6 Complex interpolation and fractional Sobolev spaces on flat space

6.1 Abstract complex interpolation for couple of Banach spaces

(see [2], [62])

For any couple $\bar{A} = (A_0, A_1)$ of Banach spaces A_0, A_1 we denote by $\Sigma(\bar{A})$ and by $\Delta(\bar{A})$ their sum and intersection respectively, i.e.

$$\Sigma(\bar{A}) = A_0 + A_1 \ , \ \Delta(\bar{A}) = A_0 \bigcap A_1 \tag{6.1.1}$$

with norms

$$\begin{aligned} \|a\|_{\Sigma(\bar{A})} &= \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} \ ; a = a_0 + a_1 \ , a_0 \in A_0, a_1 \in A_1\} \\ \|a\|_{\Delta(\bar{A})} &= \max(\|a\|_{A_0}, \|a\|_{A_1}). \end{aligned} \tag{6.1.2}$$

Then $\Sigma(A_0, A_1)$ and $\Delta(A_0, A_1)$ are Banach spaces.

The complex interpolation for the couple $\bar{A} = (A_0, A_1)$ can be associated with the space $F(\bar{A})$ of functions $f(z)$ defined, bounded and continuous in the strip

$$S = \{z \in \mathbf{C}; 0 \leq \mathrm{Re} z \leq 1\}$$

with values in $\Sigma(\bar{A})$ and satisfying the properties

$$f(it) \in A_0, \ t \in \mathbf{R} \tag{6.1.3}$$

$$f(1+it) \in A_1, \ t \in \mathbf{R} \tag{6.1.4}$$

$$f : S_0 = \{z \in \mathbf{C}; 0 < \mathrm{Re} z < 1\} \ \rightarrow \ \Sigma(\bar{A}) \quad \text{is holomorphic.} \tag{6.1.5}$$

Then $F(\bar{A})$ is a Banach space with norm

$$\|f\|_F = \max\left(\sup_{t\in\mathbf{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbf{R}} \|f(1+it)\|_{A_1}\right).$$

To show this we apply three lines lemma (see Lemma 3.2.1) with $\gamma = 0$ and see that $\|f\|_F = 0$ implies $f(z) = 0$ for $z \in S$.

To show that $F(\bar{A})$ is a Banach space, we take a Cauchy sequence

$$\{f_k(z)\}_{k=1}^{\infty}, f_k \in F(\bar{A}).$$

Then for $j = 0, 1$ and for $t \in \mathbf{R}$ fixed the sequence $f_k(j + it)$ tends to an element in A_j and we denote this element by $f(j + it)$. In a standard way, we see that $f_k(j + it)$ converges uniformly on $\mathbf{R}$ to $f(j + it)$.

Applying once more the classical three lines lemma, we see that

$$\sup_{z\in S} \|e^{z^2}(f_k(z) - f_l(z))\|_{A_0+A_1}$$

is small when k, l are large enough. This fact shows that the sequence

$$e^{z^2} f_k(z)$$

converges uniformly and tends to $e^{z^2} f(z)$. Therefore, we can extend the function $f(j+it)$ defined on the boundary of S to a holomorphic function $f(z)$ defined in S. Since

$$\|f_k - f\|_{F(\bar{A})} \to 0,$$

we see that $F(\bar{A})$ is a Banach space.

We shall mention without detailed proof the following density result.

Theorem 6.1.1 *The convex hull of the set*

$$\{e^{\delta z^2+\eta z} a : a \in A_0 \cap A_1\}$$

with $\delta > 0, \eta \in \mathbf{R}$ is dense in $F(\bar{A})$.

Proof. (case $A_1 \subset A_0$.)

Approximating $f(z)$ by $e^{\delta z^2} f(z)$, with $\delta \to 0$, $\delta > 0$, we see that without loss of generality we can assume

$$|f(x+iy)| \leq Ce^{-\delta_1 y^2} \tag{6.1.6}$$

for $0 \leq x \leq 1$ and some $\delta_1 > 0$. We can approximate further $f(z)$ by $e^{\delta z^2} f_n(z)$, where

$$f_n(z) = \sum_{j=-\infty}^{\infty} f(z + i2\pi nj),$$

where n is a sufficiently large number. To do this it is sufficient to apply (6.1.6).

Then we have to approximate $f_n(z)$. This is a continuous function in S with period $i2\pi n$. For simplicity we shall approximate functions of type $f(z)$, where $f(z)$ is holomorphic in S and periodic with period $i2\pi$. Using the Cauchy formula, we see that

$$\int_{-\pi}^{\pi} f(x+iy)e^{k(x+iy)} dy = \int_{-\pi}^{\pi} f(iy)e^{kiy} dy \tag{6.1.7}$$

for any integer k. We denote by $a(k)$ the right side of this identity. Hence, $a(k)$ (modulo constant) is the Fourier coefficient of $f(iy)$. Consider the Fourier series

$$\sum_{k=0}^{\infty} a(k, x)e^{iky},$$

where

$$a(k,x) = \int_{-\pi}^{\pi} f(x+iy)\mathrm{e}^{-iky}dy.$$

The relation (6.1.7) shows that

$$a(k,x) = \mathrm{e}^{-kx}a(k).$$

Thus we get

$$a(k,x)\mathrm{e}^{iky} = a(k)\mathrm{e}^{-kz}, \ z = x+iy,$$

so it remains to use the approximation of a periodic continuous function in the interval $(-\pi,\pi)$ using of the means

$$\Sigma_N(y) = \frac{1}{N+1}\sum_{k=0}^{N} s_k(y),$$

where

$$s_k(y) = \sum_{m=0}^{k} a_k \mathrm{e}^{iky}$$

is the Fourier partial sum for a given continuous 2π−periodic function $f(y)$, i.e.

$$a_k = \int_{-\pi}^{\pi} f(y)\mathrm{e}^{-iky}dy.$$

We know that the assumption f is continuous and periodic implies that

$$\Sigma_N(y) \to f(y)$$

uniformly as $N \to \infty$.

