Chapter 4

Quasilinear non-strictly hyperbolic systems

In Chapter 3, we discussed the global existence and the blow-up phenomenon, particularly the life span and the breakdown behaviour of classical solutions to Cauchy problem for quasilinear strictly hyperbolic systems with small and decay initial data. This chapter aims to generalize the result presented in Chapter 3 to the case that system (1.1) might be non-strictly hyperbolic.

§4.1. Generalized null condition

Consider quasilinear hyperbolic system (1.1), where we assume that the eigenvalues $\lambda_i(u)$ and left (resp. right) eigenvectors $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) of $A(u)$ have the same regularity as $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, and (1.4)-(1.6) holds. However, we do not require system (1.1) must be strictly hyperbolic.

Without loss of generality, we may suppose that

$$\lambda_0 \triangleq \lambda_1(0) = \cdots = \lambda_p(0) < \lambda_{p+1}(0) < \cdots < \lambda_n(0). \tag{4.1.1}$$

When $p = 1$, system (1.1) is strictly hyperbolic in a neighbourhood of $u = 0$; while, when $p > 1$, (1.1) is non-strictly hyperbolic.

Rewrite (1.1) as

$$u_t + A(0)u_x = \tilde{A}(u)u_x + B(u), \tag{4.1.2}$$

where

$$\tilde{A}(u) = A(0) - A(u).$$

Definition 4.1. System (1.1) satisfies the *null condition,* if each small plane wave solution $u = u(s)$ $(u(0) = 0)$, where $s = ax + bt$ (a, b constants), to the linearized system

$$u_t + A(0)\, u_x = 0 \tag{4.1.3}$$

is always a solution to system (1.1) or (4.1.2). □

Similar to the strictly hyperbolic case, we have (see [LZK1])

Lemma 4.1. The property that system (1.1) satisfies the null condition or not is invariant under any invertible linear transformation $u = Q\tilde{u}$, where Q is a nonsingular matrix with constant elements. □

By Lemma 4.1, without loss of generality we may suppose that

$$A(0) = \text{diag}\,\{\lambda_1(0), \lambda_2(0), \cdots, \lambda_n(0)\}\,. \tag{4.1.4}$$

Then, system (4.1.3) simply reduces to the following system in diagonal form:

$$\frac{\partial u_i}{\partial t} + \lambda_i(0)\,\frac{\partial u_i}{\partial x} = 0 \quad (i = 1, \cdots, n)\,, \tag{4.1.5}$$

the general solution of which can be expressed as

$$u_i = u_i\,(x - \lambda_i(0)\,t) \quad (i = 1, \cdots, n)\,, \tag{4.1.6}$$

where $u_i(\cdot)$ stands for an arbitrary C^1 function of a single variable for each $i = 1, \cdots, n$. Hence, by (4.1.1), each plane wave solutin $u = u(s)$ $(u(0) = 0)$ to system

(4.1.5) must be in the following form: either

$$u = \sum_{h=1}^{p} u_h(s)\, e_h \quad (s = x - \lambda_0 t) \tag{4.1.7}$$

or there exists an index $j \in \{p+1, \cdots, n\}$ such that

$$u = u_j(s)\, e_j \quad (s = x - \lambda_j(0)\, t), \tag{4.1.8}$$

where $e_i = \left(0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0\right)^T$ and $u_i(s)$ is a C^1 function of s with $u_i(0) = 0$ $(i = 1, \cdots, n)$.

Thus, under hypothesis (4.1.4), system (1.1) (or (4.1.2)) satisfies the null condition if and only if for any given small C^1 functions $u_i(s)$ with $u_i(0) = 0$ $(i = 1, \cdots, n)$,

$$\tilde{A}\left(\sum_{h=1}^{p} u_h(s)\, e_h\right) \sum_{h=1}^{p} u'_h(s)\, e_h \equiv 0, \tag{4.1.9}$$

$$\tilde{A}(u_j(s)\, e_j)\, u'_j(s)\, e_j \equiv 0 \quad (j = p+1, \cdots, n), \tag{4.1.10}$$

$$B\left(\sum_{h=1}^{p} u_h(s) e_h\right) \equiv 0 \tag{4.1.11}$$

and

$$B(u_j(s) e_j) \equiv 0 \quad (j = p+1, \cdots, n). \tag{4.1.12}$$

It follows from (4.1.9)-(4.1.10) that

$$\tilde{A}\left(\sum_{h=1}^{p} u_h e_h\right) e_i \equiv 0 \quad (i = 1, \cdots, p), \quad \forall\, |u_h| \text{ small} \quad (h = 1, \cdots, p) \tag{4.1.13}$$

and

$$\tilde{A}(u_j e_j)\, e_j \equiv 0, \quad \forall\, |u_j| \text{ small} \quad (j = p+1, \cdots, n). \tag{4.1.14}$$

By the definition of $\tilde{A}(u)$, (4.1.13) and (4.1.14) are equivalent to

$$A\left(\sum_{h=1}^{p} u_h e_h\right) e_i \equiv \lambda_0 e_i \quad (i = 1, \cdots, p), \quad \forall\, |u_h| \text{ small} \quad (h = 1, \cdots, p) \tag{4.1.15}$$

and

$$A(u_j e_j)\, e_j \equiv \lambda_j(0)\, e_j, \quad \forall\, |u_j| \text{ small} \quad (j = p+1, \cdots, n) \tag{4.1.16}$$

respectively. Hence we have

Lemma 4.2. Under hypotheses (4.1.1) and (4.1.4), system (1.1) satisfies the null condition if and only if

$$\lambda_i\left(\sum_{h=1}^{p} u_h e_h\right) \equiv \lambda_0 \quad (i=1,\cdots,p), \quad \forall\,|u_h| \text{ small} \quad (h=1,\cdots,p), \tag{4.1.17}$$

$$\lambda_j\left(u_j e_j\right) \equiv \lambda_j(0), \quad \forall\,|u_j| \text{ small} \quad (j=p+1,\cdots,n), \tag{4.1.18}$$

$$r_i\left(\sum_{h=1}^{p} u_h e_h\right) \equiv e_i \quad (i=1,\cdots,p), \quad \forall\,|u_h| \text{ small} \quad (h=1,\cdots,p), \tag{4.1.19}$$

$$r_j\left(u_j e_j\right) \equiv e_j, \quad \forall\,|u_j| \text{ small} \quad (j=p+1,\cdots,n), \tag{4.1.20}$$

$$B\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \quad \forall\,|u_h| \text{ small} \quad (h=1,\cdots,p) \tag{4.1.21}$$

and

$$B(u_j e_j) \equiv 0, \quad \forall\,|u_j| \text{ small} \quad (j=p+1,\cdots,n). \tag{4.1.22}$$

□

Definition 4.2. System (1.1) satisfies the *generalized null condition*, if there exists an invertible smooth transformation $u = u(\tilde{u})$ $(u(0)=0)$ such that the system for $\tilde{u}$ satisfies the null condition. □

Definition 4.3. If there exists an invertible smooth transformation $u = u(\tilde{u})$ $(u(0)=0)$ such that in $\tilde{u}$-space

$$\tilde{r}_i\left(\sum_{h=1}^{p} \tilde{u}_h e_h\right) \equiv e_i \quad (i=1,\cdots,p), \quad \forall\,|\tilde{u}_h| \text{ small} \quad (h=1,\cdots,p) \tag{4.1.23}$$

and

$$\tilde{r}_j\left(\tilde{u}_j e_j\right) \equiv e_j, \quad \forall\,|\tilde{u}_j| \text{ small} \quad (j=p+1,\cdots,n), \tag{4.1.24}$$

then the transformation is called the *normalized transformation*, and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1,\cdots,\tilde{u}_n)$ are called the *normalized variables* or *normalized coordinates*. □

Remark 4.1. The assumption that system (1.1) satisfies the generalized null condition implies the existence of the normalized transformation. □

Remark 4.2. If system (1.1) is strictly hyperbolic, then there always exists the normalized transformation (see [LZK1]). In the case that system (1.1) might be non-strictly hyperbolic, in §4.6 we will give some conditions to guarantee the existence of the normalized transformation. □

