a \in A_+$. By Lemma 6.5, $\Phi(\lambda, k; a)$ is holomorphic in λ if $\mathrm{Re}(\lambda(\alpha^\vee)) < 1 - \varepsilon$ for all $\alpha \in R_+$ and $\varepsilon > 0$. If $k_\alpha \geq 0$ for all $\alpha \in R$, then (9.1) coincides with the right hand side of (8.7) for $\mathcal{F}f = \varphi$ by analytic continuation and symmetrization.

As we have seen, the proof of Theorem 8.5 by Cherednik fails. However, the original proof of the inversion formula survives:

Theorem 9.4 (see [33, Theorem 5.4]) *Still $\mathcal{J}\mathcal{F}$ and $\mathcal{F}\mathcal{J}$ are identical.*

We will now engage a process to refine the defining formula for $\mathcal{J}$ in such a way that $\mathcal{J}$ becomes integration of λ over some subset of $\mathfrak{h}^*$, against the kernel $F(\lambda, k; a)$ multiplied by a positive measure, the Plancherel measure. This will give rise to the extension of $\mathcal{F}$ to $L_2(A, \delta_k)^W$, and eventually to an isometric isomorphism of $L_2(A, \delta_k)^W$ with the L_2 space on $\mathfrak{h}^*$ defined by the Plancherel measure. In other words, this leads to the spectral resolution of the commutative algebra of differential operators D_p, $p \in S(\mathfrak{h})^W$.

This process consists of a shift of the contour of (9.1) from $\gamma+\sqrt{-1}\mathfrak{a}^*$ to $\sqrt{-1}\mathfrak{a}^*$. The residual contours one encounters along the way also move as though they are attracted by the origin (and these again pick up residues along the way, and so on). When everybody comes to a standstill, we have contours of integration in every possible dimension. Next we have to symmetrize, and then finally we will have the integral defining $\mathcal{J}$ satisfying the properties described mentioned above.

Let us first formulate the results of all this precisely. We need some terminology:

Definition 9.5 An affine subspace $L \subset \mathfrak{a}^*$ is called *residual* if

$$\#\{\alpha \in R;\, \alpha^\vee(L) = k_\alpha\} = \#\{\alpha \in R;\, \alpha^\vee(L) = 0\} + \operatorname{codim}(L). \qquad (9.2)$$

Notice that $\mathfrak{a}^*$ itself is residual. If a residual subspace L is a point, we call it a *distinguished point.* Given L residual, let c_L denote the orthogonal projection of $0 \in \mathfrak{a}^*$ on L, and put

$$L = c_L + V^L,$$

$$L^{\text{temp}} = c_L + \sqrt{-1}V^L \subset \mathfrak{h}^*.$$

Remark 9.6 The classification of residual subspaces reduces to the classification of distinguished points by "parabolic induction". If $k_\alpha = k$ for all $\alpha \in R$, the distinguished points correspond to the distinguished nilpotent orbits in the semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}(R^\vee)$. Such orbits were classified by Carter and Bala.

The desired formula for $\mathcal{J}$ is given in the next theorem:

Theorem 9.7 ([33, Theorem 3.4])

$$\mathcal{J}\varphi(a) = \sum_L \int_{L^{\text{temp}}} \varphi(\lambda)F(\lambda, k; a)d\nu_L(\lambda, k).$$

Here

$$d\nu_L(\lambda, k) = \gamma_L(k) f_L(\lambda, k) d\mu_L(\lambda), \qquad (9.3)$$

$$f_L(\lambda, k) = \tilde{c}(\rho(k), k)^2 \frac{\prod'_L \Gamma(\lambda(\alpha^\vee) + k_\alpha)}{\prod'_L \Gamma(\lambda(\alpha^\vee))},$$

μ_L is Lebesgue measure on L^{temp} such that $\mathrm{vol}(\sqrt{-1}V^L/2\pi\sqrt{-1}(P \cap V^L)) = 1$, $\prod'_L$ is the product of the Γ-factors of the roots which do not vanish identically on L, $0 \le \gamma_L(k) \in \mathbf{Q}$, and the sum is taken over all the residual subspaces L such that $c_L \in \mathfrak{a}^*_-$.

