Chapter 6

Applications

In this Chapter we give the applications of our general theory to some physical systems and a system related to geometric problems. These systems include the quasilinear canonical system related to the Monge-Ampère equation, the system of nonlinear three-wave interaction in plasma physics, the nonlinear wave equation with higher order dissipation, the system of one-dimensional gas dynamics with higher order damping, the system of motion of an elastic string, the system of plane elastic waves for hyperelastic materials and the nonlinear wave equation with scalar operators of higher order. For these systems, we give a complete result on the global existence or the blow-up phenomenon, particularly, the life span of the C^1 solutions to their Cauchy problems.

§6.1. Quasilinear canonical system related to the Monge-Ampère equation

We consider the following quasilinear canonical system

$$\sum_{j=1}^{n} a_{ij}(u)\frac{\partial u_j}{\partial \alpha} = 0 \quad (i = 1, \cdots, m), \tag{6.1.1}$$

$$\sum_{j=1}^{n} a_{ij}(u)\frac{\partial u_j}{\partial \beta} = 0 \quad (i = m+1, \cdots, n), \tag{6.1.2}$$

where $a_{ij}(u)\,(i,j = 1, \cdots, n)$ are C^2 functions of $u = (u_1, \cdots, u_n)$ with

$$\det |a_{ij}(u)| \neq 0. \tag{6.1.3}$$

Obviously, (6.1.1)-(6.1.2) is a hyperbolic system with two families of linearly degenerate characteristics with constant multiplicity:

$$\alpha = constant \quad (\text{multiplicity} = n - m) \tag{6.1.4}$$

and

$$\beta = constant \quad (\text{multiplicity} = m). \tag{6.1.5}$$

However, is general, system (6.1.1)-(6.1.2) can not be written in a form of conservation laws and it is impossible to have the normalized coordinates for system (6.1.1)-(6.1.2).

Consider the Cauchy problem for system (6.1.1)-(6.1.2) with the following initial data:

$$\alpha = a\beta : \quad u = \tilde{u}_0 + u_0(\beta), \tag{6.1.6}$$

where $a \neq 0$ is a real number, $\tilde{u}_0$ is a constant vector, $u_0(\beta)$ is a C^1 vector function satisfying that there exists a constant $\mu > 0$ such that

$$\theta \triangleq \sup_{\beta \in \mathbf{R}} \left\{ (1 + |\beta|)^{1+\mu} |u_0'(\beta)| \right\} < \infty. \tag{6.1.7}$$

By Theorem 5.1 we have

Theorem 6.1. Under the previous assumptions, there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, Cauchy problem (6.1.1)-(6.1.2) and (6.1.6) admits a unique global C^1 solution $u = u(\alpha, \beta)$ on the whole (α, β)-plane. □

As an application of Theorem 6.1, consider the Monge-Ampère equation

$$Ar + Bs + Ct + D(rt - s^2) - E = 0, \tag{6.1.8}$$

where

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}, \tag{6.1.9}$$

A, B, C, D and E are smooth functions of (x, y, z, p, q) with

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}. \tag{6.1.10}$$

The characteristic strip method originally suggested by G.Darboux [Da], E.Goursat [Go] and used with improvement in M.Tsuji [Ts] asks to solve the following system

$$\begin{cases} \frac{\partial z}{\partial \alpha} - p\frac{\partial x}{\partial \alpha} - q\frac{\partial y}{\partial \alpha} = 0, \\ D\frac{\partial p}{\partial \alpha} + C\frac{\partial x}{\partial \alpha} + \lambda_1 \frac{\partial y}{\partial \alpha} = 0, \\ D\frac{\partial q}{\partial \alpha} + \lambda_2 \frac{\partial x}{\partial \alpha} + A\frac{\partial y}{\partial \alpha} = 0, \\ D\frac{\partial p}{\partial \beta} + C\frac{\partial x}{\partial \beta} + \lambda_2 \frac{\partial y}{\partial \beta} = 0, \\ D\frac{\partial q}{\partial \beta} + \lambda_1 \frac{\partial x}{\partial \beta} + A\frac{\partial y}{\partial \beta} = 0, \end{cases} \tag{6.1.11}$$

where λ_1 and λ_2 are two solutions of the equation

$$\lambda^2 + B\lambda + (AC + DE) = 0. \tag{6.1.12}$$

Suppose that

$$\triangle \triangleq B^2 - 4(AC + DE) > 0, \tag{6.1.13}$$

then

$$\lambda_1 \neq \lambda_2 \tag{6.1.14}$$

are real numbers. Suppose furthermore that

$$D \neq 0. \tag{6.1.15}$$

Hypotheses (6.1.13) and (6.1.15) mean that (6.1.8) is a non-degenerate Monge-Ampère equation of hyperbolic type. Noting (6.1.14)-(6.1.15), the determinant of the coefficient matrix of (6.1.11) is different from zero, therefore we can use Theorem 6.1 to get a global C^1 solution $(x, y, z, p, q) = (x(\alpha, \beta), y(\alpha, \beta), z(\alpha, \beta), p(\alpha, \beta), q(\alpha, \beta))$ to system (6.1.11) on the whole (α, β)-plane.

For the special case that

$$A = B = C \equiv 0 \quad \text{and} \quad D \equiv 1 \tag{6.1.16}$$

and $E(<0)$ depends only on x, y, p and q, instead of (6.1.11) we only need to consider the following system:

$$\begin{cases} \frac{\partial p}{\partial \alpha} + \lambda \frac{\partial y}{\partial \alpha} = 0, \\ \frac{\partial q}{\partial \alpha} - \lambda \frac{\partial x}{\partial \alpha} = 0, \\ \frac{\partial p}{\partial \beta} - \lambda \frac{\partial y}{\partial \beta} = 0, \\ \frac{\partial q}{\partial \beta} + \lambda \frac{\partial x}{\partial \beta} = 0, \end{cases} \tag{6.1.17}$$

where

$$\lambda = \lambda(x, y, p, q) \triangleq \sqrt{-E} > 0. \tag{6.1.18}$$

Similar results hold for this situation.

§6.2. System of nonlinear three-wave interaction in plasma physics

Consider the Cauchy problem for the following semilinear system

$$\begin{cases} \frac{\partial u_1}{\partial t} + c_1 \frac{\partial u_1}{\partial x} = a_1 u_2 u_3, \\ \frac{\partial u_2}{\partial t} + c_2 \frac{\partial u_2}{\partial x} = a_2 u_1 u_3, \\ \frac{\partial u_3}{\partial t} + c_3 \frac{\partial u_3}{\partial x} = a_3 u_1 u_2, \end{cases} \tag{6.2.1}$$

$$t = 0: \quad u_i = \varphi_i(x), \tag{6.2.2}$$

where c_i $(i = 1, 2, 3)$ are distinct real numbers, a_i $(i = 1, 2, 3)$ are three constants and $\varphi_i(x)$ $(i = 1, 2, 3)$ are C^1 functions satisfying that there exists a positive constant $\mu > 0$ such that

$$\theta \triangleq \sup_{x \in \mathbf{R}} \left\{ (1 + |x|)^{1+\mu} \sum_{i=1}^{3} [|\varphi_i(x)| + |\varphi_i'(x)|] \right\} < \infty. \tag{6.2.3}$$

System (6.2.1) may describe the motion of particles of a rarefied gas in a thin infinite tube. The Broadwell model of the discrete Boltzmann equation consists in discretization of the velocity of molecules, that is, the molecules take only a finite number of velocities $C_i \in R^3$ (here we assume that $i = 1, 2, 3$). The solution $u_i(t, X)$ represents the distribution function of the molecules animated with the velocity C_i, i.e., $u_i(t, X)$ is the density of molecules with the velocity C_i; at time t and at the point $X = (x, y, z)$. By virtue of the thinness of tube, we may assume that the function $u_i(t, X)$ is homogeneous with respect to the variables y and z, where we take x-axis as a variable along the axis of the tube. Let c_i be the x-component of the velocity C_i. Thus, the soltuion $u_i(t, x)$ satisfies system (6.2.1). On the other hand, system (6.2.1) can also be used to describe the nonlinear three-wave interaction arising from plasma physics (see [WW]). The Cauchy problem (6.2.1)-(6.2.2) has been studied by many people, some results on the global existence and the blow-up phenomenon of solutions have been obtained under various assumptions (see [AH], [Be], [NM], [SK], [TC], etc.). In this subsection we give the applications of our theory to system (6.2.1).

