8 Fourier transformation on manifolds with constant negative curvature

8.1 Models of manifolds with constant negative curvature

Our goal is to extend the study of the Fourier transform and Sobolev spaces to the case of a manifold of constant negative curvature. For the purpose we introduce initially some typical models of manifolds with curvature -1. For more details on the subject one can see the book of S.Helgason [20], where general symmetric spaces are studied. Probably, manifolds with curvature -1 are simplest case of symmetric spaces of rank 1. That is why we shall concentrate our attention to this case. Our secondary purpose shall be to see the case of constant curvature $-a$. Introducing the Fourier and inverse Fourier transform for this case, we would want to see the case of flat Euclidean space as a limiting case $a \to 0$.

We turn to the models of manifold with curvature $-a, a > 0$.

Example 1. For any surface $S \subset \mathbf{R}^{n+1}$ defined by

$$S: \; t = \psi(x), x = (x_1, ..., x_n) \in \mathbf{R}^n \tag{8.1.1}$$

the Minkowski metric

$$dl^2 = -dt^2 + dx^2 \tag{8.1.2}$$

induces on S a Riemannian metric provided S is spacelike, i.e. any vector tangential to S is spacelike with respect to the form (8.1.2). More precisely, the metric induced by the embedding $S \subset \mathbf{R}^{n+1}$ is

$$ds^2 = (dl_{|S})^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j, \tag{8.1.3}$$

where

$$g_{ij} = \delta_{ij} - \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j}. \tag{8.1.4}$$

One can check that

$$g = \det(g_{ij}) = 1 - |\nabla \psi|^2$$

and the condition that S is spacelike means that $g(x) > 0$. The unit vector normal to S at the point $(x, \psi(x)) \in S$ is

$$N(x) = \frac{(1, \nabla \psi(x))}{\sqrt{1 - |\nabla \psi|^2}}.$$

The second quadratic form on S is

$$\sum_{i,j=1}^{n} b_{ij} dx^i dx^j,$$

where

$$b_{ij}(x) = \frac{\partial_{x_i}\partial_{x_j}\psi(x)}{\sqrt{1-|\nabla\psi|^2}}.$$

Now the Gauss curvature $K(x)$ is

$$K(x) = -\frac{\det b_{ij}}{\det g_{ij}} = -\frac{\det(\nabla^2\psi)}{(1-|\nabla\psi|^2)^{1+n/2}}. \tag{8.1.5}$$

A special case of a spacelike surface is the upper branch of the hyperboloid

$$X_a:\ t = \sqrt{|x|^2+a^2},$$

where a is any positive number. Then we have

$$\nabla\psi(x) = \frac{x}{\sqrt{|x|^2+a^2}},$$

$$\nabla^2\psi(x) = \frac{I}{\sqrt{|x|^2+a^2}} - \frac{x\otimes x}{(\sqrt{|x|^2+a^2})^3},$$

$$\det(\nabla^2\psi(x)) = a^2(\sqrt{|x|^2+a^2})^{-n-2}\ ,\quad K(x) = -a^{-n}. \tag{8.1.6}$$

The coefficients of the Riemannian metric (8.1.3) are

$$g_{ij} = \delta_{ij} - (|x|^2+a^2)^{-1}x_ix_j \tag{8.1.7}$$

according to (8.1.4) and (8.1.6). Then we have

$$g = \det(g_{ij}) = \frac{a^2}{x^2+a^2} \tag{8.1.8}$$

and the matrix inverse to g_{ij} is

$$g^{ij} = \delta^{ij} + a^{-2}x_ix_j. \tag{8.1.9}$$

Then the Laplace-Beltrami operator on X_a is

$$\Delta_{X_a} = \sum_{i,j=1}^{n} g^{-1/2}\partial_{x_i}g^{ij}g^{1/2}\partial_{x_j} =$$

$$\sum_{i,j=1}^{n}(x^2+a^2)^{1/2}\partial_{x_i}\delta^{ij}(x^2+a^2)^{-1/2}\partial_{x_j} +$$

$$+a^{-2}\sum_{i,j=1}^{n}(x^2+a^2)^{1/2}\partial_{x_i}x_ix_j(x^2+a^2)^{-1/2}\partial_{x_j}. \tag{8.1.10}$$

Further, we shall use another parametrization of X_a

$$\Omega = (a \text{ ch}r, a\omega \text{ sh}r) \in X_a, \tag{8.1.11}$$

where $r > 0, \omega \in \mathbf{S}^{n-1}$, $\text{ch}r = (\mathrm{e}^r + \mathrm{e}^{-r})/2$ and $\text{sh}r = (\mathrm{e}^r - \mathrm{e}^{-r})/2$. Since $dt = a \text{ sh}rdr$ and $dx = a(\text{ch}r\omega dr + \text{sh}rd\omega)$, from $\omega.d\omega = 0$ we get

$$ds^2 = a^2(dr^2 + \text{sh}^2 r d\omega^2), \tag{8.1.12}$$

where $d\omega^2$ is the standard metric on $\mathbf{S}^{n-1}$. A length of a curve $\gamma(t)$ $(\alpha \le t \le \beta)$, parametrized by $\gamma(t) = a(\text{ch}r(t), \omega(t)\text{sh}r(t))$, is

$$L(\gamma) = a\int_\alpha^\beta \sqrt{|\dot r(t)|^2 + |r(t)|^2|\dot\omega(t)|^2}dt. \tag{8.1.13}$$

The distance $d(\Omega, \Omega')$ between $\Omega, \Omega' \in X_a$ is defined by

$$d(\Omega, \Omega') = \inf_\gamma L(\gamma), \tag{8.1.14}$$

where the infimum is taken over all curves connecting Ω and Ω'. Let

$$\Omega^* = a(\text{ch}r_0, \omega_0\text{sh}r_0)$$

be a fixed point on X_a. It is easy to compute the distance between Ω^* and the "origin"

$$O_a = (a, 0, ..., 0) \in X_a.$$

In fact, let $\gamma(t) = (a\text{ch}r(t), a\omega(t)\text{sh}r(t))$ be any curve connecting O_a and Ω^* and let $\gamma_0(t) = (a\text{ch}r(t), a\omega_0\text{sh}r(t))$. Then the inequality

$$|\dot r(t)|^2 \le |\dot r(t)|^2 + |r(t)|^2|\dot\omega(t)|^2$$

shows that $L(\gamma_0) \le L(\gamma)$. Since for $\dot r(t) > 0$

$$L(\gamma_0) = \int_\alpha^\beta a\dot r(t)dt = ar(\beta) - ar(\alpha) = ar_0,$$

we find

$$d(0_a, \Omega^*) = ar_0. \tag{8.1.15}$$

Since the quadratic form (8.1.2) is invariant under the action of the group $SO(1,n)$, one can derive from (8.1.15) the relation

$$d(\Omega, \Omega') = a \text{ ch}^{-1}([\Omega, \Omega']/a^2), \tag{8.1.16}$$

where

$$[\Omega, \Omega'] = \Omega_0\Omega_0' - \Omega_1\Omega_1' - ... - \Omega_n\Omega_n' \tag{8.1.17}$$

is the quadratic form associated with (8.1.2). Indeed, given $\Omega, \Omega' \in X_a$, one can find $g \in SO(1,n)$ so that

$$g(\Omega) = 0_a, \; g(\Omega') = \Omega^*.$$

Since $[\Omega, \Omega'] = [g(\Omega), g(\Omega')] = [0_a, \Omega^*]$, we get from (8.1.15) the needed relation (8.1.16). Note that the Laplace-Beltrami operator on X_a takes the form

$$\Delta_{X_a} = a^{-2}\left(\partial_r^2 + (n-1)\frac{\mathrm{ch}r}{\mathrm{sh}r}\partial_r + \frac{1}{\mathrm{sh}^2 r}\Delta_{S^{n-1}}\right), \tag{8.1.18}$$

where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on the unit sphere S^{n-1}.

