Chapter 5

Homogeneous quasilinear hyperbolic systems

In Chapter 3 and Chapter 4 we systematically study the global existence and the blow-up phenomenon of C^1 solutions to Cauchy problem for general quasilinear hyperbolic system with small and decay initial data. However, the whole discussion is based on the existence of the normalized coordinates. Unfortunately, for the non-strictly hyperbolic case, in general we do not know if there exist the normalized coordinates, and even if the normalized coordinates exist, it is still very hard to check the hypotheses given in the normalized coordinates. Therefore, a consideration without the normalized coordinates is needed. Such a discussion for homogeneous quasilinear hyperbolic systems is carried out in this Chapter. Essentially restricting our system in such a way that each characteristic is either genuinely nonlinear or linearly degenerate in the sense of P.D.Lax, we only require the assumption that $\overline{\theta}$ (see (1.14)) is small instead of the hypothesis that θ (see (1.13)) is small, eliminate the use of the normalized coordinates and obtain more results including a sharp estimate on life span of C^1 solutions. In particular, we give a lower bound of life span of the C^1 solution to the Cauchy problem (1.1) and (1.7) for the small initial data with certain monotone properties. Moreover, by construct some examples, we illustrate two mechanisms of breakdown of the C^1 solutions to the homogeneous quasilinear hyperbolic systems, and the difference between diagonalizable and non-diagonalizable systems.

§5.1. Basic assumptions and preliminaries

Consider the following quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{5.1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with C^2 elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete system of left (resp. right) eigenvectors. For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{5.1.2}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{equivalently, } \det|r_{ij}(u)| \neq 0). \tag{5.1.3}$$

All $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \cdots, n)$ are supposed to have the same regularity as $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

Without loss of generality, we suppose that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n) \tag{5.1.4}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \cdots, n), \tag{5.1.5}$$

where δ_{ij} stands for the Kronecker's symbol.

In this Chapter, we suppose that on the domain under consideration, each eigenvalue of $A(u)$ has a constant multiplicity. Without loss of generality, we may suppose that

$$\lambda(u) \triangleq \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{5.1.6}$$

where $1 \le p \le n$. When $p = 1$, system (5.1.1) is strictly hyperbolic; while, when $p > 1$, (5.1.1) is a non-strictly hyperbolic system with characteristics with constant multiplicity.

We suppose furthermore that on the domain under consideration each multiple characteristic is *linearly degenerate* in the sense of P.D.Lax. Then, by (5.1.6), when $p > 1$ we have

$$\nabla\lambda(u)r_i(u) \equiv 0 \quad (i = 1, \cdots, p). \tag{5.1.7}$$

Remark 5.1. If system (5.1.1) can be written in the form of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

where $f(u) = (f_1(u), \cdots, f_n(u))^T$, then (5.1.6) implies (5.1.7) (see [Bo] or [Fr]). □

In the present situation, (2.2.9) simply reduces to

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u)\, v_j w_k \quad (i = 1, \cdots, n), \tag{5.1.8}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u))\, l_i(u) \nabla r_j(u)\, r_k(u). \tag{5.1.9}$$

Hence, we have

$$\beta_{ijk}(u) \equiv 0, \quad \forall\ i,\ k \in \{1, \cdots, p\}, \quad \forall\ j, \tag{5.1.10}$$

$$\beta_{iji}(u) \equiv 0, \quad \forall\ i \in \{p+1, \cdots, n\}, \quad \forall\ j. \tag{5.1.11}$$

On the other hand, (2.2.19) becomes

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u)\, w_j w_k \quad (i = 1, \cdots, n), \tag{5.1.12}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u))\, l_i(u) \nabla r_k(u)\, r_j(u) - \nabla\lambda_k(u)\, r_j(u)\, \delta_{ik} + (j|k)\}, \tag{5.1.13}$$

in which $(j|k)$ denotes all the terms obtained by changing j and k in the previous terms. Hence, noting (5.1.6)-(5.1.7), we have

$$\gamma_{ijk}(u) \equiv 0, \quad \forall\, i, \quad \forall\, j,\ k \in \{1, \cdots, p\}, \tag{5.1.14}$$

$$\gamma_{ijj}(u) \equiv 0, \quad \forall\, i, \quad \forall\, j \neq i, \quad j \in \{p+1, \cdots, n\} \tag{5.1.15}$$

and

$$\gamma_{iii}(u) = -\nabla\lambda_i(u)\, r_i(u), \quad \forall\, i \in \{p+1, \cdots, n\}. \tag{5.1.16}$$

For $i = p+1, \cdots, n$, when the i-th characteristic $\lambda_i(u)$ is linearly degenerate in the sense of P.D.Lax, we have

$$\gamma_{iii}(u) \equiv 0. \tag{5.1.17}$$

Moreover, we have (see (2.2.25)-(2.2.26))

$$\mathrm{d}\left[w_i\left(\mathrm{d}x - \lambda_i(u)\,\mathrm{d}t\right)\right] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u)\, w_j w_k \mathrm{d}t \wedge \mathrm{d}x, \tag{5.1.18}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}\left(\lambda_j(u) - \lambda_k(u)\right) l_i(u) \left[\nabla r_k(u)\, r_j(u) - \nabla r_j(u)\, r_k(u)\right]. \tag{5.1.19}$$

By (5.1.6), we have

$$\Gamma_{ijk}(u) \equiv 0, \quad \forall\, i, \quad \forall\, j,\ k \in \{1, \cdots, p\} \tag{5.1.20}$$

and

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall\, i, j. \tag{5.1.21}$$

On the other hand, by (5.1.18) we have

$$\frac{\partial w_i}{\partial t} + \frac{\partial\left(\lambda_i(u)\, w_i\right)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u)\, w_j w_k \quad (i = 1, \cdots, n). \tag{5.1.22}$$

Similar to Lemma 2.3, we have

Lemma 5.1. Suppose that $u = u(t,x)$ is a C^1 solution to system (5.1.1), τ_1 and τ_2 are two C^1 arcs which are never tangent to the i-th characteristic direction, and

D is the domain bounded by τ_1, τ_2 and two i-th characteristic curves L_i^- and L_i^+, see Figure 1. Then we have

$$\int_{\tau_1} |w_i\,(\mathrm{d}x - \lambda_i\,(u)\,\mathrm{d}t)| \leq \int_{\tau_2} |w_i\,(\mathrm{d}x - \lambda_i\,(u)\,\mathrm{d}t)| + \iint_D \left| \sum_{j,k=1}^{n} \Gamma_{ijk}\,(u)\,w_j w_k \right| \mathrm{d}t\mathrm{d}x, \tag{5.1.23}$$

where w_i and $\Gamma_{ijk}(u)$ are defined by (2.2.2) and (5.1.19) respectively. □

§5.2. Main results

Consider the Cauchy problem for system (5.1.1) with the initial data (1.7), i.e.,

$$t = 0: \quad u = \varphi(x). \tag{5.2.0}$$

The main results of the Chapter are the following four Theorems.