Clearly, (6.2.1) is strictly hyperbolic system with three distinct real eigenvalues

$$\lambda_1 = c_1, \quad \lambda_2 = c_2, \quad \lambda_3 = c_3. \tag{6.2.4}$$

Moreover, system (6.2.1) is linearly degenerate in the sense of P.D.Lax and the inhomogeneous term satisfies the matching condition. Then, by Theorem 3.1 we have

Theorem 6.2. Under the hypotheses mentioned above, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, the Cauchy problem (6.2.1)-(6.2.2) admits a unique global C^1 solution $u = (u_1(t, x), u_2(t, x), u_3(t, x))$ on $t \geq 0$. □

Remark 6.1. We have a similar result for the following system describing the propagation of waves in optical fibre (see [BFJ])

$$\begin{cases} \frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} = a u_2^2, \\ \frac{\partial u_2}{\partial t} - \frac{\partial u_2}{\partial x} = -a u_1 u_2, \end{cases} \tag{6.2.5}$$

where a is a fixed real number. □

§6.3. Nonlinear wave equation with higher order dissipation

Consider the following Cauchy problem for the nonlinear wave equation

$$u_{tt} - (K(u_x))_x = -a|u_t|^{p-1}u_t, \tag{6.3.1}$$

$$t = 0: \quad u = b + \varepsilon\phi(x), \; u_t = \varepsilon\psi(x), \tag{6.3.2}$$

where $K(\nu)$ is a suitably smooth function satisfying

$$K'(0) > 0, \tag{6.3.3}$$

$p \geq 1$ is an integer, $\varepsilon > 0$ is a small parameter, a and b are two real numbers (when $a > 0$, the right-hand side of (6.3.1) stands for the dissipation), $\phi(x) \in C^2$, $\psi(x) \in C^1$ and $(\phi'(x), \psi(x))$ satisfies that there exists a positive constant $\mu > 0$ such that

$$\sup_{x\in R}\left\{(1+|x|)^{1+\mu}\left(|\varphi'(x)| + |\psi(x)| + |\varphi''(x)| + |\psi'(x)|\right)\right\} < \infty. \tag{6.3.4}$$

In 1-D elasticity, the nonlinear wave equation (6.3.1) arises with $u(t,x)+x$ the deformed location at time t of the material initially located at x, u_x the strain, and $K(u_x)$ the stress-strain function. In particular for a hard spring, or a "non-Hookian" material in a neighborhood of zero, typically $K(u_x)$ is a smooth odd function such as

$$K(u_x) = \frac{u_x}{\sqrt{1+u_x^2}}\tau(u_x),$$

where $\tau(u_x)$ is an even function representing the tension in the string corresponding to the strain u_x (see [KM] or [Ma]). Moreover, let

$$\tau = u_x, \quad v = u_t,$$

then (6.3.1) reduces

$$\begin{cases} \frac{\partial \tau}{\partial t} - \frac{\partial v}{\partial x} = 0, \\ \frac{\partial v}{\partial t} + \frac{\partial P(\tau,S)}{\partial x} = a|v|^{p-1}v, \end{cases} \tag{6.3.1a}$$

In fluid dynamics, (6.3.1a) arises as the equations of one-dimensional gas dynamics in Lagrangian coordinate. Here x is a mass coordinate, in this context, $\tau = \rho^{-1}$, the specific volume, and $v = u_t$ the fluid velocity. When $a > 0$, the term $a|v|^{p-1}v$

stands for the dissipation. The Cauchy problem (6.3.1)-(6.3.2) has been studied by T. Nishida [Ni] for the case $p = 1$ and D. Kong [K2] for the case $a > 0$.

Introduce the Riemann invariants

$$r(u_x, u_t) = \frac{1}{2}\left(u_t + \int_0^{u_x} [K'(\omega)]^{\frac{1}{2}} d\omega\right), \quad s(u_x, u_t) = \frac{1}{2}\left(u_t - \int_0^{u_x} [K'(\omega)]^{\frac{1}{2}} d\omega\right). \tag{6.3.5}$$

The Cauchy problem (6.3.1)-(6.3.2) can be rewritten as

$$\begin{cases} \frac{\partial r}{\partial t} - k(r-s)\frac{\partial r}{\partial x} = -\frac{a}{2}|r+s|^{p-1}(r+s), \\ \frac{\partial s}{\partial t} + k(r-s)\frac{\partial s}{\partial x} = -\frac{a}{2}|r+s|^{p-1}(r+s), \end{cases} \tag{6.3.6}$$

$$t = 0: \ r = r_0(\varepsilon, x), \ s = s_0(\varepsilon, x), \tag{6.3.7}$$

where

$$k(\omega) = \left[K'(H^{-1}(\omega))\right]^{\frac{1}{2}}, \tag{6.3.8}$$

$\nu = H^{-1}(\omega)$ stands for the inverse function of $\omega = H(\nu)$ in which

$$H(\nu) = \int_0^{\nu} [K'(\eta)]^{\frac{1}{2}} d\eta, \tag{6.3.9}$$

$r_0(\varepsilon, x)$ and $s_0(\varepsilon, x)$ are given by

$$\begin{aligned} r_0(\varepsilon, x) &\triangleq \tfrac{\varepsilon}{2}\left\{\psi(x) + \int_0^{\varphi'(x)} [K'(\varepsilon\omega)]^{\frac{1}{2}} d\omega\right\} = \tfrac{\varepsilon}{2}\left\{\psi(x) + \varphi'(x) + O(\varepsilon)\right\}, \\ s_0(\varepsilon, x) &\triangleq \tfrac{\varepsilon}{2}\left\{\psi(x) - \int_0^{\varphi'(x)} [K'(\varepsilon\omega)]^{\frac{1}{2}} d\omega\right\} = \tfrac{\varepsilon}{2}\left\{\psi(x) - \varphi'(x) + O(\varepsilon)\right\}. \end{aligned} \tag{6.3.10}$$

Obviously, in a neighbourhood of $(r, s) = (0, 0)$, (6.3.6) is a strictly hyperbolic system in a diagonal form. Moreover, (r, s) are just the normalized coordinates.

We assume that there exists an integer $\alpha \geq 0$ such that

$$K^{j+1}(0) = 0 \ \ (j = 1, \cdots, \alpha) \quad \text{but} \quad K^{\alpha+2}(0) \neq 0. \tag{6.3.11}$$

Moreover, without loss of generality, we may assume

$$K'(0) = 1. \tag{6.3.12}$$

Then it follows from (6.3.8) that

$$k(r-s) = 1 + \frac{1}{2}\frac{K^{\alpha+2}(0)}{(\alpha+1)!}(r-s)^{\alpha+1} + O\left((r-s)^{\alpha+2}\right), \tag{6.3.13}$$

provided that $|r|$ and $|s|$ are small.

Applying Theorem 3.3 to the Cauchy problem (6.3.6)-(6.3.7) and noting the fact that the C'^2 solution of the Cauchy problem (6.3.1)-(6.3.2) is equivalent to the C^1 solution of the Cauchy problem (6.3.6)-(6.3.7), we have

Theorem 6.3. Under the hypotheses (6.3.11)-(6.3.12), if $p > 2 + \alpha$ and $(\phi'(x), \psi(x))$ is not identically equal to zero, then there exists $\varepsilon_0 > 0$ so small that for any ε with $0 < \varepsilon \leq \varepsilon_0$, the C^2 solution $u = u(t, x)$ to the Cauchy problem (6.3.1)-(6.3.2) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ satisfies

$$\lim_{\varepsilon \to 0} \left(\varepsilon^{\alpha+1}\tilde{T}(\varepsilon)\right)^{-1} = \left(2^{\alpha+2} \cdot \alpha!\right)^{-1} \max\left\{C_1^-, C_1^+\right\} > 0, \tag{6.3.14}$$

where

$$C_1^- = \sup_{x \in R} \left\{(-1)^\alpha K^{(\alpha+2)}(0) \left[\psi(x) - \varphi'(x)\right]^\alpha \left[\psi'(x) - \varphi''(x)\right]\right\} \tag{6.3.15}$$

and

$$C_1^+ = \sup_{x \in R} \left\{K^{(\alpha+2)}(0) \left[\psi(x) + \varphi'(x)\right]^\alpha \left[\psi'(x) + \varphi''(x)\right]\right\}. \tag{6.3.16}$$

□

§6.4. System of one-dimensional gas dynamics with higher order damping

Consider the following Cauchy problem for the system of one-dimensional gas dynamics

$$\begin{cases} \frac{\partial \tau}{\partial t} - \frac{\partial v}{\partial x} = 0, \\ \frac{\partial v}{\partial t} + \frac{\partial P(\tau, S)}{\partial x} = a|v|^{p-1}v, \\ \frac{\partial S}{\partial t} = 0, \end{cases} \tag{6.4.1}$$

$$t = 0: \quad \tau = \bar{\tau}_0 + \varepsilon\tau_0(x), v = \varepsilon v_0(x), \quad S = \varepsilon S_0(x), \tag{6.4.2}$$

where τ is the specific volume, v is the velocity, S is the entropy, P is the pressure, a is a real number (when $a < 0$, the right-hand term stands for the dissipation), $p \geq 1$ is an integer, $\bar{\tau}_0 > 0$ is a constant, $\varepsilon > 0$ is a small parameter and

$(\tau_0(x), v_0(x), S_0(x))$ is a C^1 vector function satisfying that there exists a positive constant $\mu > 0$ such that

$$\sup_{x\in \boldsymbol{R}}\left\{(1+|x|)^{(1+\mu)}\left(|\tau_0(x)|+|v_0(x)|+|S_0(x)|+|\tau_0'(x)|+|v_0'(x)|+|S_0'(x)|\right)\right\} < \infty. \tag{6.4.3}$$

When $a = 0$, the Cauchy problem (6.4.1)-(6.4.2) has been studied by T.P. Liu [Lu] for the case that the initial data has a compact support and by [LZK2] for the case that (6.4.3) holds.