Example 2.(Model with a ball of radius R.) The ball

$$B_R = \{y \in \mathbf{R}^n : |y| < R\}$$

with metric

$$d\sigma^2 = \frac{4b^2}{(R^2 - |y|^2)^2} dy^2 \tag{8.1.19}$$

is also an example of a Riemannian manifold with constant (scalar) curvature. The volume element for this metric is

$$\left(\frac{2b}{R^2 - |y|^2}\right)^n dy.$$

We shall construct an isometry between the ball B_R with metric (8.1.19) and the quadratic surface X_a of Example 1 with metric (8.1.12). For the purpose we consider the parametrization (8.1.11) of the hyperboloid X_a and define the map

$$(r, \omega) \in \mathbf{R}_+ \times \mathbf{S}^{n-1} \longrightarrow y \in B_R \tag{8.1.20}$$

defined by

$$y = R\frac{\mathrm{sh}r/2}{\mathrm{ch}r/2}\omega.$$

Then we have

$$R^2 - |y|^2 = \frac{R^2}{\mathrm{ch}^2 r/2}$$

$$dy^2 = \frac{R^2 \mathrm{sh}^2 r/2}{\mathrm{ch}^2 r/2} d\omega^2 + \frac{R^2}{4\mathrm{ch}^4 r/2} dr^2$$

and from (8.1.19) we get

$$d\sigma^2 = \frac{b^2}{R^2}(dr^2 + \mathrm{sh}^2 r d\omega^2)$$

so comparing with (8.1.12) we see that the metrics of the models with a ball and the model with a hyperboloid coincide when

$$\frac{b}{R} = a.$$

From (8.1.11) and (8.1.20) we obtain

$$y = R\frac{\mathrm{sh}r}{2\mathrm{ch}^2 r/2}\omega = R\frac{(\Omega_1, ..., \Omega_n)}{a + \Omega_0}.$$

Therefore, the map

$$(8.1.21) \qquad \Omega = (\Omega_0, \Omega_1, ..., \Omega_n) \in X_a \longrightarrow y = R\frac{(\Omega_1, ..., \Omega_n)}{a + \Omega_0} \in B_R$$

is an isometry. Given any $y \in B_R$ and $0 = (0, ..., 0) \in B_R$, one can compute the distance $d(0, y)$ using the isometry (8.1.21). In fact,

$$d(0, y) = d(O_a, \Omega),$$

where Ω is given by

$$(8.1.22) \qquad \Omega_0 = a\frac{R^2 + |y|^2}{R^2 - |y|^2} \quad , \quad \Omega_i = \frac{2Ray_i}{R^2 - |y|^2}, i = 1, ..., n.$$

Then (8.1.17) implies that

$$[0_a, \Omega] = a^2\frac{R^2 + |y|^2}{R^2 - |y|^2}$$

and from (8.1.16) we get

$$d(0, y) = d(0_a, \Omega) = a\mathrm{ch}^{-1}(\frac{R^2 + |y|^2}{R^2 - |y|^2}).$$

Now the formula

$$\mathrm{ch}^{-1}(\frac{R^2 + |y|^2}{R^2 - |y|^2}) = \ln(\frac{R + |y|}{R - |y|})$$

shows that

$$(8.1.23) \qquad d(0, y) = a \ \ln(\frac{R + |y|}{R - |y|}).$$

Of special interest for the harmonic analysis on Riemannian manifolds with constant negative curvature are the horocycles S, defined as spheres contained in the ball B_R and touching tangentially the boundary ∂B_R of the ball B_R. Therefore, these spheres are

$$(8.1.24) \qquad S(\alpha, \omega) = \{y \in B_R : |y - \alpha R\omega| = (1 - \alpha)R\},$$

where $\alpha \in (0, 1), \omega \in \mathbf{S}^{n-1}$.

Definition 8.1.1 *Given any $z \in B_R$ and any $\omega \in \mathbf{S}^{n-1}$ we denote by*

$$< z, \omega >$$

the distance (with respect to the metric (8.1.14)) between 0 *and the horocycle*

$$S(\alpha, \omega) = \{y \in B_R : |y - \alpha R\omega| = (1-\alpha)R\},$$

such that $z \in S(\alpha, \omega)$. We take $< z, \omega >$ with sign minus if 0 *is in the interior of $S(\alpha, \omega)$.*

It is easy to compute $< z, \omega >$ explicitly. Indeed, we have

$$< z, \omega >= d(0, R(2\alpha - 1)\omega), \tag{8.1.25}$$

where $\alpha \in (0, 1)$ is determined by

$$|z - \alpha R\omega| = R(1 - \alpha)$$

so we have

$$\alpha = \frac{R^2 - |z|^2}{2R(R - \omega.z)}. \tag{8.1.26}$$

From (8.1.23) and (8.1.25) we obtain

$$< z, \omega >= a \ \ln \frac{\alpha}{1-\alpha} \tag{8.1.27}$$

and (8.1.26) implies that

$$< z, \omega >= a \ln \frac{R^2 - |z|^2}{|z - R\omega|^2}. \tag{8.1.28}$$

The isometry $X_a \to B_R$ defined by (8.1.21) enables one to compute $< z, \omega >$ as a function of $\Omega \in X_a$ determined by

$$z_j = \frac{R\Omega_j}{a + \Omega_0}$$

and this gives

$$< z, \omega >= -a \ln \frac{[\Omega, \Lambda(\omega)]}{a}, \tag{8.1.29}$$

where $\Lambda(\omega) = (1, \omega)$. Note that $[\Omega, \Lambda(\omega)] > 0$ for $\Omega \in X_a, \omega \in \mathbf{S}^{n-1}$.

Finally, we shall compute the volume element

$$\left(\frac{2b}{R^2 - |y|^2}\right)^n dy$$

with respect to the parametrization

$$y = y(\alpha, \theta, \omega) = R\alpha\omega + R(1-\alpha)\theta, \tag{8.1.30}$$

where $\theta, \omega \in \mathbf{S}^{n-1}, \alpha \in (0, 1)$.

Lemma 8.1.1 *For any* $\omega \in \mathbf{S}^{n-1}$ *we have*

$$\int_{B_R} f(y) \left(\frac{2b}{R^2 - |y|^2}\right)^n dy = \int_0^1 \int_{\mathbf{S}^{n-1}} f(y(\alpha, \theta)) h(\alpha, \theta, \omega) d\theta d\alpha,$$

where

$$h(\alpha, \theta, \omega) = \frac{2^{n-1}(1-\alpha)^{n-2} b^n R^{n-2}}{\alpha (R^2 - |y|^2)^{n-1}}$$

and $y = y(\alpha, \theta, \omega)$.

Proof. The key point in the proof is the fact that the volume element

$$\sqrt{g} dv_1 ... dv_n$$

is invariant with respect to any local coordinates $v_1, ..., v_n$ on the Riemannian manifold. On one hand, the volume element associated with the metric

$$\frac{4b^2 |dy|^2}{(R^2 - |y|^2)^2}$$

is

$$\left(\frac{2b}{R^2 - |y|^2}\right)^n dy.$$

Our purpose is to compute this volume element with respect to the coordinates α, θ in (8.1.30).

We lose no generality assuming $\omega = (1, 0, ..., 0)$. Then

$$\theta = (\cos\varphi, \theta' \sin\varphi)$$

where

$$\theta' \in \mathbf{S}^{n-2} \quad , \quad \varphi \in (0, \pi)$$

for $n \geq 3$. For $n = 2$ we have the coordinates

$$\theta = (\cos\varphi, \sin\varphi) \quad , \quad \varphi \in (0, 2\pi).$$

Then

$$d\theta^2 = d\varphi^2 + \sin^2\varphi |d\theta'|^2 \tag{8.1.31}$$

and from

$$(y_1, y') = R(\alpha(1 - \cos\varphi) + \cos\varphi, (1-\alpha)\theta' \sin\varphi)$$

with $y' = (y_2, ..., y_n)$ we get

$$\begin{aligned} |dy|^2 = |dy_1|^2 + |dy'|^2 = \\ = R^2((2 - 2\cos\varphi) d\alpha^2 + (1-\alpha)^2 d\varphi^2 \\ -2(1-\alpha)\sin\varphi d\varphi d\alpha + (1-\alpha)^2 \sin^2\varphi |d\theta'|^2) \end{aligned}$$

On the other hand, we have

$$R^2 - |y|^2 = 2R^2\alpha(1-\alpha)(1-\cos\varphi).$$

The volume element $\sqrt{g(\alpha,\varphi,\theta')}d\alpha d\varphi d\theta'$ is defined now with

$$g(\alpha,\varphi,\theta') = b^{2n}R^{-2n}\ \sin^{2(n-2)}\varphi\ \alpha^{-2n}\ (1-\alpha)^{-2}(1-\cos\varphi)^{-2(n-1)}.$$

From $d\theta = \sin^{n-2}\varphi d\varphi d\theta'$ we obtain

$$\sqrt{g}d\alpha d\varphi d\theta' = h d\alpha d\theta.$$

This completes the proof.

8.2 Fourier transform on space with constant negative curvature

For the flat space $\mathbf{R}^n$ with standard metric dy^2 generalized eigenfunctions of the Laplace operator are

$$\mathrm{e}^{\mu(y.\omega)}, \tag{8.2.1}$$

where $\omega \in \mathbf{S}^{n-1}$, $\mu = -i\lambda, \lambda > 0$, and $y.\omega = y_1\omega_1 + ... + y_n\omega_n$. The corresponding Fourier transform is

$$\hat{f}(\lambda,\omega) = \int_{\mathbf{R}^n} \mathrm{e}^{-i\lambda y.\omega} f(y)dy. \tag{8.2.2}$$

In a similar way we can define the corresponding exponents for the case of Riemannian manifold with constant negative curvature. For the purpose we consider the ball B_R of radius R with metric $4b^2(R^2-|y|^2)^{-2}dy$ as a model of Riemannian manifold with constant negative curvature $K = -a^{-2} = -R^2/b^2$. We shall look for eigenfunctions of the corresponding Laplace - Beltrami operator Δ_{B_R} of the form

$$\mathrm{e}^{\mu<y,\omega>}, \tag{8.2.3}$$

where $< y,\omega >$ is the distance between the origine and the horocycle $S(\alpha,\omega)$ determined in Definition 8.1.1.