This completes the proof of the Lemma.

The interpolation space $(A_0, A_1)_\theta$ for $\theta \in [0,1]$ consists of all $a \in \Sigma(\bar{A})$ such that $a = f(\theta)$ for some $f \in F(\bar{A})$. The corresponding norm is defined as follows

$$\|a\|_\theta = \inf\{\|f\|_F; a = f(\theta), f \in F(\bar{A})\}.$$

It is clear that $(A_0, A_1)_\theta$ is a Banach space.

The above Theorem 6.1.1 implies that

$$A_0 \cap A_1 \text{ is dense in } (A_0, A_1)_\theta. \tag{6.1.8}$$

Moreover, for $f \in A_0 \cap A_1$ we have $f \in (A_0, A_1)_\theta$ and the following estimate

$$\|f\|_{(A_0,A_1)_\theta} \leq C\|f\|_{A_0}^{1-\theta}\|f\|_{A_1}^{\theta} \tag{6.1.9}$$

is fulfilled.

The next Theorem gives an estimate of the norm of a bounded operator with respect to interpolation space.

Theorem 6.1.2 *Let (A_0, A_1) and (B_0, B_1) be interpolation couples and let T be a bounded operator from $A_0 + A_1$ into $B_0 + B_1$, such that $T \in L(A_j, B_j)$ with norm $\|T\|_{L(A_j,B_j)}$ for $j = 0, 1$. Then for any $\theta, 0 < \theta < 1$ we have*

$$T \in L((A_0, A_1)_\theta, (B_0, B_1)_\theta))$$

with

$$\|Tf\|_{(B_0,B_1)_\theta} \leq \|T\|^{1-\theta}_{L(A_0,B_0)} \|T\|^{\theta}_{L(A_1,B_1)} \|f\|_{(A_0,A_1)_\theta}.$$

Proof. Let $f \in (A_0, A_1)_\theta$. Then there exists a function $f(z) \in F((A_0, A_1))$ so that $f = f(\theta)$. Consider the function

$$g(z) = \|T\|^{z-\theta}_{L(A_0,B_0)} \|T\|^{-z+\theta}_{L(A_1,B_1)} Tf(z).$$

Then $g(z) \in F(B_0, B_1)$. Since

$$\|g(it)\|_{B_0} \leq \|T\|^{-\theta}_{L(A_0,B_0)} \|T\|^{\theta}_{L(A_1,B_1)} \|T\|_{L(A_0,B_0)} \|f(it)\|_{A_0}$$

and

$$\|g(1+it)\|_{B_0} \leq \|T\|^{1-\theta}_{L(A_0,B_0)} \|T\|^{-1+\theta}_{L(A_1,B_1)} \|T\|_{L(A_1,B_1)} \|f(it)\|_{A_1},$$

we see that

$$\|Tf\|_{(B_0,B_1)_\theta} \leq \|T\|^{1-\theta}_{L(A_0,B_0)} \|T\|^{\theta}_{L(A_1,B_1)} \|f\|_{(A_0,A_1)_\theta}.$$

This completes the proof.

A trivial modification in the above proof shows that we have the following.

Theorem 6.1.3 *Let (A_0, A_1) and (B_0, B_1) be interpolation couples and let $T(z)$ be a holomorphic in S_0 operator-valued function defined in the strip S and continuous there. Suppose that for $z \in S$ we have that $T(z)$ is a linear bounded operator from $A_0 + A_1$ into $B_0 + B_1$, such that $T(j + it) \in L(A_j, B_j)$ with norm*

$$\sup_{t \in \mathbf{R}} \|T(j + it)\|_{L(A_j,B_j)} < \infty$$

for $j = 0, 1$. Then for any $\theta, 0 < \theta < 1$ we have

$$T(\theta) \in L((A_0, A_1)_\theta, (B_0, B_1)_\theta)).$$

6.2 Interpolation for sequences with values in Banach spaces

(see [2], [62])

Of special interest for applications is to extend the above abstract interpolation for the space $l^s(A)$. Given any Banach space A, we denote by $l_q(A)$ the linear space of all sequences $(a_k)_{k=0}^{\infty}, a_k \in A$, such that the norm

$$\|a_k\|_{l_q(A)} = (\sum_{k=0}^{\infty} \|a_k\|_A^q)^{1/q} \tag{6.2.1}$$

is bounded. For $q = \infty$, the corresponding norm is

$$\|a_k\|_{l_\infty(A)} = \sup_k \|a_k\|_A. \tag{6.2.2}$$

For $1 \leq q \leq \infty$ the space $l_q(A)$ is a Banach space.

The main result of this section is the following interpolation result for the spaces of sequences.

Theorem 6.2.1 *(see section 5.6 in [2]) Let $A_1 \subset A_0$ be dense in A_0. Then for $1 < q, q_0, q_1 < \infty$, satisfying*

$$1/q = (1-\theta)/q_0 + \theta/q_1$$

with some $\theta \in (0, 1)$, we have

$$(l_{q_0}(A_0), l_{q_1}(A_1))_\theta = l_q((A_0, A_1)_\theta), \tag{6.2.3}$$

Proof. The property (6.1.8) shows that we can choose the space of sequences

$$d = \{d_k\}_{k=0}^{\infty}, \ d_k \in A_1 \ \ d_k = 0 \ \text{ for } \ k > N$$

as a dense subset for both sides of (6.2.3).

Let the sequence $\{d_k\}_{k=0}^{\infty}$ belongs to the left side of (6.2.3). Then there exists a function $u(z)$ defined on $S = \{0 \leq \text{Re} z \leq 1\}$ so that

$$u(z) = \{u_k(z)\}_{k=0}^{\infty},$$

with $u_k(z)$ continuous in S, bounded and holomorphic in $S_0 = \{0 < \text{Re} z < 1\}$. Further, we have the properties

a)$u(it) \in l_{q_0}(A_0)$ and $u(1+it) \in l_{q_1}(A_1)$

b) $u_k(z) = 0$ for $k > N$,

$$\sum_{k=0}^{N} \|u_k(it)\|_{A_0}^{q_0} \leq C < \infty, \tag{6.2.4}$$

$$\sum_{k=0}^{N} \|u_k(1+it)\|_{A_1}^{q_1} \leq C < \infty, \tag{6.2.5}$$

for any $t \in \mathbf{R}$. Here $C > 0$ is independent of t.