Assume that the state equation $P = P(\tau, S)$ satisfies

$$P_\tau < 0, \quad \forall\, \tau > 0 \tag{6.4.4}$$

and, without loss of generality, we may assume

$$P_\tau(\bar{\tau}_0, 0) = -1. \tag{6.4.5}$$

Obviously, on the domain under consideration, (6.4.1) is a strictly hyperbolic system with three distinct real eigenvalues

$$\lambda_1 \triangleq -\sqrt{-P_\tau} < \lambda_2 \triangleq 0 < \lambda_3 \triangleq \sqrt{-P_\tau}, \tag{6.4.6}$$

in which λ_2 is linearly degenerate in the sense of P.D.Lax, then weakly linearly degenerate.

Introduce the following transformation

$$\begin{cases} u_1 = v + \displaystyle\int_{\bar{\tau}_0}^{\tau} \sqrt{-P_\nu(\nu, S)}d\nu \triangleq v + h(\tau, S), \\ u_2 = S, \\ u_3 = v - \displaystyle\int_{\bar{\tau}_0}^{\tau} \sqrt{-P_\nu(\nu, S)}d\nu \triangleq v - h(\tau, S) \end{cases} \tag{6.4.7}$$

and let

$$u = (u_1, u_2, u_3). \tag{6.4.8}$$

Then system (6.4.1) can be equivalently rewritten as

$$\begin{cases} \frac{\partial u_1}{\partial t} - \lambda(u)\frac{\partial u_1}{\partial x} + \xi(u)\frac{\partial u_2}{\partial x} = \zeta(u), \\ \frac{\partial u_2}{\partial t} = 0, \\ \frac{\partial u_3}{\partial t} + \lambda(u)\frac{\partial u_3}{\partial x} + \xi(u)\frac{\partial u_2}{\partial x} = \zeta(u), \end{cases} \tag{6.4.9}$$

where

$$\lambda(u) = [-P_\tau(\tau, u_2)]^{\frac{1}{2}}, \tag{6.4.10}$$

$$\xi(u) = P_{u_2}(\tau, u_2) + \lambda(u) h_{u_2}(\tau, u_2), \tag{6.4.11}$$

$$\zeta(u) = 2^{-p} a |u_1 + u_3|^{p-1}(u_1 + u_3) \tag{6.4.12}$$

and

$$\tau = G(u_1 - u_3, u_2) \tag{6.4.13}$$

is the inverse function of

$$(u_1 - u_3) = 2\int_{\bar{\tau}_0}^{\tau} \sqrt{P_\nu(\nu, u_2)}d\nu. \tag{6.4.14}$$

Moreover, noting (6.4.5), we can rewrite the initial data (6.4.2) as

$$\begin{cases} u_1(0,x) = \varepsilon(v_0(x) + \tau_0(x)) + O(\varepsilon^2), \\ u_2(0,x) = \varepsilon S_0(x), \\ u_3(0,x) = \varepsilon(v_0(x) - \tau_0(x)) + O(\varepsilon^2). \end{cases} \tag{6.4.15}$$

The left and right eigenvectors of system (6.4.9) can be chosen as

$$l_1(u) = \left(1, -\frac{\xi}{\lambda}, 0\right), \quad l_2(u) = \left(1, \frac{\sqrt{\lambda^2 + 2\xi^2}}{\lambda}, 0\right), \quad l_3(u) = \left(0, \frac{\xi}{\lambda}, 1\right) \tag{6.4.16}$$

and

$$r_1(u) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad r_2(u) = \begin{pmatrix} \frac{\xi}{\sqrt{\lambda^2+2\xi^2}} \\ \frac{\lambda}{\sqrt{\lambda^2+2\xi^2}} \\ -\frac{\xi}{\sqrt{\lambda^2+2\xi^2}} \end{pmatrix}, \quad r_3(u) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{6.4.17}$$

where $\lambda = \lambda(u)$ and $\xi = \xi(u)$ are defined by (6.4.10)-(6.4.11) respectively. Clearly, in the present situation, (1.4)-(1.5) hold.

It follows from (6.4.17) that the 1st (resp. 3rd) characteristic trajectory passing through $u = 0$ is just the u_1-axis (resp. u_3-axis).

We assume that there exists an integer $\alpha \geq 1$ such that

$$P_{\tau\tau}(\bar{\tau}_0, 0) = \cdots = \frac{\partial^{1+\alpha} P}{\partial \tau^{1+\alpha}}(\bar{\tau}_0, 0) = 0 \quad \text{but} \quad \sigma_0 \triangleq \frac{\partial^{2+\alpha} P}{\partial \tau^{2+\alpha}}(\bar{\tau}_0, 0) \neq 0. \tag{6.4.18}$$

Noting (6.4.5), (6.4.7), (6.4.10) and (6.4.13)-(6.4.14), we have

$$\left.\frac{\partial^l \lambda_1(u_1,0,0)}{\partial u_1^l}\right|_{u_1=0} = 0 \quad (l = 1,\cdots,\alpha) \quad \text{but} \quad \left.\frac{\partial^{1+\alpha} \lambda_1(u_1,0,0)}{\partial u_1^{1+\alpha}}\right|_{u_1=0} = \frac{\sigma_0}{2^{2+\alpha}} \tag{6.4.19}$$

and

$$\left.\frac{\partial^l \lambda_3(0,0,u_3)}{\partial u_3^l}\right|_{u_3=0} = 0 \quad (l = 1,\cdots,\alpha) \quad \text{but} \quad \left.\frac{\partial^{1+\alpha} \lambda_3(0,0,u_3)}{\partial u_3^{1+\alpha}}\right|_{u_3=0} = \frac{(-1)^\alpha\sigma_0}{2^{2+\alpha}}. \tag{6.4.20}$$

Moreover, noting the fact that $\lambda_2(u)$ is linearly degenerate in the sense of P.D.Lax, then weakly linearly degenerate, by Theorem 3.3 we get

Theorem 6.4. Under the hypotheses (6.4.4)-(6.4.5) and (6.4.18), if $p > 2+\alpha$ and $(v_0(x),\ \tau_0(x)\ -\varrho_0 S_0(x))$ is not identically equal to zero, where

$$\varrho_0 = P_S(\bar{\tau}_0,\,0)\,, \tag{6.4.21}$$

then there exists $\varepsilon_0 > 0$ so small that for any ε with $0 < \varepsilon \le \varepsilon_0$, the C^1 solution to the Cauchy problem (6.4.1)-(6.4.2) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ satisfies

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{\alpha+1}\tilde{T}(\varepsilon)\right)^{-1} = \left(2^{2+\alpha}\cdot\alpha!\right)^{-1}\max\left\{C_2^-,\,C_2^+\right\} > 0, \tag{6.4.22}$$

where

$$C_2^- = \sup_{x\in\boldsymbol{R}}\left\{-\sigma_0\left[v_0(x)+\tau_0(x)-\varrho_0 S_0(x)\right]^\alpha\left[v_0'(x)+\tau_0'(x)-\varrho_0 S_0'(x)\right]\right\}, \tag{6.4.23}$$

$$C_2^+ = \sup_{x\in\boldsymbol{R}}\left\{(-1)^{1+\alpha}\sigma_0\left[v_0(x)-\tau_0(x)+\varrho_0 S_0(x)\right]^\alpha\left[v_0'(x)-\tau_0'(x)+\varrho_0 S_0'(x)\right]\right\}, \tag{6.4.24}$$

σ_0 and ϱ_0 are defined by (6.4.18) and (6.4.21) respectively. □

Remark 6.2. In particular, taking $S_0(x) \equiv 0$ and $K(\nu) = -P(\nu)$, from (6.4.22) we get (6.3.14) immediately. □