Proposition 8.2.1 *We have*

$$\Delta_{B_R}\mathrm{e}^{\mu<y,\omega>} = \mu(\mu - \frac{n-1}{a})\mathrm{e}^{\mu<y,\omega>}. \tag{8.2.4}$$

Proof. We shall use the isometry $y \in B_R \to \Omega \in X_a$ determined by

$$\Omega_0 = a\frac{R^2 + |y|^2}{R^2 - |y|^2}, \Omega_j = \frac{2aRy_j}{R^2 - |y|^2},$$

defined in (8.1.21) and (8.1.22). From (8.1.29) we have

$$e^{\mu<y,\omega>} = [\Omega, \Lambda(\omega)]^{-\mu a} a^{\mu a}. \tag{8.2.5}$$

Now we can use the embedding

$$i : X_a \to \mathbf{R}^{n+1}$$

and the fact that the Riemannian metric on X_a is induced by the Minkowski metric $-dx_0^2 + dx_1^2 + ... + dx_n^2$ on $\mathbf{R}^{n+1}$ and the embedding i. Further, in the interior

$$\{\sqrt{x_1^2 + ... + x_n^2} < x_0\} \tag{8.2.6}$$

of the light cone one can introduce coordinates

$$\rho = \sqrt{x_0^2 - x_1^2 - ... - x_n^2}\ ,\ \Omega = a\frac{x}{\rho} \in X_a. \tag{8.2.7}$$

Then we can use the following decomposition of the D'Alembertian

$$\Box = -\partial_{x_0}^2 + \partial_{x_1}^2 + ... + \partial_{x_n}^2. \tag{8.2.8}$$

Lemma 8.2.1 *In the interior (8.2.6) of the light cone we have the relations*

$$\rho^2\Box = -(\rho\partial_\rho)^2 - (n-1)\rho\partial_\rho + \Delta_X,$$
$$\Box = -\partial_\rho^2 - \frac{n}{\rho}\partial_\rho + \frac{a^2}{\rho^2}\Delta_{X_a}, \tag{8.2.9}$$

where Δ_{X_a} is the Laplace-Beltrami operator on X_a, $X = X_1$ and

$$\Delta_X = \sum_{j=1}^{n} Y_{0j}^2 - \sum_{j<k} Y_{jk}^2, \tag{8.2.10}$$

with

$$Y_{0j} = x_j\partial_{x_0} + x_0\partial_{x_j}\ ,\ Y_{jk} = x_j\partial_{x_k} - x_k\partial_{x_j}. \tag{8.2.11}$$

Proof of Lemma 8.2.1. For completeness we shall give the proof. Let

$$(\eta^{\alpha\beta})_{\alpha,\beta=0}^{n} = \text{diag}(-1, 1, ..., 1)$$

be the matrix associated with the Minkowski metric $-dx_0^2 + dx_1^2 + ... + dx_n^2$. Set $\partial_\beta = -\partial_{x_0}$, if $\beta = 0$ and $\partial_\beta = \partial_{x_j}$, if $\beta = j = 1, ..., n$. We shall use also the notations

$$\partial^\alpha = \sum_{\beta=0}^{n} \eta^{\alpha\beta} \partial_\beta \ , \quad x^\alpha = \sum_{\beta=0}^{n} \eta^{\alpha\beta} x_\beta .$$

Then we have

$$\rho^2 = -\sum_{\alpha=0}^{n} x^\alpha x_\alpha \ , \quad \Box = \sum_{\alpha=0}^{n} \partial^\alpha \partial_\alpha$$

and the vector fields (8.2.11) take the form

$$Y_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha \ , \quad 0 \le \alpha, \beta \le n. \tag{8.2.12}$$

Therefore, we can write the identities

$$\begin{aligned} -\rho^2 \Box &= \sum_{\alpha,\beta} x^\alpha x_\alpha \partial_\beta \partial^\beta = \\ &= \sum_{\alpha,\beta} (x^\alpha Y_{\alpha\beta} \partial^\beta + x^\alpha x_\beta \partial_\alpha \partial^\beta). \end{aligned} \tag{8.2.13}$$

On the other hand, we have the following commutator relations

$$[\partial_\alpha, x_\beta] = \eta_{\alpha\beta} = \eta^{\alpha\beta} \ , \quad [x^\alpha, Y_{\alpha\beta}] = -[Y_{\alpha\beta}, x^\alpha] = n x_\beta .$$

Here and below we use the summation convention for repeated indices. From (8.2.13) we get

$$\begin{aligned} -\rho^2 \Box &= \sum_{\alpha,\beta} (Y_{\alpha\beta} x^\alpha \partial^\beta + n x_\beta \partial^\beta + x^\alpha \partial_\alpha x_\beta \partial^\beta - x^\alpha \eta_{\alpha\beta} \partial^\beta) \\ &= \frac{1}{2} (\sum_{\alpha,\beta} Y_{\alpha\beta} Y^{\alpha\beta}) + (n-1) S + S^2, \end{aligned}$$

where

$$S = \sum_\alpha x^\alpha \partial_\alpha = \sum_\alpha x_\alpha \partial^\alpha = \rho \partial_\rho$$

and

$$Y^{\alpha\beta} = \sum_{\gamma,\delta} \eta^{\alpha\gamma} \eta^{\beta\delta} Y_{\gamma\delta} .$$

Since

$$-\Delta_X = -\frac{1}{2} (\sum_{\alpha,\beta} Y_{\alpha\beta} Y^{\alpha\beta}) = \sum_{j=1}^{n} Y_{0j} - \sum_{1 \le j,k \le n} Y_{jk}^2,$$

we get the first relation in (8.2.9). The second relation in (8.2.9) follows directly from the first one.

The Lemma is proved.

Turning back to the proof of the Proposition 8.2.1, we consider any function $\chi(x)$ of the form

$$\chi(ax) = \psi(\rho[\Omega, \Lambda(\omega)]), \tag{8.2.14}$$

where ρ, Ω are the coordinates (8.2.7) in the interior of the light cone. Since for $\omega \in \mathbf{S}^{n-1}$ we have

$$\Box(\chi(ax)) = -a^2[\Lambda(\omega), \Lambda(\omega)]\psi''(\rho[\Omega, \Lambda(\omega)] = 0.$$

Taking $\psi(s) = s^{-\mu a}$ (in accordance with (8.2.5)), we use the decomposition of D'Alembertian in Lemma 8.2.1 and get

$$\Delta_{X_a}[\Omega, \Lambda(\omega)]^{-\mu a} = \mu(\mu - \frac{n-1}{a})[\Omega, \Lambda(\omega)]^{-\mu a}. \tag{8.2.15}$$

This relation combined with (8.2.5) completes the proof of the proposition.

Setting

$$\mu = -i\lambda + \frac{n-1}{2a},$$

we get

$$-\Delta_B e^{(-i\lambda+(n-1)/(2a))<y,\omega>} =$$

$$= \left(\lambda^2 + \left(\frac{n-1}{2a}\right)^2\right) e^{(-i\lambda+(n-1)/(2a))<y,\omega>}. \tag{8.2.16}$$

The relation (8.2.16) suggests us to introduce

Definition 8.2.1 *Given any $f \in \mathbf{C}_0^\infty(B_R)$ its Fourier transform is*

$$\hat{f}(\lambda, \omega) = \int_{B_R} e^{(-i\lambda+(n-1)/(2a))<y,\omega>} f(y)\nu_{R,b}(y),$$

where $\nu_{R,b}(y)$ is the measure

$$\nu(y) = \frac{2^n b^n dy}{(R^2 - |y|^2)^n}.$$

Taking

$$\frac{b}{R} = a \ , \ \frac{2b}{R^2} = 1,$$

we see that $a = R/2, b = R^2/2$ and the measure

$$\nu_{R,b}(x) = \frac{2^n b^n dy}{(R^2 - |y|^2)^n}$$

tends to the Lebesgue measure dy as $R \to +\infty$. Moreover, from (8.1.28) we see that the quantity

$$< y, \omega >= \frac{R}{2} \ln \frac{R^2 - |y|^2}{|y - R\omega|^2}$$

tends to usual scalar product $y.\omega$ as $R \to +\infty$. This observation and the Lebesgue convergence theorem show that the Fourier transform from Definition 8.2.1 tends to the usual Fourier transform in the flat Euclidean space.

Using the isometry $B_R \to X_a$, discussed in the previous section, we determine the Fourier transform on the hyperboloid X_a as follows.