Moreover, for given positive number $\varepsilon > 0$ we can assume

$$\|u_k(it)\|_{A_0}, \|u_k(1+it)\|_{A_1} < \|d_k\|_{(A_0,A_1)_\theta} + \varepsilon.$$

Further , we construct the function

$$v(z) = \{v_k(z)\}_{k=0}^{N},$$

where

$$v_k(z) = u_k(z)\|d_k\|_{(A_0,A_1)_\theta}^{(q/q_0-q/q_1)(-z+\theta)} \|d\|_{l_q((A_0,A_1)_\theta)}^{(q/q_0-q/q_1)(z-\theta)}$$

A direct computation shows that we have the estimate

$$(\sum_{k=0}^{N} \|v_k(it)\|_{A_0}^{q_0})^{1/q_0} \leq \|d\|_{l_q((A_0,A_1)_\theta)} + \varepsilon',$$

where $\varepsilon' = \varepsilon'(\varepsilon)$ tends to 0 as ε tends to 0. In a similar way we have

$$(\sum_{k=0}^{N} \|v_k(1+it)\|_{A_1}^{q_1})^{1/q_1} \leq \|d\|_{l_q((A_0,A_1)_\theta)} + \varepsilon',$$

so

$$\|d\|_{(l_{q_0}(A_0),l_{q_1}(A_1))_\theta} \leq \|d\|_{l_q((A_0,A_1)_\theta)}.$$

To show an inequality in the opposite direction we need a modification of the classical three lines lemma. (see Lemma 3.2.1)

More precisely we want to replace $L^\infty-$ norms in Lemma 3.2.1 by L^p-norms with $1 < p < \infty$.

Lemma 6.2.1 *If $f \in F(\gamma)$ and $1 < p_0, p_1 < \infty$, then there is a positive constant C such that for any $\theta \in (0,1)$ we have*

$$|f(\theta)| \leq C\|e^{\delta(i\ \cdot)^2} f(i\ \cdot)\|_{L^{p_0}(\mathbf{R})}^{1-\theta} \|e^{\delta(1+i\ \cdot)^2} f(1+i\ \cdot)\|_{L^{p_1}(\mathbf{R})}^{\theta}.$$

Proof. Again we consider the function

$$g(z) = e^{\delta z^2} f(z) a_0^{z-1} a_1^{-z},$$

where

$$a_j = \|e^{\delta(j+i\ \cdot)^2} f(j+i\ \cdot)\|_{L^{p_j}(\mathbf{R})},\ j = 0, 1.$$

Again a_j are positive numbers and we have the estimate

$$|g(z)| \leq C e^{-\delta_1 |\mathrm{Im} z|}$$

for $\mathrm{Re} z \in [0,1]$.

Using the fact that

$$|\theta - \zeta| > c > 0$$

for $\mathrm{Re}\zeta = 0, 1$, we see that the Cauchy identity implies that $g(\theta + iy)$ is bounded and we have

$$|g(\theta)| \leq C\|g(i\cdot)\|_{L^{p_0}} + C\|g(1+i\cdot)\|_{L^{p_1}}.$$

Our choice of g guarantees that $\|g(i\cdot)\|_{L^{p_0}}$ and $\|g(1+i\cdot)\|_{L^{p_1}}$ are bounded, so the same is true for $|g(\theta)|$. This completes the proof.

Turning back to the proof of the Theorem, we take again

$$d = \{d_k\}_{k=0}^{\infty},\ d_k \in A_0 \cap A_1 \ \ d_k = 0 \ \text{ for } \ k > N.$$

For any $k = 1, ..., N$, there exists a function $u_k(z)$ with $u_k(z)$ continuous in S, holomorphic in $S_0 = \{0 < \mathrm{Re} z < 1\}$. For given positive number $\varepsilon > 0$ we can assume

$$\|u_k(it)\|_{A_0}, \|u_k(1+it)\|_{A_1} < \|d_k\|_{(A_0,A_1)_\theta} + \varepsilon.$$

Applying the estimate of the above Lemma 6.2.1, we derive

$$\|d\|^q_{l_q((A_0,A_1)_\theta)} \leq$$

$$C\sum_{k=0}^{N}\Big(\int e^{\delta(it)^2}\|u_k(it)\|^{q_0}_{A_0}dt\Big)^{(1-\theta)q/q_0}\Big(\int e^{\delta(1+it)^2}\|u_k(1+it)\|^{q_1}_{A_1}dt\Big)^{\theta q/q_1}.$$

This estimate shows that

$$\|d\|_{l_q((A_0,A_1)_\theta)} \leq C\|d\|_{(l_{q_0}(A_0), l_{q_1}(A_1))_\theta}$$

and completes the proof.

Further, given any real number s, we denote by $l^s_q(A)$ the linear space of all sequences $(a_k)_{k=0}^{\infty}, a_k \in A$, such that the norm

$$\|a_k\|_{l_q(A)} = \Big(\sum_{k=0}^{\infty} 2^{ksq}\|a_k\|^q_A\Big)^{1/q} \tag{6.2.6}$$

is bounded. For $1 \leq q < \infty$ the space $l^s_q(A)$ is a Banach space.

Then we have the following result for the complex interpolation (see section 5.6 in [2]).

$$(l^{s_0}_{q_0}(A_0), l^{s_1}_{q_1}(A_1))_\theta = l^s_q((A_0, A_1)_\theta), \tag{6.2.7}$$

where

$$1/q = (1-\theta)/q_0 + \theta/q_1 \quad , \quad s = (1-\theta)s_0 + \theta s_1$$

and moreover $1 \le q_0, q_1 < \infty$. The proof is standard and we omit it.