Corollary 9.8 ([33, Theorem 5.7, Corollary 5.8]) *$\nu_L(\lambda, k)$ is a positive measure (if nonzero). The W-invariant square integrable eigenfunctions of $L(k)$ are $F(\lambda(k), k; \cdot)$ with $\lambda(k)$ distinguished in $\mathfrak{a}^*_-$ and $\gamma_{\lambda(k)}(k) > 0$. For these we have*

$$\int_A F(\lambda(k), k; a)^2 \delta(k, a) da$$
$$= \pm\gamma_L^{-1} |W\lambda(k)|^{-1} \frac{\prod_{\alpha\in R_+} \Gamma(\rho(k)(\alpha^\vee) + k_\alpha)^2 \prod_{\alpha\in R\setminus R_z} \Gamma(\lambda(k)(\alpha^\vee))}{\prod_{\alpha\in R_+} \Gamma(\rho(k)(\alpha^\vee))^2 \prod_{\alpha\in R\setminus R_p} \Gamma(\lambda(k)(\alpha^\vee) + k_\alpha)},$$

where

$$R_z = \{\alpha \in R;\ \lambda(k)(\alpha^\vee) = 0 \text{ for all } k\},$$
$$R_p = \{\alpha \in R;\ \lambda(k)(\alpha^\vee) + k_\alpha = 0\}.$$

The parameters $\lambda(k)$ in Corollary 9.8 are classified in [15, Section 4].

Example 9.9 (see [1]) If $k_\alpha = k$ for all $\alpha \in R$ then $\lambda(k) = \rho(k)$ is distinguished and for $F(\rho(k), k; \cdot) = 1$, we have

$$\int_A \delta(k, a) da = \prod_{i=1}^n \binom{d_i k}{k} \frac{\pi}{\sin(-m_i \pi k)},$$

where m_i are the exponents and $d_i = m_i + 1$ are the degrees.

In the rest of the section, we will give an outline of the proof of Theorem 9.7.

9.2 Residues

Given a finite arrangement of affine hyperplanes $\mathcal{H}$ in a Euclidean space V, we choose for each $H \in \mathcal{H}$ a vector $\alpha_H \in V$, and a number $k_H \in \mathbf{R}$ such that

$$H = \{\lambda \in V;\ (\alpha_H, \lambda) = k_H\}.$$

Let $\mathcal{L}$ be the lattice of intersections of elements of $\mathcal{H}$, ordered by inclusion (and $V \in L$ by definition). Let ω be a rational n-form on $V_{\mathbf{C}}$, with poles possibly at the hyperplanes of $\mathcal{H}$, but nowhere else. Let PW denote the space of Paley-Wiener functions, with rapid decay in the imaginary direction.

GOAL Study the functional

$$X_{V,\gamma} : PW \to \mathbf{C}, \varphi \mapsto \int_{\gamma+\sqrt{-1}V} \varphi\omega,$$

in particular what happens when γ moves from chamber to chamber.

We may rewrite $X_{V,\gamma}$ in many different ways as a sum of $X_{V,\gamma'}$'s and residual integrations over lower dimensional contours. In fact, we will describe a systematic way of pointing out a special chamber in each $L \in \mathcal{L}$, to which we want to move γ. The point is that this defines a unique way of rewriting $X_{V,\gamma}$.

Given $L \in \mathcal{L}$, let c_L be the orthogonal projection of $O \in V$ onto L. Write $L = c_L + V^L$, where $V^L \subset V$ is a linear subspace and $\mathcal{C} = \{c_L \,;\, L \in \mathcal{L}\}$, the set of centers. The next lemma is elementary, but very effective.

Lemma 9.10 ([33, Lemma 3.1]) *There exists a unique collection of tempered distributions on X_c, $c \in \mathcal{C}$ such that*

(a) $\operatorname{supp}(X_c) \subset \cup_{L;\, c_L=c}\sqrt{-1}V^L$,

(b) X_c *has finite order,*

(c) $X_{V,\gamma}(\varphi) = \sum_{c\in\mathcal{C}} X_c(\varphi(c+\cdot))$ *for all* $\varphi \in PW$.

The distributions X_c play a crucial role. We refer to X_c as "the local contribution of $X = X_{V,\gamma}$ at the center c".

Remark 9.11 The value of X_c does not change when either O or γ passes a hyperplane that does not contain c. Hence, when computing X_c, we may always assume that both O and γ are in chambers which contain c in their closure. In other words, we reduce in this way to consider the central arrangement of hyperplanes that contain the center c.