Definition 8.2.2 *Given any $f(\Omega) \in \mathbf{C}_0^\infty(X_a)$ its Fourier transform is*

$$\hat{f}(\lambda, \omega) = a^{(-ia\lambda+(n-1)/2)} \int_{X_a} [\Omega, \Lambda(\omega)]^{(ia\lambda-(n-1)/2)} f(\Omega) d\Omega,$$

where $\Lambda(\omega) = (1, \omega)$.

8.3 Spherical transform on space with constant negative curvature

In this section we shall study Fourier transform of spherical functions, i.e. functions defined in the ball $B_R = \{y; |y| \leq R\}$ and depending on $|y|$. From the definition 8.2.1 we have

$$\hat{f}(\lambda, \omega) = \int_{B_R} \mathrm{e}^{(-i\lambda+(n-1)/(2a))<y,\omega>} f(|y|) \left(\frac{2b}{R^2 - y^2}\right)^n dy.$$

Since the function $< y, \omega >$ is invariant under the action of the group of rotations $SO(n)$ and since the measure $(\frac{2b}{R^2-y^2})^n dy$ is also invariant under the action of the same group, we see that

$$\hat{f}(\lambda, \omega) = \hat{f}(\lambda, \omega_0) \ , \ \omega_0 = (1, 0, ..., 0) \in \mathbf{S}^{n-1}. \tag{8.3.1}$$

Taking a mean average on the unit ball, we consider the following spherical transform

$$\tilde{f}(\lambda) = \frac{1}{\mu(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} \hat{f}(\lambda, \omega) d\omega =$$
$$= \int_{B_R} \varphi_{-\lambda}(y) f(|y|) (\frac{2b}{R^2 - y^2})^n dy, \tag{8.3.2}$$

where $\mu(\mathbf{S}^{n-1})$ is the surface of the unit ball $\mathbf{S}^{n-1}$, i.e.

$$\mu(\mathbf{S}^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{8.3.3}$$

and $\varphi_\lambda(y)$ is the "spherical exponent"

$$\varphi_\lambda(y) = \frac{1}{\mu(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} e^{(i\lambda + (n-1)/(2a)) \langle y, \omega \rangle} d\omega. \tag{8.3.4}$$

The invariance of the product $\langle y, \omega \rangle$ with respect to the action of the group of rotations guarantees that the "spherical exponents" are spherical functions. From (8.3.1) it is clear that

$$\tilde{f}(\lambda) = \hat{f}(\lambda, \omega_0) \tag{8.3.5}$$

Our next step is to compute explicitly the "spherical exponents". For the purpose we use the parametrization (8.1.20)

$$y = R\frac{\mathrm{sh} r/2}{\mathrm{ch} r/2}(\cos\varphi, \omega' \sin\varphi) = R\omega\frac{\mathrm{sh} r/2}{\mathrm{ch} r/2}, \tag{8.3.6}$$

where $\varphi \in (0, \pi), \omega' \in \mathbf{S}^{n-2}$ for $n \geq 3$ and

$$y = R\frac{\mathrm{sh} r/2}{\mathrm{ch} r/2}(\cos\varphi, \sin\varphi) \tag{8.3.7}$$

with $\varphi \in (0, 2\pi)$ for $n = 2$. Then the relation (8.1.28) implies that

$$\varphi_\lambda(R\,\mathrm{th} r/2) = \frac{\mu(\mathbf{S}^{n-2})}{\mu(\mathbf{S}^{n-1})} \int_0^\pi (\mathrm{ch} r - \cos\varphi\, \mathrm{sh} r)^{(-ia\lambda - (n-1)/2)} \sin^{n-2}\varphi\, d\varphi \tag{8.3.8}$$

for $n \geq 3$. Choosing $\mu(\mathbf{S}^0) = 2$, we see that (8.3.8) is valid also for $n = 2$. Then the integral representation (8.5.11) of the Legendre function and (8.3.3) imply

$$\begin{aligned} \varphi_\lambda(R\,\mathrm{th} r/2) &= \frac{2^{(n-2)/2}\Gamma(n/2)}{(\mathrm{sh} r)^{(n-2)/2}} P^{-(n-2)/2}_{-1/2 - i\lambda a}(\mathrm{ch} r) = \\ &= 2^{(n-2)/2}\Gamma(n/2) L^n_{\lambda a}(r), \end{aligned} \tag{8.3.9}$$

where

$$L^n_\sigma(r) = (\mathrm{sh} r)^{-(n-2)/2} P^{-(n-2)/2}_{-1/2 - i\sigma}(\mathrm{ch} r).$$

For $n \geq 3$ odd one can apply Lemma 8.5.2 from the Appendix of this Chapter and find

$$\begin{aligned} &\varphi_\lambda(R\,\mathrm{th} r/2) = \\ &= (-2)^{(n-1)/2} \frac{\Gamma(n/2)}{\Gamma(1/2)} \frac{|\Gamma(i\lambda a)|^2}{|\Gamma((n-1)/2 + i\lambda a)|^2} \left(\frac{1}{\mathrm{sh} r}\partial_r\right)^{(n-1)/2} \cos r\lambda a. \end{aligned} \tag{8.3.10}$$

For $n \geq 2$ even Lemma 8.5.3 gives

$$\varphi_\lambda(R\text{th}r/2) = \tag{8.3.11}$$
$$= \frac{2^{(n-1)/2}}{(-1)^{n/2}} \frac{\Gamma(n/2)}{\pi} \frac{|\Gamma(i\lambda a)|^2}{|\Gamma((n-1)/2 + i\lambda a)|^2} I_{\lambda,R}(r),$$

where

$$I_{\lambda,R}(r) = \int_r^\infty \frac{\text{sh}s}{\sqrt{\text{ch}s - \text{ch}r}} (\frac{1}{\text{sh}s}\partial_s)^{n/2} \cos(s\lambda a) ds.$$

After this preparation we can state the main result of this section. This result gives an inverse formula for the spherical transform defined according to (8.3.2).

Theorem 8.3.1 *For any spherical function* $f(y) = f(|y|), f \in \mathbf{C}_0^\infty(B_R)$ *we have*

$$f(|y|) = \tag{8.3.12}$$
$$= c_n \int_{-\infty}^\infty \varphi_\lambda(|y|) \tilde{f}(\lambda) \frac{|\Gamma((n-1)/2 + i\lambda a)|^2}{|\Gamma(i\lambda a)|^2} \frac{d\lambda}{a^{n-1}},$$

where

$$c_n = (2\pi)^{-n} \frac{\mu(\mathbf{S}^{n-1})}{2} = \frac{1}{2^n \pi^{n/2} \Gamma(n/2)}.$$

Proof. Our starting point is an application of Lemma 8.1.1 for the integral over the ball B_R representing $\tilde{f}(\lambda) = \hat{f}(\lambda, \omega_0)$ according to the Definition 8.2.1. For the purpose we use the parametrization

$$y(\alpha, \theta) = R(\alpha\omega_0 + (1-\alpha)\theta) \tag{8.3.13}$$

of the ball B_R by a family of horocycles. From (8.1.27) we have

$$< y(\alpha, \theta), \omega_0 >= a \ln \frac{\alpha}{1-\alpha}.$$

Then Lemma 8.1.1 yields

$$\tilde{f}(\lambda) = 2^{n-1} b^n R^{n-2} \mu(\mathbf{S}^{n-2}) \int_0^1 \int_0^\pi (\frac{\alpha}{1-\alpha})^{-i\lambda a + (n-1)/2} \times$$
$$\times \frac{f(|y(\alpha, \theta)|)}{(R^2 - |y(\alpha, \theta)|^2)^{n-1}} (1-\alpha)^{n-2} \sin^{n-2} \varphi d\varphi \frac{d\alpha}{\alpha} \tag{8.3.14}$$

where $\cos\varphi = \theta.\omega_0$. Assuming $\mu(\mathbf{S}^0) = 2$, we see that the above relation is valid for $n = 2$ too.

