completing this case.

Lastly, we show that the invariant $\hat{W}(\hat{Z}(L))$ satisfies the third formula (3.10), using formulas obtained above. We have

$$(q^{1/2} - q^{-1/2})\hat{W}(\hat{Z}(\uparrow))\hat{W}(\hat{Z}(\bigcirc))$$

$$= (q^{1/2} - q^{-1/2})\hat{W}(\hat{Z}(\uparrow \bigcirc))$$

$$= q^{1/2N}\hat{W}(\hat{Z}(\overset{\uparrow}{\frown\bigcirc})) - q^{-1/2N}\hat{W}(\hat{Z}(\overset{\uparrow}{\backslash\bigcirc}))$$

$$= q^{1/2N}q^{(N-1/N)/2}\hat{Z}(\uparrow) - q^{-1/2N}q^{-(N-1/N)/2}\hat{Z}(\uparrow)$$

$$= (q^{N/2} - q^{-N/2})\hat{Z}(\uparrow).$$

Hence we have $\hat{W}(\hat{Z}(\bigcirc)) = [N]$. □

4 The modified Kontsevich invariant and Vassiliev invariants

4.1 Vassiliev invariants of framed knots

We denote framed knots with even[5] framings simply by knots in this section. Let $\mathcal{K}$ be the vector space freely spanned by knots over $\mathbb{C}$. A *singular knot* is an immersion of S^1 into S^3 whose singularities are transversal double points. We regard a singular knot as an element in $\mathcal{K}$ by linearly removing each singularity by the following relation

for example, see Figure 4.1. We define the subspace $\mathcal{K}_d$ of $\mathcal{K}$ by

$$\mathcal{K}_d = span\{\text{the singular knots with } d \text{ singular points}\}.$$

---

[5]The framing of a framed knot usually changes by even, by a crossing change. Hence we consider framed knots only with even framings. If we considered framed knots only with odd framings, we obtain the same results as in this section. This suggestion is due to Thang Le.
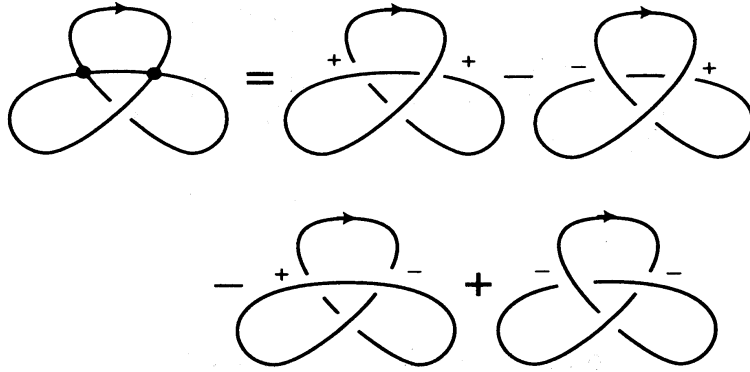
Figure 4.1: A singular knot belongs to $\mathcal{K}$

Definition 4.1. A map $v : \mathcal{K} \to \mathbb{C}$ is called a Vassiliev invariant of degree d, if $v|_{\mathcal{K}_{d+1}} = 0$.

Remark 4.2. By the above definition, we have the following identification

$$\{\text{the Vassiliev invariants of degree } d\} = (\mathcal{K}/\mathcal{K}_{d+1})^*.$$

We denote by $\mathcal{A}(S^1)^{(d)}$ the subspace of $\mathcal{A}(S^1)$ spanned by the chord diagrams of degree d.