For the case of Lebesgue L^p spaces we have the following result

$$(L^{p_0}, L^{p_1})_\theta = L^p, \tag{6.2.8}$$

where

$$\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}.$$

The proof is similar to the proof of the discrete case l_p and we shall omit it.

Let $\chi(x)$ be a smooth positive function on $\mathbf{R}^n$. Then the weighted space $L^p(\chi)$ for $1 \le p \le \infty$ by definition is formed by all measurable functions f, such that $\chi f \in L^p(\mathbf{R}^n)$. The norm in this space is

$$\|\chi f\|_{L^p(\mathbf{R}^n)}.$$

For the case of weighted L^p spaces we have the following interpolation result

$$(L^{p_0}(\chi_0), L^{p_1}(\chi_1))_\theta = L^p(\chi), \tag{6.2.9}$$

where

$$\chi = \chi_0^{p(1-\theta)/p_0}\chi_1^{p\theta/p_1}, \frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}.$$

Applying Theorem 6.1.2, we obtain the following interpolation result.

Lemma 6.2.2 *Let*

$$\chi_0(x), \chi_1(x), \chi(x) \text{ and } \sigma_0(x), \sigma_1(x), \sigma(x)$$

be smooth positive functions. Suppose

$$T : L^{p_0}(\chi_0) \to L^{q_0}(\sigma_0)$$

and

$$T : L^{p_1}(\chi_1) \to L^{q_1}(\sigma_1)$$

is bounded with corresponding norms M_0 and M_1 respectively. Then the operator

$$T : L^p(\chi) \to L^q(\sigma)$$

is bounded and its norm is not greater than constant times

$$M_0^{1-\theta}M_1^\theta.$$

for

$$\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1} , \frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1} ,$$

$$\sigma = \sigma_0^{q(1-\theta)/q_0}\sigma_1^{q\theta/q_1}, \chi = \chi_0^{p(1-\theta)/p_0}\chi_1^{p\theta/p_1}.$$

6.3 Interpolation for semigroups in Banach spaces

(see [2], [62])

To have a possibility to introduce some other equivalent norms in the interpolation spaces, we shall consider the case when a strongly continuos semigroup $\{G(t)\}$ of operators acts in a Banach space A. For more detail of the proofs one can consider section 6.7 in [2] or section 1.15 in [62].

Recall that a family of bounded linear operators $\{G(t)\}$ defined for $t > 0$ is uniformly bounded and strongly continuous semigroup if the following three conditions are fulfilled

(6.3.1) $$G(s+t)a = G(s)G(t)a \ , \ (s,t > 0 \ , \ a \in A),$$

(6.3.2) $$\|G(t)a\|_A \leq C\|a\|_A \ \ (t > 0 \ , \ a \in A),$$

(6.3.3) $$\lim_{t\to 0} \|G(t)a - a\|_A = 0 \ \ a \in A.$$

The generator Λ of this semigroup is defined in dense domain $D(\Lambda)$ in A, such that the limit

(6.3.4) $$\lim_{t\to 0} t^{-1}(G(t)a - a)$$

in A exists for $a \in D(\Lambda)$. The limit in (6.3.4) defines Λa. One can see that $D(\Lambda)$ is a Banach space with respect to the norm

(6.3.5) $$\|a\|_{D(\Lambda)} = \|a\|_A + \|\Lambda a\|_A.$$

Moreover, for $a \in D(\Lambda)$ we have the relations

(6.3.6) $$\frac{d(G(t)a)}{dt} = \Lambda G(t)a = G(t)\Lambda a$$

and

(6.3.7) $$G(t)a - a = \int_0^t G(s)\Lambda a \, ds.$$

Given any $\theta \in (0,1)$ one can consider the interpolation space $(A, D(\Lambda))_\theta$.

For simplicity, we consider the special case, when Λ is a positive operator in a Hilbert space H. This means the Λ is a self-adjoint operator in H such that

$$(u, \Lambda u)_H > 0$$

for any $u \in D(\Lambda), u \neq 0$. As before, $(u,v)_H$ denotes the scalar product in H.

Again we assume that the operator Λ has a dense domain $D(\Lambda)$ in H.

Lemma 6.3.1 *(see [62]) If $\theta \in (0,1)$, then*

$$(H, D(\Lambda))_\theta = D(\Lambda^\theta).$$

Proof. Let $f \in D(\Lambda)$. Then

$$f(z) = \Lambda^{-z+\theta} f$$

belongs to $F(H, D(\Lambda))$. Further, we have the estimate

$$\|f\|_{(H,D(\Lambda))_\theta} \leq C\|e^{(z-\theta)^2}\Lambda^{-z+\theta} f\|_{F(H,D(\Lambda))} \leq C\|\Lambda^\theta f\|_H.$$

Since $D(\Lambda)$ is dense in $(H, D(\Lambda))_\theta$, we conclude that

$$D(\Lambda^\theta) \subset (H, D(\Lambda))_\theta.$$

To show the opposite inclusion we take $f \in D(\Lambda)$ such that $f \in (H, D(\Lambda))_\theta$. Then there exists $f(z) \in F(H, D(\Lambda))$, so that $f = f(\theta)$. Consider the function $g(z) = \Lambda^z f(z)$. Using the three lines lemma we find

$$|\Lambda^\theta f| \leq (\sup_t \|e^{(it)^2} f(it)\|_H)^{1-\theta} (\sup_t \|e^{(1+it)^2} f(1+it)\|_{D(\Lambda)})^\theta,$$

so we have

$$\|\Lambda f(\theta)\| \leq \|f\|_{(H,D(\Lambda))_\theta}.$$

This completes the proof.