Lemma 9.12 ([33, Lemma 3.3]) *Let $\mathcal{H}$ be a central arrangement with center c. If $X_c \neq 0$, then O must be in the closure of the anti-dual chamber of the chamber in which γ lies. Explicitly,*

$$O \in \overline{\sum_{H\in\mathcal{H}'} \mathbf{R}_+ c_H + \sum_{H\in\mathcal{H}''} \mathbf{R}_- c_H} + c,$$

where $\mathcal{H}'$ is the set of non-separating hyperplanes for γ and O, and $\mathcal{H}'' = \mathcal{H} \setminus \mathcal{H}'$.

The above result follows from the next example, the special case of normal crossings, since every arrangement of hyperplanes can be approximated by arrangements with normal crossings only. In this normal crossing case it is a simple exercise using the geometry of simplicial cones.

Example 9.13 (normal crossing case) Suppose $(\gamma, \alpha_H) < k_H$ for all $H \in \mathcal{H}$, and $\mathcal{H}$ is divisor with normal crossings at $c = \cap_{H \in \mathcal{H}} H$. Assume

$$\omega = \prod_H ((\lambda, \alpha_H) - k_H)^{-1} d\lambda$$

and assume that O is in the anti-dual of γ. Then

$$\begin{aligned} X_c(\varphi(c + \cdot)) &= (-2\pi\sqrt{-1})^n \det(\alpha_H, \alpha_{H'})^{-1/2} \varphi(c) \\ &= (-2\pi\sqrt{-1})^n \frac{1}{\mathrm{vol}(V / \sum_H \mathbf{Z}\alpha_H)} \varphi(c). \end{aligned}$$

9.3 The arrangement of shifted root hyperplanes

Assume that we have a root system R, irreducible, reduced, in $V = \mathfrak{a}^*$, and root multiplicities $k_\alpha \in \mathbf{R}_-$. Let $R^\vee \subset \mathfrak{a}$ be the set of coroots, and normalize the Lebesgue measure dx (resp. $d\lambda$) on $\mathfrak{a}$ (resp. $\sqrt{-1}\mathfrak{a}^*$) such that $\mathrm{covol}(Q^\vee) = 1$ (resp. $\mathrm{covol}(2\pi\sqrt{-1}P) = 1$). Denote by $c'(\lambda, k)$ the rational function

$$c'(\lambda, k) = \prod_{\alpha \in R_+} \frac{\lambda(\alpha^\vee) + k_\alpha}{\lambda(\alpha^\vee)}.$$

Consider

$$\begin{aligned} X_{\mathfrak{a}^*,\gamma}(\varphi) &= \int_{\gamma + \sqrt{-1}\mathfrak{a}^*} \varphi(\lambda) \frac{d\lambda}{c'(-\lambda, k)}, \\ Y_{\mathfrak{a}^*,\gamma}(\varphi) &= \int_{\gamma + \sqrt{-1}\mathfrak{a}^*} \varphi(\lambda) \frac{d\lambda}{c'(-\lambda, k) c'(\lambda, k)}, \end{aligned}$$

where $\gamma \in \mathfrak{a}^*_-$ such that $\gamma(\alpha^\vee) - k_\alpha < 0$ for all $\alpha \in R_+$. Let

$$H_\alpha = \{\lambda \in \mathfrak{a}^* \,;\, \lambda(\alpha^\vee) = k_\alpha \, {}^\forall \alpha \in R\}$$

and let $\mathcal{C}$ be the set of centers of the corresponding intersection lattice $\mathcal{L}$. For $c \in \mathcal{C}$, denote by X_c and Y_c the local contribution of $X_{\mathfrak{a}^*,\gamma}$ and $Y_{\mathfrak{a}^*,\gamma}$. Given $c \in \mathcal{C}$, denote by W_c the stabilizer in W of c, and let A_c denote the symmetrization operation

$$A_c \varphi(\lambda) = |W_c|^{-1} \sum_{w \in W_c} c'(w\lambda, k) \varphi(w\lambda).$$

Notice that this is holomorphic in a neighborhood of $c + \sqrt{-1}\mathfrak{a}^*$ if φ is so.

