9 Sobolev spaces on manifolds with constant negative curvature

9.1 Sobolev spaces on the upper branch of the hyperboloid

The upper branch of the unit hyperboloid

$$X : \; t = \sqrt{|x|^2 + 1}$$

is a Riemannian manifold with constant negative curvature $K(x) = -1$. With respect to the parametrization of X

$$\Omega = (\mathrm{ch} r, \omega \,\mathrm{sh} r) \in X, \tag{9.1.1}$$

where $r > 0, \omega \in \mathbf{S}^{n-1}$ the metric on X is

$$ds^2 = dr^2 + \mathrm{sh}^2 r d\omega^2,$$

where $d\omega^2$ is the standard metric on $\mathbf{S}^{n-1}$. In particular, if $f : X \to \mathbb{R}$ and $d\Omega$ is the standard measure on X, we have

$$\int_X f(\Omega) d\Omega = \int_0^\infty \int_{\mathbf{S}^{n-1}} f(\mathrm{ch} r, \omega \mathrm{sh} r)(\mathrm{sh} r)^{n-1} dr d\omega.$$

With respect to this parametrization, the Laplace-Beltrami operator on X takes the form (8.1.18).

If f is a real integrable function defined on X, then we observe that

$$\int_X f(\Omega) d\Omega = \int_{\mathbb{R}^n} f(< x >, x) \frac{dx}{< x >}, \tag{9.1.2}$$

where $< x >= \sqrt{1 + |x|^2}$.

Further, in the interior of the positive light cone

$$\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \; : \; |x| < t\}, \tag{9.1.3}$$

one can introduce the coordinates

$$\rho = \sqrt{t^2 - |x|^2}, \quad \Omega = \left(\frac{t}{\rho}, \frac{x}{\rho} \right) \in X$$

obtaining the following decomposition of the D'Alembertian operator(see Lemma 8.2.1)

$$\Box = -\partial_t^2 + \Delta$$

$$\rho^2 \Box = -(\rho \partial_\rho)^2 - (n-1)\rho \partial_\rho + \Delta_X,$$

$$\Box = -\partial_\rho^2 - \frac{n}{\rho} \partial_\rho + \frac{a^2}{\rho^2} \Delta_{X_a},$$

where Δ_{X_a} is the Laplace-Beltrami operator on X_a, $X = X_1$ and

$$\Delta_X = \sum_{j=1}^{n} Y_{0j}^2 - \sum_{j<k} Y_{jk}^2,$$

with

$$Y_{0j} = x_j \partial_{x_0} + x_0 \partial_{x_j} \ , \ Y_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j}.$$

It is not difficult to see that the operators Y_{0j} are skew–selfadjoint on $L^2(X)$. In fact, from (9.1.2) it follows that

$$\int_X Y_{0j} f(\Omega)\overline{g(\Omega)}d\Omega = \int_{\mathbb{R}^n} \partial_{x_j} f(< x >, x)\overline{g(< x >, x)}dx \tag{9.1.4}$$

and the assertion follows from the fact that the partial derivatives ∂_{x_j} are skew–selfadjoint operators on $L^2(\mathbb{R}^n)$.

Also in a standard way one can check that the operator Δ_X is selfadjoint on $L^2(X)$. Moreover, there is a well– developed harmonic analysis of Δ_X on the manifold X of curvature -1. (see [57] §4, §5b).

We shall avoid a direct application of the Fourier transform on X, since we shall see later on that our study is connected with the operator

$$\Delta_X + \Delta_{\mathbf{S}^{n-1}}, \tag{9.1.5}$$

where

$$\Delta_{\mathbf{S}^{n-1}} = \sum_{j<k} Y_{jk}^2 \tag{9.1.6}$$

is the Laplace-Beltrami operator on the unit sphere. From (8.2.10) we see that

$$\Delta_X + \Delta_{\mathbf{S}^{n-1}} = \sum_{j=1}^{n} Y_{0j}^2.$$

Moreover , if f, g are smooth compactly supported functions on X, then

$$\int_X Y_{0j}^2 f(\Omega)\overline{g(\Omega)}d\Omega =$$

$$= \int_{\mathbb{R}^n} \partial_{x_j} < x > \partial_{x_j} f(< x >, x)\overline{g(< x >, x)}dx \tag{9.1.7}$$

First we have to mention, that the operator in (9.1.5) is symmetric. Moreover, (9.1.7) shows it is sufficient to show that the operator

$$P = \sum_{j=1}^{n} < x > \partial_{x_j} < x > \partial_{x_j}$$

is self-adjoint on $L^2(< x >^{-1}\, dx)$. This is already established as an application of Theorem 4.2.1.