Proposition 4.3. There exists the following natural surjection φ,

$$\varphi : \mathcal{A}(S^1)^{(d)} \twoheadrightarrow \mathcal{K}_d/\mathcal{K}_{d+1}.$$

Proof. Before giving the required map, we define a map

$$\hat{\varphi} : \{\text{the chord diagrams with } d \text{ chords and no dashed trivalent vertices}\}$$
$$\longrightarrow \mathcal{K}_d/\mathcal{K}_{d+1}$$

Let D be a chord diagram with d chords and no trivalent vertices. By collapsing each chord in D to a point, we obtain a singular S^1. Let K_D be an embedding of the singular S^1 in S^3; for example, see Figure 4.2. Any singular knots obtained from D in such a way are equivalent to each other by finite sequence of crossing changes. Note that, if two singular knots with d singular points are different by a crossing change, they are equivalent to each other in $\mathcal{K}_d$ modulo $\mathcal{K}_{d+1}$. Hence
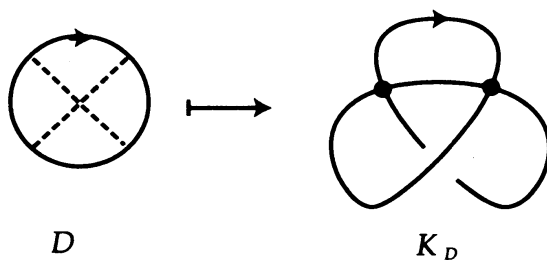
Figure 4.2: Collapsing the chords

the equivalence class $[K_D]$ in $\mathcal{K}_d/\mathcal{K}_{d+1}$ does not depend on embeddings of the singular S^1, and we obtain a well-defined map $\hat{\varphi}$.

Further we show that $\hat{\varphi}$ induces the required map φ as follows. Let D be a chord diagram of degree d. By using the STU relation, D is equivalent to a linear sum of chord diagrams with d chords and no trivalent vertices. We put $\varphi(D)$ to be the linear sum of the images of such chord diagrams by $\hat{\varphi}$. It is sufficient to show that $\varphi(D)$ does not depend on the way of removing trivalent vertices in D by the STU relation. If two chord diagrams without trivalent vertices are related by finite sequence of the STU relation, they are related by the 4T relation shown in Figure 4.3, see [3]. The embedding of singular knots corresponding to the terms in the 4T relation cancel with each other as in Figure 4.4. Therefore the map φ is well defined. $\qquad\square$
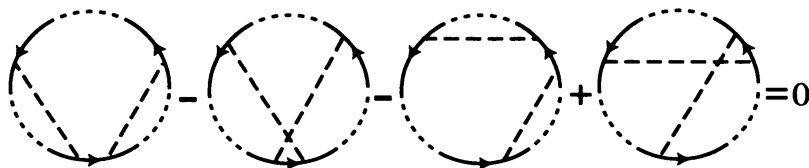

Figure 4.3: The 4T relation

We obtain the following corollary by taking the dual of the map φ.

Corollary 4.4. There exists a natural injection φ^*,

$$(\mathcal{A}(S^1)^{(d)})^* \xrightarrow{\varphi^*} \frac{\{\text{the Vassiliev invariants of degree } d\}}{\{\text{the Vassiliev invariants of degree } d-1\}}.$$

Figure 4.4: The 4T relation vanishes by $\hat{\varphi}$.

Proof. As in Remark 4.2, we identify {the Vassiliev invariants of degree d} and {the Vassiliev invariants of degree $d-1$} with $(\mathcal{K}/\mathcal{K}_{d+1})^*$ and $(\mathcal{K}/\mathcal{K}_d)^*$ respectively. Hence the right hand side is equal to $(\mathcal{K}/\mathcal{K}_{d+1})^*/(\mathcal{K}/\mathcal{K}_d)^* = (\mathcal{K}_d/\mathcal{K}_{d+1})^*$. By taking the dual of φ, we have the required injection. $\square$

For a Vassiliev invariant v of degree d, we call its image in $(\mathcal{A}(S^1)^{(d)})^*$ by the above injection φ^* the *weight system* of v.

