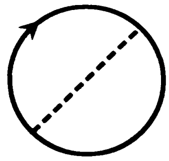


Figure 2.1: The chord diagram Θ

Invariance under RII. The invariance is derived from $R\cdot R^{-1} = R^{-1}\cdot R = 1$.
Invariance under RIII. We obtain the invariance by the hexagon relations (2.6).

□

3 The modified Kontsevich invariant and quantum invariants

In this section we show that quantum invariants recover from the modified Kontsevich invariant; we expect the recovery because of the following historical development from quantum invariants to the modified Kontsevich invariant.

T. Kohno [16] gave an expression of quantum invariants using an iterated integral solution of the Knizhnik-Zamolodchikov equation [15]. Based on Kohno's work, Drinfeld [7] led the universal version of the Knizhnik-Zamolodchikov equation; the solution of it consists of chord diagrams, not depending on a Lie algebra and its representation, and the ordinary solution recovers from the "universal" solution by substituting a Lie algebra and its representation to chord diagrams. After that, Kontsevich gave a definition of an invariant (the Kontsevich invariant) of knots using the universal solution written by the iterated integral. Further J. Murakami and Le [25] gave a combinatorial construction of the invariant (the modified Kontsevich invariant), modifying the invariant for framed links; we denote it by $\hat{Z}(L)$ for an oriented framed link L. Therefore quantum invariants should recover from the modified Kontsevich invariant by substituting a Lie algebra and its representation into chord diagrams.

3.1 Substitution of Lie algebra and its representation into a chord diagram

In this section we define a map from the set of chord diagrams to the complex number field $\mathbb{C}$ by substitution of a Lie algebra and its representation into chord diagrams. For the definition of the map, see also [2]. We call this map the *weight system* derived from substitution of the Lie algebra and the representation.

Let $\mathfrak{g}$ be a simple Lie algebra, R its irreducible representation and X a closed 1-manifold. We define a map $W_{\mathfrak{g},R} : \mathcal{A}(X) \to \mathbb{C}$ as follows. For a chord diagram D depicted in the plane, we draw horizontal lines (after deforming the chord diagram into a suitable position in the plane, if necessary) so that each part between two adjacent horizontal lines includes one vertex or one maximal or minimal point of the chord diagram as shown in the formulas (3.1) to (3.5); for an example see Figure 3.1. We put two lines on the top and the bottom of the diagram.

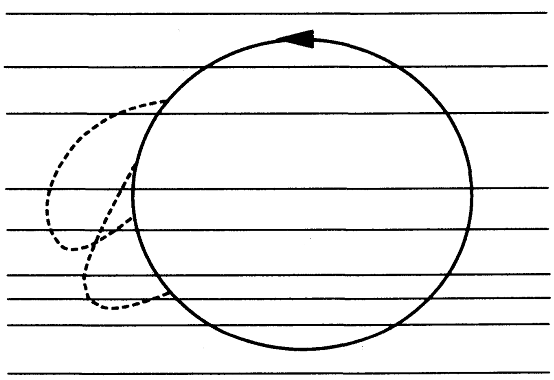
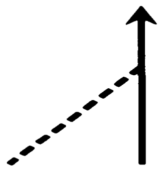


Figure 3.1: Horizontal lines on a chord diagram

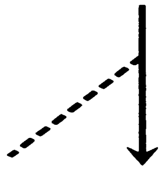
We associate a tensor product of copies of $\mathfrak{g}$, R and R^*, where R^* denotes the dual representation of R, to each horizontal line as follows. On the horizontal line the chord diagram D has sections of dashed chords and solid lines. We associate $\mathfrak{g}$ to a section of a dashed chord, R a section of an upward solid line and R^* a section of a downward solid line, and associate the tensor product of them to the horizontal line. Further we associate the trivial representation $\mathbb{C}$

to the horizontal line which has no sections of the chord diagram.

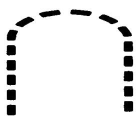
We give a linear map between two tensor products corresponding to two adjacent horizontal lines as follows,



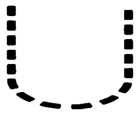
$$\begin{array}{c} R \\ \uparrow \\ \mathfrak{g} \otimes R \end{array} \tag{3.1}$$



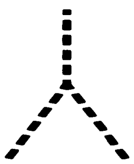
$$\begin{array}{c} R^* \\ \uparrow \\ \mathfrak{g} \otimes R^* \end{array} \tag{3.2}$$



$$\begin{array}{c} \mathbb{C} \\ \uparrow \\ \mathfrak{g} \otimes \mathfrak{g} \end{array} \tag{3.3}$$



$$\begin{array}{c} \mathfrak{g} \otimes \mathfrak{g} \\ \uparrow \\ \mathbb{C} \end{array} \tag{3.4}$$



$$\begin{array}{c} \mathfrak{g} \\ \uparrow \\ \mathfrak{g} \otimes \mathfrak{g} \end{array} \tag{3.5}$$

where the first map is defined to be the representation, the second map the dual representation, the third map the Killing form, the fourth map its dual[4] and the last map the Lie bracket.