Applying the spectral Theorem it is easy to see that the norm in this interpolation space is equivalent to

$$\|a\|_{(H,D(\Lambda))_\theta} \sim \|a\|_H + (\int_0^1 t^{-2\theta}\|G(t)a - a\|_H^2 dt/t)^{1/2}. \tag{6.3.8}$$

A minor modification in the above scheme is needed when A is a skew self-adjoint operator in a Hilbert space H. Then the operator

$$\Lambda = I - A^2 = I + A^* A$$

is a positive operator in H. Then we have

Lemma 6.3.2 *If $\theta \in (0,1)$ and A is a skew self-adjoint operator, then*

$$(H, D(A))_\theta = D((I - A^2)^{\theta/2})$$

and the norm in $H, D(A))_\theta$ is equivalent to

$$\|f\|_H + (\int_0^1 t^{-2\theta}\|e^{At} f - f\|_H^2 dt/t)^{1/2}. \tag{6.3.9}$$

Proof. Let $f \in D(A)$.Then

$$f(z) = (I - A^2)^{(-z+\theta)/2} f$$

belongs to $F(H, D(A))$ and

$$\|f\|_{(H,D(A))_\theta} \leq C\|(I - A^2)^{\theta/2} f\|_H.$$

This inequality leads to

$$D((I - A^2)^{\theta/2}) \subset (H, D(A))_\theta.$$

To show the opposite inclusion we use the density property (6.1.8).

To show the equivalence of the norm in $(H, D(A))_\theta$ and (6.3.9) we use the spectral theorem (see Theorem 2.2.4) and see that $f \in D((I - A^2)^{\theta/2})$ means that the integral

$$\int_{-\infty}^{\infty} (1 + \lambda^2)^{\theta/2} d(f, P_\lambda f)$$

is convergent. Note also that

$$\begin{aligned} \|e^{At} f - f\|_{H^2} &= \int_{-\infty}^{\infty} (2 - 2\cos t\lambda) d(f, P_\lambda f) \\ &= 4 \int_{-\infty}^{\infty} \sin^2((t\lambda)/2) d(f, P_\lambda f) \end{aligned}$$

and we see that

$$\int_0^1 t^{-2\theta} \|e^{At} a - a\|_H^2 dt/t = 4 \int_0^1 t^{-2\theta} \int_{-\infty}^{\infty} \sin^2((t\lambda)/2) d(f, P_\lambda f) dt/t.$$

On the other hand, for $0 < \theta < 1$ the integral

$$\int_0^1 t^{-2\theta} \sin^2((t\lambda)/2) dt/t$$

is equivalent to $C\lambda^{2\theta}$ so the square of the norm in (6.3.9) is equivalent to

$$\|f\|_H^2 + \int_{-\infty}^{\infty} \lambda^{2\theta} d(f, P_\lambda f)$$

and this quantity is equivalent to

$$\|(I - A^2)^{\theta/2} f\|_H^2.$$

This completes the proof.

A simple generalization of the above argument works, when $A_1, ..., A_N$ are skew self-adjoint operators in a Hilbert space H with commuting resolvents, i.e.

$$[(I - A_j^2)^{-1}, (I - A_k^2)^{-1}] = 0$$

for $j, k = 1, ..., N$. Then one can see that

$$\cap_j D((I - A_j^2)^{\theta/2}) = \cap_j (H, D(A_j))_\theta$$
$$= (H, \cap_j D(A_j))_\theta = (H, D((I - A_1^2 - ... - A_N^2)^{1/2})_\theta =$$
$$= D((I - A_1^2 - ... - A_N^2)^{\theta/2}). \tag{6.3.10}$$

An equivalent norm in the interpolation space $\cap_j (H, D(A_j))_\theta$ is

$$\|f\|_H + \sum_{j=1}^N (\int_0^1 t^{-2\theta} \|e^{A_j t} f - f\|_H^2 dt/t)^{1/2}.$$

As an application let us consider the case

$$H = L^2(\mathbf{R}^n), A_j = \partial_{x_j}, j = 1, ..., n.$$

Then we have

$$e^{A_j t} f(x) = f(x + te_j),$$

$e_j = (0, ..., 0, 1, 0..., 0)$ with 1 on jth place. For $0 < \theta < 1$ the norm in $H^\theta(\mathbf{R}^n)$ is

$$\|f\|_{L^2(\mathbf{R}^n)} + \sum_{j=1}^n (\int_0^1 \int_{\mathbf{R}^n} t^{-2\theta} |f(x + te_j) - f(x)|^2 dx dt/t)^{1/2}.$$

Any positive s can be represented in the form

$$s = k + \theta,$$

where $k \geq 0$ is an integer and $0 < \theta < 1$. Then the norm in $H^s(\mathbf{R}^n)$ is equivalent to

$$\sum_{|\alpha| \leq k} \|\partial_x^\alpha f\|_{L^2(\mathbf{R}^n)} +$$
$$\sum_{|\alpha|=k} \sum_{j=1}^n (\int_0^1 \int_{\mathbf{R}^n} t^{-2\theta} |\partial_x^\alpha f(x + te_j) - \partial_x^\alpha f(x)|^2 dx dt/t)^{1/2}. \tag{6.3.11}$$

In fact, Strichartz ([58]) established the following equivalence.

Lemma 6.3.3 *Let $s = k + \theta$, where $k \geq 0$ is an integer and $0 < \theta < 1$. Then for any multiindex α, $|\alpha| = k$, and any integer $j = 1, ..., n$, we have*

$$\|f\|_{H^k(\mathbf{R}^n)} + \|S_{k,\theta}(f)\|_{L_2(\mathbf{R}^n)} \sim \|f\|_{H_2^s(\mathbf{R}^n)}. \tag{6.3.12}$$

where

$$S_{k,\theta}(x) = \sum_{|\alpha|\le k} \left(\int_0^\infty \int_{|y|\le 1} t^{-2\theta} |\partial_x^\alpha f(x+ty) - \partial_x^\alpha f(x)|^2 dy dt/t \right)^{1/2} .$$

6.4 Fourier multipliers

To study Sobolev space $H_p^s(\mathbf{R}^n)$ for fractional values of s we shall study convolution type operators

$$A(f)(x) = \int_{\mathbf{R}^n} e^{ix\xi} a(\xi) \hat{f}(\xi) d\xi, \tag{6.4.1}$$

where $a(\xi)$ belongs to Hörmander's class S^0, i.e. $a(\xi)$ is a smooth function, satisfying

$$|\partial_\xi^\alpha a(\xi)| \le C < \xi >^{-|\alpha|} . \tag{6.4.2}$$

for any $\xi \in \mathbf{R}^n$. As we know from the general theory of pseudodifferential operators, this is a bounded operator from L^p into L^p for $1 < p < \infty$.(see (5.2.9) and (5.2.10) in the section devoted to pseudodifferential operators.)