Definition 4.4. The i-th characteristic $\lambda_i(u)$ is *weakly linearly degenerate*, if there exists the normalized transformation and in the normalized coordinates

$$\lambda_i\left(\sum_{h=1}^{p} u_h e_h\right) \equiv \lambda_0, \quad \forall\, |u_h| \text{ small} \quad (h=1,\cdots,p), \quad \text{when } i \in \{1,\cdots,p\}; \tag{4.1.25}$$

$$\lambda_i(u_i e_i) \equiv \lambda_i(0), \quad \forall\, |u_i| \text{ small}, \quad \text{when } i \in \{p+1,\cdots,n\}. \tag{4.1.26}$$

If all characteristics $\lambda_i(u)$ $(i=1,\cdots,n)$ are weakly linearly degenerate, then system (1.1) is called to be *weakly linearly degenerate.* □

Definition 4.5. The inhomogeneous term $B(u)$ is said to satisfy the *matching condition*, if there exists the normalized transformation and in the normalized coordinates

$$B\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \quad \forall\, |u_h| \text{ small} \quad (h=1,\cdots,p) \tag{4.1.27}$$

and

$$B(u_j e_j) \equiv 0, \quad \forall\, |u_j| \text{ small} \quad (j=p+1,\cdots,n). \tag{4.1.28}$$

□

Thus, we have

Lemma 4.3. System (1.1) satisfies the generalized null condition if and only if (1.1) is weakly linearly degenerate and $B(u)$ satisfies the matching condition. □

§4.2. Some relations in the normalized coordinates

Similar to §3.2, in this section we give some relations on the decomposition of waves in the normalized coordinates.

Noting (4.1.23)-(4.1.24) and using (2.2.10)-(2.2.11), we observe that in the normalized coordinates (if any!)

$$\beta_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \qquad \forall\, i \in \{1,\cdots,n\}, \quad \forall\, j,k \in \{1,\cdots,p\}, \quad \forall\, |u_h| \text{ small} \quad (h=1,\cdots,p), \tag{4.2.1}$$

$$\beta_{ijj}\,(u_j e_j) \equiv 0, \quad \forall\, i \in \{1,\cdots,n\}, \quad \forall\, |u_j| \text{ small}, \quad \forall\, j \in \{p+1,\cdots,n\}, \tag{4.2.2}$$

$$\nu_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \qquad \forall\, i \in \{1,\cdots,n\}, \quad \forall\, j,k \in \{1,\cdots,p\}, \quad \forall\, |u_h| \text{ small} \quad (h=1,\cdots,p) \tag{4.2.3}$$

and

$$\nu_{ijj}\,(u_j e_j) \equiv 0, \quad \forall\, i \in \{1,\cdots,n\}, \quad \forall\, |u_j| \text{ small}, \quad \forall\, j \in \{p+1,\cdots,n\}. \tag{4.2.4}$$

When $B(u)$ satisfies the matching condition, it follows from (1.6), (2.2.3) and (4.1.27)-(4.1.28) that in the normalized coordinates (if any!)

$$b_i(u) = \sum_{\lambda_j(0)\neq\lambda_k(0)} b_{ijk}(u) u_j u_k, \quad \forall\, i \in \{1,\cdots,n\}, \quad \forall\, |u| \text{ small}, \tag{4.2.5}$$

where $b_{ijk}(u)$ are continuous functions of u, which are produced by Taylor's formula.

Noting (4.2.1)-(4.2.2) and using (2.2.16), in the normalized coordinates (if any!) we have

$$\tilde{\beta}_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \qquad \forall\, i \in \{p+1,\cdots,n\}, \quad \forall\, j,k \in \{1,\cdots,p\}, \quad \forall\, |u_h| \text{ small} \quad (h=1,\cdots,p) \tag{4.2.6}$$

and

$$\tilde{\beta}_{ijj}\,(u_j e_j) \equiv 0, \quad \forall\, |u_j| \text{ small}, \quad \forall\, j \in \{p+1,\cdots,n\} \quad \text{and} \quad j \neq i. \tag{4.2.7}$$

Furthermore, when the i-th characteristic $\lambda_i(u)$ is weakly linearly degenerate, in the normalized coordinates we have

$$\tilde{\beta}_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \qquad \forall\, j,k \in \{1,\cdots,p\}, \quad \forall\, |u_h| \text{ small} \quad (h=1,\cdots,p), \quad \text{if} \quad i \in \{1,\cdots,p\}; \tag{4.2.8}$$

$$\tilde{\beta}_{iii}(u_i e_i) \equiv 0, \quad \forall\, |u_i| \text{ small}, \quad \text{if} \quad i \in \{p+1,\cdots,n\}. \tag{4.2.9}$$

Moreover, by (2.2.12) we have

$$\tilde{\beta}_{iji}(u) \equiv 0, \quad \forall\, j \neq i; \tag{4.2.10}$$

while

$$\tilde{\beta}_{iii}(u) = \nabla\lambda_i(u)\, r_i(u) \tag{4.2.11}$$

which identically vanishes only in the case that $\lambda_i(u)$ is linearly degenerate in the sense of P.D.Lax.

Moreover, by (4.1.23)-(4.1.24), in the normalized coordinates (if any!) we have

$$\gamma_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \qquad \forall\, i \in \{p+1,\cdots,n\}, \quad \forall\, j,k \in \{1,\cdots,p\}, \quad \forall\, |u_h| \text{ small} \quad (h=1,\cdots,p). \tag{4.2.12}$$

Furthermore, when $\lambda_i(u)$ is weakly linearly degenerate, in the normalized coordinates we have

$$\gamma_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \qquad \forall\, j,k \in \{1,\cdots,p\}, \quad \forall\, |u_h| \text{ small} \quad (h=1,\cdots,p), \quad \text{if} \quad i \in \{1,\cdots,p\}; \tag{4.2.13}$$

$$\gamma_{iii}(u_i e_i) \equiv 0, \quad \forall\, |u_i| \text{ small}, \quad \text{if} \quad i \in \{p+1,\cdots,n\}. \tag{4.2.14}$$

In the present situation, (3.2.7) is still valid, namely, we have

$$(b_i(u))_x = \sum_{k=1}^{n} \tilde{b}_{ik}(u) w_k, \tag{4.2.15}$$

where

$$\tilde{b}_{ik}(u) = \sum_{l=1}^{n} \frac{\partial b_i(u)}{\partial u_l} r_{kl}(u). \tag{4.2.16}$$

In the normalized coordinates (if any!), by (4.1.23) and (4.1.24) we have

$$\tilde{b}_{ik}\left(\sum_{h=1}^{p} u_h e_h\right) = \frac{\partial b_i\left(\sum_{h=1}^{p} u_h e_h\right)}{\partial u_k}, \quad \forall\, |u_h|,\ |u_k| \text{ small} \quad (h = 1, \cdots, p),$$

$$\forall\, i \in \{1, \cdots, n\}, \text{ when } \quad k \in \{1, \cdots, p\}; \tag{4.2.17}$$

$$\tilde{b}_{ik}(u_k e_k) = \frac{\partial b_i(u_k e_k)}{\partial u_k}, \quad \forall\, i \in \{1, \cdots, n\}, \quad \forall\, |u_k| \text{ small}, \quad \text{when} \quad k \in \{p+1, \cdots, n\}. \tag{4.2.18}$$

When $B(u)$ satisfies the matching condition, noting (2.2.3) and (4.1.27)-(4.1.28) we observe that in the normalized coordinates

$$b_i\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \quad \forall\, i \in \{1, \cdots, n\}, \quad \forall\, |u_h| \text{ small} \quad (h = 1, \cdots, p) \tag{4.2.19}$$

and

$$b_i(u_k e_k) \equiv 0, \quad \forall\, i \in \{1, \cdots, n\}, \quad \forall\, |u_k| \text{ small} \quad (k = p+1, \cdots, n), \tag{4.2.20}$$

then

$$\frac{\partial b_i\left(\sum_{h=1}^{p} u_h e_h\right)}{\partial u_k} \equiv 0, \quad \forall\, i \in \{1, \cdots, n\}, \quad \forall\, k \in \{1, \cdots, p\}, \quad \forall\, |u_h|,\ |u_k| \text{ small} \quad (h = 1, \cdots, p) \tag{4.2.21}$$

and

$$\frac{\partial b_i(u_k e_k)}{\partial u_k} \equiv 0, \quad \forall\, i \in \{1, \cdots, n\}, \quad \forall\, k \in \{p+1, \cdots, n\}, \quad \forall\, |u_k| \text{ small}, \tag{4.2.22}$$

and then

$$\tilde{b}_{ik}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \quad \forall\, i \in \{1, \cdots, n\}, \quad \forall\, k \in \{1, \cdots, p\}, \quad \forall\, |u_h| \text{ small} \quad (h = 1, \cdots, p) \tag{4.2.23}$$

and

$$\tilde{b}_{ik}(u_k e_k) \equiv 0, \quad \forall\, i \in \{1, \cdots, n\}, \quad \forall\, k \in \{p+1, \cdots, n\}, \quad \forall\, |u_k| \text{ small.} \quad (4.2.24)$$