For any function $f(\Omega) \in C_0^\infty(X)$ we define the norm

$$\|f\|_{\tilde{H}^s(X)} = \|(1-\Delta_X)^{s/2} f\|_{L^2(X)}. \tag{9.1.8}$$

On the other hand, for $s \geq 0$ integer one can consider the norm

$$\sum_{|\alpha|\leq s} \|Y^\alpha f\|_{L^2(X)}, \tag{9.1.9}$$

where $Y^\alpha = Y_1^{\alpha_1} \cdots Y_N^{\alpha_N}$ and $Y_1, \ldots, Y_N$ are the vector fields $\{Y_{0j}, Y_{jk}\}_{j,k=1,\ldots,n}$ given by (8.2.11).

In order to make an interpolation and define the corresponding Sobolev spaces of fractional order we start with the following:

Lemma 9.1.1 *For $s \geq 0$ integer the norm* (9.1.9) *is equivalent to*

$$\|f\|_{H^s(X)} = \|(1-\Delta_X - \Delta_{\mathbf{S}^{n-1}})^{s/2} f\|_{L^2(X)}. \tag{9.1.10}$$

Proof. It is sufficient to verify the assertion for $s = 1$ and then to proceed inductively. The square of the norm in (9.1.10) for $s = 1$ is

$$\int_X (1-\Delta_X - \Delta_{\mathbf{S}^{n-1}}) f(\Omega)\overline{f(\Omega)}d\Omega. \tag{9.1.11}$$

Further, we use the parametrization $\Omega = (\text{ch}r, \omega\ \text{sh}r) \in X$, and from (8.1.18) we have

$$\Delta_X = \partial_r^2 + (n-1)\frac{\text{ch}r}{\text{sh}r}\partial_r + \frac{1}{\text{sh}^2 r}\Delta_{\mathbf{S}^{n-1}},$$

where

$$\Delta_{\mathbf{S}^{n-1}} = \sum_{j<k} Y_{jk}^2. \tag{9.1.12}$$

From the fact that the volume element on X is $d\Omega = (\text{sh}r)^{n-1} dr d\omega$ we see that the integral in (9.1.11) can be represented as

$$\int_0^\infty \int_{\mathbf{S}^{n-1}} \left(|f|^2 + |\partial_r f|^2\right)\ (\text{sh}r)^{n-1} d\omega dr +$$

$$+ \int_0^\infty \int_{\mathbf{S}^{n-1}} \left[\frac{1}{\text{sh}^2 r}\left(\sum_{j<k} |Y_{jk} f|^2\right) + \sum_{j<k} |Y_{jk} f|^2\right]\ (\text{sh}r)^{n-1} d\omega dr.$$

Any vector field Y_{0j} is a linear combination of ∂_r and $(\mathrm{ch}r/\mathrm{sh}r)\ Y_{jk}$. Since

$$Y_{0j}(f) = -\omega_j \partial_r f - \frac{\mathrm{ch}r}{\mathrm{sh}r} \sum_{k=1}^{n} \omega_k Y_{jk} f, \tag{9.1.13}$$

we conclude that

$$\|Y_{0j} f\|^2_{L^2(X)} \leq C \|(1 - \Delta_X - \Delta_{\mathbf{S}^{n-1}})^{1/2} f\|^2_{L^2(X)}.$$

To establish the opposite inequality we take advantage of the relations (8.2.10) and (9.1.12); we find

$$\Delta_X + \Delta_{\mathbf{S}^{n-1}} = \sum_{j=0}^{n} Y_{0j}^2.$$

Therefore, we have the relation

$$\int_X (1 - \Delta_X - \Delta_{\mathbf{S}^{n-1}}) f(\Omega) \overline{f(\Omega)} d\Omega = \|f\|^2_{L^2(X)} - \sum_{j=1}^{n} \int_X Y_{0j}^2 f(\Omega) \overline{f(\Omega)} d\Omega$$

and using the fact that Y_{0j} are skew-selfadjoint operators, we get

$$\|(1 - \Delta_X - \Delta_{\mathbf{S}^{n-1}})^{1/2} f\|^2_{L^2(X)} \leq \|f\|^2_{L^2(X)} + \sum_{j=1}^{n} \|Y_{0j} f\|^2_{L^2(X)}.$$