Theorem 5.1. Suppose that in a neighbourhood of $u = 0$, $A\,(u) \in C^2$, (5.1.1) is a hyperbolic system and (5.1.4)-(5.1.7) hold. Suppose furthermore that in a neighbourhood of $u = 0$, system (5.1.1) is linearly degenerate, namely, all characteristics are linearly degenerate in the sense of P.D.Lax. Suppose finally that $\varphi\,(x)$ is a C^1 vector function satisfying that there exists a constant $\mu > 0$ such that

$$\theta \triangleq \sup_{x \in \boldsymbol{R}} \left\{ (1 + |x|)^{1+\mu}\,|\varphi'(x)| \right\} < \infty. \tag{5.2.1}$$

Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, the Cauchy problem (5.1.1) and (5.2.0) admits a unique global C^1 solution $u = u\,(t, x)$ on $\boldsymbol{R} \times \boldsymbol{R}$. □

Theorem 5.2. Suppose that in a neighbourhood of $u = 0$, $A\,(u) \in C^2$, (5.1.1) is a hyperbolic system and (5.1.4)-(5.1.7) hold. Suppose furthermore that in a neighbourhood of $u = 0$,

$$\gamma_{ijk}\,(u) \equiv 0, \quad \forall\, i \in I, \quad \forall\, j,\ k \notin I, \tag{5.2.2}$$

where γ_{ijk} is defined by (5.1.13) and $I \subseteq \{1, \cdots, n\}$ is an index set such that $\lambda_i(u)$ is not linearly degenerate if and only if $i \in I$. If $\varphi(x)$ satisfies (5.2.1) and

$$l_i(\varphi(x))\varphi'(x) \equiv 0, \quad \forall\, x \in \boldsymbol{R}, \quad \forall\, i \in I, \tag{5.2.3}$$

where $l_i(u)$ denotes the i-th left eigenvector, then the conclusion of Theorem 5.1 holds. □

Remark 5.2. (5.2.2) is essentially T.P.Liu's hypothesis that "linear waves do not generate nonlinear waves" (see [Lu]), however, we do not ask $\lambda_i(u)$ to be genuinely nonlinear for $i \in I$. □

Remark 5.3. Noting (5.1.14)-(5.1.15), we have

$$\gamma_{ijk}(u) \equiv 0, \quad \forall\, i \in I, \quad \forall\, j,\ k \notin I \quad \text{and} \quad \lambda_j(u) \equiv \lambda_k(u). \tag{5.2.4}$$

Hence, in order to have (5.2.2), it suffices to check

$$\gamma_{ijk}(u) \equiv 0, \quad \forall\, i \in I, \quad \forall\, j,\ k \notin I \quad \text{and} \quad \lambda_j(u) \not\equiv \lambda_k(u). \tag{5.2.5}$$

For $i \in I$, j, $k \notin I$ with $\lambda_j(u) \not\equiv \lambda_k(u)$, by (5.1.13) we have

$$\gamma_{ijk}(u) = \frac{1}{2}\left(\lambda_j(u) - \lambda_k(u)\right) l_i(u)\left[\nabla r_k(u)\, r_j(u) - \nabla r_j(u)\, r_k(u)\right]. \tag{5.2.6}$$

It follows from (5.2.6) that if all the right eigenvectors corresponding to linearly degenerate characteristics can be chosen to be constant vectors, then (5.2.2) is automatically satisfied. □

Theorem 5.3. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, (5.1.1) is a hyperbolic system and (5.1.4)-(5.1.7) hold. Suppose furthermore that there exists a nonempty index set $J \subseteq \{1, \cdots, n\}$ such that in a neighbourhood of $u = 0$, when $i \in J$, $\lambda_i(u)$ is genuinely nonlinear; while, when $i \notin J$, $\lambda_i(u)$ is linearly degenerate. Suppose finally that $\phi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying that there exists a constant $\mu > 0$ such that

$$\sup_{x \in \boldsymbol{R}} \left\{(1 + |x|)^{1+\mu}\, |\psi'(x)|\right\} < \infty. \tag{5.2.7}$$

If there exist $i_0 \in J$ and $x_0 \in \boldsymbol{R}$ such that

$$\operatorname{sgn}\left(\nabla\lambda_{i_0}(0) r_{i_0}(0)\right) \cdot l_{i_0}(0)\, \psi'(x_0) < 0, \tag{5.2.8}$$

where $l_{i_0}(u)$ (resp. $r_{i_0}(u)$) stands for the i_0-th left (resp. right) eigenvector corresponding to the eigenvalue $\lambda_{i_0}(u)$, then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$ the first order derivatives of the C^1 solution $u = u(t, x)$ to the Cauchy problem (5.1.1) and (5.2.0) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ of $u = u(t, x)$ satisfies

$$\lim_{\varepsilon \to 0} \left(\varepsilon \tilde{T}(\varepsilon) \right) = \left[\max_{i \in J} \sup_{x \in \boldsymbol{R}} \{ -\nabla \lambda_i(0) r_i(0) \cdot l_i(0) \psi_i'(x) \} \right]^{-1}. \tag{5.2.9}$$

□

Remark 5.4. Obviously, if $\psi(x)$ satisfies

$$|\psi(x)| \longrightarrow const. \quad \text{as} \quad |x| \to \infty, \tag{5.2.10}$$

then the condition that there exists $i_0 \in J$ such that

$$l_{i_0}(0)\psi(x) \not\equiv const. \tag{5.2.11}$$

implies that there exists $x_0 \in \boldsymbol{R}$ such that (5.2.8) holds. In this case, the hypothesis (5.2.8) can be replaced by (5.2.11). In particular, if the system (5.1.1) is strictly hyperbolic and $\psi(x)$ has a compact support, Theorem 5.3 gives the same result as in [Jo], [Lu] and [Ho1], but we only ask the C^1 regularity for the solution and the initial data; moreover, T.P.Liu's hypothesis that "linear waves do not generate nonlinear waves" is not required. □

Theorem 5.4. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, (5.1.1) is a hyperbolic system and (5.1.4)-(5.1.7) hold. Suppose furthermore that there exists an index set $J \subseteq \{1, \cdots, n\}$ such that in a neighbourhood of $u = 0$, $\lambda_i(u)$ is genuinely nonlinear if and only if $i \in J$. Suppose finally that $\varphi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying (5.2.7). If

$$\operatorname{sgn}(\nabla \lambda_i(0) r_i(0)) \cdot l_i(0)\psi'(x) \geq 0, \quad \forall\, x \in \boldsymbol{R}, \quad \forall\, i \in J, \tag{5.2.12}$$

then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in [0, \varepsilon_0]$, the life span $\tilde{T}(\varepsilon)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (5.1.1) and (5.2.0) satisfies

$$\tilde{T}(\varepsilon) \geq C_* \varepsilon^{-2}, \tag{5.2.13}$$

where C_* is a positive constant independent of ε. □

Remark 5.5. In Theorem 5.4, the index set J might be empty, in which case (5.2.12) is eliminated. On the other hand, when $i \notin J$, $\lambda_i(u)$ might be neither genuinely nonlinear nor linearly degenerate, or might be linearly degenerate. □

Remark 5.6. Under the hypotheses of Theorem 5.4, by (5.2.13), we have

$$\lim_{\varepsilon \to 0} \left(\varepsilon \tilde{T}(\varepsilon) \right)^{-1} = 0, \tag{5.2.14}$$

namely,

$$\lim_{\varepsilon \to 0} \left(\varepsilon \tilde{T}(\varepsilon) \right) = \infty. \tag{5.2.15}$$

This is compatible with (5.2.9) because (5.2.12) holds. □

§5.3. Two mechanisms of breakdown of C^1 solutions

In this section, by constructing some quasilinear hyperbolic systems with constant characteristics, we illustrate two mechanisms of breakdown of the classical solutions to the homogeneous quasilinear hyperbolic systems, and the difference between diagonalizable and non-diagonalizable quasilinear hyperbolic systems.

It is well-known that Cauchy problem for a hyperbolic system of linear partial differential equations with smooth coefficients always admits a unique global classical solution on the whole domain, provided that the initial data is suitably smooth. This situation is due to the "linearity" of system. However, the situation for quasilinear hyperbolic equations is quite different. Generally speaking, in the quasilinear case the classical solution to the Cauchy problem exists only locally in time and singularities may occur after a finite time, even for the sufficiently smooth and small initial data.

Consider the following Cauchy problem for quasilinear hyperbolic system in diagonal form

$$\frac{\partial u_i}{\partial t} + \lambda_i(u) \frac{\partial u_i}{\partial x} = 0 \quad (i = 1, \cdots, n), \tag{5.3.1}$$

$$t = 0: \quad u = \varphi(x), \tag{5.3.2}$$

where $\lambda_i(u)$ $(i = 1, \cdots, n)$ are C^1 function satisfying

$$\lambda_1(u) < \cdots < \lambda_n(u), \tag{5.3.3}$$

and $\varphi(x)$ is a C^1 vector function with bounded C^1 norm.