4.2 Universality of the modified Kontsevich invariant among Vassiliev invariants

Recall that the modified Kontsevich invariant $\hat{Z}$ was constructed as the universal invariant among quantum invariants. On the other hand, the map φ is defined above, independently of $\hat{Z}$, though φ is also related to chord diagrams. In Theorem 4.5 below, we show $\hat{Z}$ induces the inverse of φ as

$$\mathcal{A}(S^1)^{(d)} \underset{[\hat{Z}]}{\overset{\varphi}{\rightleftarrows}} \mathcal{K}_d/\mathcal{K}_{d+1}.$$

The theorem is remarkable in a viewpoint of giving a direct connection between the "universal quantum invariant" and Vassiliev invariants via chord diagrams.

Theorem 4.5 ([20]). Let $\hat{Z}$ be the modified Kontsevich invariant and φ the map defined in Proposition 4.3.

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(1) $\hat{Z}$ induces a well-defined linear map $[\hat{Z}] : \mathcal{K}_d/\mathcal{K}_{d+1} \to \mathcal{A}(S^1)^{(d)}$.

(2) $[\hat{Z}] \circ \varphi$ is equal to the identity map on $\mathcal{A}(S^1)^{(d)}$.

(3) $\varphi \circ [\hat{Z}]$ is equal to the identity map on $\mathcal{K}_d/\mathcal{K}_{d+1}$.

To prove this theorem, we need the following lemma.

Lemma 4.6. Let D be a chord diagram with d chords and no trivalent vertices, and K_D the element of $\mathcal{K}_d$ obtained by collapsing chords into singular points as in the proof of Proposition 4.3. Then we have

$$\hat{Z}(K_D) = D + (\text{terms of degree} > d\ ).$$

Proof. By the definition of $\hat{Z}$, we have

We obtain $\hat{Z}(K_D)$ as

where the chord diagram D appears again as the first term in the right hand side, and the other terms have degrees more than d. This implies the required formula. $\square$

Proof of Theorem 4.5. We define $\mathcal{A}(S^1)^{(\geq d)}$ to be the subspace of $\mathcal{A}(S^1)$ spanned by the chord diagrams of degree $\geq d$. The image of $\mathcal{K}_d$ by $\hat{Z}$ is in $\mathcal{A}(S^1)^{(\geq d)}$ by Lemma 4.6; note that we remove trivalent vertices in a chord diagram before applying the lemma. Hence $\hat{Z}$ induces a map $\mathcal{K}_d \to \mathcal{A}(S^1)^{(\geq d)}$. By composing the projection $\mathcal{A}(S^1)^{(\geq d)} \to \mathcal{A}(S^1)^{(d)}$, we obtain a map $\mathcal{K}_d \to \mathcal{A}(S^1)^{(d)}$. This map takes $\mathcal{K}_{d+1}$ to 0, since $\hat{Z}(\mathcal{K}_{d+1}) \subset \mathcal{A}(S^1)^{(\geq d+1)}$ by Lemma 4.6. It follows that $\hat{Z}$ induces a well-defined map $\mathcal{K}_d/\mathcal{K}_{d+1} \to \mathcal{A}(S^1)^{(d)}$; we denote it by $[\hat{Z}]$, completing the proof of (1).

We have

$$\hat{Z}(\varphi(D)) = D + (\text{terms of degree} > d),$$

by applying Lemma 4.6 after removing trivalent vertices in D. Further we project it into $\mathcal{A}(S^1)^{(d)}$ when defining $[\hat{Z}]$. Hence we have $[\hat{Z}](\varphi(D)) = D$, completing the proof of (2).

As for (3), since φ is surjective, it suffices to show $\varphi \circ [\hat{Z}] \circ \varphi = \varphi$. It holds by (2). $\qquad\square$

There exists the inverse of φ by Theorem 4.5. Hence we have

Corollary 4.7. The natural surjection $\varphi : \mathcal{A}(S^1)^{(d)} \twoheadrightarrow \mathcal{K}_d/\mathcal{K}_{d+1}$ is an isomorphic linear map.

Further we have

Corollary 4.8. For any positive integer d and any Vassiliev invariant v of degree d, there exists the map $W : \mathcal{A}(S^1) \to \mathbb{C}$ satisfying $v(K) = W(\hat{Z}(K))$ for any knot K and $W|_{\mathcal{A}(S^1)^{(\geq d+1)}} = 0$.