Lemma 9.14 ([33, Proposition 3.6]) *For $c \in \mathcal{C} \cap \overline{\mathfrak{a}^*_-}$ and $w \in W$, we have*

$$X_{wc} = Y_c \circ w^{-1} \circ A_{wc}.$$

This has the following application, which is of substance when c is singular. Suppose that λ is in the support of some Y_c with $c \in \mathfrak{a}^*_-$. If $w\lambda$ is not in the support of X_{wc} then $A_{wc}\varphi(w\lambda)$ must be zero. By Lemma 9.12 this is always the case when $wc = Re(w\lambda) \notin \overline{{}_-\mathfrak{a}^*}$. This argument will show that the hypergeometric function $F(\lambda, k, \cdot)$ has all its leading exponents in $\overline{{}_-\mathfrak{a}^*}$ for such λ, hence is tempered by a well known criterion of Casselman and Milicić. This is the content of Corollary 9.19. Let us now formulate this argument on a technical level. The next result is a direct application of Lemma 9.12.

Corollary 9.15 ([33, Corollary 3.7]) *For $c \in \mathcal{C}$, write*

$${}_-\mathfrak{a}^{*c} = \sum_{\alpha \in R_+, c(\alpha^\vee) = k_\alpha} \mathbf{R}_- \alpha \subset \overline{{}_-\mathfrak{a}^*},$$

where $\overline{{}_-\mathfrak{a}^*}$ is the closure of anti-dual of ${\mathfrak{a}^*}_+$. Let $c \in \mathcal{C} \cap \overline{{\mathfrak{a}^*}_-}$ and $w \in W$ with $wc \notin {}_-\mathfrak{a}^{*wc}$. If $\lambda \in c + \operatorname{supp}(Y_c)$ then $A_{wc}\varphi(w\lambda) = 0$ for all $\varphi \in PW({\mathfrak{a}^*}_c)$.

First of all, recall that in this attractive case $k_\alpha < 0$, we are interested only in the situation where $\delta_k(a)$ is integrable on A, and we have seen that this means that condition (b) in Theorem 9.1 holds. It means geometrically that

$$C_{\rho(k)} \subset \{\lambda \in \mathfrak{a}^* \, ; \, |\lambda(\alpha^\vee)| < 1 + k_\alpha \, {}^\forall \alpha \in R\}.$$

Choose an open convex W-invariant set U between these sets.

Lemma 9.16 ([33, Proposition 2.2]) *Let $a \in A_+$. Then $\lambda \mapsto \Phi(\lambda, k; a)$ is holomorphic on $\mathfrak{a}_- + U + \sqrt{-1}\mathfrak{a}^*$, and uniformly bounded there.*

Lemma 9.17 ([33, Lemma 3.3]) *Write $c(\lambda, k) = c'(\lambda, k)c''(\lambda, k)$. Then $c''(\lambda, k)^{\pm}$ are holomorphic on $U + \sqrt{-1}a^*$, and $c''(\lambda, k)^{-1}$ bounded, $c''(\lambda, k)$ of moderate growth. Also $c''(-\lambda, k)^{-1}$ is holomorphic in $\mathfrak{a}_- + U + \sqrt{-1}\mathfrak{a}^*$ and $c''(\lambda, k)c''(-\lambda, k)$ and $c'(\lambda, k)c'(-\lambda, k)$ are W-invariant.*

Lemma 9.18 ([33, Lemma 3.2]) *All centers $c \in \mathcal{C}$ lie in U.*

Corollary 9.15 contains important information about the hypergeometric function, because the operator A_{wc} plays a role in its definition. If $c = Re(\lambda)$ then

$$\begin{aligned} F(\lambda, k; a) &= \sum_{w \in W} c(w\lambda, k)\Phi(w\lambda, k; a) \\ &= \sum_{w \in W/W_c} |W_c| A_{wc}(c''(\cdot, k)\Phi(\cdot, k; a))(w\lambda). \end{aligned}$$

Together with the above results concerning the good behavior of Φ and c'' on $U + i\mathfrak{a}^*$ this finally leads to the desired result:

Corollary 9.19 ([33, Corollary 3.7]) *If* $\lambda \in c + \mathrm{supp}(Y_c)$, $c \in \mathcal{C} \cap \mathfrak{a}^*_-$, *and* $w \in W$ *such that* $wc \notin {}_-\mathfrak{a}^{*wc}$, *then* $a \mapsto F(\lambda, k; a)$ *is tempered on* A. *If* $L = c$, *and* $Y_c \neq 0$, *then* $F(c, k; a)$ *has exponential decay; such* F *are called cuspidal.*

Now we need to say more about the shifted root hyperplane arrangement. There are two very special geometric peculiarities of this arrangement that make everything work properly. It is obvious that the local contributions of $Y_{\mathfrak{a}^*,\gamma}$ have support at subspaces that are residual in the following sense.