Let us make change of variables

$$(\alpha, \varphi) \in (0,1) \times (0, \pi) \longrightarrow (t, \rho) \in (-\infty, \infty) \times (0, \infty)$$

defined by

$$\frac{\alpha}{1-\alpha} = e^t , \quad 1-\cos\varphi = \frac{2}{1+\phi^{-2}\rho^2}, \tag{8.3.15}$$

where the function $\phi = \phi(t)$ will be chosen later. Then we have the relations

$$\begin{aligned} \alpha = \frac{e^t}{1+e^t} = \frac{e^{t/2}}{2\mathrm{ch}t/2}, \quad 1-\alpha = \frac{1}{1+e^t} = \frac{e^{-t/2}}{2\mathrm{ch}t/2} \\ d\alpha = \frac{e^t}{(1+e^t)^2}dt = \frac{dt}{4\mathrm{ch}^2 t/2}. \end{aligned} \tag{8.3.16}$$

From the second relation in (8.3.15) and

$$R^2 - |y(\alpha,\theta)|^2 = 2R^2\alpha(1-\alpha)(1-\cos\varphi) \tag{8.3.17}$$

we find

$$\begin{aligned} 1+\phi^{-2}\rho^2 = \frac{4R^2\alpha(1-\alpha)}{R^2-|y|^2} = \frac{R^2}{\mathrm{ch}^2 t/2(R^2-|y|^2)}, \\ \sin\varphi = \frac{2\phi^{-1}\rho}{1+\phi^{-2}\rho^2} = \frac{2\rho\mathrm{ch}^2 t/2(R^2-|y|^2)}{\phi R^2}, \\ \rho = \phi\sqrt{\frac{1+\cos\varphi}{1-\cos\varphi}}, \\ d\varphi = -\frac{2\phi^{-1}d\rho}{1+\phi^{-2}\rho^2} = -\frac{2(\mathrm{ch}^2 t/2)(R^2-|y|^2)}{\phi R^2}d\rho, \end{aligned} \tag{8.3.18}$$

where $y = y(\alpha,\theta)$. Since

$$\frac{D(\alpha,\varphi)}{D(t,\rho)} = -\frac{R^2-|y|^2}{2\phi R^2},$$

and $a = b/R = R/2$, taking $\phi(t) = \mathrm{ch}t/2$, from (8.3.14) we get

$$\tilde{f}(\lambda) = 2^{n-1}a^n\mu(S^{n-2})\int_{-\infty}^{\infty}\int_0^{\infty} e^{-it\lambda a} f(|y(t,\rho)|)\rho^{n-2}d\rho dt.$$

Let $F(s)$ be the function defined by

$$F\left(\frac{R^2}{R^2-y^2}\right) = f(|y|). \tag{8.3.19}$$

From (8.3.17), (8.3.15) and (8.3.16) we have

$$\frac{R^2}{R^2-y^2} = \mathrm{ch}^2 t/2 + \rho^2$$

so

$$\tilde f(\lambda) = d_n a^n \int_{-\infty}^{\infty}\int_0^{\infty} \mathrm{e}^{-it\lambda a} F(\rho^2 + \mathrm{ch}^2 t/2)\rho^{n-2} d\rho dt =$$

$$= d_n a^n \int_{-\infty}^{\infty}\int_0^{\infty} \cos(t\lambda a) F(\rho^2 + \mathrm{ch}^2 t/2)\rho^{n-2} d\rho dt \tag{8.3.20}$$

where

$$d_n = \frac{2^n \pi^{(n-1)/2}}{\Gamma((n-1)/2)}. \tag{8.3.21}$$

Now we are in situation to apply the inverse formula for 1-dimensional Fourier transform. Namely, the relation (8.3.20) can be written in the form

$$\tilde f(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{-it\lambda a}\psi(t)dt,$$

where

$$\psi(t) = \psi_n(t) = d_n a^n \int_0^{\infty} F(\rho^2 + \mathrm{ch}^2 t/2)\rho^{n-2} d\rho. \tag{8.3.22}$$

The inverse transform on $\mathbf{R}^1$ gives

$$\psi(t) = \frac{a}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{it\lambda a}\tilde f(\lambda)d\lambda = \frac{a}{2\pi}\int_{-\infty}^{\infty} \cos(t\lambda a)\tilde f(\lambda)d\lambda,$$

since $\tilde f$ is an even function due to (8.3.20). Thus we arrive at

$$\int_0^{\infty} F(\rho^2 + \mathrm{ch}^2 t/2)\rho^{n-2} d\rho =$$

$$= \frac{\Gamma((n-1)/2)}{2^{n+1}\pi^{(n+1)/2}a^{n-1}}\int_{-\infty}^{\infty} \cos(t\lambda a)\tilde f(\lambda)d\lambda. \tag{8.3.23}$$

Our next step is to study the integral transform

$$F \to H(F) = H_n(F),$$

defined by

$$H_n(F)(x) = \int_0^{\infty} F(\rho^2 + x)\rho^{n-2} d\rho. \tag{8.3.24}$$

For the purpose we use the recurrent relations

$$\partial_t(H_n(F)(\mathrm{ch}^2 t/2)) =$$

$$-\frac{n-3}{4}\mathrm{sh} t H_{n-2}(F)(\mathrm{ch}^2 t/2)\ ,\quad n \geq 4, \tag{8.3.25}$$

and

$$\partial_t(H_3(F)(\mathrm{ch}^2 t/2)) = -\frac{1}{4}\mathrm{sh}t F(\mathrm{ch}^2 t/2). \tag{8.3.26}$$

Our goal is to solve the integral equation

$$H_n(F) = H_n \tag{8.3.27}$$

with respect to F.

First, we consider the case $n \geq 3$ odd. Then combining the above relations, we get

$$F(\mathrm{ch}^2 t/2) = \frac{(-1)^{(n-1)/2}2^{(n+1)/2}}{\Gamma((n-1)/2)}\left(\frac{1}{\mathrm{sh}t}\partial_t\right)^{(n-1)/2} H_n(\mathrm{ch}^2 t/2) \tag{8.3.28}$$

for $n \geq 3$ odd.

If $n \geq 4$ is even, then the relation (8.3.25) gives

$$H_2(F)(\mathrm{ch}^2 t/2) = \\ = \frac{(-1)^{(n-2)/2}\sqrt{\pi}2^{(n-2)/2}}{\Gamma((n-1)/2)}\left(\frac{1}{\mathrm{sh}t}\partial_t\right)^{(n-2)/2} H_n(\mathrm{ch}^2 t/2) \tag{8.3.29}$$

and the problem to solve the integral equation (8.3.27) is reduced to the case $n = 2$, i.e. we have to solve the equation

$$\int_0^\infty F(\rho^2 + \mathrm{ch}^2 t/2)d\rho = H_2(\mathrm{ch}^2 t/2) \tag{8.3.30}$$

with respect to F. For this we shall transform in a suitable way the quantity

$$I(t) = \int_0^\infty \left(\frac{2}{\mathrm{sh}t}\partial_t\right) H_2(\sigma^2 + \mathrm{ch}^2 t/2)d\sigma = \int_0^\infty \int_0^\infty F'(\rho^2 + \sigma^2 + \mathrm{ch}^2 t/2)d\rho d\sigma.$$

Using polar coordinates $r = \sqrt{\rho^2 + \sigma^2}, \cos\varphi = \rho/r$, we obtain

$$I(t) = \frac{\pi}{2}\int_0^\infty F'(r^2 + \mathrm{ch}^2 t/2)rdr = \\ = \frac{\pi}{4}\int_0^\infty \partial_r(F(r^2 + \mathrm{ch}^2 t/2))dr = -\frac{\pi}{4}F(\mathrm{ch}^2 t/2)$$

so

$$F(\mathrm{ch}^2 t/2) = -\frac{8}{\pi}\int_0^\infty \frac{1}{\mathrm{sh}t}\partial_t(H_2(\sigma^2 + \mathrm{ch}^2 t/2))d\sigma.$$

Applying (8.3.29), we get

$$F(\mathrm{ch}^2 t/2) = \\ = \frac{(-1)^{n/2}2^{(n+4)/2}}{\sqrt{\pi}\Gamma((n-1)/2)}\int_0^\infty \left(\frac{1}{\mathrm{sh}t}\partial_t\right)^{n/2}(H_n(\sigma^2 + \mathrm{ch}^2 t/2))d\sigma \tag{8.3.31}$$

for any $n \geq 2$ even.

After this preparation we can solve (8.3.23) taking for H_n the right side of (8.3.23).

For $n \geq 3$ odd we use (8.3.28) and find

$$F(\mathrm{ch}^2 t/2) = \frac{(-1)^{(n-1)/2}}{(2\pi)^{(n+1)/2} a^{n-1}} \int_{-\infty}^{\infty} \tilde{f}(\lambda)(\frac{1}{\mathrm{sh}t}\partial_t)^{(n-1)/2} \cos(t\lambda a) d\lambda.$$

An application of (8.3.10) gives (8.3.12) for $n \geq 3$ odd.