By composing the linear maps, we have a linear map as

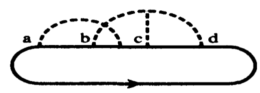
$$\mathbb{C} \longrightarrow \cdots \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes R \otimes R^* \longrightarrow \cdots \longrightarrow \mathbb{C}.$$

We define $W_{\mathfrak{g},R}(D)$ to be the image of 1 in $\mathbb{C}$ by the composed map $\mathbb{C} \to \mathbb{C}$.

In order to see that $W_{\mathfrak{g},R}(D)$ does not depend on the position of D in the plane in the above definition, we give another definition of $W_{\mathfrak{g},R}$ as follows. Let

[4]We take the dual with respect to the Killing form, which is a non-degenerate bilinear form on the simple Lie algebra.

$\{\mathfrak{g}_a\}_{a\in I}$ be an orthonormal basis with respect to the Killing form on $\mathfrak{g}$ and f_{bcd} the structure constants of $\mathfrak{g}$ relative to the basis $\{\mathfrak{g}_a\}$. For a chord diagram D, we label each dashed edge by an element of I. We associate the structure constant f_{bcd} to a trivalent dashed vertex such that the three edges around the vertex are labeled by b, c and d, and associate the base $\mathfrak{g}_a$ to a univalent dashed vertex labeled by a. We define $W_{\mathfrak{g},R}(D)$ to be the sum of the product of all labeled structure constants and the traces on R of the product of the labeled basis along solid lines, where we take the sum over all labeling. For example, we have

$W_{\mathfrak{g},R}($ 

$) = \sum_{a,b,c,d=1}^{dim\mathfrak{g}} f_{bcd}\mathrm{trace}\big(\rho(\mathfrak{g}_a)\rho(\mathfrak{g}_b)\rho(\mathfrak{g}_a)\rho(\mathfrak{g}_c)\rho(\mathfrak{g}_d)\big),$

where ρ denotes the representation $\mathfrak{g} \to End(R)$.

Lemma 3.1. The map $W_{\mathfrak{g},R} : \mathcal{A}(X) \to \mathbb{C}$ is well defined.

Proof. The above two definitions of $W_{\mathfrak{g},R}$ are equivalent; it is shown by computing the composed linear map in the first definition by using the basis in the second definition. Hence, for a given chord diagram D, the value $W_{\mathfrak{g},R}(D)$ is well defined. It is sufficient to show that $W_{\mathfrak{g},R}$ is invariant under the AS, IHX and STU relations.

Firstly, we get invariance under the AS relation by the anti-symmetry of the Lie bracket, $[X, Y] = -[Y, X]$.

Secondly, we get invariance under the IHX relation by the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0, \tag{3.6}$$

as follows. By taking in $W_{\mathfrak{g},R}$ both sides of the IHX relation, we have linear maps $\mathfrak{g}^{\otimes 3} \to \mathfrak{g}$;

$$X \otimes Y \otimes Z \mapsto [[X,Y],Z]$$
$$X \otimes Y \otimes Z \mapsto [X,[Y,Z]] - [Y,[X,Z]],$$

with respect to $X, Y, Z \in \mathfrak{g}$ shown in Figure 3.2. These two maps are equal to each other by (3.6).

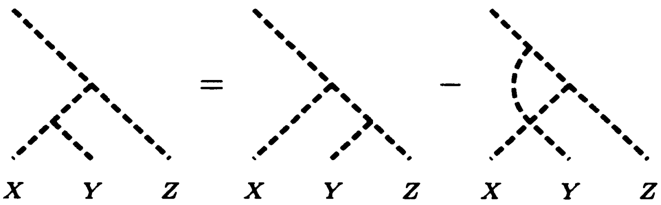


Figure 3.2: The IHX relation derived from the Jacobi identity

Lastly, we get invariance under the STU relation by the defining relation of representations $\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$; for a corresponding picture see Figure 3.3. $\square$

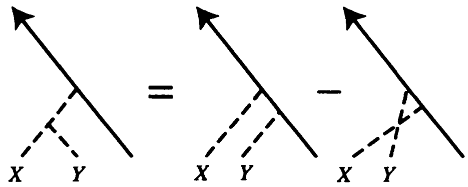


Figure 3.3: The STU relation derived from the definition of representation

Remark 3.2. If X is not closed (for example, as in Figure 3.4), for a chord diagram D on X, the map $W_{\mathfrak{g},R}(D)$ is defined to be an intertwiner between two representations of $\mathfrak{g}$ corresponding to the top and the bottom horizontal lines in the first definition of $W_{\mathfrak{g},R}$. Here we call a linear map between two representations compatible with the action of $\mathfrak{g}$ an *intertwiner*.