This completes the proof of the Lemma. □

Finally, we are in position to make complex interpolation, and we obtain the space $H^s(X)$ with fractional order $s \geq 0$:

$$H^s(X) := (H^{s_0}(X), H^{s_1}(X))_\theta\ , \quad s = s_0(1-\theta) + s_1\theta. \tag{9.1.14}$$

In particular, an equivalent norm for these spaces is given by (see [62] 1.15.3)

$$\|(1 - \Delta_X - \Delta_{\mathbf{S}^{n-1}})^{s/2} f\|_{L^2(X)}. \tag{9.1.15}$$

The following inequality between the norms in $H^s(X)$ and $\tilde{H}^s(X)$ is fulfilled:

$$\|f\|_{\tilde{H}^s(X)} \leq \|f\|_{H^s(X)}. \tag{9.1.16}$$

In fact we can check the equivalent inequality

$$\|(1 - \Delta_X)^{s/2} f\|_{L^2(X)} \leq C \|(1 - \Delta_X - \Delta_{\mathbf{S}^{n-1}})^{s/2} f\|_{L^2(X)}.$$

Since the operator $-\Delta_{\mathbf{S}^{n-1}}$ can be extended as non–negative operator on X such that its resolvent commutes with the resolvent of $-\Delta_X$, we can use the following general statement from functional analysis: if $A \geq 0, B \geq 0$ are selfadjoint operators with commuting resolvents in a Hilbert space H, then

$$\|(1+A)^{s/2} f\|_H \leq C\|(1+A+B)^{s/2} f\|_H.$$

This follows from the functional calculus based on the spectral theorem (see [62] for example) and completes the proof of (9.1.16).

Further, from the general functional calculus, if $A \geq 0, B \geq 0$ and if their resolvents are commuting, then we have the inequality

$$\|(1+A)^{s_1/2}(1+B)^{s_2/2} f\|_H \leq C\|(1+A+B)^{s/2} f\|_H, \quad s = s_1 + s_2.$$

This leads to

$$\begin{aligned} &\|(1-\Delta_X)^{s_1/2}(1-\Delta_{\mathbf{S}^{n-1}})^{s_2/2} f\|_{L^2(X)} \leq \\ &\leq \|(1-\Delta_X - \Delta_{\mathbf{S}^{n-1}})^{s/2} f\|_{L^2(X)} \end{aligned} \tag{9.1.17}$$

provided $s_1 + s_2 = s$.

Now we are in position to state the corresponding Sobolev inequality:

Theorem 9.1.1 *For any $s > n/2$ we have*

$$\Omega_0^{(n-1)/2} |f(\Omega)| \leq C\|f\|_{H^s(X)}. \tag{9.1.18}$$

We shall postpone the proof in the next sections.

9.2 Weighted Sobolev spaces on the upper branch of the hyperboloid

Our next step is to introduce weighted Sobolev spaces on X, fixing our attention on the weight $\chi(\Omega) = \Omega_0^\beta$, where β is a real number.

As usual, we denote by $L^2(X; \Omega_0^\beta)$ the L^2–space on X with measure $\Omega_0^{2\beta} d\Omega$, where $d\Omega$ is the standard measure on X; this means:

$$\|f\|_{L^2(X;\Omega_0^\beta)} = \|\Omega_0^\beta f\|_{L^2(X)}.$$

The corresponding weighted Sobolev spaces $H^{N,\beta}(X)$ for any integer $N \geq 0$ have the norm

$$\|f\|_{H^{N,\beta}(X)} = \sum_{|\alpha| \leq N} \|Y^\alpha \Omega_0^\beta f\|_{L^2(X)} \tag{9.2.1}$$

Y^α being defined by in Lemma 8.2.1.