Finally, noting (4.1.23) and (2.2.26) we obtain that in the normalized coordinates (if any!) we have

$$\tilde{\gamma}_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \quad \forall\, i \in \{1, \cdots, n\},\ \forall\, j, k \in \{1, \cdots, p\}, \quad \forall\, |u_h| \text{ small} \quad (h = 1, \cdots, p). \quad (4.2.25)$$

§4.3. Main results

Consider the Cauchy problem

$$u_t + A(u)u_x = B(u), \quad (4.3.1)$$

$$t = 0: \quad u = \varphi(x), \quad (4.3.2)$$

where $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$, $B(u) = (B_1(u), \cdots, B_n(u))^T$ is a suitably smooth vector function of u, and $\varphi(x)$ is a C^1 vector function of x. Suppose that in a neighbourhood of $u = 0$, (4.3.1) is a hyperbolic system with

$$\lambda_0 \triangleq \lambda_1(0) = \cdots = \lambda_p(0) < \lambda_{p+1}(0) < \cdots < \lambda_n(0) \quad (p \geq 1). \quad (4.3.3)$$

Without loss of generality, we may suppose that in a neighbourhood of $u = 0$, the following normalized conditions hold

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n) \quad (4.3.4)$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \cdots, n). \quad (4.3.5)$$

Furthermore, we suppose that all $\lambda_i(u)$, $l_i(u)$ and $r_i(u)$ $(i = 1, \cdots, n)$ have the same regularity as $A(u)$ in a neighbourhood of $u = 0$. Finally, we suppose that $B(u)$ satisfies

$$B(0) = 0 \quad \text{and} \quad \nabla B(0) = 0. \tag{4.3.6}$$

In §4.4 we shall prove the following theorem similar to Theorem 3.1.

Theorem 4.1. Under the hypotheses mentioned above, suppose that $A(u)$ and $B(u)$ are C^2 in a neighbourhood of $u = 0$. Suppose furthermore that system (4.3.1) is weakly linearly degenerate and $B(u)$ satisfies the matching conditon. Suppose finally that $\varphi(x)$ is a C^1 vector function satisfying that there exists a constant $\mu > 0$ such that

$$\theta \triangleq \sup_{x \in \boldsymbol{R}} \left\{ (1 + |x|)^{1+\mu} \left(|\varphi(x)| + |\varphi'(x)| \right) \right\} < \infty. \tag{4.3.7}$$

Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, the Cauchy problem (4.3.1)-(4.3.2) admits a unique global C^1 solution $u = u(t, x)$ on $t \geq 0$. □

In particular, we have

Corollary 4.1. If, in a neighbourhood of $u = 0$, system (4.3.1) is linearly degenerate in the sense of Lax and $B(u)$ satisfies the matching condition, then the conclusion of Theorem 4.1 holds. □

In the case that system (4.3.1) is not weakly linearly degenerate but all multiple characteristics are weakly linearly degenerate, we will show that for a quite large class of initial data, the first order derivatives of C^1 solution to the Cauchy problem (4.3.1)-(4.3.2) must blow up in a finite time and we will give a sharp estimate on life span of the C^1 solution.

In the present situation, there exists a nonempty set $J \subseteq \{1, 2, \cdots, n\}$ such that $\lambda_i(u)$ is not weakly linearly degenerate if and only if $i \in J$.

Similar to Chapter 3, we observe that for any fixed $i \in J$, either there exists an integer $\alpha_i \geq 0$ such that

$$\left.\frac{d^l \lambda_i\left(u^{(i)}(s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, \cdots, \alpha_i) \quad \text{but} \quad \left.\frac{d^{\alpha_i+1} \lambda_i\left(u^{(i)}(s)\right)}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0 \tag{4.3.8}$$

or

$$\left.\frac{d^l \lambda_i\left(u^{(i)}(s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, 2, \cdots), \tag{4.3.9}$$

where $u = u^{(i)}(s)$ is defined by (3.1.2). In the case that (4.3.9) holds, we define $\alpha_i = +\infty$.

In the normalized coordinates, conditions (4.3.8)-(4.3.9) simply reduce to

$$\frac{\partial^l \lambda_i}{\partial u_i^l}(0) = 0 \quad (l = 1, \cdots, \alpha_i) \quad \text{but} \quad \frac{\partial^{\alpha_i+1} \lambda_i}{\partial u_i^{\alpha_i+1}}(0) \neq 0 \tag{4.3.10}$$

and

$$\frac{\partial^l \lambda_i}{\partial u_i^l}(0) = 0 \quad (l = 1, 2, \cdots) \tag{4.3.11}$$

respectively.

Similar to Theorem 3.2, the following theorem will be proved in §4.5.

Theorem 4.2. Under the assumptions mentioned at the beginning of this section, suppose that $A(u)$ is suitably smooth and $B(u) \in C^2$ in a neighbourhood of $u = 0$. Suppose furthermore that $\phi(x) = \varepsilon \psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying that there exists a constant $\mu > 0$ such that

$$\sup_{x \in \mathbb{R}} \left\{ (1 + |x|)^{1+\mu} \left(|\psi(x)| + |\psi'(x)| \right) \right\} < \infty. \tag{4.3.12}$$

Suppose finally that $B(u)$ satisfies the matching condition, system (4.3.1) is not weakly linearly degenerate, but all multiple characteristics are weakly linearly degenerate. Set

$$\alpha = \min\{\alpha_i \mid i \in J\} < \infty, \tag{4.3.13}$$

where α_i is defined by (4.3.8)-(4.3.9). Let

$$J_1 = \{i \mid i \in J,\ \alpha_i = \alpha\}. \tag{4.3.14}$$

If there exists $i_0 \in J_1$ such that

$$l_{i_0}(0)\,\psi(x) \not\equiv 0, \tag{4.3.15}$$

where $l_{i_0}(u)$ stands for the i_0-th left eigenvector, then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$ the first order derivatives of the C^1 solution $u = u(t,x)$ to the Cauchy problem (4.3.1)-(4.3.2) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ of $u = u(t,x)$ satisfies

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\varepsilon^{\alpha-1}\tilde{T}(\varepsilon)\right)^{-1} = \\ &\max_{i\in J_1}\sup_{x\in \mathbf{R}}\left\{-\frac{1}{\alpha!}\left.\frac{d^{\alpha+1}\lambda_i(u^{(i)}(s))}{ds^{\alpha+1}}\right|_{s=0}[l_i(0)\psi(x)]^{\alpha}\, l_i(0)\psi'(x)\right\},\end{aligned} \tag{4.3.16}$$

where $u = u^{(i)}(s)$ is defined by (3.1.2). □

Similarly, Theorem 3.3 and Theorem 3.4 can be generalized to the present case, and similar conclusions are valid.

§4.4. Global existence of C^1 solution — Proof of Theorem 4.1

The main results in this chapter can be proved in a way similar to the proof of Theorem 3.1 and Theorem 3.2 in Chapter 3. In what follows we only point out the essentially different part in the proof.

Without loss of generality, we may suppose that

$$0 < \lambda_0 \triangleq \lambda_1(0) = \cdots = \lambda_p(0) < \lambda_{p+1}(0) < \cdots < \lambda_n(0). \tag{4.4.1}$$

Moreover, we have

$$\begin{aligned}&\lambda_{p+1}(u) - \lambda_i(v) \geq 4\delta_0, \quad \forall\, |u|,\ |v| \leq \delta \quad (i = 1,\cdots,p),\\ &\lambda_{j+1}(u) - \lambda_j(v) \geq 4\delta_0, \quad \forall\, |u|,\ |v| \leq \delta \quad (j = p+1,\cdots,n-1)\end{aligned} \tag{4.4.2}$$

and

$$|\lambda_i(u) - \lambda_i(v)| \leq \frac{\delta_0}{2}, \quad \forall\, |u|,\ |v| \leq \delta \quad (i = 1,\cdots,n), \tag{4.4.3}$$

where δ and δ_0 are two suitably small positive constants.