B.L.Rozdesstvenskii and A.P.Sidorenko [RS] and D.Hoff [Hf] proved

Proposition 5.1. If

$$\frac{\partial \lambda_i(\varphi_1(\alpha_1), \cdots, \varphi_{i-1}(\alpha_{i-1}), \varphi_i(\alpha_i), \varphi_{i+1}(\alpha_{i+1}), \cdots, \varphi_n(\alpha_n))}{\partial \alpha_i} \geq 0 \quad (i = 1, \cdots, n), \tag{5.3.4}$$

where α_j $(j = 1, \cdots, n)$ satisfying $\alpha_1 \geq \cdots \geq \alpha_n$, then the Cauchy problem (5.3.1) and (5.3.2) admits a unique global C^1 solution on $t \geq 0$. □

Corollary 5.1. In particular, if system (5.3.1) is linear degenerate in the sense of P.D.Lax, i.e.,

$$\frac{\partial \lambda_i(u)}{\partial u_i} \equiv 0 \quad (i = 1, \cdots, n), \tag{5.3.5}$$

then the Cauchy problem (5.3.1) and (5.3.2) always has a unique global C^1 solution on $t \geq 0$. □

On the other hand, if system (5.3.1) is genuinely nonlinear in the sense of P.D.Lax, i.e.,

$$\frac{\partial \lambda_i(u)}{\partial u_i} \neq 0 \quad (i = 1, \cdots, n), \tag{5.3.6}$$

then (5.3.4) is necessary to guarantee the global existence on $t \geq 0$ of the C^1 solution to the Cauchy problem (5.3.1) and (5.3.2) (see [K4]).

By the argument mentioned above, we observe that the breakdown of the C^1 solution to the Cauchy problem (5.3.1) and (5.3.2) is due to the "non-constant" of eigenvalues $\lambda_i(u)$ $(i = 1, \cdots, n)$ of system (5.3.1), because in this case, eigenvectors of system (5.3.1) can be chosen as constant vectors, i.e., left eigenvectors are

$$l_i(u) = \left(0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0\right) \quad (i = 1, \cdots, n) \tag{5.3.7}$$

and the right eigenvectors read

$$r_i(u) = \left(0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0\right)^T \quad (i = 1, \cdots, n). \tag{5.3.8}$$

They do not produce the singularity of solutions. The situation for non-diagonal systems is quite different, the conclusion of Proposition 5.1 might be false.

Example 5.1. Consider the following quasilinear system[1]

$$u_t + A(u)u_x = 0, \tag{5.3.9}$$

where

$$A(u) = \begin{pmatrix} -1 & 0 & 0 \\ -e^{u_2} & 0 & 0 \\ -2e^{-u_2} & 0 & 1 \end{pmatrix}, \tag{5.3.10}$$

It is easy to see that (5.3.9) is a strictly hyperbolic system with three distinct real eigenvalues:

$$\lambda_1(u) \equiv -1, \quad \lambda_2(u) \equiv 0, \quad \lambda_3(u) \equiv 1. \tag{5.3.11}$$

Clearly, $\lambda_i(u)$ $(i = 1, 2, 3)$ are linearly degenerate in the sense of P.D.Lax. Right eigenvectors can be chosen as follows

$$r_1(u) = \begin{pmatrix} 1 \\ e^{u_2} \\ e^{-u_2} \end{pmatrix}, \quad r_2(u) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad r_3(u) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.3.12}$$

Consider the Cauchy problem(5.3.9) and (5.3.2). The solution is given by

$$\begin{cases} u_1(t,x) = \varphi_1(x+t), \\ u_2(t,x) = -\ln\left(e^{-\varphi_2(x)} + \varphi_1(x) - \varphi_1(x+t)\right), \\ u_3(t,x) = \varphi_3(x-t) + G(t,x), \end{cases} \tag{5.3.13}$$

where

$$G(t,x) = 2\int_0^t \left[e^{-\varphi_2(\tau+x-t)} + \varphi_1(\tau+x-t) - \varphi_1(x+\tau)\right]\varphi_1'(x+\tau)d\tau. \tag{5.3.14}$$

In particular, we take

$$\varphi_1(x) = \sin x, \quad \varphi_2(x) \equiv \varphi_3(x) \equiv 0, \tag{5.3.15}$$

[1] Examples 5.1-5.2 are provided by Ta-tsien Li and Fa-gui Liu.

then, it follows from (5.3.13) that

$$u_2(t,x) = -\ln\left(1+\sin x - \sin(x+t)\right). \tag{5.3.16}$$

From (5.3.16), we see that

$$u_2(t,x) \to \infty \quad \text{as} \quad (t,x) \to \left(\frac{\pi}{2}, 0\right). \tag{5.3.17}$$

(5.3.17) implies that the Cauchy problem (5.3.9) and (5.3.2) with (5.3.10) and (5.3.15) does not have any C^1 solution on $t \geq 0$. □

Example 5.2. In system (5.3.9), we take

$$A(u) = \begin{pmatrix} -1 & 0 & 0 \\ -(1+u_2^2) & 0 & 0 \\ u_2 & 0 & 1 \end{pmatrix}. \tag{5.3.18}$$

In this case, (5.3.9) is still a strictly hyperbolic system with three distinct real eigenvalues:

$$\lambda_1(u) \equiv -1, \quad \lambda_2(u) \equiv 0, \quad \lambda_3(u) \equiv 1. \tag{5.3.19}$$

Moreover, right eigenvectors can be taken as follows

$$r_1(u) = \begin{pmatrix} 1 \\ 1+u_2^2 \\ -\frac{1}{2}u_2 \end{pmatrix}, \quad r_2(u) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad r_3(u) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.3.20}$$

In the present situation, the solution to the Cauchy problem (5.3.9) and (5.3.2) is given by

$$\begin{cases} u_1(t,x) = \varphi_1(x+t), \\ u_2(t,x) = \mathrm{tg}\left(\varphi_1(x+t) - \varphi_1(x)\right), \\ u_3(t,x) = -\int_0^t \varphi_1'(\tau+x)\mathrm{tg}\left(\varphi_1(\tau+x-t) - \varphi_1(\tau+x)\right) d\tau. \end{cases} \tag{5.3.21}$$

A discussion similar to Example 5.1 can be carried out, and a result on the breakdown of the C^1 solution (5.3.21) can be obtained. □

Example 5.3 (Jeffrey's model). As in [Je], we consider system (5.3.9) with

$$A(u)=\begin{bmatrix} -\cosh(2u_2) & 0 & -\sinh(2u_2) \\ \cosh u_2 & 0 & \sinh u_2 \\ \sinh(2u_2) & 0 & \cosh(2u_2) \end{bmatrix}. \tag{5.3.22}$$

For this model, we can discuss similarly and obtain the corresponding conclusion. □

By these examples mentioned above, we see that the breakdown of the C^1 solutions to the Cauchy problem (5.3.9) and (5.3.2) is due to "non-constant" eigenvectors instead of eigenvalues, because in this case, eigenvalues are constants, certainly, they are linearly degenerate in the sense of P.D.Lax. Thus, we have

Conclusion 5.1. There exist two kinds of mechanisms to produce the singularity of C^1 solutions to quasilinear hyperbolic systems. One is due to the "non-constant" of eigenvalues. This case generally corresponds to the formation of envelope of characteristics of the same family (see §3.8 and [K4]). The other is owing to the "non-constant" of eigenvectors. For this case, the envelope of characteristics of the same family might never appear (for instance, see Example 5.1-5.3). It is very possible that the two mechanisms occur simultaneously. □

Conclusion 5.2. The C^1 solutions to non-diagonalizable quasilinear hyperbolic systems with linearly degenerate characteristics (even for constant characteristics) may blow up in a finite time. This is quite different from quasilinear hyperbolic systems in diagonal form. □

§5.4. Global existence of C^1 solution —Proof of Theorems 5.1-5.2

For simplicity and without loss of generality, we may suppose that

$$\begin{gathered} \varphi(0)=0, \\ 0<\lambda(0)<\lambda_{p+1}(0)<\cdots<\lambda_n(0), \end{gathered} \tag{5.4.1}$$

where $\lambda(u) \triangleq \lambda_1(u) \equiv \cdots \equiv \lambda_p(u)$.