Thus, we know that its norm is

$$\|A\|_{L(L^p, L^p)} \le \beta C, \tag{6.4.3}$$

where C is the constant from (6.4.2) and β is an universal constant depending only on p and n. This fact follows also from the classical Michlin theorem, established by Hörmander (see [60], Theorem 1.1 in Chapter XI for example).

Further, we shall construct a Paley-Littlewood partition of unity $\phi_j(x)$ on $\mathbf{R}^n$, so that the following properties are fulfilled

$$1 = \sum_{j=0}^{\infty} \phi_j(x),$$

$$\phi_j(x) \ge 0, \ \phi_j(x) \in C_0^\infty, \quad \text{for } \ j \ge 0,$$

$$C^{-1} 2^{-j} \le |x| \le C 2^j \text{ for } x \in \text{ supp } \phi_j(x), \ j \ge 1. \tag{6.4.4}$$

To construct this partition of unity we choose a smooth function $\psi(x)$ supported in $\{x : 1/2 \le |x| \le 2\}$ and such that $\psi(x) = 1$ for $x \in \{x : 1/\sqrt{2} \le |x| \le \sqrt{2}\}$. Setting

$$\phi(x) = \psi(x) (\sum_{j=-\infty}^{\infty} \psi(2^{-j} x))^{-1},$$

we see that

$$\sum_{j=-\infty}^{\infty} \phi(2^{-j} x) = 1,$$

for $x \neq 0$. Finally, setting

$$\phi_j(x) = \phi(2^{-j}x),\ j \geq 1,\ \phi_0(x) = \sum_{j=-\infty}^{0} \phi(2^{-j}x),$$

we see that this partition of unity satisfies (6.4.4).

In addition to these properties we have the important relation

$$\phi_j(x) = \phi(2^{-j}x),\ j \geq 1, \tag{6.4.5}$$

where $\phi(x)$ is a smooth compactly supported function.

Once the partition of unity satisfying (6.4.4) and (6.4.5) is constructed, we can consider the operators

$$\phi_j(f)(x) = \int_{\mathbf{R}^n} e^{ix\xi}\phi_j(\xi)\hat{f}(\xi)d\xi. \tag{6.4.6}$$

The corresponding kernels for $j \geq 1$ are

$$k_j(x) = \int_{\mathbf{R}^n} e^{ix\xi}\phi(2^{-j}\xi)d\xi = 2^{nj}k(2^j x),$$

where

$$k(x) = \int_{\mathbf{R}^n} e^{ix\xi}\phi(\xi)d\xi$$

is a smooth rapidly decreasing function. Then we have (see [62] , [2])

$$\int (\sum_{j=0}^{\infty} |\phi_j(f)(x)|^2)^{p/2} dx \leq C\|f\|_{L^p}^p. \tag{6.4.7}$$

It is not difficult to establish an estimate in the opposite direction. Indeed, for $f, g \in C_0^\infty$ we have

$$\begin{aligned}\int f(x)\overline{g(x)}dx &= \sum_{j,l=0}^{\infty} \int \phi_j(f)(x)\overline{\phi_l(g)(x)}dx \\ &= \sum_{j,l=0}^{\infty} \int \phi_j(\xi)\hat{f}(\xi)\overline{\phi_l(\xi)\hat{f}(\xi)}d\xi.\end{aligned}$$

Since the elements of the partition of unity have finite overlap, there exists N so that $\phi_j(\xi)\phi_l(\xi) = 0$ for $|j-l| > N$. Therefore, applying this property and the Cauchy inequality we get

$$\left|\int f(x)\overline{g(x)}dx\right| \leq \int F(x)G(x)dx,$$

where

$$F(x) = (\sum_{j=0}^{\infty} |\phi_j(f)(x)|^2)^{1/2}$$

and

$$G(x) = (\sum_{j=0}^{\infty} |\phi_j g(x)|^2)^{1/2}.$$

From (6.4.7) we know that

$$\|F\|_{L^p} \leq C\|f\|_{L^p}.$$

Thus applying the Hölder inequality, we get

$$|\int f(x)\overline{g(x)}dx| \leq C\|f\|_{L^p}\|G\|_{L^{p'}},$$

where $1/p + 1/p' = 1$. This estimate implies

$$\|g\|_{L^{p'}} \leq C\|G\|_{L^{p'}}$$

so we have

Lemma 6.4.1 *Let $\{\phi_j(x)\}, j = 0, 1, ...,$ be a Paley-Littlewood partition of unity satisfying (6.4.4) and (6.4.5). Then for any p with $1 < p < \infty$ the norms*

$$(\int (\sum_{j=0}^{\infty} |\phi_j(f)(x)|^2)^{p/2} dx)^{1/p}$$

and

$$\|f\|_{L^p}$$

are equivalent.

The following generalization of the above Lemma is due to H.Triebel.

Let $A_j, j = 0, 1, ...$ be a sequence of convolution type operators

$$A_j(f)(x) = \int_{\mathbf{R}^n} e^{ix\xi} a_j(\xi)\hat{f}(\xi)d\xi, \tag{6.4.8}$$

Let

$$k_j(x) = \int_{\mathbf{R}^n} e^{ix\xi} a_j(\xi)d\xi$$

be the corresponding kernel of the operator K_j.