For the time being it is supposed that on the existence domain of the C^1 solution $u = u(t,x)$ we have

$$|u(t,x)| \le \delta. \tag{4.4.4}$$

At the end of the proof of Lemma 4.6, we shall explain the reasonableness of this hypothesis.

By (4.4.1) and (4.4.4), on the existence domain of the C^1 solution we have

$$0 < \lambda_1(u), \cdots, \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{4.4.5}$$

provided that $\delta > 0$ is suitably small.

Similar to §3.4, for any fixed $T > 0$, let

$$D_-^T = \{(t,x) \,|\, 0 \le t \le T,\ x \le -t\}, \tag{4.4.6}$$

$$D_0^T = \{(t,x) \,|\, 0 \le t \le T,\ -t \le x \le (\lambda_0 - \delta_0)t\}, \tag{4.4.7}$$

$$D^T = \{(t,x) \,|\, 0 \le t \le T,\ (\lambda_0 - \delta_0)t \le x \le (\lambda_n(0) + \delta_0)t\}, \tag{4.4.8}$$

$$D_+^T = \{(t,x) \,|\, 0 \le t \le T,\ x \ge (\lambda_n(0) + \delta_0)t\} \tag{4.4.9}$$

and for $i = 1, \cdots, n$,

$$\begin{aligned} D_i^T = \{(t,x) \,|\, 0 \le t \le T,& \\ -[\delta_0 + \eta(\lambda_i(0) - \lambda_0)]t \le x - \lambda_i(0)t &\le [\delta_0 + \eta(\lambda_n(0) - \lambda_i(0))]t\}, \end{aligned} \tag{4.4.10}$$

where $\eta > 0$ is suitably small.

Noting (4.4.1)-(4.4.2), when $\eta > 0$ is suitably small, we have

$$D_1^T \equiv D_2^T \equiv \cdots \equiv D_p^T \triangleq D_m^T, \tag{4.4.11}$$

$$D_i^T \cap D_j^T = \emptyset, \quad \forall\, i \ne j, \quad i,j \in \{m, p+1, \cdots, n\} \tag{4.4.12}$$

and

$$D_m^T \bigcup \bigcup_{i=p+1}^{n} D_i^T \subset D^T. \tag{4.4.13}$$

Let

$$V\left(D_{\pm}^{T}\right)=\max_{i=1,\cdots,n}\|(1+|x|)^{1+\mu}v_{i}(t,x)\|_{L^{\infty}\left(D_{\pm}^{T}\right)},\tag{4.4.14}$$

$$V\left(D_{0}^{T}\right)=\max_{i=1,\cdots,n}\|(1+t)^{1+\mu}v_{i}(t,x)\|_{L^{\infty}\left(D_{0}^{T}\right)},\tag{4.4.15}$$

$$W\left(D_{\pm}^{T}\right)=\max_{i=1,\cdots,n}\|(1+|x|)^{1+\mu}w_{i}(t,x)\|_{L^{\infty}\left(D_{\pm}^{T}\right)},\tag{4.4.16}$$

$$W\left(D_{0}^{T}\right)=\max_{i=1,\cdots,n}\|(1+t)^{1+\mu}w_{i}(t,x)\|_{L^{\infty}\left(D_{0}^{T}\right)},\tag{4.4.17}$$

$$U_{\infty}^{c}(T)=\max_{i=1,\cdots,n}\sup_{(t,x)\in D^{T}\setminus D_{i}^{T}}(1+|x-\lambda_{i}(0)t|)^{1+\mu}|u_{i}(t,x)|,\tag{4.4.18}$$

$$V_{\infty}^{c}(T)=\max_{i=1,\cdots,n}\sup_{(t,x)\in D^{T}\setminus D_{i}^{T}}(1+|x-\lambda_{i}(0)t|)^{1+\mu}|v_{i}(t,x)|,\tag{4.4.19}$$

$$W_{\infty}^{c}(T)=\max_{i=1,\cdots,n}\sup_{(t,x)\in D^{T}\setminus D_{i}^{T}}(1+|x-\lambda_{i}(0)t|)^{1+\mu}|w_{i}(t,x)|,\tag{4.4.20}$$

$$U_{1}(T)=\max_{i=1,\cdots,n}\sup_{0\le t\le T}\int_{D_{i}^{T}(t)}|u_{i}(t,x)|dx,\tag{4.4.21}$$

$$V_{1}(T)=\max_{i=1,\cdots,n}\sup_{0\le t\le T}\int_{D_{i}^{T}(t)}|v_{i}(t,x)|dx,\tag{4.4.22}$$

$$W_{1}(T)=\max_{i=1,\cdots,n}\sup_{0\le t\le T}\int_{D_{i}^{T}(t)}|w_{i}(t,x)|dx,\tag{4.4.23}$$

$$V_{\infty}(T)=\max_{i=1,\cdots,n}\sup_{\substack{0\le t\le T\\ x\in\mathbb{R}}}|v_{i}(t,x)|\tag{4.4.24}$$

and

$$W_{\infty}(T)=\max_{i=1,\cdots,n}\sup_{\substack{0\le t\le T\\ x\in\mathbb{R}}}|w_{i}(t,x)|,\tag{4.4.25}$$

where $D_i^T(t)$ $(t\ge 0)$ denotes the t-section of D_i^T:

$$D_{i}^{T}(t)=\left\{(\tau,x)\mid\tau=t,(\tau,x)\in D_{i}^{T}\right\}.\tag{4.4.26}$$

Noting (4.4.11), we get

$$D_{1}^{T}(t)\equiv D_{2}^{T}(t)\equiv\cdots\equiv D_{p}^{T}(t)\triangleq D_{m}^{T}(t).\tag{4.4.27}$$

Therefore, by (4.4.1) and (4.4.11) we have

$$U_\infty^c(T) = \max\left\{\max_{i=1,\cdots,p}\sup_{(t,x)\in D^T\setminus D_m^T}(1+|x-\lambda_0 t|)^{1+\mu}|u_i(t,x)|,\right. \left.\max_{i=p+1,\cdots,n}\sup_{(t,x)\in D^T\setminus D_i^T}(1+|x-\lambda_i(0)t|)^{1+\mu}|u_i(t,x)|\right\}, \tag{4.4.18a}$$

$$V_\infty^c(T) = \max\left\{\max_{i=1,\cdots,p}\sup_{(t,x)\in D^T\setminus D_m^T}(1+|x-\lambda_0 t|)^{1+\mu}|v_i(t,x)|,\right. \left.\max_{i=p+1,\cdots,n}\sup_{(t,x)\in D^T\setminus D_i^T}(1+|x-\lambda_i(0)t|)^{1+\mu}|v_i(t,x)|\right\}, \tag{4.4.19a}$$

$$W_\infty^c(T) = \max\left\{\max_{i=1,\cdots,p}\sup_{(t,x)\in D^T\setminus D_m^T}(1+|x-\lambda_0 t|)^{1+\mu}|w_i(t,x)|,\right. \left.\max_{i=p+1,\cdots,n}\sup_{(t,x)\in D^T\setminus D_i^T}(1+|x-\lambda_i(0)t|)^{1+\mu}|w_i(t,x)|\right\}, \tag{4.4.20a}$$

$$U_1(T) = \max\left\{\max_{i=1,\cdots,p}\sup_{0\le t\le T}\int_{D_m^T(t)}|u_i(t,x)|dx,\right. \left.\max_{i=p+1,\cdots,n}\sup_{0\le t\le T}\int_{D_i^T(t)}|u_i(t,x)|dx\right\}, \tag{4.4.21a}$$

$$V_1(T) = \max\left\{\max_{i=1,\cdots,p}\sup_{0\le t\le T}\int_{D_m^T(t)}|v_i(t,x)|dx,\right. \left.\max_{i=p+1,\cdots,n}\sup_{0\le t\le T}\int_{D_i^T(t)}|v_i(t,x)|dx\right\}, \tag{4.4.22a}$$

$$W_1(T) = \max\left\{\max_{i=1,\cdots,p}\sup_{0\le t\le T}\int_{D_m^T(t)}|w_i(t,x)|dx,\right. \left.\max_{i=p+1,\cdots,n}\sup_{0\le t\le T}\int_{D_i^T(t)}|w_i(t,x)|dx\right\}. \tag{4.4.23a}$$