For the case $n \geq 2$ even we have to compute the action of the operator

$$(\frac{1}{\mathrm{sh}t}\partial_t)^{n/2}$$

on the function

$$\begin{aligned} &H_n(\sigma^2 + \mathrm{ch}^2 t/2) = H_n(\mathrm{ch}^2 s/2) = \\ &= \frac{\Gamma((n-1)/2)}{2^{n+1}\pi^{(n+1)/2} a^{n-1}} \int_{-\infty}^{\infty} \cos(t\lambda a)\tilde{f}(\lambda) d\lambda \end{aligned} \tag{8.3.32}$$

with $s = s(t, \sigma)$ determined by

$$\mathrm{ch}^2 s/2 = \mathrm{ch}^2 t/2 + \sigma^2. \tag{8.3.33}$$

Then for any integer $k \geq 0$ we have

$$\begin{aligned} &(\frac{1}{\mathrm{sh}t}\partial_t)^k H_n(\sigma^2 + \mathrm{ch}^2 t/2) = (\frac{1}{\mathrm{sh}s}\partial_s)^k H_n(\mathrm{ch}^2 s/2)_{|s=s(t,\sigma)} \\ &= \frac{\Gamma((n-1)/2)}{2^{n+1}\pi^{(n+1)/2} a^{n-1}} \int_{-\infty}^{\infty} (\frac{1}{\mathrm{sh}s}\partial_s)^k (\cos(s\lambda a)\tilde{f}(\lambda))_{|s=s(t,\sigma)} d\lambda. \end{aligned}$$

so from (8.3.31), making the change of variables $\sigma \to s = s(t, \sigma)$ defined by (8.3.33), we get

$$\begin{aligned} f(|y|) = F(\mathrm{ch}^2 t/2) = \frac{(-1)^{n/2}}{2^{(n+1)/2}\pi^{(n+2)/2} a^{n-1}} \times \\ \times \int_{-\infty}^{\infty} \tilde{f}(\lambda) \int_t^{\infty} \frac{\mathrm{sh}s}{\sqrt{\mathrm{ch}s - \mathrm{ch}t}} (\frac{1}{\mathrm{sh}s}\partial_s)^{n/2} \cos(s\lambda a) ds d\lambda. \end{aligned}$$

From (8.3.11) we derive the needed identity (8.3.12) when $n \geq 2$ is an even integer.

This completes the proof of the Theorem .

8.4 Inverse Fourier transform

Once the inverse formula (8.3.12) for spherical transformations is established, one can establish easy a formula for the inverse Fourier transform on any Riemannian manifold with constant negative curvature (see [20]). More precisely, we have the following.

Theorem 8.4.1 *For any function $f(y) \in \mathbf{C}_0^\infty(B_R)$ we have*

(8.4.1)
$$f(y) = \int_{-\infty}^{\infty}\int_{\mathbf{S}^{n-1}} e^{(i\lambda+(n-1)/(2a))<y,\omega>}\hat{f}(\lambda,\omega)|c_a(\lambda)|^{-2}\frac{d\omega}{a^{n-1}}d\lambda,$$

where

$$c_a(\lambda) = \sqrt{2}(2\pi)^{n/2}\frac{\Gamma(i\lambda a)}{\Gamma((n-1)/2+i\lambda a)}.$$

Especially for the model on the hyperboloid

$$X_a = \{\Omega \in \mathbf{R}^{n+1} : \Omega_0^2 - \Omega_1^2 - ... - \Omega_n^2 = a^2, \Omega_0 > 0\}$$

the inverse formula (8.4.1) takes the form

(8.4.2)
$$f(\Omega) = \int_{-\infty}^{\infty}\int_{\mathbf{S}^{n-1}} a^{(i\lambda a-(n-1)/2)}[\Omega,\Lambda(\omega)]^{(-i\lambda a-(n-1)/2)}\hat{f}(\lambda,\omega)|c_a(\lambda)|^{-2}d\omega d\lambda.$$

For simplicity we shall consider only the case of curvature -1 and therefore we can take $a = b = R = -K = 1$ for the parameters connected with the models of manifolds considered at the beginning of this Chapter.

The key idea of the proof is to use the inverse Fourier transform (8.3.12), obtained for spherical functions together with a convolution in the space of functions defined on manifold with constant negative curvature. For the purpose, consider the group $G = SO(1,n)$ and denote by dg a Haar measure on G normalized by the condition

(8.4.3)
$$\int_G f(gO)dg = \int_X f(\Omega)d\Omega,$$

where $O = (1,0,...,0) \in X$ and $g\Omega$ denotes the action of the element $g \in G = SO(1,n)$ on $\Omega \in X$.

Definition 8.4.1 *If $f_1, f_2 \in \mathbf{C}_0^\infty(X)$, then the convolution $f_1 * f_2$ is defined by*

(8.4.4)
$$f_1 * f_2(\Omega) = \int_G f_1(gO)f_2(g^{-1}\Omega)dg.$$

It is not difficult to see that the requirement $f_1, f_2 \in \mathbf{C}_0^\infty(X)$ is rather strong. For example, it is sufficient to take $f_1 \in \mathbf{C}_0^\infty(X)$ and $f_2 \in \mathbf{C}^\infty(X)$.

An important step to prove the inverse formula (8.4.1) is the following.

Proposition 8.4.1 *For $f \in \mathbf{C}_0^\infty(X)$ we have*

$$f * \varphi_\lambda(\Omega) =$$

$$= \frac{1}{\mu(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} [\Omega, \Lambda(\omega)]^{(-i\lambda-(n-1)/2)} \hat{f}(\lambda, \omega) d\omega. \tag{8.4.5}$$

Proof of Proposition 8.4.1. We have

$$f * \varphi_\lambda(\Omega) = \int_{SO(1,n)} f(gO) \varphi_\lambda(|g^{-1}\Omega|) dg \tag{8.4.6}$$

according to the Definition 8.4.1. The crucial step in the proof is to show that for any $g \in SO(1,n)$ there exists a unique diffeomorphism

$$\omega \in \mathbf{S}^{n-1} \longrightarrow \tilde{\omega} = \tilde{g}(\omega) \in \mathbf{S}^{n-1} \tag{8.4.7}$$

satisfying the relations

$$\widetilde{g_1 g_2} = \tilde{g}_1 \tilde{g}_2, \tag{8.4.8}$$

and

$$g\Lambda(\omega) = \mu \Lambda(\tilde{g}\omega) \tag{8.4.9}$$

for some real number μ. Multiplying (8.4.9) by O and using the fact that $[O, \Lambda(\omega)] = 1$, we see that (8.4.9) is equivalent to

$$g\Lambda(\omega) = \Lambda(\tilde{g}(\omega))[O, g\Lambda(\omega)] \tag{8.4.10}$$

This observation enables one to verify the following property: if for g_1, g_2 one can find $\tilde{g}_1, \tilde{g}_2$ so that (8.4.10) is fulfilled for $g_1, \tilde{g}_1$ and $g_2, \tilde{g}_2$ respectively, then for $g = g_1 g_2$ and $\tilde{g} = \tilde{g}_1 \tilde{g}_2$ the property (8.4.10) is also fulfilled. Moreover, if the property (8.4.10) is fulfilled for $g, \tilde{g}$ then the same property is fulfilled for g^{-1} and $\tilde{g}^{-1}$. This observation shows that we can rewrite (8.4.10) in the form

$$g\Lambda(\omega)[O, g^{-1}\Lambda(\tilde{g}\omega)] = g\Lambda(\omega)[gO, \Lambda(\tilde{g}\omega)] = \Lambda(\tilde{g}\omega). \tag{8.4.11}$$

Our purpose is to compute the Jacobian of the diffeomorphism

$$\omega \to \tilde{g}(\omega).$$

We can use the following observation. For any $g \in SO(1,n)$ one can find a basis in $\mathbf{R}^{n+1}$ so that

$$g = g_r k, \tag{8.4.12}$$

where

$$(8.4.13) \qquad g_r = \begin{pmatrix} \mathrm{ch}r & \mathrm{sh}r & 0 \\ \mathrm{sh}r & \mathrm{ch}r & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}, \ k = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}$$

with $K \in SO(n)$. It is clear that the diffeomorphism $\tilde{k}$ corresponding to k according to (8.4.3) is $K \in SO(n)$. This observation shows that it is sufficient to find the diffeomorphism $\tilde{g}_r$ corresponding to g_r in (8.4.13). Applying (8.4.10) with $g = g_r$ we obtain that $\tilde{\omega} = \tilde{g}_r(\omega)$ is defined by

$$(8.4.14) \qquad \tilde{\omega}_1 = \frac{\mathrm{sh}r + \omega_1 \mathrm{ch}r}{\mathrm{ch}r + \omega_1 \mathrm{sh}r}, \ \tilde{\omega}_j = \frac{\omega_j}{\mathrm{ch}r + \omega_1 \mathrm{sh}r}, \ j = 2, ..., n.$$

A simple computation shows the following relation between the volume elements $d\tilde{\omega}$ and $d\omega$ on $\mathbf{S}^{n-1}$.

$$d\tilde{\omega} = \frac{d\omega}{(\mathrm{ch}r + \mathrm{sh}r\omega_1)^{n-1}} = \frac{d\omega}{[g_{-r}O, \Lambda(\omega)]^{n-1}}.$$

Multiplying (8.4.10) by gO and using the identity $[gO, g\Lambda(\omega)] = [O, \Lambda(\omega)] = 1$, we get

$$[gO, \Lambda(\tilde{\omega})][O, g\Lambda(\omega)] = 1$$

so from $KO = O$ for any $K \in SO(n)$, the decomposition (8.4.12) shows that

$$(8.4.15) \qquad d\tilde{g}(\omega) = [gO, \Lambda(\tilde{\omega})]^{n-1} d\omega.$$