§6.5. System of motion of an elastic string

Consider the following Cauchy problem

$$\begin{cases} u_t - v_x = 0, \\ v_t - \left(\frac{T(r)}{r}u\right)_x = 0, \end{cases} \tag{6.5.1}$$

$$t = 0: \ u = \tilde{u}^0 + \varepsilon u^0(x), \ v = \varepsilon v^0(x) \tag{6.5.2}$$

where $u = (u_1, u_2)^T$, $v = (v_1, v_2)^T$, $r = |u| = \sqrt{u_1^2 + u_2^2}$, $T(r)$ is a given smooth function of the stretch $r(> 1)$ satisfying

$$T'(\tilde{r}_0) > \frac{T(\tilde{r}_0)}{\tilde{r}_0} > 0, \tag{6.5.3}$$

where $\tilde{r}_0 = |\tilde{u}_0| = \sqrt{(\tilde{u}_1^0)^2 + (\tilde{u}_2^0)^2} > 1$, $\varepsilon > 0$ is a small parameter, $u^0(x)$ and $v^0(x)$ are C^1 vector functions satisfying that there exists a positive constant μ such that

$$\sup_{x \in \mathbf{R}} \left\{ (1 + |x|)^{1+\mu} \left(\left| \frac{du^0(x)}{dx} \right| + \left| \frac{dv^0(x)}{dx} \right| \right) \right\} < \infty. \tag{6.5.4}$$

System (6.5.1) describes the planar motion of an elastic string (see Chapter 3 in [An]), where u stands for the displacement of a point from the position x in the natural state, v is the velocity, and $T(r)$ is the tension of the string. The generalized Riemann problem for system (6.5.1) was studied by [LSZ]. Later, Li, Kong and Zhou [LKZ] discussed the blow-up phenomenon of classical solution to the Cauchy problem (6.5.1)-(6.5.2). In this subsection, we shall give the asymptotic behaviour of life span of C^1 solution to the Cauchy problem (6.5.1)-(6.5.2).

Let

$$U = (u_1, u_2, v_1, v_2)^T = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{6.5.5}$$

By (6.5.3), in a neighbourhood of

$$U = \begin{pmatrix} \tilde{u}^0 \\ 0 \end{pmatrix} \triangleq U^0,$$

(6.5.1) is a strictly hyperbolic system with four distinct real eigenvalues

$$\lambda_1 \triangleq -\sqrt{T'(r)} < \lambda_2 \triangleq -\sqrt{\frac{T(r)}{r}} < \lambda_3 \triangleq \sqrt{\frac{T(r)}{r}} < \lambda_4 \triangleq \sqrt{T'(r)}, \tag{6.5.6}$$

in which λ_2 and λ_3 are linearly degenerate in the sense of P.D.Lax. The corresponding left and right eigenvectors can be taken as follows

$$r_1 = \frac{1}{r\sqrt{1+T'(r)}}\begin{pmatrix} u_1 \\ u_2 \\ \sqrt{T'(r)}u_1 \\ \sqrt{T'(r)}u_2 \end{pmatrix}, \quad r_2 = \frac{1}{r\sqrt{1+\frac{T(r)}{r}}}\begin{pmatrix} -u_2 \\ u_1 \\ -\sqrt{\frac{T(r)}{r}}u_2 \\ \sqrt{\frac{T(r)}{r}}u_1 \end{pmatrix},$$

$$r_3 = \frac{1}{r\sqrt{1+\frac{T(r)}{r}}}\begin{pmatrix} u_2 \\ -u_1 \\ -\sqrt{\frac{T(r)}{r}}u_2 \\ \sqrt{\frac{T(r)}{r}}u_1 \end{pmatrix}, \quad r_4 = \frac{1}{r\sqrt{1+T'(r)}}\begin{pmatrix} -u_1 \\ -u_2 \\ \sqrt{T'(r)}u_1 \\ \sqrt{T'(r)}u_2 \end{pmatrix} \tag{6.5.7}$$

and

$$\begin{aligned} l_1 &= \frac{\sqrt{1+T'(r)}}{2r\sqrt{T'(r)}}\left(\sqrt{T'(r)}u_1,\ \sqrt{T'(r)}u_2,\ u_1,\ u_2\right), \\ l_2 &= \frac{\sqrt{1+\frac{T(r)}{r}}}{2r\sqrt{\frac{T(r)}{r}}}\left(-\sqrt{\frac{T(r)}{r}}u_2,\ \sqrt{\frac{T(r)}{r}}u_1,\ -u_2,\ u_1\right), \\ l_3 &= \frac{\sqrt{1+\frac{T(r)}{r}}}{2r\sqrt{\frac{T(r)}{r}}}\left(\sqrt{\frac{T(r)}{r}}u_2,\ -\sqrt{\frac{T(r)}{r}}u_1,\ -u_2,\ u_1\right), \\ l_4 &= \frac{\sqrt{1+T'(r)}}{2r\sqrt{T'(r)}}\left(-\sqrt{T'(r)}u_1,\ -\sqrt{T'(r)}u_2,\ u_1,\ u_2\right). \end{aligned} \tag{6.5.8}$$

We first consider a special but important case[1]

$$T(r) = r - 1. \tag{6.5.9}$$

In this case, by (6.5.6) we have

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_4 = 1. \tag{6.5.10}$$

Obviously, λ_1 and λ_4 are linearly degenerate in the sense of P.D.Lax. Thus, by Theorem 5.1 we have

Theorem 6.5. Under the hypotheses (6.5.3)-(6.5.4) and (6.5.9), there exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in [0, \varepsilon_0]$, the Cauchy problem (6.5.1)-(6.5.2) admits a unique global C^1 solution $U = U(t,x)$ on the whole (t,x)-plane. □

[1] In fact, for the case that $T''(r) \equiv 0$, Theorem 6.5 is still valid.

In what follows, we consider another important case that $T(r)$ is a nonlinear function satisfying that

$$T''(\tilde{r}_0) \neq 0. \tag{6.5.11}$$

In this case, λ_1 and λ_4 are genuinely nonlinear in the sense of P.D.Lax, moreover,

$$\nabla\lambda_1(U^0)r_1(U^0) = \nabla\lambda_4(U^0)r_4(U^0) = -\frac{T''(\tilde{r}_0)}{2\sqrt{T'(\tilde{r}_0)(1+T'(\tilde{r}_0))}} \triangleq \Theta. \tag{6.5.12}$$

We assume that there exists a point $x_0 \in R$ such that

$$\Xi_-(x_0) < 0 \tag{6.5.13}$$

or

$$\Xi_+(x_0) < 0, \tag{6.5.14}$$

where

$$\Xi_-(x) = \tilde{\Theta}\left[\tilde{u}_1^0\frac{du_1^0(x)}{dx} + \tilde{u}_2^0\frac{du_2^0(x)}{dx} + (T'(\tilde{r}_0))^{-\frac{1}{2}}\left(\tilde{u}_1^0\frac{dv_1^0(x)}{dx} + \tilde{u}_2^0\frac{dv_2^0(x)}{dx}\right)\right], \tag{6.5.15}$$

$$\Xi_+(x) = \tilde{\Theta}\left[(T'(\tilde{r}_0))^{-\frac{1}{2}}\left(\tilde{u}_1^0\frac{dv_1^0(x)}{dx} + \tilde{u}_2^0\frac{dv_2^0(x)}{dx}\right) - \left(\tilde{u}_1^0\frac{du_1^0(x)}{dx} + \tilde{u}_2^0\frac{du_2^0(x)}{dx}\right)\right], \tag{6.5.16}$$

in which $\tilde{\Theta} = (2\tilde{r}_0)^{-1}\sqrt{1+T'(\tilde{r}_0)}\Theta$. Thus, by Theorem 5.3 we have

Theorem 6.6. Under the hypotheses (6.5.3)-(6.5.4), (6.5.11) and (6.5.13) (or (6.5.14)), then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$ the first order derivatives of the C^1 solution $U = U(t,x)$ to the Cauchy problem (6.5.1)-(6.5.2) must blow up in a finite time and the life-span $\tilde{T}(\varepsilon)$ satisfies

$$\lim_{\varepsilon\to 0}\left(\varepsilon\tilde{T}(\varepsilon)\right)^{-1} = \max\left\{\sup_{x\in R}\{-\Xi_-(x)\}, \sup_{x\in R}\{-\Xi_+(x)\}\right\}. \tag{6.5.17}$$

□

For the following Cauchy problem of the system for the general motion of an elastic string:

$$\begin{cases} u_t - v_x = 0, \\ v_t - \left(\frac{T(r)}{r}u\right)_x = 0, \end{cases} \tag{6.5.18}$$

$$t = 0: \ u = \tilde{u}^0 + \varepsilon u^0(x), \ v = \varepsilon v^0(x), \tag{6.5.19}$$

where $u = (u_1, \cdots, u_n)^T$, $v = (v_1, \cdots, v_n)^T$, $r = |u| = \sqrt{u_1^2 + \cdots + u_n^2}$, $T(r)$ is a suitably smooth function of $r(> 1)$ such that

$$T'(\tilde{r}_0) > \frac{T(\tilde{r}_0)}{\tilde{r}_0} > 0, \tag{6.5.20}$$

where $\tilde{r}_0 = |\tilde{u}^0| = \sqrt{(\tilde{u}_1^0)^2 + \cdots + (\tilde{u}_n^0)^2} > 1$, $\varepsilon > 0$ is a small parameter, $\tilde{u}^0 = (\tilde{u}_1^0, \cdots, \tilde{u}_n^0)^T$ is a constant vector, $u^0(x)$ and $v^0(x)$ are C^1 vector functions satisfying (6.5.4), we have similar results.