3.2 Recovery of the quantum $(\mathfrak{g}, R)$ invariant from the modified Kontsevich invariant

For each representation R of $\mathfrak{g}$ and an oriented framed link L, we have the quantum $(\mathfrak{g}, R)$ invariant $Q^{\mathfrak{g},R}(L)$ in $\mathbb{Z}[q^{1/2N}, q^{-1/2N}]$ where N is the rank of $\mathfrak{g}$; for example, see [40] for its definition. In this section, we show recovery of the quantum invariant by the map $\hat{W}_{\mathfrak{g},R} : \mathcal{A}(\sqcup^l S^1) \to \mathbb{C}[[h]]$ defined by $\hat{W}_{\mathfrak{g},R}(D) = W_{\mathfrak{g},R}(D)h^{deg(D)}$ for a chord diagram D, where $W_{\mathfrak{g},R}$ is the weight system derived from substitution of $\mathfrak{g}$ and R; recall that the degree $deg(D)$ is

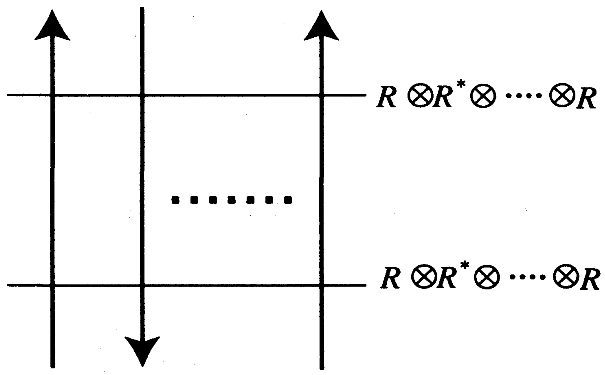


Figure 3.4: The linear map between the top and bottom horizontal lines becomes an intertwiner

half the number of univalent and trivalent vertices of the chord diagram D, *i.e.* $deg(D)$ is half the number of the chords after removing trivalent vertices from D by the STU relation.

Theorem 3.3. For any framed link L, we have

$$\hat{W}_{\mathfrak{g},R}(\hat{Z}(L)) = Q^{\mathfrak{g},R}(L)|_{q=e^h}.$$

We have a conceptual proof of the theorem, along the historical development of the modified Kontsevich invariant, see the beginning of Section 3. This approach is justified by using the uniqueness of category of quasi-Hopf algebras. See [26, Theorem 10] and [11, Theorem XX.8.3] for rigorous proofs of Theorem 3.3 in this approach.

By a direct combinatorial approach using the skein relation, J. Murakami and T. T. Q. Le [25] showed Theorem 3.3 in the case that R is the vector representation V of sl_N. In this section we give a combinatorial proof of Theorem 3.3 in that case along the proof in [25]. We denote $W_{sl_N,V}$ simply by W in this section.

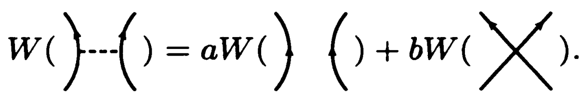
We show the following lemma before we prove the theorem.

Lemma 3.4. The following equation holds,

$$W(\,)\text{- - -}(\,) = -\frac{1}{N}W(\,)\ (\,) + W(\times),$$

where both sides belong to $\mathrm{Hom}_{sl_N}(V \otimes V \to V \otimes V)$.

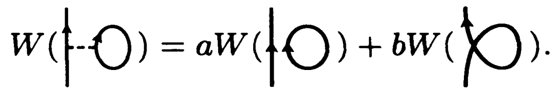
Proof. The dimension of $\mathrm{Hom}_{sl_N}(V \otimes V \to V \otimes V)$ is 2, and $W($ $)$ and $W($ $)$ are linearly independent; it is shown by the same method as below. Hence we put



(3.7)

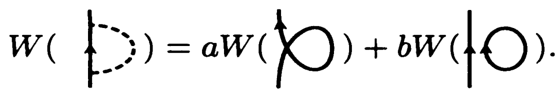
We derive two relations from the above equation as follows.