From (8.4.11) we see that

$$(8.4.16) \qquad [\Omega, \Lambda(\tilde{g}\omega)] = [g^{-1}\Omega, \Lambda(\omega)][gO, \Lambda(\tilde{g}\omega)]$$

Then (8.4.15) implies that

$$\varphi_\lambda(|g^{-1}\Omega|) = \frac{1}{\mu(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} [g^{-1}\Omega, \Lambda(\omega)]^{(-i\lambda-(n-1)/2)} d\omega$$
$$= \frac{1}{\mu(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} [\Omega, \Lambda(\tilde{\omega})]^{(-i\lambda-(n-1)/2)} [gO, \Lambda(\tilde{\omega})]^{(i\lambda+(n-1)/2)} d\omega$$
$$= \frac{1}{\mu(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} [\Omega, \Lambda(\tilde{\omega})]^{(-i\lambda-(n-1)/2)} [gO, \Lambda(\tilde{\omega})]^{(i\lambda-(n-1)/2)} d\tilde{\omega}.$$

Note that with $\Omega = O$ this relation gives

$$(8.4.17) \qquad \varphi_\lambda(|g^{-1}O|) = \varphi_{-\lambda}(|gO|).$$

Thus (8.4.6) leads to

$$\mu(\mathbf{S}^{n-1}) f * \varphi_\lambda(\Omega) =$$

$$= \int_{\mathbf{S}^{n-1}} \int_G f(gO)[gO, \Lambda(\tilde{\omega})]^{(i\lambda-(n-1)/2)} dg [\Omega, \Lambda(\tilde{\omega})]^{(-i\lambda-(n-1)/2)} d\tilde{\omega} =$$

$$= \int_{\mathbf{S}^{n-1}} \int_X f(\Omega')[\Omega', \Lambda(\tilde{\omega})]^{(i\lambda-(n-1)/2)} d\Omega' [\Omega, \Lambda(\tilde{\omega})]^{(-i\lambda-(n-1)/2)} d\tilde{\omega}$$

$$= \int_{\mathbf{S}^{n-1}} \hat{f}(\lambda, \tilde{\omega}) [\Omega, \Lambda(\tilde{\omega})]^{(-i\lambda-(n-1)/2)} d\tilde{\omega}$$

and this completes the proof of the proposition.

After this preparation we can turn to the proof of the formula for inverse Fourier transform.

Proof of Theorem 8.4.1. As it was mentioned before, for simplicity we shall consider only the case $K = -1$.

Let $H = SO(n)$ and dh be the corresponding Haar measure satisfying

$$\int_H dh = 1.$$

Then for any fixed $g \in G = SO(1, n)$ we consider the spherical function

$$f_1(\Omega) = \int_H f(gh\Omega) dh.$$

We know from Theorem 8.3.1 and the isomorphism between the models B_1 and X of a manifold with curvature -1 that the inverse formula (8.4.1) is valid for f_1. In particular, we have

$$f_1(O) = \frac{(2\pi)^{-n}}{2} \int_{-\infty}^{\infty} \tilde{f}_1(\lambda) \left| \frac{\Gamma((n-1)/2 + i\lambda)}{\Gamma(i\lambda)} \right|^2 d\lambda. \tag{8.4.18}$$

On the other hand, we have the relations

$$\tilde{f}_1(\lambda) = \int_X \varphi_{-\lambda}(\Omega) \int_H f(gh\Omega) dh d\Omega$$

$$= \int_X f(\Omega) \varphi_{-\lambda}(g^{-1}\Omega) d\Omega = \int_G f(g'O) \varphi_{-\lambda}(g^{-1}g'O) dg'.$$

Applying now (8.4.17), we get

$$\tilde{f}_1(\lambda) = \int_G f(g'O) \varphi_\lambda((g')^{-1} gO) dg' = f * \varphi_\lambda(gO).$$

From this relation, Proposition 8.4.1 and (8.4.18) we get the desired relation, since

$$f_1(O) = f(gO)$$

This completes the proof of Theorem 8.4.1.

It is not difficult to derive a Plancherel identity

$$\int_X |f(\Omega)|^2 d\Omega = \tag{8.4.19}$$

$$\frac{(2\pi)^{-n}}{2}\int_{-\infty}^{\infty}\int_{S^{n-1}} |\hat{f}(\lambda,\omega)|^2 \frac{|\Gamma((n-1)/2+ia\lambda)|^2}{|\Gamma(ia\lambda)|^2} d\omega \frac{d\lambda}{a^{n-1}}$$

by the aid of the inverse formula from Theorem 8.4.1.

8.5 Appendix: Some facts about the Euler gamma function and the Legendre function

The Euler function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{8.5.1}$$

for Re$z > 0$. Using the recurrent relation

$$\Gamma(z+1) = z\Gamma(z), \tag{8.5.2}$$

one can extend the definition of $\Gamma(z)$ for $z \in \mathbf{C}, z \neq 0, -1-, 2, ...$ Since $\Gamma(1) = 1$, we get for any integer $n, n \geq 1$ the relation $\Gamma(n+1) = n!$. Some other relations for the function Γ are given below ([1])

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}, \tag{8.5.3}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{8.5.4}$$

$$\Gamma(z+1/2)\Gamma(z-1/2) = \frac{\pi}{\cos(\pi z)}. \tag{8.5.5}$$

The function $\Gamma(z)$ is closely related to the function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \tag{8.5.6}$$

defined for Re$x > 0$, Re$y > 0$. Namely, we have

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{8.5.7}$$

Another integral representation is the following.

$$B(x,y) = 2\int_0^{\pi/2} (\sin t)^{2x-1}(\cos t)^{2y-1} dt. \tag{8.5.8}$$

It is not difficult to compute the surface of the unit sphere $\mathbf{S}^{n-1}$. In fact for $n = 2$ we have $\mu(\mathbf{S}^1) = 2\pi$. For $n \geq 3$ we can introduce polar coordinates

$$(\cos\varphi,\ \omega'\sin\varphi)\ ,\omega' \in \mathbf{S}^{n-2}.$$

Then

$$\mu(\mathbf{S}^{n-1}) = \mu(\mathbf{S}^{n-2})\int_0^\pi \sin^{n-2}\varphi d\varphi =$$

$$= \mu(\mathbf{S}^{n-2})B((n-1)/2, 1/2) = \mu(\mathbf{S}^{n-2})\frac{\Gamma((n-1)/2)\Gamma(1/2)}{\Gamma(n/2)}.$$

From the recurrent relation

$$\mu(\mathbf{S}^{n-1}) = \mu(\mathbf{S}^{n-2})\sqrt{\pi}\frac{\Gamma((n-1)/2)}{\Gamma(n/2)}$$

we get

$$\mu(\mathbf{S}^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{8.5.9}$$

The Legendre function $P_\nu^\mu(z)$ satisfies the equation (see [1], volume 1, relation (3.2.1))

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]w = 0. \tag{8.5.10}$$

We shall use the following integral representation of these special functions ([1] volume 1, relation (3.7.7))

$$P_\nu^\mu(\mathrm{ch}r) =$$

$$= c_\mu(\mathrm{sh}r)^{-\mu}\int_0^\pi (\mathrm{ch}r - \mathrm{sh}r\cos\varphi)^{\nu+\mu}\sin^{-2\mu}\varphi d\varphi, \tag{8.5.11}$$

where

$$c_\mu = \frac{2^\mu}{\sqrt{\pi}\Gamma(1/2-\mu)}$$

and $\mathrm{Re}\mu < 1/2$.

Another solution of (8.5.10) are the associated Legendre functions $Q_\nu^\mu(z)$ of second kind. They are also solutions of (8.5.10) and satisfy the following relation

$$P_\nu^{-\mu}(z) =$$

$$= \frac{e^{-i\mu\pi}\Gamma(\nu-\mu+1)}{\pi\cos(\nu\pi)\Gamma(\nu+\mu+1)}\sin[\pi(\nu-\mu)][Q_\nu^\mu(z) - Q^\mu_{-\nu-1}(z)]. \tag{8.5.12}$$

An integral representation of Q_ν^μ is given by (see [1] volume 1, relation (3.7.4))

$$Q_\nu^\mu(\mathrm{ch}r) = d_\mu(\mathrm{sh}r)^\mu\int_r^\infty e^{-(\nu+1/2)s}(\mathrm{ch}s - \mathrm{ch}r)^{-\mu-1/2}ds, \tag{8.5.13}$$

where

$$d_\mu = \sqrt{\frac{\pi}{2}}\,\frac{e^{i\mu\pi}}{\Gamma(1/2-\mu)}$$

and the above representation is valid for $\mathrm{Re}(\nu+\mu+1) > 0, \mathrm{Re}\mu < 1/2$. A differential equation satisfied by $P_\nu^\mu(z)$ is

$$\frac{dP_\nu^\mu(z)}{dz} = \frac{(\nu+\mu)(\nu-\mu+1)}{\sqrt{z^2-1}}P_\nu^{\mu-1}(z) - \frac{\mu z}{z^2-1}P_\nu^\mu(z). \tag{8.5.14}$$

From these relations we shall establish the following.