Remark 6.3. For the system modeling the dynamics of a moving threadline (see [ALZ]), we can obtain similar results. □

§6.6. System of plane elastic waves for hyperelastic materials

Now we consider the time dependent deformation of an elastic medium from the natural state, in which the position vector of a particle is denoted by $X = (X_1, X_2, X_3)^T$. At the time t, the same particle has a position vector $Y = Y(t, X) = (Y_1, Y_2, Y_3)^T$. For homogeneous hyperelastic materials there exists a stored energy function $W = W(p)$, where

$$p = (p_{ik}) = \left(\frac{\partial Y_i}{\partial X_k}\right) \tag{6.6.1}$$

is the strain tensor.

In the isotropic case, $W = W(p)$ satisfies

$$W(Qp) = W(pQ) = W(p), \tag{6.6.2}$$

where Q is an arbitrary orthogonal matrix.

The system of motion is given by

$$\rho \frac{\partial^2 Y_i}{\partial t^2} = \sum_{k,r,s=1}^{3} c_{ikrs}(p) \frac{\partial^2 Y_r}{\partial X_k \partial X_s}, \tag{6.6.3}$$

where ρ denotes the density, without loss generality, we may suppose that $\rho \equiv 1$, and

$$c_{ikrs}(p) = \frac{\partial^2 W(p)}{\partial p_{ik} \partial p_{rs}}. \tag{6.6.4}$$

Let π be a non-singular square matrix of order 3. For any given vector $\omega = (\omega_1, \omega_2, \omega_3)$ with $|\omega| = 1$, the solution of *plane elastic waves* will be given by

$$Y = \pi X + f(t, x), \tag{6.6.5}$$

where

$$x = \omega X \tag{6.6.6}$$

and $f = (f_1, f_2, f_3)^T$. Then

$$p = \pi + f_x \omega. \tag{6.6.7}$$

Without loss of generality, we may suppose that $\pi = I$. In fact, taking the following invertible linear transformtion:

$$\overline{Y} = \pi^{-1} Y, \quad \overline{f} = \pi^{-1} f, \tag{6.6.8}$$

we have $\pi = I$ in new variables $(\overline{Y}, \overline{f})$, however, the initial data of f should be changed according to the second equation of (6.6.8).

For the solution of plane elastic waves (6.6.5), system (6.6.3) reduces to

$$\frac{\partial^2 f}{\partial t^2} - V''(f_x) \frac{\partial^2 f}{\partial x^2} = 0, \tag{6.6.9}$$

where $V'' = (V_{ij})$, in which

$$V_{ij}(\eta) = V_{ji}(\eta) = \frac{\partial^2 V(\eta)}{\partial \eta_i \partial \eta_j} \quad (i, j = 1, 2, 3) \tag{6.6.10}$$

with

$$V(\eta) = W(I + \eta \omega), \tag{6.6.11}$$

where $\eta = (\eta_1, \eta_2, \eta_3)^T$.

Let

$$u_i = \frac{\partial f_i}{\partial x}, \quad u_{i+3} = \frac{\partial f_i}{\partial t} \quad (i = 1, 2, 3). \tag{6.6.12}$$

$u = (u_1, \cdots, u_6)^T$ satisfies the following system

$$u_t + A(u)\, u_x = 0, \tag{6.6.13}$$

where

$$A(u) = \begin{pmatrix} 0 & -I \\ -V'' & 0 \end{pmatrix}, \tag{6.6.14}$$

in which 0 and I stand for the zero matrix and the unit matrix of order 3 respectively. Obviously, $A(u)$ depends only on the components $u_1,\ u_2,\ u_3$ of u.

In what follows we consider so-called Ciarlet-Geymonat material, the stored energy function of which is given by

$$W(p) = a \parallel p \parallel^2 + b \parallel \mathrm{Cof}\ p \parallel^2 + \Gamma(\det p) + e, \tag{6.6.15}$$

where a, b are positive constants, e is a real number and

$$\parallel p \parallel = \left(\mathrm{tr}\ p^T p\right)^{\frac{1}{2}}, \quad \mathrm{Cof}\ p = (\det p)\left(p^{-1}\right)^T, \quad \Gamma(\delta) = c\delta^2 - d\log\delta, \qquad \forall\, \delta > 0, \tag{6.6.16}$$

where c, d are two positive constants (see [Ci]).

In this case it is easy to see that

$$V''(u) = 2(a+b)\, I + \left[2(b+c) + d(1+\omega\hat{u})^{-2}\right] \omega^T \omega, \tag{6.6.17}$$

where $\hat{u} = (u_1, u_2, u_3)^T$, then the corresponding matrix $A(u)$ has the following six real eigenvalues:

$$\lambda_1 = -\lambda, \quad \lambda_{2,3} = -\lambda_0, \quad \lambda_{4,5} = \lambda_0, \quad \lambda_6 = \lambda, \tag{6.6.18}$$

where

$$\lambda = \sqrt{2(a+b) + 2(b+c) + d(1+\omega\hat{u})^{-2}}, \quad \lambda_0 = \sqrt{2(a+b)}. \tag{6.6.19}$$

Moreover, the corresponding right and left eigenvectors can be taken as follows:

$$r_1 = \frac{1}{\sqrt{1+\lambda^2}}\begin{pmatrix} \omega^T \\ \lambda\omega^T \end{pmatrix}, \qquad r_2 = \frac{1}{\sqrt{(1+\lambda_0^2)(\omega_1^2+\omega_2^2)}}\begin{pmatrix} (\omega^3)^T \\ \lambda_0(\omega^3)^T \end{pmatrix},$$

$$r_3 = \frac{1}{\sqrt{(1+\lambda_0^2)(\omega_1^2+\omega_3^2)}}\begin{pmatrix} (\omega^2)^T \\ \lambda_0(\omega^2)^T \end{pmatrix}, \quad r_4 = \frac{1}{\sqrt{(1+\lambda_0^2)(\omega_1^2+\omega_2^2)}}\begin{pmatrix} (\omega^3)^T \\ -\lambda_0(\omega^3)^T \end{pmatrix},$$

$$r_5 = \frac{1}{\sqrt{(1+\lambda_0^2)(\omega_1^2+\omega_3^2)}}\begin{pmatrix} (\omega^2)^T \\ -\lambda_0(\omega^2)^T \end{pmatrix}, \quad r_6 = \frac{1}{\sqrt{1+\lambda^2}}\begin{pmatrix} \omega^T \\ -\lambda\omega^T \end{pmatrix} \tag{6.6.20}$$

and

$$l_1 = \frac{\sqrt{1+\lambda^2}}{2\lambda}(\lambda\omega, \omega), \qquad l_2 = \frac{\sqrt{1+\lambda_0^2}}{\lambda_0\sqrt{\omega_1^2+\omega_2^2}}(\lambda_0\omega^3, \omega^3),$$

$$l_3 = \frac{\sqrt{1+\lambda_0^2}}{\lambda_0\sqrt{\omega_1^2+\omega_3^2}}(\lambda_0\omega^2, \omega^2), \quad l_4 = \frac{\sqrt{1+\lambda_0^2}}{\lambda_0\sqrt{\omega_1^2+\omega_2^2}}(\lambda_0\omega^3, -\omega^3), \tag{6.6.21}$$

$$l_5 = \frac{\sqrt{1+\lambda_0^2}}{\lambda_0\sqrt{\omega_1^2+\omega_3^2}}(\lambda_0\omega^2, -\omega^2), \quad l_6 = \frac{\sqrt{1+\lambda^2}}{2\lambda}(\lambda\omega, -\omega),$$

respectively, where

$$\omega^2 = (\omega_3, 0, -\omega_1), \quad \omega^3 = (\omega_2, -\omega_1, 0), \tag{6.6.22}$$

provided that $\omega_1 \neq 0$. Hence, (6.6.13) is a hyperbolic system of conservation laws, and λ_2, λ_3, λ_4, λ_5 are linearly degenerate in the sense of P.D.Lax; while both λ_1 and λ_6 are genuinely nonlinear in the sense of P.D.Lax, moreover

$$\nabla\lambda_1(u)r_1(u) = d(1+\omega\hat{u})^{-3}\lambda^{-1}\left(1+\lambda^2\right)^{-\frac{1}{2}} = -\nabla\lambda_6(u)r_6(u). \tag{6.6.23}$$