Lemma 9.1.1 shows that this norm is equivalent to

$$\|f\|_{H^{N,\beta}(X)} = \|(1 - \Delta_X - \Delta_{\mathbf{S}^{n-1}})^{N/2}\Omega_0^\beta f\|_{L^2(X)}. \tag{9.2.2}$$

On the other hand, using the parametrization

$$\Omega = (< x >, x) \in X,$$

where $x \in \mathbb{R}^n$, by (9.1.2) we have

$$\|f\|^2_{L^2(X)} = \int_{\mathbb{R}^n} |f(< x >, x)|^2 \frac{dx}{< x >}; \tag{9.2.3}$$

then we can express the norms (9.2.1) and (9.2.2) as L^2 norms with measure

$$\frac{dx}{< x >}.$$

Moreover, in what follows we put $\tilde{f}(x) = f(< x >, x)$.

Our next step is to establish the following.

Lemma 9.2.1 *For any integer $N \geq 0$ we have*

$$\|f\|_{H^{N,\beta}(X)} \simeq \|\tilde{f}\|_{H^{N,\beta-1/2}(\mathbb{R}^n)}. \tag{9.2.4}$$

Proof. Since the quantities

$$\sum_{|\alpha| \leq N} |(< x > \partial_x)^\alpha \tilde{f}(x)|$$

and

$$\sum_{|\alpha| \leq N} |(Y^\alpha f)(< x >, x)|$$

are equivalent, to obtain (9.2.4) it suffices to use (9.2.3).

□.

This Lemma gives us a possibility to define the corresponding weighted Sobolev spaces on X by the aid of the norms

$$\|f\|_{H^{s,\beta}(X)} = \|\tilde{f}\|_{H^{s,\beta-1/2}(\mathbb{R}^n)}. \tag{9.2.5}$$

It is clear also that the map

$$f \in H^{s,\beta}(X) \to < x >^{-1/2} \tilde{f} \in H^{s,\delta}(\mathbb{R}^n) \tag{9.2.6}$$

is an isometry. Using this fact we obtain the following particular properties of these spaces

$$(H^{s,\nu}(X))' = H^{-s,-\nu}(X), \tag{9.2.7}$$

$$(9.2.8) \qquad (H^{s_0,\nu_0}(X), H^{s_1,\nu_1}(X))_\theta = H^{s,\nu}(X)$$

for $s = (1-\theta)s_0 + \theta s_1$, and $\nu = \nu_0(1-\theta) + \nu_1\theta$. Finally, from the above definition and the equivalence (7.5.22) we can take any real numbers ν, s and we can assert that

$$(9.2.9) \|(1 - \Delta_X - \Delta_{\mathbf{S}^{n-1}})^{s/2}\Omega_0^\nu f\|_{L^2(X)} \simeq \|\Omega_0^\nu(1 - \Delta_X - \Delta_{\mathbf{S}^{n-1}})^{s/2} f\|_{L^2(X)}.$$

From Lemma 7.5.2 with $p = 2$ and the isometry (9.2.6) we get the following weighted Sobolev inequality

$$(9.2.10) \qquad \Omega_0^{\delta+(n-1)/2}|u(\Omega)| \le C\|u\|_{H^{s,\delta}(X)}.$$

Applying Theorem 7.5.1 in combination with the isometry (9.2.6) we obtain the following .

Theorem 9.2.1 *For*

$$f, g \in H^s(X; \Omega_0^{\kappa_1}) \cap L^\infty(X; \Omega_0^{\kappa_2})$$

and any non-negative s we have

$$\|fg\|_{H^{s,\kappa}(X)} \le$$
$$(9.2.11) \quad \le C(\|f\|_{H^{s,\kappa_1}(X)}\|\Omega_0^{\kappa_2} g\|_{L^\infty(X)} + \|\Omega_0^{\kappa_2} f\|_{L^\infty(X)}\|g\|_{H^{s,\kappa_1}(X)})$$

for $\kappa = \kappa_1 + \kappa_2$.

Applying (7.5.47) together with the isometry (9.2.6) we arrive at

Theorem 9.2.2 *For*

$$f \in H^{s,\beta}(X) \cap L^\infty(X; \Omega_0^\gamma),$$

any $\lambda > 1$ and any non-negative s with $s < \lambda$ we have

$$(9.2.12) \qquad \||f|^\lambda\|_{H^{s,\alpha}(X)} \le C\|f\|_{H^{s,\beta}(X)}\|\Omega_0^\gamma f\|_{L^\infty(X)}^{\lambda-1}$$

for $\alpha = \beta + (\lambda - 1)\gamma$.