Lemma 8.5.1 *The function*

$$L^n(r) = L_\sigma^n(r) = (\mathrm{sh}r)^{-(n-2)/2}\; P_{-1/2-i\sigma}^{-(n-2)/2}(\mathrm{ch}r) \tag{8.5.15}$$

satisfies the recurrent relation

$$L^n(r) = \frac{-1}{\sigma^2+(\frac{n-3}{2})^2}\left(\frac{1}{\mathrm{sh}r}\partial_r\right)L^{n-2}(r). \tag{8.5.16}$$

Proof. From (8.5.14) we have

$$\partial_r\left(\mathrm{sh}^\mu r P_\nu^\mu(\mathrm{ch}r)\right) = (\nu+\mu)(\nu-\mu+1)\mathrm{sh}^\mu r P_\nu^{\mu-1}(\mathrm{ch}r).$$

With $\mu = -(n-4)/2$ and $\nu = -1/2 - i\sigma$ we obtain the desired relation.

Further, we shall obtain

Lemma 8.5.2 *For $n \geq 1$ odd the function*

$$L^n(r) = L_\sigma^n(r) = \mathrm{sh}^{-(n-2)/2} r P_{-1/2-i\sigma}^{-(n-2)/2}(\mathrm{ch}r)$$

takes the form

$$L^n(r) = (-1)^{(n-1)/2}\sqrt{\frac{2}{\pi}}\frac{|\Gamma(i\sigma)|^2}{|\Gamma(i\sigma+\frac{n-1}{2})|^2}\left(\frac{1}{\mathrm{sh}r}\partial_r\right)^{(n-1)/2}\cos\sigma r. \tag{8.5.17}$$

Proof. For $n = 1$ we use the relation (see [1]), identity (3.6.12))

$$P_\nu^{1/2}(z) = \frac{1}{\sqrt{2\pi}}(z^2-1)^{-1/4}[(z+\sqrt{z^2-1})^{\nu+1/2} + (z+\sqrt{z^2-1})^{-\nu-1/2}]$$

and we find

$$L^1(r) = \sqrt{\frac{2}{\pi}}\cos\sigma r. \tag{8.5.18}$$

Thus, the assertion is verified for $n = 1$. In case $n \geq 3$ odd from (8.5.16) we get

$$L^n(r) = \frac{(-1)^{(n-1)/2}}{\prod_{k=0}^{(n-3)/2}(\sigma^2 + k^2)}(\frac{1}{\mathrm{sh}r}\partial_r)^{(n-1)/2}L^1(r). \tag{8.5.19}$$

On the other hand, we have

$$\prod_{k=0}^{(n-3)/2}(\sigma^2 + k^2) = \frac{|\Gamma(i\sigma + \frac{n-1}{2})|^2}{|\Gamma(i\sigma)|^2} \tag{8.5.20}$$

and hence this relation combined with (8.5.18) and (8.5.19) imply the desired identity (8.5.17).

The Lemma is proved.

Lemma 8.5.3 *For $n \geq 2$ even the function*

$$L^n(r) = L^n_\sigma(r) = (\mathrm{sh}r)^{-(n-2)/2}\, P^{-(n-2)/2}_{-1/2-i\sigma}(\mathrm{ch}r)$$

takes the form

$$L^n(r) = \frac{(-1)^{n/2}\sqrt{2}}{\pi}\frac{|\Gamma(i\sigma)|^2}{|\Gamma(i\sigma + \frac{n-1}{2})|^2}\int_r^\infty \frac{\mathrm{sh}s}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}}(\frac{1}{\mathrm{sh}s}\partial_s)^{n/2}(\cos\sigma s)ds. \tag{8.5.21}$$

Proof. Set

$$K^n(r) = K^n_\sigma(r) = \int_r^\infty \frac{\mathrm{sh}s}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}}(\frac{1}{\mathrm{sh}s}\partial_s)^{n/2}(\cos\sigma s)ds. \tag{8.5.22}$$

Integrating by parts, we find

$$K^n(r) = 2\int_r^\infty \partial_s(\sqrt{\mathrm{ch}s - \mathrm{ch}r})(\frac{1}{\mathrm{sh}s}\partial_s)^{n/2}(\cos\sigma s)ds =$$
$$= -2\int_r^\infty (\sqrt{\mathrm{ch}s - \mathrm{ch}r})\partial_s(\frac{1}{\mathrm{sh}s}\partial_s)^{n/2}(\cos\sigma s)ds.$$

Hence, for $r \neq 0$ we have

$$\frac{1}{\mathrm{sh}r}\partial_r K^n(r) = \int_r^\infty \frac{1}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}}\partial_s(\frac{1}{\mathrm{sh}s}\partial_s)^{n/2}(\cos\sigma s)ds = K^{n+2}(r).$$

Thus for $n \geq 2$ even we have

$$K^n(r) = (\frac{1}{\mathrm{sh}r}\partial_r)^{(n-2)/2}K^2(r). \tag{8.5.23}$$

Now we have to compute

$$K^2(r) = -\sigma \int_r^\infty \frac{\sin s\sigma}{\sqrt{\text{ch}s - \text{ch}r}} ds.$$

If we apply (8.5.13) with $\mu = 0, \nu = -1/2 + i\sigma$, then we get

$$\int_r^\infty \frac{e^{-is\sigma}}{\sqrt{\text{ch}s - \text{ch}r}} ds = \sqrt{2} Q^0_{-1/2+i\sigma}(\text{ch}r).$$

Applying (8.5.12), we obtain further

$$\int_r^\infty \frac{\sin s\sigma}{\sqrt{\text{ch}s - \text{ch}r}} ds = \frac{\sqrt{2}}{2} \pi \ \text{th}(\sigma\pi) P^0_{-1/2+i\sigma}(\text{ch}r).$$

Therefore, we have

$$\begin{aligned} K^2(r) &= -\frac{\pi}{\sqrt{2}} \sigma \ \text{th}(\sigma\pi) P^0_{-1/2+i\sigma}(\text{ch}r) = \\ &= -\frac{\pi}{\sqrt{2}} \sigma \ \text{th}(\sigma\pi) L^2(r). \end{aligned} \tag{8.5.24}$$

Now we can apply Lemma 8.5.1 and get for $n \geq 2$ even

$$\begin{aligned} \frac{(-1)^{n/2}}{\sqrt{2}} \pi\sigma \ \text{th}(\sigma\pi)(\sigma^2 + (\frac{n-3}{2})^2)...(\sigma^2 + (\frac{1}{2})^2) L^n_\sigma(r) = \\ = (\frac{1}{\text{sh}r} \partial_r)^{(n-2)/2} K^2_\sigma(r). \end{aligned} \tag{8.5.25}$$

Comparing this relation with (8.5.23), we obtain

$$\begin{aligned} \frac{(-1)^{n/2}}{\sqrt{2}} \pi\sigma \ \text{th}(\sigma\pi)(\sigma^2 + (\frac{n-3}{2})^2)...(\sigma^2 + (\frac{1}{2})^2) L^n_\sigma(r) = \\ = K^n_\sigma(r). \end{aligned} \tag{8.5.26}$$

On the other hand, for $n \geq 2$ even we can apply the following relations for the Gamma function (see [1], vol.1, relations (1.2.5) and (1.2.7))

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin \pi z} \quad , \quad \Gamma(1/2 + z)\Gamma(1/2 - z) = \frac{\pi}{\cos \pi z}.$$

Then we get

$$\frac{\Gamma(1/2 + i\sigma)}{\Gamma(i\sigma)} = \sigma \text{th}(\pi\sigma) \frac{\Gamma(-i\sigma)}{\Gamma(1/2 - i\sigma)}$$

so

$$\left| \frac{\Gamma(1/2 + i\sigma)}{\Gamma(i\sigma)} \right|^2 = \sigma \text{th}(\pi\sigma). \tag{8.5.27}$$

From $\Gamma(1+z) = z\Gamma(z)$ together with (8.5.27) we derive

(8.5.28)
$$\left|\frac{\Gamma((n-1)/2+i\sigma)}{\Gamma(i\sigma)}\right|^2 =$$

$$= \sigma \mathrm{th}(\pi\sigma)\left(\left(\frac{n-3}{2}\right)^2 + \sigma^2\right)\cdots\left(\left(\frac{1}{2}\right)^2 + \sigma^2\right).$$

From (8.5.25) we get

$$L^n(r) = \frac{(-1)^{n/2}\sqrt{2}}{\pi}\frac{|\Gamma(i\sigma)|^2}{|\Gamma(i\sigma+\frac{n-1}{2})|^2}K_\sigma^n(r).$$

This completes the proof of the Lemma.