For Ciarlet-Geymonat material, consider the Cauchy problem for system (6.6.9) of plane elastic waves with the initial data

$$t = 0: \ f = \overline{f}_0 + \varepsilon f_0(x), \ \ f_t = \varepsilon g_0(x), \tag{6.6.24}$$

or, equivalently, the Cauchy problem for system (6.6.13) with the initial data

$$t = 0: \ u = \left((\varepsilon f_0'(x))^T, (\varepsilon g_0(x))^T\right)^T, \tag{6.6.25}$$

where $\overline{f_0}$ is a constant vector, $f_0(x) \in C^2$ and $g_0(x) \in C^1$ are two given vector functions and $\varepsilon > 0$ is a small parameter. By Theorem 5.3, we have

Theorem 6.7. Suppose that there exists a constant $\mu > 0$ such that

$$\sup_{x \in \boldsymbol{R}} \left\{ (1+|x|)^{1+\mu} \left(|f_0''(x)| + |g_0'(x)| \right) \right\} < \infty. \tag{6.6.26}$$

Suppose furthermore that there exists a point $x_0 \in R$ such that

$$\triangle_-(x_0) < 0 \tag{6.6.27}$$

or

$$\triangle_+(x_0) < 0, \tag{6.6.28}$$

where

$$\triangle_-(x) = \frac{d}{2[2(a+b)+2(b+c)+d]} \left(\omega\sqrt{2(a+b)+2(b+c)+d}f_0''(x) + \omega g_0'(x) \right) \tag{6.6.29}$$

and

$$\triangle_+(x) = \frac{d}{2[2(a+b)+2(b+c)+d]} \left(\omega g_0''(x) - \omega\sqrt{2(a+b)+2(b+c)+d}f_0''(x) \right). \tag{6.6.30}$$

Then, there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the second order derivatives of the C^2 solution to the Cauchy problem (6.6.9) and (6.6.24) must blow up in a finite time and the life-span $\tilde{T}(\varepsilon)$ satisfies

$$\lim_{\varepsilon \to 0} \left(\varepsilon \tilde{T}(\varepsilon) \right)^{-1} = \max \left\{ \sup_{x \in \boldsymbol{R}} \{ - \triangle_- (x) \}, \sup_{x \in \boldsymbol{R}} \{ - \triangle_+ (x) \} \right\}. \tag{6.6.31}$$

□

On the other hand, noting that r_2, r_3, r_4 and r_5 are all constant vectors, by Theorem 5.2 and Remark 5.3 we can easily get the following.

Theorem 6.8. Suppose that (6.6.26) holds and

$$\omega\sqrt{1+\wedge^2(x)}f_0'(x) \equiv constant \quad \text{and} \quad \omega \wedge^{-1}(x)\sqrt{1+\wedge^2(x)}g_0(x) \equiv constant \tag{6.6.32}$$

where $\wedge(x) = \sqrt{2(a+b)+2(b+c)+d(1+\varepsilon\omega f_0'(x))^{-2}}$. Then, there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in [0, \varepsilon_0]$, the Cauchy problem (6.6.9) and (6.6.24) admits a unique global C^2 solution for all $t \in R$. □

Remark 6.4. For some other classical hyperelastic materials such as compressible Mooney-Rivlin material, Hadamard-Green material, neo-Hookean material, Burgess-Leginson material, etc. (see [Ci]), we can obtain similar results. □

§6.7. Nonlinear wave equation with scalar operators of higher order

L.Hörmander [Ho1] studied the following nonlinear wave equation with scalar operators of higher order in one space dimension

$$\sum_{j=0}^{m} c_j(\partial^{m-1}u)\partial_t^{m-j}\partial_x^j u = 0 \tag{6.7.1}$$

with Cauchy boundary condition

$$u - \varepsilon\varphi = O(t^m) \quad \text{as} \quad t \to 0, \tag{6.7.2}$$

where $u = u(t,x)$ is the unknown function, $c_j = c_j(\nu)$ $(j = 0, 1, \cdots, m)$ are C^∞ smooth functions of $\nu = (\nu_1, \cdots, \nu_m)^T$ and $c_0 \neq 0$, $\partial^{m-1}u = (\partial_t^{m-1}u, \partial_t^{m-2}\partial_x u, \cdots, \partial_x^{m-1}u)^T$, $\varphi \in C_0^\infty(R^2)$ and $\varepsilon > 0$ is a small parameter. Under the assumption that system is *genuinely nonlinear*, he proves that the C^∞ solution must blow up in a finite time and gives a limit formula on the life span.

Since he assumes that the equation is genuinely nonlinear and the initial data has a compact support, a general consideration is needed for the nonlinear wave equation (6.7.1). In this section, we shall introduce the concept of *weak linear degeneracy* and present a quite complete result on the global existence and the life span of C^m solution to the Cauchy problem (6.7.1)-(6.7.2) for small initial data with certain decay properties as $|x| \to \infty$.

Consider equation (6.7.1) and introduce

$$U = \partial^{m-1}u = \left(\partial_t^{m-1}u, \partial_t^{m-2}\partial_x u, \cdots, \partial_x^{m-1}u\right)^T. \tag{6.7.3}$$

Without loss of generality, we may suppose that $c_0 \equiv 1$.

Define

$$P(U;\xi) = \sum_{j=0}^{m} c_j(U)\xi_t^{m-j}\xi_x^j, \tag{6.7.4}$$

where $\xi = (\xi_t, \xi_x)$.

Definition 6.1. Equation (6.7.1) is *strictly hyperbolic*, if, for any given u on the domain under consideration, the polynomial $P(U;\xi)$ is strictly hyperbolic, i.e., there exist m real eigenvalues:

$$\lambda_1(U) < \cdots < \lambda_m(U) \tag{6.7.5}$$

such that

$$P(U;\xi) = \prod_{k=1}^{m}\left(\xi_t + \lambda_k(U)\xi_x\right). \tag{6.7.6}$$

□

Let

$$\xi_k(U) = \left(-\lambda_k(U), 1\right) \quad \text{and} \quad \Xi_k(U) = \left(\left(-\lambda_k(U)\right)^{m-1}, \cdots, -\lambda_k(U), 1\right)^T. \tag{6.7.7}$$

Definition 6.2. The k-th eigenvalue $\lambda_k(U)$ is *genuinely nonlinear*, if, for any given u on the domain under consideration, along the curve $U^{(k)} = U^{(k)}(s,U)$ passing through the point $U = \partial^{m-1}u$, defined by

$$\begin{cases} \frac{dU^{(k)}}{ds} = \Xi_k(U^{(k)}), \\ s = 0: \; U^{(k)} = U, \end{cases} \tag{6.7.8}$$

we have

$$\left.\frac{\partial P\left(U^{(k)}(s,U); \xi_k(U)\right)}{\partial s}\right|_{s=0} \neq 0, \quad \forall\, U; \tag{6.7.9}$$

The k-th eigenvalue $\lambda_k(U)$ is *linearly degenerate*, if

$$\left.\frac{\partial P\left(U^{(k)}(s,U);\xi_k(U)\right)}{\partial s}\right|_{s=0} \equiv 0; \quad \forall\, U. \tag{6.7.10}$$

Equation (6.7.1) is called to be *genuinely nonlinear* (resp. *linearly degenerate*), if all eigenvalues are genuinely nonlinear (resp. linearly degenerate). □

Another definition on the genuine nonlinearity and linear degeneracy can be given as follows.

Definition 6.3. The k-th eigenvalue $\lambda_k(U)$ is *genuinely nonlinear*, if, for any given u on the domain under consideration

$$\partial_\varepsilon P\left(U + \varepsilon \Xi_k\left(U\right);\, \xi_k\left(U\right)\right)\big|_{\varepsilon=0} \neq 0, \quad \forall\, U; \tag{6.7.11}$$

The k-th eigenvalue $\lambda_k(U)$ is *linearly degenerate*, if

$$\partial_\varepsilon P\left(U + \varepsilon \Xi_k\left(U\right);\, \xi_k\left(U\right)\right)\big|_{\varepsilon=0} \equiv 0, \quad \forall\, U. \tag{6.7.12}$$

If all eigenvalues are genuinely nonlinear (resp. linearly degenerate), equation (1.1) is said to be *genuinely nonlinear* (resp. *linearly degenerate*). □

It follows easily from (6.7.6)-(6.7.8) that

Lemma 6.1. Definition 6.2 is equivalent to Definition 6.3. Particularly, (6.7.9) is equivalent to (6.7.11), and they are all equivalent to

$$< \nabla\lambda_k(U),\, \Xi_k(U) >\neq 0, \quad \forall\, U; \tag{6.7.13}$$

while, (6.7.10) is equivalent to (6.7.12), and they are all equivalent to

$$< \nabla\lambda_k(U),\, \Xi_k(U) >\equiv 0, \quad \forall\, U. \tag{6.7.14}$$

□

As in Chapter 3, we can also define the concept of *weak linear degeneracy* for the k-th eigenvalue $\lambda_k = \lambda_k(U)$.