Lemma 6.4.2 *(see [62]) Let A_j be the pseudodifferential operators defined in above and such that for any integer $N \geq 0$ there exists a constant $C > 0$, so that*

$$\sum_{|\alpha|\leq N} \sum_{j=0}^{\infty} |\partial_\xi^\alpha a_j(\xi)|^2 \leq C(1 + |\xi|)^{-2|\alpha|}.$$

Then for any p with $1 < p < \infty$ we have

$$\int (\sum_{j=0}^{\infty} |A_j(f)(x)|^2)^{p/2} dx \leq C\|f\|^p_{L^p}.$$

6.5 Complex interpolation in H^s_p.

An application of Lemma 6.4.1 enables us to give the following equivalent norm in the Sobolev space $H^s_p(\mathbf{R}^n)$, defined as a completition of smooth compactly supported functions $f(x)$ with respect to the norm

$$\|f\|_{H^s_p(\mathbf{R}^n)} = \|(1-\Delta)^{s/2} f\|_{L^p(\mathbf{R}^n)}. \tag{6.5.1}$$

Theorem 6.5.1 *For $1 < p < \infty$ and $s \geq 0$ the norm in $H^s_p(\mathbf{R}^n)$ is equivalent to*

$$(\int (\sum_{j=0}^{\infty} 2^{2js} |\phi_j(f)(x)|^2)^{p/2} dx)^{1/p} \tag{6.5.2}$$

where

$$\phi_j(f)(x) = \int_{\mathbf{R}^n} e^{ix\xi} \phi_j(\xi) \hat{f}(\xi) d\xi \tag{6.5.3}$$

and $\{\phi_j(\xi)\}, j = 0, 1, ...,$ is a Paley-Littlewood partition of unity.

Proof. Consider the convolution type operator $A_j = \phi_j(D_x)2^{js}$ defined as follows

$$A_j(f)(x) = \int e^{ix\xi} \phi_j(\xi) 2^{js} \hat{f}(\xi) d\xi.$$

Taking a smooth compactly supported function $g(x)$, we have

$$\sum_{j=0}^{\infty} \int A_j(f)(x) \overline{\phi_j(g)(x)} dx = \sum_{j=0}^{\infty} \int \phi_j A_j(f)(x) \overline{g(x)} dx.$$

Now a direct computation shows that the convolution type operator

$$\sum_{j=0}^{\infty} \phi_j A_j$$

has a symbol of order s so the L^p- boundedness of pseudodifferential operators of order s gives

$$\left| \sum_{j=0}^{\infty} \int A_j(f)(x) \overline{\phi_j(g)(x)} dx \right| \leq C\|f\|_{H^s_p} \|g\|_{L^{p'}}.$$

Applying Lemma 6.4.1, we see that

$$\|g\|_{L^{p'}} \leq (\int (\sum_{j=0}^{\infty} |\phi_j(g)(x)|^2)^{p'/2} dx)^{1/p'}$$

so we arrive at

$$(\int (\sum_{j=0}^{\infty} 2^{2js} |\phi_j(f)(x)|^2)^{p/2} dx)^{1/p} \leq C\|f\|_{H_p^s}.$$

To show the opposite estimate we follow the line of the proof of Lemma 6.4.1. For $f, g \in C_0^\infty$ we have

$$\int ((1-\Delta)^{s/2} f(x))\overline{g(x)} dx = \sum_{j,l=0}^{\infty} \int \phi_j((1-\Delta)^{s/2} f)(x) \overline{\phi_l(g)(x)} dx$$
$$= \sum_{j,l=0}^{\infty} \int \phi_j(\xi)(1+|\xi|^2)^{s/2} \hat{f}(\xi) \overline{\phi_l(\xi)\hat{f}(\xi)} d\xi.$$

Since the elements of the partition of unity have finite overlap, there exists N so that $\phi_j(\xi)\phi_l(\xi) = 0$ for $|j-l| > N$. Therefore, applying this property and the Cauchy inequality we get

$$\left| \int (1-\Delta)^{s/2} f(x) \overline{g(x)} dx \right| \leq \int F(x)G(x) dx,$$

where

$$F(x) = (\sum_{j=0}^{\infty} 2^{2js} |\phi_j(f)(x)|^2)^{1/2}$$

and

$$G(x) = (\sum_{j=0}^{\infty} |\phi_j 2^{-2js} (1-\Delta)^{s/2} g(x)|^2)^{1/2}.$$

From Lemma 6.4.2 we know that

$$\|G\|_{L^{p'}} \leq C\|g\|_{L^{p'}}.$$

Thus applying the Hölder inequality, we get

$$\left| \int (1-\Delta)^{s/2} f(x) \overline{g(x)} dx \right| \leq C\|F\|_{L^p} \|g\|_{L^{p'}},$$

where $1/p + 1/p' = 1$. This estimate implies

$$\|f\|_{H_p^s} \leq C\|F\|_{L^p}$$

and completes the proof of the theorem.

Applying the result for complex interpolation in the space of sequences and in L^p spaces we obtain

$$(H^{s_0}_{p_0}, H^{s_1}_{p_1})_\theta = H^s_p, \tag{6.5.4}$$

where $\theta \in (0,1)$, $1 < p, p_0, p_1 < \infty$ and

$$s = (1-\theta)s_0 + \theta s_1 \quad , \quad \frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}.$$

In particular, with $s_1 = 0$ we have

$$(H^{s_0}_{p_0}, L_{p_1})_\theta = H^s_p, \tag{6.5.5}$$

where $\theta \in (0,1)$, $1 < p, p_0, p_1 < \infty$ and

$$s = (1-\theta)s_0 \quad , \quad \frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}. \tag{6.5.6}$$

Applying the estimate (6.1.9), we see that this interpolation result leads to the following interpolation inequality

$$\|u\|_{H^s_p} \le C\|u\|^{1-\theta}_{H^{s_0}_{p_0}} \|u\|^\theta_{L_{p_1}} \tag{6.5.7}$$

assuming the conditions (6.5.6) are fulfilled. For the limiting case $p_1 = \infty$ the above estimate is still true in view of the result in [45].