Noting (4.4.4), $V_\infty(T)$ is obviously equivalent to

$$U_\infty(T) = \max_{i=1,\cdots,n}\sup_{\substack{0\le t\le T\\ x\in R}}|u_i(t,x)|. \tag{4.4.28}$$

In the present situation, Lemma 3.2 and Lemma 3.3 are still valid and can be stated as follows:

Lemma 4.4. For each $i = 1, \cdots, n$, on the domain $D^T \backslash D_i^T$ we have

$$ct \leq |x - \lambda_i(0)\, t| \leq Ct, \tag{4.4.29}$$

$$cx \leq |x - \lambda_i(0)\, t| \leq Cx, \tag{4.4.30}$$

where c and C are positive constants independent of (t, x) and T. □

Lemma 4.5. Suppose that (4.3.3) and (4.3.6) hold, and $A(u), B(u) \in C^2$ in a neighbourhood of $u = 0$. Suppose furthermore that $\phi(x)$ is a C^1 vector function satisfying (4.3.7). There exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (4.3.1)-(4.3.2) there exist positive constants k_1 and k_2 independent of θ and T, such that the following uniform *a priori* estimates hold:

$$V\left(D_{\pm}^T\right),\ W\left(D_{\pm}^T\right) \leq k_1\theta \tag{4.4.31}$$

and

$$V\left(D_0^T\right),\ W\left(D_0^T\right) \leq k_2\theta. \tag{4.4.32}$$

□

Lemma 4.6. Suppose that (4.3.3) and (4.3.6) hold, $A(u), B(u) \in C^2$ in a neighbourhood of $u = 0$, and (4.3.4)-(4.3.5) hold. Suppose furthermore that system (4.3.1) is weakly linearly degenerate and $B(u)$ satisfies the matching condition. Suppose finally that $\phi(x)$ is a C^1 vector function satisfying (4.3.7). In the normalized coordinates there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (4.3.1)-(4.3.2) there exist positive constants $k_i\ (i = 3, \cdots, 9)$ independent of θ and T, such that the following uniform *a priori* estimates hold:

$$U_\infty^c(T) \leq k_3\theta, \tag{4.4.33}$$

$$V_\infty^c(T) \leq k_4\theta, \tag{4.4.34}$$

$$W_\infty^c(T) \leq k_5\theta, \tag{4.4.35}$$

$$V_1(T) \le k_6\theta, \tag{4.4.36}$$

$$W_1(T) \le k_7\theta, \tag{4.4.37}$$

$$V_\infty(T) \le k_8\theta \tag{4.4.38}$$

and

$$W_\infty(T) \le k_9\theta. \tag{4.4.39}$$

□

Proof. This lemma will be proved in a way similar to the proof of Lemma 3.4. In what follows we only point out the essentially different part in the proof. Without loss of generality, the following discussion is always carried out in the normalized coordinates.

We first prove that when $\delta > 0$ is suitably small, we have

$$U_\infty^c(T) \le C_1 V_\infty^c(T) + C_2 V_\infty(T)\, U_\infty^c(T), \tag{4.4.40}$$

henceforth C_j $(j = 1, 2, \cdots)$ will denote positive constants independent of θ and T.

In fact, the proof of (4.4.40) is basically the same as that of (3.4.73) in Chapter 3 and all we have to supply is the following: For any given point $(t, x) \in D^T \backslash D_i^T$, if $(t, x) \in D_m^T$, then $i \notin \{1, \cdots, p\}$ and $(t, x) \notin D_k^T$ $(k = p+1, \cdots, n)$. Noting (4.1.23), we have

$$\begin{aligned} u_i(t,x) &= u^T(t,x)\, e_i = \sum_{k=1}^{n} v_k r_k^T(u)\, e_i \\ &= \sum_{j=1}^{p} v_j \left(r_j^T(u) - r_j^T\left(\sum_{h=1}^{p} u_h e_h\right)\right) e_i + \sum_{k=p+1}^{n} v_k r_k^T(u)\, e_i. \end{aligned} \tag{4.4.41}$$

By Hadamard's formula, for $j = 1, \cdots, p$ we have

$$r_j(u) - r_j\left(\sum_{h=1}^{p} u_h e_h\right) = \int_0^1 \sum_{k=p+1}^{n} \frac{\partial r_j}{\partial u_k}(u_1, \cdots, u_p, \tau u_{p+1}, \cdots, \tau u_n)\, u_k d\tau, \tag{4.4.42}$$

we get

$$(1 + |x - \lambda_i(0)\, t|)^{1+\mu}\, |u_i(t,x)| \le C_3 V_\infty^c(T) + C_4 V_\infty(T)\, U_\infty^c(T). \tag{4.4.43}$$

This leads to (4.4.40).

Similar to (3.4.74), we have

$$U_1(T) \le C_5 V_1(T) + C_6 V_\infty^c(T). \tag{4.4.44}$$

Let

$$\begin{aligned}\tilde{U}_1(T) = \max\Bigg\{ & \max_{i=1,\cdots,p}\ \max_{j\in\{p+1,\cdots,n\}}\ \sup_{\tilde{C}_j}\int_{\tilde{C}_j}|u_i(t,x)|dt\ ,\\ & \max_{i=p+1,\cdots,n}\ \max_{j\neq i}\ \sup_{\tilde{C}_j}\int_{\tilde{C}_j}|u_i(t,x)|dt\Bigg\},\end{aligned} \tag{4.4.45}$$

$$\begin{aligned}\tilde{V}_1(T) = \max\Bigg\{ & \max_{i=1,\cdots,p}\ \max_{j\in\{p+1,\cdots,n\}}\ \sup_{\tilde{C}_j}\int_{\tilde{C}_j}|v_i(t,x)|dt\ ,\\ & \max_{i=p+1,\cdots,n}\ \max_{j\neq i}\ \sup_{\tilde{C}_j}\int_{\tilde{C}_j}|v_i(t,x)|dt\Bigg\},\end{aligned} \tag{4.4.46}$$

$$\begin{aligned}\tilde{W}_1(T) = \max\Bigg\{ & \max_{i=1,\cdots,p}\ \max_{j\in\{p+1,\cdots,n\}}\ \sup_{\tilde{C}_j}\int_{\tilde{C}_j}|w_i(t,x)|dt\ ,\\ & \max_{i=p+1,\cdots,n}\ \max_{j\neq i}\ \sup_{\tilde{C}_j}\int_{\tilde{C}_j}|w_i(t,x)|dt\Bigg\},\end{aligned} \tag{4.4.47}$$

where, when $i \in \{1,\cdots,p\}$, $\tilde{C}_j$ stands for any given j-th characteristic in D_m^T ($j \in \{p+1,\cdots,n\}$); while, when $i \in \{p+1,\cdots,n\}$, $\tilde{C}_j$ stands for any given j-th characteristic in D_i^T ($j \neq i$) .

Similar to (3.4.101), we have

$$\tilde{U}_1(T) \le C_7\tilde{V}_1(T) + C_8 V_\infty^c(T). \tag{4.4.48}$$

In the present situation, (3.4.84) can be rewritten as

$$\begin{aligned}
&\int_{t_0}^{t_2} |w_i(t, x_j(t))| \, |\lambda_j(u(t, x_j(t))) - \lambda_i(u(t, x_j(t)))| dt \\
&\quad \le \int_0^{\frac{y_2}{\lambda_n(0)+\delta_0}} |w_i(t, (\lambda_n(0) + \delta_0)t)| \, (\lambda_n(0) + \delta_0 - \lambda_i(t, (\lambda_n(0) + \delta_0)t)) dt \\
&\quad + \iint_{P_0 O A_2 P_2} \left| \sum_{\substack{j,k=1 \\ j \text{ or } k \notin \{1,\cdots,p\}}}^{n} \tilde{\gamma}_{ijk}(u) w_j w_k \right| dt dx \\
&\quad + \iint_{P_0 O A_2 P_2} \left| \sum_{j,k=1}^{p} \tilde{\gamma}_{ijk}(u) w_j w_k + (b_i(u))_x \right| dt dx.
\end{aligned} \tag{4.4.49}$$

By the corresponding estimates given in §3.4, we only need to estimate the last term on the right-hand side of (4.4.49).