By the existence and uniqueness of local C^1 solution to the Cauchy problem (see Chapter 1 in [LY]), in order to prove Theorem 5.1 it suffices to establish a uniform *a priori* estimate on the C^0 norm of u and u_x on the existence domain of the C^1 solution $u = u(t,x)$.

By (5.4.1), there exist positive constants δ and δ_0 so small that

$$\begin{gathered} \lambda(0) > \delta_0, \\ \lambda_{p+1}(u) - \lambda_i(v) \geq 4\delta_0, \quad \forall\, |u|,\, |v| \leq \delta \quad (i = 1, \cdots, p), \\ \lambda_{j+1}(u) - \lambda_j(v) \geq 4\delta_0, \quad \forall\, |u|,\, |v| \leq \delta \quad (j = p+1, \cdots, n-1), \end{gathered} \tag{5.4.2}$$

and

$$|\lambda_i(u) - \lambda_i(v)| \leq \frac{\delta_0}{2}, \quad \forall\, |u|,\, |v| \leq \delta \quad (i = 1, \cdots, n). \tag{5.4.3}$$

For the time being, we assume that on the existence domain of the C^1 solution $u = u(t,x)$

$$|u(t,x)| \leq \delta. \tag{5.4.4}$$

After Lemma 5.4, we will explain that this hypothesis is reasonable.

By (5.4.1) and (5.4.4), on the existence domain of C^1 solution we have

$$0 < \lambda(u) \triangleq \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{5.4.5}$$

provided that $\delta > 0$ is suitably small.

Similar to §4.4, for any fixed $T > 0$, let

$$D_-^T = \{(t,x) \,|\, 0 \leq t \leq T,\ x \leq -t\}, \tag{5.4.6}$$

$$D_0^T = \{(t,x) \,|\, 0 \leq t \leq T,\ -t \leq x \leq (\lambda(0) - \delta_0)t\}, \tag{5.4.7}$$

$$D^T = \{(t,x) \,|\, 0 \leq t \leq T,\ (\lambda(0) - \delta_0)t \leq x \leq (\lambda_n(0) + \delta_0)t\}, \tag{5.4.8}$$

$$D_+^T = \{(t,x) \,|\, 0 \leq t \leq T,\ x \geq (\lambda_n(0) + \delta_0)t\} \tag{5.4.9}$$

and for $i = 1, \cdots, n$,

$$\begin{aligned} D_i^T = \{&(t,x) \,|\, 0 \leq t \leq T, \\ &-[\delta_0 + \eta(\lambda_i(0) - \lambda(0))]t \leq x - \lambda_i(0)t \leq [\delta_0 + \eta(\lambda_n(0) - \lambda_i 0))]t\}, \end{aligned} \tag{5.4.10}$$

where $\eta > 0$ is suitably small.

Noting (5.4.1)-(5.4.2), when $\eta > 0$ is suitably small, we have

$$D_1^T \equiv D_2^T \equiv \cdots \equiv D_p^T \triangleq D_m^T, \tag{5.4.11}$$

$$D_i^T \cap D_j^T = \emptyset, \quad \forall\, i \neq j, \quad i, j \in \{m, p+1, \cdots, n\} \tag{5.4.12}$$

and

$$D_m^T \bigcup \bigcup_{i=p+1}^{n} D_i^T \subset D^T. \tag{5.4.13}$$

Let

$$W\left(D_\pm^T\right) = \max_{i=1,\cdots,n} \|\left(1+|x|\right)^{1+\mu} w_i(t,x)\|_{L^\infty\left(D_\pm^T\right)}, \tag{5.4.14}$$

$$W\left(D_0^T\right) = \max_{i=1,\cdots,n} \|\left(1+t\right)^{1+\mu} w_i(t,x)\|_{L^\infty\left(D_0^T\right)}, \tag{5.4.15}$$

$$W_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T\setminus D_i^T} \left(1+|x-\lambda_i(0)\,t|\right)^{1+\mu} |w_i(t,x)| \tag{5.4.16}$$

and

$$W_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\in \mathbb{R}}} |w_i(t,x)|, \tag{5.4.17}$$

where $D_i^T(t)$ $(t \geq 0)$ denotes the t-section of D_i^T:

$$D_i^T(t) = \left\{(\tau, x) \mid \tau = t, (\tau, x) \in D_i^T\right\}. \tag{5.4.18}$$

Noting (5.4.11), we get

$$D_1^T(t) \equiv D_2^T(t) \equiv \cdots \equiv D_p^T(t) \triangleq D_m^T(t). \tag{5.4.19}$$

Therefore, by (5.4.1) and (5.4.11) we have

$$\begin{aligned} W_\infty^c(T) = \; & \max\{\max_{i=1,\cdots,p} \sup_{(t,x)\in D^T\setminus D_m^T} \left(1+|x-\lambda(0)\,t|\right)^{1+\mu} |w_i(t,x)|, \\ & \max_{i=p+1,\cdots,n} \sup_{(t,x)\in D^T\setminus D_i^T} \left(1+|x-\lambda_i(0)\,t|\right)^{1+\mu} |w_i(t,x)|\}. \end{aligned} \tag{5.4.20}$$

By the definition of D_i^T and D^T, it is easy to get the following

Lemma 5.2. For each $i = 1, \cdots, n$, on the domain $D^T \backslash D_i^T$ we have

$$ct \le |x - \lambda_i(0)\, t| \le Ct, \tag{5.4.21}$$

$$cx \le |x - \lambda_i(0)\, t| \le Cx, \tag{5.4.22}$$

where c and C are positive constants independent of T. □

Similar to Lemma 3.3, we have

Lemma 5.3. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and (5.1.1) is a hyperbolic system. Suppose furthermore that $\varphi(x)$ is a C^1 vector function satisfying (5.2.1) and the first equality in (5.4.1). There exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (5.1.1) and (5.2.0) there exist positive constants k_1 and k_2 independent of θ and T, such that the following uniform *a priori* estimates hold:

$$W\left(D_{\pm}^T\right) \le k_1 \theta \tag{5.4.23}$$

and

$$W\left(D_0^T\right) \le k_2 \theta. \tag{5.4.24}$$

□

Remark 5.7. By the definition of D_0^T, for any given $(t, x) \in D_0^T$ we have

$$|x| \le \max\{1,\ \lambda(0) - \delta_0\}\, t. \tag{5.4.25}$$

Introduce

$$W\left(D_-^T \cup D_0^T\right) = \max_{i=1,\cdots,n} \|(1 + |x|)^{1+\mu}\, w_i(t, x)\|_{L^\infty(D_-^T \cup D_0^T)}. \tag{5.4.26}$$

By (5.4.23)-(5.4.25), we obtain

$$W\left(D_-^T \cup D_0^T\right) \le k_3 \theta, \tag{5.4.27}$$

where k_3 is a positive constant independent of θ and T. Thus, under the hypotheses of Lemma 5.3, on any given existence domain $0 \le t \le T$ of the C^1 solution

$u = u(t, x)$ to the Cauchy problem (5.1.1) and (5.2.0), for any fixed $t \in [0, T]$,

$$|w(t,x)| \longrightarrow 0 \quad \text{as} \quad |x| \to \infty. \tag{5.4.28}$$

Moreover, $w(t, x)$ is integrable in space, i.e.,

$$\int_{-\infty}^{\infty} |w_i(t,x)| dx \quad (i = 1, \cdots, n) \tag{5.4.29}$$

make sense for any fixed $t \in [0, T]$. □

The following Lemma can be found in Appendix 3.