Definition 6.4. The k-th eigenvalue $\lambda_k = \lambda_k(U)$ is *weakly linearly* degenerate at a point $\bar{U}$ on the domain under consideration, if, along the curve $U^{(k)} = U^{(k)}(s, \bar{U})$ passing through the point $U = \bar{U}$, defined by

$$\begin{cases} \frac{dU^{(k)}}{ds} = \Xi_k(U^{(k)}), \\ s = 0 : \ U^{(k)} = \bar{U}, \end{cases} \tag{6.7.15}$$

we have

$$\left.\frac{\partial P\left(U^{(k)}\left(\tau, U^{(k)}\left(s, \bar{U}\right)\right) ; \xi_k\left(U^{(k)}\left(s, \bar{U}\right)\right)\right)}{\partial \tau}\right|_{\tau=0} \equiv 0, \quad \forall\ |s| \text{ small}, \tag{6.7.16}$$

where $U^{(k)} = U^{(k)}\left(\tau, U^{(k)}\left(s, \bar{U}\right)\right)$ is given by

$$\begin{cases} \frac{dU^{(k)}}{d\tau} = \Xi_k(U^{(k)}), \\ \tau = 0 : U^{(k)} = U^{(k)}(s, \bar{U}). \end{cases} \tag{6.7.17}$$

Equation (6.7.1) is called to be *weakly linearly degenerate* at $\bar{U}$, if all eigenvalues are weakly linearly degenerate at $\bar{U}$. □

Similar to Definition 6.3, another definition on the weak linear degeneracy is given as follows.

Definition 6.5. The k-th eigenvalue $\lambda_k = \lambda_k(U)$ is *weakly linearly degenerate* at a point $\bar{U}$, if, along the curve $U^{(k)} = U^{(k)}(s, \bar{U})$ defined by (6.7.15) we have

$$\partial_\varepsilon P\left(U^{(k)}\left(s, \bar{U}\right) + \varepsilon \Xi_k\left(U^{(k)}\left(s, \bar{v}\right)\right) ; \xi_k\left(U^{(k)}\left(s, \bar{v}\right)\right)\right)\Big|_{\varepsilon=0} \equiv 0, \quad \forall\ |s| \text{ small}. \tag{6.7.18}$$

Equation (6.7.1) is called to be *weakly linearly degenerate* at $\bar{U}$, if all eigenvalues are weakly linearly degenerate at $\bar{U}$. □

Similar to Lemma 6.1, by (6.7.6)-(6.7.7) and (6.7.15) we have

Lemma 6.2. Definition 6.4 is equivalent to Definition 6.5. Particularly, (6.7.16) is equivalent to (6.7.18), and they are all equivalent to

$$< \lambda_k(U),\ \Xi_k(U) > \equiv 0 \tag{6.7.19}$$

along $U^{(k)} = U^{(k)}(s, \bar{U})$ for small $|s|$, namely

$$\lambda_k\left(U^{(k)}\left(s, \bar{U}\right)\right) \equiv \lambda_k(\bar{U}), \quad \forall\ |s| \text{ small}, \tag{6.7.20}$$

where $U^{(k)} = U^{(k)}(s, \bar{U})$ is defined by (6.7.15). □

Obviously, it follows from the definition of linear degeneracy and definition of weak linear degeneracy that

Lemma 6.3. The k-th eigenvalue $\lambda_k(U)$ is linear degenerate if and only if it is weakly linearly degenerate at all points U on the domain under consideration. □

In this section, we will discuss the "small" solutions for equation (6.7.1), so we pay more attention whether eigenvalue $\lambda_k = \lambda_k(U)$ is weakly linearly degenerate at $U = 0$ or not. For simplicity, in what follows, we say the eigenvalue $\lambda_k = \lambda_k(U)$ is weakly linearly degenerate means that it is weakly linearly degenerate at $U = 0$, namely,

$$\lambda_k\left(U^{(k)}(s)\right) \equiv \lambda_k(0), \quad \forall\ |s| \text{ small}, \tag{6.7.21}$$

where $U^{(k)} = U^{(k)}(s)$ denotes $U^{(k)} = U^{(k)}(s, 0)$, namely, $U^{(k)} = U^{(k)}(s)$ satisfies

$$\begin{cases} \frac{dU}{ds} = \Xi_k(U), \\ s = 0:\ U = 0. \end{cases} \tag{6.7.22}$$

By a direct calculation, we have

Lemma 6.4. There exists an integer $\alpha_i \geq 0$ such that

$$\left.\frac{d^l P\left(U^{(i)}(s); (-\lambda_i(0), 1)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, \cdots, \alpha_i) \quad \text{but}$$

$$\left.\frac{d^{\alpha_i+1} P\left(U^{(i)}(s); (-\lambda_i(0), 1)\right)}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0 \tag{6.7.23}$$

if and only if

$$\left.\frac{d^l \lambda_i\left(U^{(i)}(s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, \cdots, \alpha_i) \quad \text{but} \quad \left.\frac{d^{\alpha_i+1} \lambda_i\left(U^{(i)}(s)\right)}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0; \tag{6.7.24}$$

moreover, we have

$$\left.\frac{d^{\alpha_i+1}P\left(U^{(i)}(s);\left(-\lambda_i\left(0\right),1\right)\right)}{ds^{\alpha_i+1}}\right|_{s=0}=\prod_{k\neq i}\left(\lambda_k\left(0\right)-\lambda_i\left(0\right)\right)\left.\frac{d^{\alpha_i+1}\lambda_i\left(U^{(i)}\left(s\right)\right)}{ds^{\alpha_i+1}}\right|_{s=0}. \tag{6.7.25}$$

Furthermore,

$$\left.\frac{d^lP\left(U^{(i)}\left(s\right);\left(-\lambda_i\left(0\right),1\right)\right)}{ds^l}\right|_{s=0}=0\quad(l=1,2,\cdots) \tag{6.7.26}$$

if and only if

$$\left.\frac{d^l\lambda_i\left(U^{(i)}\left(s\right)\right)}{ds^l}\right|_{s=0}=0\quad(l=1,2,\cdots). \tag{6.7.27}$$

□

In the case that (6.7.26) or (6.7.27) hold, we define $\alpha_i=+\infty$.

Definition 6.6. Equation (6.7.1) is critical, if equation (6.7.1) is not weakly linearly degenerate but (6.7.26) or (6.7.27) always hold for all $i=1,\cdots,m$, i.e., $\alpha_i=+\infty$ $(i=1,\cdots,n)$. □

Recall (6.7.3), i.e.,

$$U=(U_1,\cdots,U_m)^T=\partial^{m-1}u=(\partial_t^{m-1}u,\partial_t^{m-2}\partial_xu,\cdots,\partial_x^{m-1}u)^T. \tag{6.7.28}$$

Equation (6.7.1) can be then written as

$$\partial_tU_1+\sum_{j=1}^mc_j(U)\partial_xU_j=0$$

and together with the compatibility conditions

$$\partial_tU_{j+1}-\partial_xU_j=0,\quad 1\leq j\leq m-1$$

for any classical solution, i.e., C^m solution. Then we obtain

$$\partial_tU+a(U)\partial_xU=0, \tag{6.7.29}$$

where

$$a(U) = \begin{pmatrix} c_1(U) & c_2(U) & \cdots & c_{m-1}(U) & c_m(U) \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}. \tag{6.7.30}$$

The determinant of $a(U) - \lambda I$ is $\sum_{j=0}^{m}(-\lambda)^{(m-j)}c_j(U)$. Obviously, the roots of $a(U) - \lambda I$, i.e., the eigenvalues of system (6.7.29) are nothing but the eigenvalues of the polynomial $P = P(U;\xi)$: $\lambda_1(U), \cdots, \lambda_m(U)$. It follows that (6.7.6) holds. On the other hand, if we assume that equation (6.7.1) is strictly hyperbolic, then (6.7.5) implies that system (6.7.29) is also strictly hyperbolic. In the present situation, the right eigenvector $r_k = r_k(U)$ (corresponding to the eigenvalue $\lambda_k = \lambda_k(U)$) ($k = 1, \cdots, m$) can be chosen as $\Xi_k = \Xi_k(U)$, that is, $r_k = \Xi_k$. Thus, the definitions of genuine nonlinearity, linear degeneracy and weak linear degeneracy for equation (6.7.1) are nothing but the corresponding concepts for strictly quasilinear hyperbolic system (6.7.29). Therefore, all the results on quasilinear hyperbolic systems can be applied to equation (6.7.1).