Noting (4.2.25) and using Hadamard's formula, for $j, k \in \{1, \cdots, p\}$ we have

$$\begin{aligned}
\tilde{\gamma}_{ijk}(u) &= \tilde{\gamma}_{ijk}(u) - \tilde{\gamma}_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \\
&= \int_0^1 \sum_{l=p+1}^{n} \frac{\partial \tilde{\gamma}_{ijk}}{\partial u_l}(u_1, \cdots, u_p, \tau u_{p+1}, \cdots, \tau u_n) u_l d\tau.
\end{aligned} \tag{4.4.50}$$

On the other hand, noting (4.2.23)-(4.2.24) and using Hadamard's formula again, from (4.2.15) we obtain

$$\begin{aligned}
(b_i(u))_x &= \sum_{k=1}^{p} \left(\tilde{b}_{ik}(u) - \tilde{b}_{ik}\left(\sum_{h=1}^{p} u_h e_h\right)\right) w_k + \sum_{k=p+1}^{n} \left(\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_k e_k)\right) w_k \\
&= \sum_{k=1}^{p} \int_0^1 \sum_{l=p+1}^{n} \frac{\partial \tilde{b}_{ik}}{\partial u_l}(u_1, \cdots, u_p, \tau u_{p+1}, \cdots, \tau u_n) u_l d\tau w_k \\
&\quad + \sum_{k=p+1}^{n} \int_0^1 \sum_{l \ne k} \frac{\partial \tilde{b}_{ik}}{\partial u_l}(\tau u_1, \cdots, \tau u_{k-1}, u_k, \tau u_{k+1}, \cdots, \tau u_n) u_l d\tau w_k.
\end{aligned} \tag{4.4.51}$$

Hence, similar to (3.4.88), using (4.4.48) we obtain from (4.4.49) that

$$\begin{aligned}
\tilde{W}_1(T) \le\ & C_9 \Big\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T) + U_\infty^c(T) W_\infty^c(T) \\
& + W_\infty^c(T) V_\infty^c(T) + W_\infty^c(T) V_1(T) + U_\infty^c(T) W_1(T)\Big\}.
\end{aligned} \tag{4.4.52}$$

Similarly, we have

$$\begin{aligned} W_1(T) \le & C_{10}\left\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T)W_1(T) + U^c_\infty(T)W^c_\infty(T)\right. \\ & + W^c_\infty(T)V^c_\infty(T) + W^c_\infty(T)V_1(T) + U^c_\infty(T)W_1(T)\} \end{aligned} \tag{4.4.53}$$

Moreover, similar to (3.4.103), using (2.2.21), (4.2.12) and (4.4.48) we obtain

$$\begin{aligned} W^c_\infty(T) \le & C_{11}\left\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T)\tilde W_1(T) + U^c_\infty(T)W^c_\infty(T)\right. \\ & \left. + \tilde V_1(T)W^c_\infty(T) + V^c_\infty(T)W^c_\infty(T) + U^c_\infty(T)\tilde W_1(T)\right\}. \end{aligned} \tag{4.4.54}$$

Noting (4.2.5)-(4.2.9), we may rewrite (3.4.90) as

$$\begin{aligned} &\int_{t_0}^{t_2} |v_i(t, x_j(t))|\,|\lambda_j(u(t,x_j(t))) - \lambda_i(u(t,x_j(t)))|dt \\ &\le \int_0^{\frac{y_2}{\lambda_n(0)+\delta_0}} |v_i(t,(\lambda_n(0)+\delta_0)t)|(\lambda_n(0)+\delta_0 - \lambda_i(t,(\lambda_n(0)+\delta_0)t))dt \\ &+ \iint_{P_0OA_2P_2} \left| \sum_{\substack{j,k=1 \\ j \text{ or } k \notin\{1,\cdots,p\}}}^n \tilde\beta_{ijk}(u)v_jw_k \right| dtdx \\ &+ \iint_{P_0OA_2P_2} \left| \sum_{j,k=1}^n \sum_{\lambda_l(0)\neq\lambda_q(0)} \nu_{ijk}(u)b_{klq}(u)v_ju_lu_q \right| dtdx \\ &+ \iint_{P_0OA_2P_2} \left| \sum_{\lambda_j(0)\neq\lambda_k(0)} b_{ijk}(u)u_ju_k \right| dtdx + \iint_{P_0OA_2P_2} \left| \sum_{j,k=1}^p \tilde\beta_{ijk}(u)v_jw_k \right| dtdx. \end{aligned} \tag{4.4.55}$$

Noting the corresponding estimates given in Chapter 3, we only need to estimate the last term on the right-hand side of (4.4.55). Since system (4.3.1) is weakly linearly degenerate, for $j,k \in \{1,\cdots,p\}$, noting (4.2.6) and (4.2.8) and using Hadamard's formula we have

$$\begin{aligned} \tilde\beta_{ijk}(u) &= \tilde\beta_{ijk}(u) - \tilde\beta_{ijk}\left(\sum_{h=1}^p u_ke_k\right) \\ &= \int_0^1 \sum_{l=p+1}^n \frac{\partial\tilde\beta_{ijk}}{\partial u_l}(u_1,\cdots,u_p,\tau u_{p+1},\cdots,\tau u_n)\,u_ld\tau, \\ &\qquad \forall\, i\in\{1,\cdots,n\}, \quad \forall\, j,k\in\{1,\cdots,p\}. \end{aligned} \tag{4.4.56}$$

Then, using (4.4.48) we can also obtain a similar estimate for the last term on the right-hand side of (4.4.55). Finally, we get

$$\begin{aligned}\tilde{V}_1(T) \leq\ & C_{12}\{\theta + V_\infty^c(T)W_\infty^c(T) + V_1(T)W_\infty^c(T) \\ & + V_\infty^c(T)W_1(T) + U_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & + (U_\infty^c(T))^2 + U_\infty^c(T)V_1(T) + U_\infty^c(T)V_\infty^c(T)\Big\}.\end{aligned} \tag{4.4.57}$$

Similarly, we have

$$\begin{aligned}V_1(T) \leq\ & C_{13}\{\theta + V_\infty^c(T)W_\infty^c(T) + V_1(T)W_\infty^c(T) \\ & + V_\infty^c(T)W_1(T) + U_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & + (U_\infty^c(T))^2 + U_\infty^c(T)V_1(T) + U_\infty^c(T)V_\infty^c(T)\Big\}.\end{aligned} \tag{4.4.58}$$

Moreover, similar to (3.4.104), noting (4.2.1) and (4.2.5) we obtain

$$\begin{aligned}V_\infty^c(T) \leq\ & C_{14}\Big\{\theta + V_\infty^c(T)W_\infty^c(T) + V_\infty^c(T)\tilde{W}_1(T) + \tilde{V}_1(T)W_\infty^c(T) \\ & + U_\infty^c(T)W_\infty^c(T) + U_\infty^c(T)\tilde{W}_1(T) + \tilde{V}_1(T)W_\infty^c(T) \\ & + (U_\infty^c(T))^2 + U_\infty^c(T)\tilde{V}_1(T) + U_\infty^c(T)V_\infty^c(T)\Big\}.\end{aligned} \tag{4.4.59}$$

Furthermore, similar to (3.4.109) we have

$$V_\infty(T) \leq C_{15}\{\theta + W_\infty^c(T) + W_1(T)\}. \tag{4.4.60}$$

Thus, using a procedure similar to that in Chapter 3, by (4.4.40), (4.4.52)-(4.4.54) and (4.4.57)-(4.4.60), we can easily obtain (4.4.33)-(4.4.38).

We finally prove (4.4.39).