Remark 4.9. The above corollary implies that any Vassiliev invariant v factors $\hat{Z}$ with some W, i.e., we have the following commutative diagram.

$$
\begin{array}{ccc}
\{\text{knots}\} & \xrightarrow{\;v\;} & \mathbb{C} \\
{}_{\hat{Z}}\searrow & & \nearrow_{W} \\
& \mathcal{A}(S^1) &
\end{array}
$$

Conversely, if we have $\hat{Z}$ and a weight system W which vanishes in $\mathcal{A}(S^1)^{(\geq d+1)}$, we obtain a Vassiliev invariant as the composition of them. Hence we can call $\hat{Z}$ the *universal Vassiliev invariant*.

Proof of Corollary 4.8. Let v be a Vassiliev invariant of degree d. We give $W_k : \mathcal{A}(S^1)^{(k)} \to \mathbb{C}$ by induction on k as follows, and obtain the required weight system by putting $W = \sum_k W_k$.

For $k > d$, we put $W_k = 0$.

For $k = d$, we put W_k to be the weight system of the Vassiliev invariant v.

For $k = d - 1$, we make a Vassiliev invariant of degree $d - 1$ from v and W_d as follows. As a restriction of the Vassiliev invariant v, we have a linear map

$v : \mathcal{K}_d \to \mathbb{C}$. Further this map induces a linear map $\mathcal{K}_d/\mathcal{K}_{d+1} \to \mathbb{C}$, since v is of degree d i.e., $v|_{\mathcal{K}_{d+1}} = 0$. By composing φ, we obtain the weight system W_k of v. Further we have the inverse $[\hat{Z}]$ of φ by Theorem 4.5. These maps are written in a diagram as:

$$
\begin{array}{ccccc}
\mathcal{K} & \supset & \mathcal{K}_d & \overset{v}{\longrightarrow} & \mathbb{C} \\
& & \text{projection} \downarrow & \circlearrowleft \nearrow \circlearrowleft & \uparrow W_d \\
& & \mathcal{K}_d/\mathcal{K}_{d+1} & \underset{\circlearrowleft}{\overset{\varphi}{\longleftarrow}} & \mathcal{A}(S^1)^{(d)} \\
& & & \underset{[\hat{Z}]}{\longrightarrow} &
\end{array}
$$

where we obtain the commutativity of this diagram by Theorem 4.5 and the definition of the weight system. Thus we have $(v - W_d \circ [\hat{Z}])|_{\mathcal{K}_d} = 0$ in the diagram. Hence the map

$$
v - W_d \circ [\hat{Z}] : \{\text{knots}\} \to \mathbb{C}
$$

is a Vassiliev invariant of degree $d - 1$. We put W_{d-1} to be the weight system of $v - W_d \circ [\hat{Z}]$.

For $k = d - 2$, we put W_{d-2} to be the weight system of $v - (W_d + W_{d-1}) \circ [\hat{Z}]$; it is a Vassiliev invariant of degree $d - 2$ by the same argument as above.

For $k = d - 3, d - 4, \cdots$, we can go on similarly for the rest. $\square$

5   Vassiliev invariants and quantum invariants

We have seen the relations between quantum invariants and the modified Kontsevich invariant in Section 3, and between Vassiliev invariants and the modified Kontsevich invariant in Section 4. In this section, we see a relation between quantum invariants and Vassiliev invariants.

Theorem 5.1 ([4]). For a framed knot K, the coefficient of h^d in $Q^{\mathfrak{g},R}(K)|_{q=e^h}$ is a Vassiliev invariant of degree d as an invariant of K.

Proof. In a construction of $Q^{\mathfrak{g},R}(K)|_{q=e^h}$, we associate positive and negative crossings with R-matrices $\mathcal{R}_+$ and $\mathcal{R}_-$, respectively. These two R-matrices