Lemma 5.4. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, (5.1.1) is a hyperbolic system and (5.1.4)-(5.1.6) hold. Suppose furthermore that $\varphi(x)$ is a C^1 vector function satisfying (5.2.1) and the first equality in (5.4.1). There exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (5.1.1) and (5.2.0) there exist positive constants k_4 and k_5 independent of θ and T, such that the following uniform *a priori* estimates hold:

$$||u(t,x)||_{C^0([0,T]\times R)} \le k_4 \theta \tag{5.4.30}$$

and

$$\int_{L_j} |w_i(t,x)| dx \le k_5 \theta, \tag{5.4.31}$$

where L_j stands for arbitrary j-th characteristic: $x = x_j(t)$ ($t \in [0, T]$) defined by

$$\frac{dx}{dt} = \lambda_j(u(t, x(t))), \tag{5.4.32}$$

in which $j \in \{p+1, \cdots, n\}$ if $i \in \{1, \cdots, p\}$; while $j \neq i$ if $i \in \{p+1, \cdots, n\}$; moreover, k_5 is independent of L_j. □

When system is quasilinear strictly hyperbolic, Lemma 5.4 is due to [Sc] and [Ho2].

Taking θ_0 so small that

$$k_4 \theta_0 \le \frac{\delta}{2}, \tag{5.4.33}$$

we obtain from (5.4.30) that

$$\|u(t,x)\|_{C^0([0,T]\times R)} \le k_4\theta \le k_4\theta_0 \le \frac{\delta}{2}. \tag{5.4.34}$$

This implies that the hypothesis (5.4.4) is reasonable.

Lemma 5.5. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, (5.1.1) is a hyperbolic system and (5.1.4)-(5.1.7) hold. Suppose furthermore that $\varphi(x)$ is a C^1 vector function satisfying (5.2.1) and the first equality in (5.4.1). There exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (5.1.1) and (5.2.0) there exists a positive constant k_6 independent of θ and T, such that the following uniform *a priori* estimate holds:

$$W_\infty^c(T) \le k_6\theta. \tag{5.4.35}$$

□

Proof. Let

$$\begin{aligned}\tilde{W}_1(T) &= \max\{\max_{i=1,\cdots,p}\ \max_{j\in\{p+1,\cdots,n\}}\ \sup_{\tilde{C}_j}\int_{\tilde{C}_j}|w_i(t,x)|dt,\\ &\quad \max_{i=p+1,\cdots,n}\ \max_{j\ne i}\ \sup_{\tilde{C}_j}\int_{\tilde{C}_j}|w_i(t,x)|dt\},\end{aligned} \tag{5.4.36}$$

where, when $i \in \{1,\cdots,p\}$, $\tilde{C}_j$ denotes any given j-th characteristic in D_m^T (where $j \in \{p+1,\cdots,n\}$); while, when $i \in \{p+1,\cdots,n\}$, $\tilde{C}_j$ stands for any given j-th characteristic in D_i^T (where $j \ne i$).

By (5.4.31) we have

$$\tilde{W}_1(T) \le k_5\theta. \tag{5.4.37}$$

Using (5.4.37) and repeating the proof of (5.4.26) in [LK], we get (5.4.35) directly. Q.E.D.

As in [LK], we have

Lemma 5.6. Under the assumptions of Lemma 5.5, suppose furthermore that in a neighbourhood of $u = 0$, system (5.1.1) is linearly degenerate. Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (5.1.1) and (5.2.0), there exists a positive constant k_7 independent of θ and T, such that the following uniform *a priori* estimate holds:

$$W_\infty(T) \leq k_7\theta. \tag{5.4.38}$$

□

Using Lemmas 5.3-5.6 and completely repeating the proof of Theorems 3.1-3.2 in [LK], we can easily obtain Theorem 5.1 and Theorem 5.2. The details are omitted.

§5.5. Blow-up phenomenon and life span of C^1 solution — Proof of Theorems 5.3-5.4

Under the hypotheses of Theorem 5.3, Lemma 5.3-5.5 are still valid and can be stated as the following three lemmas.

Lemma 5.7. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and (5.1.1) is a hyperbolic system. Suppose furthermore that $\varphi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying (5.2.7). There exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (5.1.1) and (5.2.0) there exist positive constants k_1 and k_2 independent of ε and T, such that the following uniform *a priori* estimates hold:

$$W\left(D_\pm^T\right) \leq k_1\varepsilon \tag{5.5.1}$$

and

$$W\left(D_0^T\right) \leq k_2\varepsilon. \tag{5.5.2}$$

□

Lemma 5.8. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, (5.1.1) is a hyperbolic system and (5.1.4)-(5.1.6) hold. Suppose furthermore that $\varphi = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function safisfying (5.2.7). There exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (5.1.1) and (5.2.0) there exist positive constants k_3 and k_4 independent of ε and T, such that the following uniform *a priori* estimates hold:

$$||u(t,x)||_{C^0([0,T]\times \boldsymbol{R})} \leq k_3\varepsilon \tag{5.5.3}$$

and

$$\int_{L_j} |w_i(t,x)|dt \leq k_4\varepsilon, \tag{5.5.4}$$

where L_j stands for arbitrary j-th characteristic, in which $j \in \{p+1, \cdots, n\}$ if $i \in \{1, \cdots, p\}$; while $j \neq i$ if $i \in \{p+1, \cdots, n\}$; moreover, k_4 is independent of L_j. □

Lemma 5.9. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, (5.1.1) is a hyperbolic system and (5.1.4)-(5.1.7) hold. Suppose furthermore that $\varphi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function safisfying (5.2.7). There exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (5.1.1) and (5.2.0) there exist positive constants k_5 and k_6 independent of ε and T, such that the following uniform *a priori* estimates hold:

$$W_\infty^c(T) \leq k_5\varepsilon \tag{5.5.5}$$

and

$$\tilde{W}_1(T) \leq k_6\varepsilon. \tag{5.5.6}$$

□

Proof of Theorem 5.3. It suffices to prove (5.2.9). Obviously, it follows from (5.2.8) that the right-hand side of (5.2.9) is a positive number. We denote this number by M_0. Noting the fact that $\lambda_i(u)$ is linear degenerate when $i \notin J$, we have

$$\begin{aligned} M_0 &\triangleq \left[\max_{i\in J} \sup_{x\in R} \{-\nabla\lambda_i(0)r_i(0)l_i(0)\psi'(x)\}\right]^{-1} \\ &= \left[\max_{i\in\{1,\cdots,n\}} \sup_{x\in R} \{-\nabla\lambda_i(0)r_i(0)l_i(0)\psi'(x)\}\right]^{-1}. \end{aligned} \tag{5.5.7}$$

Moreover, it follows from (5.2.7)-(5.2.8) that there exist an index $i_* \in J$ and a point $x_* \in R$ such that

$$M_0 = [-\nabla\lambda_{i_*}(0)r_{i_*}(0)l_{i_*}(0)\psi'(x_*)]^{-1}. \tag{5.5.8}$$

We first estimate $w_i(t,x)$ $(i = 1, \cdots, p)$.

Let

$$W_\infty^m(T) = \max_{i=1,\cdots,p} \sup_{\substack{0\le t\le T \\ x\in R}} |w_i(t,x)|. \tag{5.5.9}$$

For each $i = 1, \cdots, p$, integrating (5.1.12) along the i-th characteristic, noting (5.1.7) and (5.1.14)-(5.1.16), and using Lemma 5.2, we get

$$\begin{aligned} |w_i(t,x)| \le\ & c_1 \Big\{W\left(D_\pm^T\right) + W\left(D_0^T\right) + (W_\infty^c(T))^2 + \\ & \tilde{W}_1(T)W_\infty^c(T) + W_\infty^c(T)W_\infty^m(T)\Big\}, \end{aligned} \tag{5.5.10}$$

henceforth c_j $(j = 1, 2)$ will denote positive constants independent of ε, (t,x) and T. Then, using (5.5.1)-(5.5.2) and (5.5.5)-(5.5.6), from (5.5.10) we obtain

$$W_\infty^m(T) \le c_2\varepsilon. \tag{5.5.11}$$

(5.5.11) gives an estimate of $w_i(t,x)$ $(i = 1, \cdots, p)$. This estimate implies that $w_i(t,x)$ $(i = 1, \cdots, p)$ remain bounded on any given existence domain $0 \le t \le T$ of the C^1 solution to the Cauchy problem (5.1.1) and (5.2.0).