Noting that system (4.3.1) is weakly linearly degenerate and using (4.2.13)-(4.2.14) and (4.2.23)-(4.2.24), corresponding to (3.4.110) we have

$$\begin{aligned}w_i(t,x) =\ & w_i\left(\frac{y}{\lambda_n(0)+\delta_0}, y\right) + \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \sum_{\substack{j,k=1 \\ j \text{ or } k\notin\{1,\cdots,p\}}}^n \gamma_{ijk}(u)\, w_j w_k(s, x_i(s;t,x))ds \\ & + \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \sum_{j,k=1}^p \left(\gamma_{ijk}(u) - \gamma_{ijk}\left(\sum_{h=1}^p u_h e_h\right)\right) w_i w_k(s, x_i(s;t,x))ds \\ & + \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \sum_{k=p+1}^n \left(\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_k e_k)\right) w_k(s, x_i(s;t,x))ds \\ & + \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \sum_{k=1}^p \left(b_{ik}(u) - \tilde{b}_{ik}\left(\sum_{h=1}^p u_h e_h\right)\right) w_k(s, x_i(s;t,x))ds.\end{aligned} \tag{4.4.61}$$

Noting the corresponding estimates given in Chapter 3 and using Hadamard's formula we can still obtain the same estimate as (3.4.112):

$$\begin{aligned} W_\infty(T) \le\ & C_{16}\Big\{\theta + W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)\,W_\infty(T) \\ & + U_\infty(T)\,(W_\infty^c(T))^2 + U_\infty^c(T)\,(W_\infty(T))^2 + U_\infty^c(T)\,W_\infty^c(T) \\ & + U_\infty(T)\,W_\infty^c(T) + U_\infty^c(T)W_\infty(T)\Big\}. \end{aligned} \tag{4.4.62}$$

As in Chapter 3, from (4.4.62) we get (4.4.39) immediately.

Finally we point out that (4.4.38) implies the reasonableness of hypotheses (4.4.4), provided that $\theta_0 > 0$ is suitably small. The proof of this Lemma is finished. Q.E.D.

By Lemma 4.6 we get Theorem 4.1 immediately.

§4.5. Blow-up phenomenon and life span of C^1 solution — Proof of Theorem 4.2

Under the hypotheses of Theorem 4.2, Lemma 4.5 is still valid and can be stated as

Lemma 4.7. Suppose that (4.3.3) and (4.3.6) hold, and $A(u), B(u) \in C^2$ in a neighbourhood of $u = 0$. Suppose furthermore that $\varphi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying (4.3.12). There exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (4.3.1)-(4.3.2) there exists a positive constant k_1 independent of ε and T, such that the following uniform *a priori* estimate holds:

$$V(D_0^T),\ V(D_\pm^T),\ W(D_0^T),\ W(D_\pm^T) \le k_1\varepsilon. \tag{4.5.1}$$

□

Let

$$W_{\infty}^{p}(T) = \max_{i \notin J} \sup_{\substack{0 \le t \le T \\ x \in \mathbb{R}}} |w_i(t,x)|, \tag{4.5.2}$$

where J is defined in §4.3. We have

Lemma 4.8. Under the assumptions of Theorem 4.2, in the normalized coordinates there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (4.3.1)-(4.3.2) there exist positive constants k_i $(i = 2, \cdots, 10)$ independent of ε and T, such that the following uniform *a priori* estimates hold:

$$U_{\infty}^{c}(T) \le k_2 \varepsilon, \tag{4.5.3}$$

$$V_{\infty}^{c}(T) \le k_3 \varepsilon, \tag{4.5.4}$$

$$W_{\infty}^{c}(T) \le k_4 \varepsilon, \tag{4.5.5}$$

$$W_1(T), \tilde{W}_1(T) \le k_5 \varepsilon, \tag{4.5.6}$$

$$V_1(T), \tilde{V}_1(T) \le k_6 \varepsilon + k_7 \varepsilon^{2+\alpha} T, \tag{4.5.7}$$

$$U_{\infty}(T), V_{\infty}(T) \le k_8 \varepsilon, \tag{4.5.8}$$

$$W_{\infty}^{p}(T) \le k_9 \varepsilon, \tag{4.5.9}$$

where

$$T \varepsilon^{\frac{3}{2}+\alpha} \le 1. \tag{4.5.10}$$

Moreover,

$$W_{\infty}(T) \le k_{10} \varepsilon, \tag{4.5.11}$$

where

$$T \varepsilon^{1+\alpha} \le k_{11}. \tag{4.5.12}$$

□

Proof. This lemma will be proved in a way similar to the proof of Lemma 4.6. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$

is always supposed to be suitably small. As before, all discussions are carried out in the normalized coordinates.

In the present situation, (4.4.40) and (4.4.48) are still valid.

Noting that the proofs of (4.4.52)-(4.4.54) and (4.4.59)-(4.4.60) are not based on the hypothesis of weak linear degeneracy and that $\{1,\cdots,p\}\cap J=\emptyset$, in the present situation we still have

$$\begin{aligned}\tilde{W}_1(T) \le\ & C_1\Big\{\varepsilon+(W_\infty^c(T))^2+W_\infty^c(T)W_1(T)+U_\infty^c(T)W_\infty^c(T)\\ &+W_\infty^c(T)V_\infty^c(T)+W_\infty^c(T)V_1(T)+U_\infty^c(T)W_1(T)\},\end{aligned} \tag{4.5.13}$$

$$\begin{aligned}W_1(T) \le\ & C_2\Big\{\varepsilon+(W_\infty^c(T))^2+W_\infty^c(T)W_1(T)+U_\infty^c(T)W_\infty^c(T)\\ &+W_\infty^c(T)V_\infty^c(T)+W_\infty^c(T)V_1(T)+U_\infty^c(T)W_1(T)\},\end{aligned} \tag{4.5.14}$$

$$\begin{aligned}W_\infty^c(T) \le\ & C_3\Big\{\varepsilon+(W_\infty^c(T))^2+W_\infty^c(T)\tilde{W}_1(T)+U_\infty^c(T)W_\infty^c(T)\\ &+\tilde{V}_1(T)W_\infty^c(T)+V_\infty^c(T)W_\infty^c(T)+U_\infty^c(T)\tilde{W}_1(T)\Big\},\end{aligned} \tag{4.5.15}$$

$$\begin{aligned}V_\infty^c(T) \le\ & C_4\Big\{\varepsilon+V_\infty^c(T)W_\infty^c(T)+V_\infty^c(T)\tilde{W}_1(T)+\tilde{V}_1(T)W_\infty^c(T)\\ &+U_\infty^c(T)W_\infty^c(T)+U_\infty^c(T)\tilde{W}_1(T)+\tilde{V}_1(T)W_\infty^c(T)\\ &+(U_\infty^c(T))^2+U_\infty^c(T)\tilde{V}_1(T)+U_\infty^c(T)V_\infty^c(T)\Big\},\end{aligned} \tag{4.5.16}$$

$$U_\infty(T),\ V_\infty(T)\le C_5\{\varepsilon+W_\infty^c(T)+W_1(T)\}, \tag{4.5.17}$$

henceforth $C_j\,(j=1,2,\cdots,)$ will denote positive constants independent of ε and T.

For $i\notin J$, we can estimate (4.4.55) just as in the proof of Lemma 4.6; while, for $i\in J$, noting (4.2.6)-(4.2.7) and the fact that $J\cap\{1,\cdots,p\}=\emptyset$, instead of (4.4.55) we have

$$\int_{t_0}^{t_2} |v_i(t,x_j(t))|\;|\lambda_j(u(t,x_j(t)))-\lambda_i(u(t,x_j(t)))|dt$$
$$\le \int_0^{\frac{y_2}{\lambda_n(0)+\delta_0}} |v_i(t,(\lambda_n(0)+\delta_0)t)|\;(\lambda_n(0)+\delta_0-\lambda_i(t,(\lambda_n(0)+\delta_0)t))dt$$
$$+\iint_{P_0OA_2P_2}\left|\sum_{\substack{j\neq k\\ j \text{ or } k\notin\{1,\cdots,p\}}}^{n}\tilde{\beta}_{ijk}(u)v_jw_k\right|dtdx+\iint_{P_0OA_2P_2}\left|\sum_{j,k=1}^{p}\tilde{\beta}_{ijk}(u)v_jw_k\right|dtdx$$
$$+\iint_{P_0OA_2P_2}\left|\sum_{j=1}^{n}\left(\tilde{\beta}_{ijj}(u)-\tilde{\beta}_{ijj}(u_je_j)\right)v_jw_j\right|dtdx$$
$$+\iint_{P_0OA_2P_2}\left|\sum_{j,k=1}^{n}\sum_{\lambda_l(0)\neq\lambda_q(0)}\nu_{ijk}(u)b_{klq}(u)v_ju_lu_q\right|dtdx$$
$$+\iint_{P_0OA_2P_2}\left|\sum_{\lambda_j(0)\neq\lambda_k(0)}b_{ijk}(u)u_ju_k\right|dtdx+\iint_{P_0OA_2P_2}\left|\tilde{\beta}_{iii}(u_ie_i)v_iw_i\right|dtdx, \tag{4.5.18}$$

hence, we only need to estimate the last term of the right-hand side of (4.5.18).