6.6 Multiplicative inequalities in H^s_p.

First, we shall establish the following inequality due to Coifman and Meyer (see [7]).

Theorem 6.6.1 *If $s \ge 0$ and $1 < p_0, p_1, p < \infty$, then for $u, v \in H^s_{p_0} \cap L^{p_1}$ we have*

$$\|uv\|_{H^s_p} \le C(\|u\|_{H^s_{p_0}} \|v\|_{L^{p_1}} + \|v\|_{H^s_{p_0}} \|u\|_{L^{p_1}})$$

for

$$\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}.$$

Proof. We shall use a dyadic partition of unity of type

$$1 = \sum_{j=0}^{\infty} \phi_j(D_x),$$

where

$$\phi_j(D_x)u(x) = \int e^{ix\xi}\phi_j(\xi)\hat{u}(\xi)d\xi.$$

Here $\{\phi_j(\xi)\}$ is a Paley - Litlewood partition of unity constructed in (6.4.4).

From Theorem 6.5.1 we know that the norm in $\|uv\|_{H^s_p(\mathbf{R}^n)}$ on power p is equivalent to

(6.6.1) $$\int (\sum_{j=0}^{\infty} 2^{2js}|\phi_j(uv)(x)|^2)^{p/2}dx.$$

Since

(6.6.2) $$\phi_j(uv)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{ix\xi}\phi_j(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta,$$

we can decompose this term into the form

$$\phi_j(uv)(x) = I + II + III,$$

where

$$I = \sum_{k\le j-N} \phi_j(u\phi_k(v))(x),$$

$$II = \sum_{|k-j|\le N} \phi_j(u\phi_k(v))(x),$$

$$III = \sum_{k\ge j+N} \phi_j(u\phi_k(v))(x).$$

Setting

$$v_j = \sum_{k\le j-N} \phi_k(v),$$

$$u_j = \sum_{|k-j|\le N} \phi_k(v),$$

and choosing $N \geq 1$ sufficiently large we have

$$I = \phi_j(uv_j)(x) = \phi_j(u_jv_j)(x)$$

and this leads to

$$\int (\sum_{j=0}^{\infty} 2^{2js}|I|^2)^{p/2}dx \le C\int (\sum_{j=0}^{\infty} 2^{2js}|u_j(x)|^2)^{p/2} \max_j |v_j(x)|^p dx$$

Now we combine the Hölder inequality and the estimate

$$\|\max_j |v_j(.)|\|_{L^{p_1}} \le C\|v\|_{L^{p_1}}$$

valid in view of the estimate (6.4.3). In this way we obtain

$$\int (\sum_{j=0}^{\infty} 2^{2js} |I|^2)^{p/2} dx$$

$$\leq C \|v\|^p_{L^{p_1}} \left(\int (\sum_{j=0}^{\infty} 2^{2js} |u_j(x)|^2)^{p_0/2} dx \right)^{p/p_0} . \tag{6.6.3}$$

Now we are in situation to apply Theorem 6.5.1 and so we see that the left side of (6.6.3) is bounded from above by constant times

$$\|v\|^p_{L^{p_1}} \|u\|^p_{H^s_{p_0}} .$$

In the same way we obtain the estimate

$$\int (\sum_{j=0}^{\infty} 2^{2js} |III|^2)^{p/2} dx$$

$$\leq C \|u\|^p_{L^{p_1}} \|v\|^p_{H^s_{p_0}} . \tag{6.6.4}$$

Further, applying the Cauchy inequality, we get

$$\int (\sum_{j=0}^{\infty} 2^{2js} |II|^2)^{p/2} dx$$

$$\int (\sum_{j=0}^{\infty} 2^{2js} |U_j|^2)^{p/2} (\sum_{j=0}^{\infty} |V_j|^2)^{p/2} dx,$$

where

$$U_j = \sum_{|k-j| \leq N} \phi_k u \ , \ V_j = \sum_{|k-j| \leq N} \phi_k v.$$

Applying the Hölder inequality and Theorem 6.5.1, we see that

$$\int (\sum_{j=0}^{\infty} 2^{2js} |II|^2)^{p/2} dx$$

$$\leq C \|v\|^p_{L^{p_1}} \|u\|^p_{H^s_{p_0}} .$$

This completes the proof.

In the limiting case $p_1 = \infty$ we have the estimate (see the Appendix in [27])

$$\|uv\|_{H^s_p} \leq C(\|u\|_{H^s_p} \|v\|_{L^\infty} + \|v\|_{H^s_p} \|u\|_{L^\infty}) \tag{6.6.5}$$

The Sobolev embedding

$$\|u\|_{L^q} \leq C \|u\|_{H^s_p} \tag{6.6.6}$$

is valid for $1 < p < q < \infty$ and

$$\frac{s}{n} \geq \frac{1}{p} - \frac{1}{q}.$$

To verify this estimate we represent u in the form

$$u = K * v,$$

where

$$v = (1 - \Delta)^{s/2} u$$

and

$$K(x) = c \int e^{ix\xi} (1 + |\xi|^2)^{-s/2} d\xi.$$

From Lemma 5.2.3 we know that the oscillatory integral $K(x)$ satisfies the estimate

$$|K(x)| \leq C|x|^{-n+s}$$

so an application of Hardy-Sobolev estimate in Lemma 2.4.1 leads to the Sobolev embedding.

Our next step is to present a Moser type estimate.

Lemma 6.6.1 *(see [46]5.4.3) Let λ, s be real numbers such that $1 < s < \lambda$. Then we have*

$$\| |u|^\lambda \|_{H^s} \leq C \|u\|_{H^s} \|u\|^{\lambda-1}_{L^\infty}.$$