Hence in what follows, it suffices to consider $w_i(t,x)$ $(i = p+1, \cdots, n)$.

On any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (5.1.1) and (5.2.0), we consider equation (5.1.12) again along the i-th characteristic $x = x_i(t,y)$ $(i = p+1, \cdots, n)$ passing through an arbitrary fixed point $(0,y)$ on x-axis. Noting (5.1.14)-(5.1.15), we may write (5.1.12) as

$$\frac{dw_i}{d_it} = a_0(t;i,y)w_i^2 + a_1(t;i,y)w_i + a_2(t;i,y), \quad \forall\, i \in \{p+1, \cdots, n\}, \tag{5.5.12}$$

where

$$a_0(t;i,y) = \gamma_{iii}(u), \quad a_1(t;i,y) = \sum_{j\neq i} (\gamma_{iij}(u) + \gamma_{iji}(u))\, w_j,$$
$$a_2(t;i,y) = \sum_{j,k\neq i} \gamma_{ijk}(u) w_j w_k, \tag{5.5.13}$$

in which $u = u\,(t, x_i(t,y))$ and $w_j = w_j\,(t, x_i(t,y))$ $(j = 1, \cdots, n)$.

Noting (5.1.14)-(5.1.16) and the fact that $i \in \{p+1, \cdots, n\}$, and using Lemma 5.2 and Lemma 5.7-5.9, by (5.5.13) we have

$$a_0(t;i,y) = \gamma_{iii}(0) + O(\varepsilon) = -\nabla\lambda_i(0) r_i(0) + O(\varepsilon), \quad \forall\, t \in [0,T], \tag{5.5.14}$$

$$\int_0^T |a_1(t;i,y)|dt \le C_1 \left\{ W(D_{\pm}^T) + W(D_0^T) + W_\infty^c(T) + \tilde{W}_1(T) \right\} \le C_2 \varepsilon \tag{5.5.15}$$

and

$$\begin{aligned}\int_0^T |a_2(t;i,y)|dt \; &\le \; C_3 \left\{ \left(W\,(D_{\pm}^T) + W(D_0^T)\right)^2 + (W_\infty^c(T))^2 + W_\infty^c(T)\tilde{W}_1(T) \right\} \\ &\le \; C_4 \varepsilon^2,\end{aligned} \tag{5.5.16}$$

provided that $\varepsilon > 0$ is small enough, henceforth C_j $(j = 1, 2, \cdots)$ will denote positive constants independent of ε, y and T. Hence,

$$K(i,y) \triangleq \int_0^T |a_2(t;i,y)|ds \cdot \exp\left(\int_0^T |a_1(t;i,y)|dt \right) \le C_5 \varepsilon^2, \tag{5.5.17}$$

provided that $\varepsilon > 0$ is small enough.

(I) Upper bound of the life span — Estimate on $\overline{\lim\limits_{\varepsilon\to 0}} \left(\varepsilon \tilde{T}(\varepsilon)\right)$

Without loss of generality, we may suppose that

$$-\nabla\lambda_{i_*}(0) r_{i_*}(0) > 0. \tag{5.5.18}$$

Otherwise, replacing u by $-u$ we can always realize (5.5.18).

By (5.1.7) we observe that $J \cap \{1, \cdots, p\} = \emptyset$, and then we see that $i_* \in \{p+1, \cdots, n\}$. Thus, it follows from (5.5.14) that

$$a_0(t; i_*, x_*) > 0, \quad \forall\, t \in [0,T], \tag{5.5.19}$$

provided that $\varepsilon > 0$ is small. Noting (5.5.8) and (5.5.18), we have

$$\begin{aligned} w_{i_*}(0,x_*) &= \varepsilon l_{i_*}(\varepsilon\psi(x_*))\,\psi'(x_*) \\ &= \varepsilon l_{i_*}(0)\psi'(x_*) + O\left(\varepsilon^2\right) > 0, \end{aligned} \tag{5.5.20}$$

provided that $\varepsilon > 0$ is suitably small. Then it follows from (5.5.17) and (5.5.20) that

$$w_{i_*}(0,x_*) > K(i_*,x_*), \tag{5.5.21}$$

provided that $\varepsilon > 0$ is suitably small. We now consider equation (5.5.12) along the characteristic $x = x_{i_*}(t,x_*)$. Noting (5.5.19)-(5.5.21), we observe that Lemma 2.1 can be applied to the initial value problem for (5.5.12) with the initial condition

$$t = 0: \ w_{i_*} = w_{i_*}(0,x_*). \tag{5.5.22}$$

It follows from (2.1.4) that

$$(w_{i_*}(0,x_*) - K(i_*,x_*))\int_0^T a_0(t;i_*,x_*)dt \cdot \exp\left(-\int_0^T |a_1(t;i_*,x_*)|dt\right) < 1, \tag{5.5.23}$$

provided that $\varepsilon > 0$ is suitably small.

Taking $T = \tilde{T}(\varepsilon) - 1$ in (5.5.23) and noting (5.5.14)-(5.5.15), (5.5.17), (5.5.20) and (5.5.8), from (5.5.23) we obtain

$$\overline{\lim_{\varepsilon\to 0}}\left(\varepsilon\tilde{T}(\varepsilon)\right) \leq M_0. \tag{5.5.24}$$

This gives an upper bound of the life span $\tilde{T}(\varepsilon)$.

(II) Lower bound of the life span — Estimate on $\underline{\lim}_{\varepsilon\to 0}\left(\varepsilon\tilde{T}(\varepsilon)\right)$

By (5.5.24), in order to prove (5.2.9) it remains to show

$$\underline{\lim_{\varepsilon\to 0}}\left(\varepsilon\tilde{T}(\varepsilon)\right) \geq M_0. \tag{5.5.25}$$

To prove (5.5.25), it suffices to show that, for any given constant M satisfying that

$$0 < M < M_0 \tag{5.5.26}$$

we have

$$\tilde{T}(\varepsilon) > M\varepsilon^{-1} \tag{5.5.27}$$

for small $\varepsilon > 0$, where M_0 is defined by (5.5.7).

To do so, it suffices to establish a uniform *a priori* estimate on the C^1 norm of the C^1 solution $u = u(t,x)$ on any given existence domain $[0,\ T] \times R$ with $T \le M\varepsilon^{-1}$. A uniform *a priori* estimate on the C^0 norm of $u = u(t,x)$ has been established in Lemma 5.8, hence it remains to establish a uniform *a priori* estimate on the C^0 norm of the first derivatives of $u = u(t,x)$, namely, a uniform *a priori* estimate on the C^0 norm of $w_i = w_i(t,x)\,(i = 1,\cdots,n)$. By (5.5.11), it suffices to estimate w_i $(i = p+1,\cdots,n)$. To estimate w_i $(i = p+1,\cdots,n)$, we still consider equation (5.5.12) along the i-th characteristic $x = x_i(t,y)$ passing through an arbitrary fixed point $(0,y)$ on x-axis. However we will make use of Lemma 2.2 instead of Lemma 2.1.