Noting the fact that in the normalized coordinates

$$\tilde{\beta}_{iii}(u_ie_i)=\frac{\partial\lambda_i(0,\cdots,0,u_i,0,\cdots,0)}{\partial u_i} \tag{4.5.19}$$

and the difinitions of α_i and α, we have

$$|\tilde{\beta}_{iii}(u_ie_i)|\le C_6|u_i|^{\alpha}. \tag{4.5.20}$$

Thus, similar to (3.6.17) we have

$$\iint_{P_0OA_2P_2}\left|\tilde{\beta}_{iii}(u_ie_i)v_iw_i\right|dtdx\le C_7\left(V_\infty(T)\right)^{1+\alpha}\left(W_\infty^c(T)+W_1(T)\right)T, \tag{4.5.21}$$

then we have

$$\begin{aligned}\tilde{V}_1(T)\;\le\;& C_8\{\varepsilon+V_\infty^c(T)W_\infty^c(T)+V_1(T)W_\infty^c(T)\\ &+V_\infty^c(T)W_1(T)+U_\infty^c(T)W_1(T)+U_\infty^c(T)W_\infty^c(T)\\ &+(U_\infty^c(T))^2+U_\infty^c(T)V_1(T)+U_\infty^c(T)V_\infty^c(T)\\ &+(V_\infty(T))^{1+\alpha}(W_\infty^c(T)+W_1(T))T\Big\}.\end{aligned} \tag{4.5.22}$$

Similar to (4.4.58), we obtain

$$\begin{aligned} V_1(T) \leq\ & C_9 \{\varepsilon + V_\infty^c(T)W_\infty^c(T) + V_1(T)W_\infty^c(T) \\ & + V_\infty^c(T)W_1(T) + U_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & + (U_\infty^c(T))^2 + U_\infty^c(T)V_1(T) + U_\infty^c(T)V_\infty^c(T) \\ & + (V_\infty(T))^{1+\alpha}(W_\infty^c(T) + W_1(T))T \Big\}. \end{aligned} \tag{4.5.23}$$

Since $\lambda_i(u)$ $(i \notin J)$ is weakly linearly degenerate, similar to (4.4.62), we get

$$\begin{aligned} W_\infty^p(T) \leq\ & C_{10}\Big\{\varepsilon + W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)W_\infty(T) \\ & + U_\infty(T)(W_\infty^c(T))^2 + U_\infty^c(T)(W_\infty^p(T))^2 + U_\infty^c(T)W_\infty^c(T) \\ & + U_\infty(T)W_\infty^c(T) + U_\infty^c(T)W_\infty^p(T)\}. \end{aligned} \tag{4.5.24}$$

Thus, using a procedure similar to that in the proof of Lemma 4.6, by (4.4.40), (4.5.13)-(4.5.17) and (4.5.22)-(4.5.24) we can easily prove (4.5.3)-(4.5.9), provided that (4.5.10) holds.

We finally show (4.5.11).

In the present situation, if $i \notin J$, then $|w_i(t,x)|$ can be bounded by $W_\infty^p(T)$. Otherwise, noting the fact that $J \cap \{i, \cdots, p\} = \emptyset$, similar to (3.6.54) we have

$$\begin{aligned} |w_i(t,x)| \leq\ & C_{11}\Big\{\varepsilon + (W_\infty^c(T))^2 + W_\infty^c(T)W_\infty(T) \\ & + U_\infty(T)(W_\infty^c(T))^2 + U_\infty^c(T)(W_\infty(T))^2 \\ & + U_\infty^c(T)W_\infty^c(T) + U_\infty(T)W_\infty^c(T) \\ & + U_\infty^c(T)W_\infty(T) + (V_\infty(T))^\alpha (W_\infty(T))^2 T\Big\}. \end{aligned} \tag{4.5.25}$$

Then, noting (4.5.3), (4.5.5), (4.5.8) and (4.5.9), and using Lemma 4.5, we get

$$W_\infty(T) \leq C_{12}\Big\{\varepsilon\left(1 + W_\infty(T) + (W_\infty(T))^2\right) + \varepsilon^\alpha T (W_\infty(T))^2\Big\}, \tag{4.5.26}$$

where T satisfies (4.5.10).

As in Chapter 3, (4.5.11) follows from (4.5.26) immediately, provided that (4.5.12) holds.

Thus, the proof of Lemma 4.8 is completed. Q.E.D.

Restricting the domain under consideration such that $0 \le t \le \varepsilon^{-(\frac{3}{2}+\alpha)}$, using (4.5.3)-(4.5.8), noting the fact that $J \cap \{1, \cdots, p\} = \emptyset$, repeating completely the procedure of proving Theorem 3.2 in Chapter 3, we can prove Theorem 4.2 easily.

§4.6. Quasilinear hyperbolic systems of conservation laws with characteristics with constant multiplicity

In order to apply Theorems 4.1-4.2, we have to consider the following problem: under what conditions does system (4.3.1) possess the normalized coordinates?

Using Frobenius' Theorem we can easily prove the following.

Lemma 4.9. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^k$, where k is an integer ≥ 1, and (4.3.3)-(4.3.5) hold. Suppose furthermore that $p > 1$ and the right eigenvectors $r_i(u)$ $(i = 1, \cdots, p)$ corresponding to the multiple eigenvalues $\lambda_i(u)$ $(i = 1, \cdots, p)$ satisfy the following completely integrable condition:

$$[r_i, r_j] \in \operatorname{span}\{r_1(u), \cdots, r_p(u)\}, \quad \forall\, i, j = 1, \cdots, p, \tag{4.6.1}$$

where span $\{r_1(u), \cdots, r_p(u)\}$ stands for the linear space spanned by the right eigenvectors $r_1(u), \cdots, r_p(u)$, and $[\cdot, \cdot]$ denotes Poisson's bracket defined by

$$[r_i, r_j] = \left(r_i^T \cdot \nabla\right) r_j - \left(r_j^T \cdot \nabla\right) r_i. \tag{4.6.2}$$

Then there exists an invertible C^{k+1} transformation $u = u(\tilde{u})$ $(u(0) = 0)$ such that in $\tilde{u}$-space

$$\tilde{r}_i\left(\sum_{h=1}^{p} \tilde{u}_h e_h\right) \equiv e_i \quad (i = 1, \cdots, p), \quad \forall\, |\tilde{u}_h| \text{ small} \quad (h = 1, \cdots, p) \tag{4.6.3}$$

and

$$\tilde{r}_j(\tilde{u}_j e_j) \equiv e_j, \quad \forall\, |\tilde{u}_j| \text{ small} \quad (j = p+1, \cdots, n), \tag{4.6.4}$$

namely, system (4.3.1) possesses the normalized coordinates. □

Now we investigate an important special case.

Consider the following quasilinear hyperbolic system of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{4.6.5}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown function and $f(u) = (f_1(u), \cdots, f_n(u))^T$ is suitably smooth.

Suppose that all the eigenvalues of $A(u) = \nabla f(u)$ has constant multiplicity. Without loss of generality, we suppose that on the domain under consideration

$$\lambda(u) \stackrel{\triangle}{=} \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{4.6.6}$$

where $p \geq 1$ is an integer.

By [Bo]-[Fr] we have

Lemma 4.10. The eigenvalue $\lambda(u)$ with constant multiplicity p (> 1) must be linearly degenerate in the sense of P.D.Lax, i.e., on the domain under consideration we have

$$\nabla\lambda(u) \cdot r_i(u) \equiv 0 \quad (i = 1, \cdots, p); \tag{4.6.7}$$

moreover, the completely integrable condition (4.6.1) holds. □

By Lemma 4.9 and Lemma 4.10, the quasilinear hyperbolic system of conservation laws with eigenvalues with constant multiplicity must have the normalized coordinates. Therefore, Theorems 4.1-4.2 can be applied to obtain the corresponding results.