To do so, we consider equation (6.7.1) with the following Cauchy boundary condition[2]

$$u - \varepsilon\varphi = O(t^m), \quad \text{as} \quad t \to 0, \tag{6.7.31}$$

where $\varphi \in C^m(\boldsymbol{R} \times \boldsymbol{R})$ is given and satisfies that there exists a constant $\mu > 0$ such that

$$\sup_{x\in\boldsymbol{R}} \left\{(1 + |x|)^{1+\mu} \left(\left|\partial^{m-1}\varphi(0,x)\right| + \left|\partial_x\partial^{m-1}\varphi(0,x)\right|\right)\right\} < \infty, \tag{6.7.32}$$

where $\partial^{m-1} = \left\{\partial_t^{m-j}\partial_x^{j-1}\right\}_{1\leq j\leq m}$.

[2] Usually, the Cauchy data for equation (6.7.1) should be given by the following form

$$t = 0: \ u = \varphi_0, \ u_t = \varphi_1, \ \cdots, \ u_t^{m-1} = \varphi_{m-1}.$$

Here, we use the form (6.7.31) only for the simplicity of statement. We can discuss the Cauchy problem for (6.7.1) with usual initial data with the above form in a similar way.

To apply the results on quasilinear hyperbolic systems, we must write $U = \sum_{k=1}^{m} v_k r_k(0)$ when $t = 0$. Let

$$Q_k = \prod_{l \neq k} \left(\partial_t + \lambda_l (0) \partial_x\right),$$

and note that if we take $U = r_j(0) = \Xi_j(0) = \left((-\lambda_j (0))^{m-1}, \cdots, -\lambda_j (0), 1\right)^T$ at $(0, x)$, then

$$Q_k u = \prod_{l \neq k} \left(\lambda_l (0) - \lambda_j (0)\right). \tag{6.7.33}$$

This is 0 if $j \neq k$. Hence we have when $t = 0$

$$v_k(x) \prod_{l \neq k} \left(\lambda_l (0) - \lambda_k (0)\right) = Q_k u = \varepsilon \prod_{l \neq k} \left(\partial_t + \lambda_l (0) \partial_x\right) \varphi. \tag{6.7.34}$$

First of all, by Theorem 3.1 we have

Theorem 6.9. Suppose that in a neighbourhood of $U = 0$, equation (6.7.1) is strictly hyperbolic and weakly linearly degenerate. Suppose furthermore that $c_j(U) \in C^2$ $(j = 1, \cdots, m)$ in a neighbourhood of $U = 0$ and $\varphi = \varphi(t, x)$ is a C^m function satisfying (6.7.32). Then there exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in [0, \varepsilon_0]$, the Cauchy problem (6.7.1) and (6.7.31) admits a unique global C^m solution $u = u(t, x)$ on $t \geq 0$. □

When equation is not weakly linearly degenerate, there exists a nonempty set $J \subseteq \{1, 2, \cdots, n\}$ such that $\lambda_i(u)$ is not weakly linearly degenerate if and only if $i \in J$.

Noting (6.7.21), for any fixed $i \in J$, either there exists an integer $\alpha_i \geq 0$ such that

$$\left.\frac{d^l \lambda_i \left(U^{(i)} (s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, \cdots, \alpha_i) \quad \text{but} \quad \left.\frac{d^{\alpha_i+1} \lambda_i \left(U^{(i)} (s)\right)}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0 \tag{6.7.35}$$

or

$$\left.\frac{d^l \lambda_i \left(U^{(i)} (s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, 2, \cdots), \tag{6.7.36}$$

where $U = U^{(i)}(s)$ is defined by (6.7.22). In the case that (6.7.36) holds, we define $\alpha_i = +\infty$.

By Theorem 3.2 we have

Theorem 6.10. Suppose that (6.7.5) holds and $c_j(U)$ $(j = 1, \cdots, m)$ are suitably smooth in a neighbourhood of $U = 0$. Suppose furthermore that $\varphi = \varphi(t,x) \in C^m(R \times R)$ and satisfies (6.7.32). Suppose finally that equation (6.7.1) is not weakly linearly degenerate and

$$\alpha = \min\{\alpha_i, i \in J\} < \infty, \tag{6.7.37}$$

where α_i is defined by (6.7.35)-(6.7.36). Let

$$J_1 = \{i \mid i \in J, \alpha_i = \alpha\}. \tag{6.7.38}$$

If there exists $i_0 \in J_1$ such that

$$v_{i_0}(x) = \prod_{l \neq i_0} (\lambda_l(0) - \lambda_{i_0}(0))^{-1} (\partial_t + \lambda_l(0)\partial_x)\varphi \not\equiv 0. \tag{6.7.39}$$

Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$ the m order derivatives of the C^m solution $u = u(t,x)$ to the Cauchy problem (6.7.1) and (6.7.31) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ of $u = u(t,x)$ satisfies

$$\lim_{\varepsilon \to 0} \left(\varepsilon^{\alpha+1}\tilde{T}(\varepsilon)\right) = C, \tag{6.7.40}$$

where

$$\begin{aligned} C^{-1} &= \max_{i \in J_1} \left\{ \sup_{x \in R} \left\{ -\frac{1}{\alpha!} \left. \frac{d^{\alpha+1}\lambda_i(U^{(i)}(s))}{ds^{\alpha+1}} \right|_{s=0} [v_i(x)]^\alpha v_i'(x) \right\} \right\} \\ &= \max_{i \in J_1} \left\{ \sup_{x \in R} \left\{ -\frac{1}{\alpha!} \prod_{l \neq i} (\lambda_l(0) - \lambda_i(0))^{-1} \times \right.\right. \\ &\qquad \left.\left. \left. \frac{d^{\alpha+1} P(U^{(i)}(s); (-\lambda_i(0), 1))}{ds^{\alpha+1}} \right|_{s=0} [v_i(x)]^\alpha v_i'(x) \right\} \right\}, \end{aligned} \tag{6.7.41}$$

in which $U = U^{(i)}(s)$ is defined by (6.7.22) and $v_i(x)$ is given by

$$v_i(x) = \prod_{l \neq i} (\lambda_l(0) - \lambda_i(0))^{-1} (\partial_t + \lambda_l(0)\partial_x)\varphi \Big|_{t=0}. \tag{6.7.42}$$

□

For the critical case, corresponding to Proposition 3.2, we have

Theorem 6.11. Suppose that (6.7.5) holds and $c_j(U)$ $(j = 1, \cdots, m)$ are suitably smooth in a neighbourhood of $U = 0$. Suppose furthermore that $\varphi(t, x) \in C^m(\mathbf{R} \times \mathbf{R})$ is a C^m function satisfying (6.7.32). Suppose finally that equation (6.7.1) is critical. Then, for any fixed integer $N \geq 1$ there exists $\varepsilon_0 = \varepsilon_0(N) > 0$ such that for any fixed $\varepsilon \in (0, \varepsilon_0]$,

$$\tilde{T}(\varepsilon) \geq C_N \varepsilon^{-N}. \tag{6.7.43}$$

□

Corollary 6.1. If equation (6.7.1) is genuinely nonlinear in a neighbourhood of $U = 0$, then, under the assumption that $c_j(U)$ $(j = 1, \cdots, m)$ are C^2 functions of U, for any nontrivial C^m initial data $\varepsilon\varphi(t, x)$ with compact support, the m order derivatives of the C^m solution to (6.7.1) and (6.7.31) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ satisfies

$$\lim_{\varepsilon \to 0} \left(\varepsilon \tilde{T}(\varepsilon)\right) = C_0, \tag{6.7.44}$$

where

$$C_0^{-1} = \max_{i \in \{1, \cdots, m\}} \left\{ \sup_{x \in \mathbf{R}} \{-\nabla\lambda_i \cdot r_i \,|_{U=0}\, v_i'(x)\} \right\} =$$
$$\max_{i \in \{1, \cdots, m\}} \left\{ \sup_{x \in \mathbf{R}} \left\{ -\prod_{l \neq i} (\lambda_l(0) - \lambda_i(0))^{-1} \left. \frac{dP\left(U^{(i)}(s); (-\lambda_i(0), 1)\right)}{ds} \right|_{s=0} v_i'(x) \right\} \right\}. \tag{6.7.45}$$

□

Remark 6.5. Corollary 6.1 gives the corresponding result given in [Hol]. □

Remark 6.6. If system (6.7.29) is non-strictly hyperbolic, then we have the corresponding results similar to those in Chapter 4 and Chapter 5. □

Remark 6.7. If we multiply all coefficients by the same function $c(U) \neq 0$, we see that the right-hand side of (6.7.41) is invariant. So it is unnecessary to assume that c_0 is normalized to be equal to 1 in our theory. □