Without loss of generality, we may assume that

$$w_i(0,y) = \varepsilon l_i\,(\varepsilon\psi(y))\,\psi'(y) \ge 0. \tag{5.5.28}$$

Otherwise, replacing w_i by $-w_i$ we can draw the same conclusion. Noting (5.5.3), (5.1.16), (5.5.7) and (5.5.26), when $\varepsilon > 0$ is small enough we have

$$\gamma_{iii}(u)w_i(0,y) < \frac{M_0^{-1} + M^{-1}}{2}\varepsilon \tag{5.5.29}$$

on the i-th characteristic $x = x_i(t,y)$. Thus we get

$$w_i(0,y)\int_0^T a_0^+(t;i,y)dt \le \frac{M_0^{-1} + M^{-1}}{2}T\varepsilon \le \frac{M_0^{-1} + M^{-1}}{2}M < 1, \tag{5.5.30}$$

where $a_0^+(t;i,y) = \max\{a_0(t;i,y),\ 0\}$. In (5.5.30) we have made use of the fact that $T \le M\varepsilon^{-1}$. Noting (5.5.14) and the fact that $T \le M\varepsilon^{-1}$, we obtain

$$\int_0^T |a_0(t;i,y)|dt \le C_6 T \le C_6 M\varepsilon^{-1} \le C_7\varepsilon^{-1}. \tag{5.5.31}$$

By (5.5.15) and (5.5.17), it follows from (5.5.30)-(5.5.31) that (2.1.7)-(2.1.8) hold for small $\varepsilon > 0$. Therefore, using Lemma 2.2 we get

$$(w_i(T, x_i(T, y)))^{-1} \geq (w_i(0, y) + K(i, y))^{-1} - \int_0^T a_0^+(t; i, y)dt \times \exp\left(\int_0^T |a_1(t; i, y)|dt\right), \quad \text{if} \quad w_i(T, x_i(T, y)) > 0 \tag{5.5.32}$$

and

$$|w_i(T, x_i(T, y))|^{-1} \geq (K(i, y))^{-1} - \int_0^T |a_0(t; i, y)|dt \cdot \exp\left(\int_0^T |a_1(t; i, y)|dt\right), \quad \text{if} \quad w_i(T, x_i(T, y)) > 0. \tag{5.5.33}$$

Noting (5.5.15), (5.5.17) and (5.5.30)-(5.5.31), from (5.5.32)-(5.5.33) we obtain

$$(w_i(T, x_i(T, y)))^{-1} \geq \frac{1}{2}(w_i(0, y) + K(i, y))^{-1}\left(1 - \frac{M_0^{-1} + M^{-1}}{2}M\right), \quad \text{if } w_i(T, x_i(T, y)) > 0 \tag{5.5.34}$$

and

$$|w_i(T, x_i(T, y))|^{-1} \geq \frac{1}{2}(K(i, y))^{-1}, \quad \text{if} \quad w_i(T, x_i(T, y)) < 0, \tag{5.5.35}$$

provided that $\varepsilon > 0$ is small enough. For $w_i(t, x_i(t, y))$ ($\forall\, t \in [0, T]$) we have the same estimate. Thus, on the strip $0 \leq t \leq T$ we get the following uniform *a priori* estimate

$$|w_i(t, x)| \leq C_8\varepsilon \quad (i = p + 1, \cdots, n), \quad \forall\, t \in [0, T], \quad \forall\, x \in R, \tag{5.5.36}$$

where $T \leq M\varepsilon^{-1}$. Combining (5.5.11) and (5.5.36), we obtain

$$||w(t, x)||_{C^0[0,T]\times R} \leq C_9\varepsilon, \tag{5.5.37}$$

where $T \leq M\varepsilon^{-1}$. (5.5.37) implies that (5.5.27) holds for small $\varepsilon > 0$, and then (5.5.25) holds. (5.5.25) gives a lower bound of the life span $\tilde{T}(\varepsilon)$.

Combining (5.5.24) and (5.5.25) yields (5.2.9). The proof is completed. Q.E.D.

Proof of Theorem 5.4. Theorem 5.4 can be proved in a way similar to the proof of (5.5.27). In what follows, we only point out the essentially different part in the proof.

Under the hypotheses of Theorem 5.4, Lemma 5.7-5.9 are still valid. Moreover, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t,x)$, (5.5.11) and (5.5.14)-(5.5.17) (in which $i = p+1, \cdots, n$) also hold when $\varepsilon_0 > 0$ is suitably small. In order to prove Theorem 5.4, it suffices to show that there exist a small constant $\varepsilon_0 > 0$ and a positive constant C_* independent of ε such that for any given $\varepsilon \in [0, \varepsilon_0]$, the Cauchy problem (5.1.1) and (5.2.0) admits a unique C^1 solution $u = u(t,x)$ on the strip $0 \leq t \leq C_* \varepsilon^{-2}$. To do so, by (5.5.3) and (5.5.11), it suffices to establish a uniform *a priori* estimate on the C^0 norm of w_i $(i = p+1, \cdots, n)$ on any given existence domain $0 \leq t \leq T$ with $T \leq C_* \varepsilon^{-2}$. In order to estimate w_i $(i = p+1, \cdots, n)$, we still use equation (5.5.12) satisfied by w_i along the i-th characteristic $x = x_i(t,y)$.

Noting that

$$w_i(0,y) = \varepsilon l_i(\varepsilon\psi(y))\psi'(y) \quad (i = 1, \cdots, n), \tag{5.5.38}$$

we have

$$|w_i(0,y) - \varepsilon l_i(0)\psi'(y)| \leq C_{10}\varepsilon^2 \quad (i = 1, \cdots, n), \quad \forall\, y \in \mathbf{R}. \tag{5.5.39}$$

We first ask the positive constant C_* to satisfy

$$0 < C_* \leq \left[2(C_5 + 2C_{10}) \max_{i \in J} |\nabla\lambda_i(0) r_i(0)|\right]^{-1} \triangleq \overline{C}, \tag{5.5.40}$$

where C_5 and C_{10} are two positive constants given by (5.5.17) and (5.5.39) respectively.

Case I: $i \in J_L \triangleq \{j \in \{p+1, \cdots, n\} \mid \lambda_j(u)$ is linearly degenerate$\}$. In this case, $\lambda_i(u)$ is linearly degenerate. Let

$$W_L(T) = \max_{i \in J_L} \sup_{\substack{0 \leq t \leq T \\ x \in \mathbf{R}}} |w_i(t,x)|. \tag{5.5.41}$$

Integrating (5.1.12) along the i-th characteristic and noting (5.1.14)-(5.1.15) and (5.1.17), we have

$$\begin{aligned} |w_i(t,x)| \leq\ & C_{11} \Big\{ W(D_\pm^T) + W(D_0^T) + (W_\infty^c(T))^2 + \tilde{W}_1(T)\, W_\infty^c(T) + \\ & W_\infty^c(T) W_L(T) \Big\}, \quad \forall\, (t,x) \in [0,T] \times \mathbf{R}, \quad \forall\, i \in J_L. \end{aligned} \tag{5.5.42}$$

Using (5.5.1)-(5.5.2) and (5.5.5)-(5.5.6), on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ we obtain from (5.5.42) that

$$W_L(T) \le C_{12}\varepsilon. \tag{5.5.43}$$

Case II: $i \in J$. In this case, $\lambda_i(u)$ is genuinely nonlinear. As before, without loss of generality, we may suppose that

$$\nabla\lambda_i(0)r_i(0) > 0. \tag{5.5.44}$$

Otherwise, replacing u by $-u$, we can always realize (5.5.44).

In this case, by (5.2.12) we have

$$l_i(0)\psi'(y) \ge 0, \quad \forall\, y \in \boldsymbol{R}. \tag{5.5.45}$$

Moreover, it follows from (5.5.14) that

$$a_0^+(t;i,y) \triangleq \max\{a_0(t;i,y),\ 0\} \equiv 0, \tag{5.5.46}$$

provided that $\varepsilon_0 > 0$ is suitably small.