Definition 9.20 L is called residual if

$$\#\{\alpha \in R;\, \alpha^\vee(L) = k_\alpha\} \geq \#\{\alpha \in R;\, \alpha^\vee(L) = 0\} + \mathrm{codim}(L).$$

However, as we have seen in Definition 9.5, whenever the above inequality holds it has to be an equality! This is of crucial importance because this shows that the local contributions of $Y_{\mathfrak{a}^*,\gamma}$ are in fact densities (distributions of order 0). Another important point is that a residual subspace L of dimension k is determined by a distinguished point of a parabolic subsystems of rank $n - k$. In fact L^{temp} is the space of the corresponding unitary parabolic induction parameters, as embedded in the parameter space of the minimal principal series. This structure makes it possible to work with "unitary parabolic induction". The second peculiarity has to do with the positivity of the relative Plancherel measures on L^{temp} needed in this inductive process. Here one needs the property that $-c_L$ and c_L are in the same orbit of the fixator group of $\mathfrak{a}^{*L}$ in W.

The following theorem is proved by the classification (!) of distinguished points.

Theorem 9.21 ([15, Theorem 3.9, Theorem 3.10, Remark 3.11]) *If* L *is residual in the sense of Definition 9.20, then*

$$\#\{\alpha \in R;\, \alpha^\vee(L) = k_\alpha\} = \#\{\alpha \in R;\, \alpha^\vee(L) = 0\} + \mathrm{codim}(L).$$

If L *is residual, then its center* c_L *is a distinguished point for* $R_L = \{\alpha \in R;\, L(\alpha^\vee) = \text{constant}\}$ *and* $-c_L \in W(R_L)c_L$.

As indicated, this leads to:

Corollary 9.22 *If* L *is residual,* $c_L \in \mathcal{C} \cap \mathfrak{a}^*_-$ *and* $Y_{c_L} \neq 0$, *then it is in fact a measure, namely integration over* $c_L + \sqrt{-1}\mathfrak{a}^{*L}$ *against the density*

$$d\nu'_L(\lambda, k) = \gamma_L(k) \frac{\prod' |\lambda(\alpha^\vee)|}{\prod' |\lambda(\alpha^\vee) + k_\alpha|} d_L(\lambda),$$

where $\prod'$ *denotes the product over all* $\alpha \in R$, *omitting zero factors.*

The Corollary 9.19 makes it possible to show (by induction, starting with the distinguished points) that all densities involved are in fact positive measures (and Theorem 9.21 is crucially needed in the inductive process):

Corollary 9.23 *The function* $(c''(\lambda,k)c''(-\lambda,k))^{-1}$ *is positive, bounded and real analytic on* $c_L+\sqrt{-1}\mathfrak{a}^{*L}$, *and* $\nu_L(\lambda,k)=(c''(\lambda,k)c''(-\lambda,k))^{-1}\nu'_L(\lambda,k)$ *is given by formula (9.3). It is a positive, real analytic measure when* $\gamma_L(k)\neq 0$).

Corollary 9.24 *If* φ *is a* W*-invariant, PW-function and* $\gamma(\alpha^\vee)<k_\alpha$ *for all* $\alpha\in R_+$, *then*

$$\begin{aligned}\mathcal{J}_\gamma(\varphi) &= X_{\mathfrak{a}^*,\gamma}(\varphi\Phi(\cdot,k,a)c''(-\lambda,k)^{-1})\\ &= \int_{\gamma+\sqrt{-1}\mathfrak{a}^*}\varphi(\lambda)\Phi(\lambda,k,a)\frac{d\lambda}{c(-\lambda,k)}\\ &= \sum_{L:\text{residual},c_L\in C\cap\mathfrak{a}^*_-}\int_{L^{\text{temp}}}\varphi(\lambda)F(\lambda,k;a)d\nu_L(\lambda,k)\\ &:= \mathcal{J}(\varphi).\end{aligned}$$

Theorem 9.7 follows from this corollary. We finish with the main result, the Plancherel Theorem.

Theorem 9.25 ([33, Theorem 5.5]) *A residual subspace* L *is called spherically tempered when* $\nu_L\neq 0$. *The map* $\mathcal{F}$ *extends naturally to an isometric isomorphism*

$$\mathcal{F}: L_2(A,\delta_k da)^W \to \left\{\bigoplus_{L\,sph.temp.} L_2(L^{\text{temp}},\nu_L(k))\right\}^W,$$

with inverse $\mathcal{J}$ *as in Theorem 9.7.*

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