If

$$w_i(0,y) = \varepsilon l_i(\varepsilon\psi(y))\psi'(y) \ge 0, \tag{5.5.47}$$

then (2.1.7) always holds. On the other hand, noting (5.5.14)-(5.5.15), (5.5.17), (5.5.40) and the fact that $T \le C_*\varepsilon^{-2}$, we see that, when $\varepsilon_0 > 0$ is small enough, (2.1.8) holds. Then, using (5.5.46), (5.5.14)-(5.5.15), (5.5.17) and (5.5.40), by Lemma 2.2 we get

$$(w_i(T,x_i(T,y)))^{-1} \ge (w_i(0,y) + K(i,y))^{-1}, \quad \text{if} \quad w_i(T,x_i(T,y)) > 0 \tag{5.5.48}$$

and

$$|w_i(T,x_i(T,y))|^{-1} \ge \frac{1}{2}(K(i,y))^{-1}, \quad \text{if} \quad w_i(T,x_i(T,y)) < 0, \tag{5.5.49}$$

provided that $\varepsilon_0 > 0$ is small enough, where we have made use of the fact that $T \le C_*\varepsilon^{-2}$.

If

$$w_i(0,y) = \varepsilon l_i(\varepsilon\psi(y))\psi'(y) < 0, \tag{5.5.50}$$

then, by (5.5.39) and (5.5.45) we have

$$|w_i(0,y)| \le 2C_{10}\varepsilon^2. \tag{5.5.51}$$

Let

$$\tilde{w}_i(t) = -w_i\left(t, x_i(t,y)\right). \tag{5.5.52}$$

It follows from (5.5.12) that

$$\frac{d\tilde{w}_i(t)}{dt} = -a_0(t;i,y)\tilde{w}_i^2 + a_1(t;i,y)\tilde{w}_i - a_2(t;i,y). \tag{5.5.53}$$

Noting (5.5.14), (5.5.44), (5.5.50) and (5.5.51), we get

$$\tilde{a}_0^+(t;i,y) = \max\{-a_0(t;i,y),0\} = \nabla\lambda_i(u)r_i(u) > 0, \tag{5.5.54}$$

$$\tilde{w}_i(0) = -w_i(0,y) > 0 \tag{5.5.55}$$

and

$$\tilde{w}_i(0) < 2C_{10}\varepsilon^2, \tag{5.5.56}$$

provided that $\varepsilon_0 > 0$ is small enough. On the other hand, using (5.5.14)-(5.5.17), (5.5.40), (5.5.56) and the fact that $T \le C_*\varepsilon^{-2}$, we observe that, when $\varepsilon_0 > 0$ is suitably small, (2.1.7)-(2.1.8) hold. Then, we can apply Lemma 2.2 to the initial value problem (5.5.53) and (5.5.55). It follows from (2.1.9) and (2.1.10) that

$$\left(\tilde{w}_i(T)\right)^{-1} \ge \frac{1}{4}\left(\tilde{w}_i(0) + K(i,y)\right)^{-1}, \quad \text{if} \quad \tilde{w}_i(T) > 0 \tag{5.5.57}$$

and

$$|\tilde{w}_i(T)|^{-1} \ge \frac{1}{2}\left(K(i,y)\right)^{-1}, \quad \text{if} \quad \tilde{w}_i(T) < 0, \tag{5.5.58}$$

provided that $\varepsilon_0 > 0$ is small enough, where we have made use of (5.5.40) and the fact that $T \le C_*\varepsilon^{-2}$. Thus, noting (5.5.17) and (5.5.38), from (5.5.48)-(5.5.49) and (5.5.57)-(5.5.58) we get immediately

$$|w_i(T, x_i(T,y))| \le C_{13}\varepsilon, \quad \forall\, i \in J,$$

where $T \le C_* \varepsilon^{-2}$.

Similarly, for $w_i(t, x_i(t, y))$ ($\forall\, t \in [0, T]$, $\forall\, i \in J$) we have the same estimate

$$|w_i(t, x_i(t, y))| \le C_{14}\varepsilon, \quad \forall\, t \in [0, T], \quad \forall\, i \in J, \tag{5.5.59}$$

where $T \le C_* \varepsilon^{-2}$.

Let

$$W_G(T) = \max_{i \in J} \sup_{\substack{0 \le t \le T \\ x \in \mathbf{R}}} |w_i(t, x)|.$$

Noting the fact that $(0, y)$ is an arbitrary point on x-axis, by (5.5.59) we have

$$W_G(T) \le C_{14}\varepsilon, \tag{5.5.60}$$

where $T \le C_* \varepsilon^{-2}$.

Case III: $i \in J_0 \triangleq \{p+1, \cdots, n\} \setminus (J \cup J_L)$. In this case, it follows that

$$\nabla\lambda_i(0) r_i(0) = 0, \quad \forall\, i \in J_0. \tag{5.5.61}$$

On any given existence domain of the C^1 solution $u = u(t, x)$, by (5.5.3) we have

$$|\gamma_{iii}(u)| \le C_{15}\varepsilon. \tag{5.5.62}$$

Integrating (5.5.12) along the i-th characteristic $x = x_i(t, y)$ and noting (5.5.62) and (5.5.38), we obtain

$$\begin{aligned} |w_i(t, x_i(t, y))| \quad \le \quad & |w_i(0, y)| + C_{15}\varepsilon \int_0^t w_i^2 d\tau + \\ & \int_0^t |a_1(t; i, y)||w_i| dt + \int_0^t |a_2(t; i, y)| dt \\ \le \quad & C_{16}\varepsilon + C_{15}\varepsilon \int_0^t w_i^2 d\tau + \\ & \int_0^t |a_1(t; i, y)||w_i| dt + \int_0^t |a_2(t; i, y)| dt. \end{aligned} \tag{5.5.63}$$

Let

$$W_0^i(t) = \sup_{\substack{0 \le \tau \le t \\ y \in \mathbf{R}}} |w_i(\tau, x_i(\tau, y))|. \tag{5.5.64}$$

It follows from (5.5.59) that

$$\begin{aligned} |w_i(t, x_i(t, y))| \leq\ & C_{16}\varepsilon + C_{15}\varepsilon \textstyle\int_0^t \left(W_0^i(\tau)\right)^2 d\tau + \\ & W_0^i(t) \textstyle\int_0^t |a_1(\tau; i, y)| d\tau + \int_0^t |a_2(\tau; i, y)| d\tau. \end{aligned} \tag{5.5.65}$$

Noting the fact that $(0, y)$ is an arbitrary point on x-axis and using (5.5.15)-(5.5.16), we get

$$W_0^i(t) \leq C_{17}\varepsilon + C_{18}\varepsilon \int_0^t \left(W_0^i(\tau)\right)^2 d\tau. \tag{5.5.66}$$

It follows from (5.5.66) that

$$W_0^i(t) \leq 2C_{17}\varepsilon, \quad \forall\, t \in \left[0, \left(2C_{17}C_{18}\varepsilon^2\right)^{-1}\right]. \tag{5.5.67}$$

Let

$$W_0(t) = \max_{i \in J_0} \left\{W_0^i(t)\right\}. \tag{5.5.68}$$

It follows from (5.5.67) that

$$W_0(T) \leq C_{19}\varepsilon, \tag{5.5.69}$$

where T satisfies that $T \leq C^*\varepsilon^{-2}$, in which $C^* = (2C_{17}C_{18})^{-1}$.

Furthermore, the constant C_* is required to satisfy

$$0 < C_* \leq C^*.$$

Then, from (5.5.11), (5.5.43), (5.5.60) and (5.5.69) it follows that

$$W_\infty(T) \leq W_\infty^m(T) + W_L(T) + W_G(T) + W_0(T) \leq C_{20}\varepsilon, \tag{5.5.70}$$

where T satisfies that $T \leq C_*\varepsilon^{-2}$, in which C_* satisfies that $0 < C_* \leq \min\left\{\overline{C}, C^*\right\}$.

Thus, (5.5.70) gives a uniform *a priori* estimate on $w = w(t, x)$ on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t, x)$, where T satisfies that $T \leq C_*\varepsilon^{-2}$. Hence, the proof of Theorem 5.4 is finished. Q.E.D.