On one hand, by closing the right string in the pictures in (3.7), we obtain



If a chord diagram includes a solid circle with one dashed univalent vertex, it vanishes by W, since any intertwiner from the adjoint representation of the Lie algebra sl_N to the trivial representation is equal to the zero map; recall the definition of the weight system W given in Section 3.1. Hence we have $W($ $) = 0$. Further we have $W($ $) = NW($ $)$, since $W($ $)$ is equal to the dimension of the vector representation V. Therefore we have $aN + b = 0$.

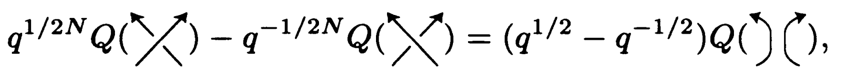
On the other hand, by closing the right string after changing two bottom ends in the pictures in (3.7), we obtain



The left hand side equals $W($ $) = (N - 1/N)W($ $)$; note that $N - 1/N$ is the eigenvalue of the Casimir element of sl_N on the vector representation V. Therefore, we have $N - 1/N = a + bN$.

Hence we have $a = -1/N$ and $b = 1$. Substituting them into (3.7) we obtain the required formula. □

Proof of Theorem 3.3. We denote $Q^{sl_N, V}(L)$ simply by $Q(L)$ in this proof. The quantum invariant $Q(L)$ is characterized by the following three relations,



(3.8)

$$Q(\uparrow\!\!\!\bigcirc) = q^{(N-1/N)/2}Q(\uparrow), \tag{3.9}$$

$$Q(\bigcirc) = [N] = \frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}}, \tag{3.10}$$

where we call the first relation ***skein relation.*** It is sufficient to show that $\hat{W}(\hat{Z}(L))$ also satisfies the above three formulas.

Firstly, we show the skein relation (3.8) for $\hat{W}(\hat{Z}(L))$. We put

$$H =)\text{- - -}(, \quad P = \nwarrow\!\!\!\!\nearrow \quad \text{and } 1 = \uparrow\ \uparrow.$$

By the definition of $\hat{W}$ and $\hat{Z}$ we have

$$\begin{aligned}
\hat{W}(\hat{Z}(\nwarrow\!\!\!\!\nearrow)) &= \hat{W}\big(P(1+\frac{1}{2}H+\frac{1}{8}H^2+\cdots+\frac{1}{n!\cdot 2^n}H^n+\cdots)\big) \\
&= W\big(P(1+\frac{h}{2}H+\frac{h^2}{8}H^2+\cdots+\frac{h^n}{n!\cdot 2^n}H^n+\cdots)\big) \\
&= W(Pe^{hH/2}).
\end{aligned}$$

By the above lemma, this equals $W(Pe^{h(P-1/N)/2})$. Then we have

$$e^{h/2N}\hat{W}(\hat{Z}(\nwarrow\!\!\!\!\nearrow)) = W(Pe^{hP/2}).$$

In the same way as above, we have

$$e^{-h/2N}\hat{W}(\hat{Z}(\nwarrow\!\!\!\!\nearrow)) = W(Pe^{-hP/2}).$$

From the above two equations, the left hand side of the skein relation is equal to $W(Pe^{hP/2} - Pe^{-hP/2})$. Note that every term in the expansion has even power of P. Since $P^2 = 1$, the left hand side is equal to $W(e^{h/2} - e^{-h/2})$. This is equal to the right hand side of the skein relation.

Secondly, we show that the invariant $\hat{W}(\hat{Z}(L))$ satisfies the second formula (3.9). Applying the following relation

$$W(\uparrow\!\!\!\bigcirc) = (N - \frac{1}{N})W(\uparrow)$$

to the formula (2.7), we obtain

$$\hat{W}(\hat{Z}(\uparrow\!\!\!\bigcirc)) = e^{h(N-1/N)/2}\hat{W}(\hat{Z}(\uparrow)),$$

completing this case.

Lastly, we show that the invariant $\hat{W}(\hat{Z}(L))$ satisfies the third formula (3.10), using formulas obtained above. We have

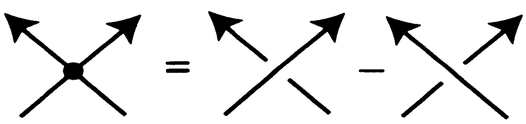
$$
\begin{aligned}
&(q^{1/2}-q^{-1/2})\hat{W}(\hat{Z}(\uparrow))\hat{W}(\hat{Z}(\bigcirc))\\
&=(q^{1/2}-q^{-1/2})\hat{W}(\hat{Z}(\uparrow\bigcirc))\\
&=q^{1/2N}\hat{W}(\hat{Z}(\ )) - q^{-1/2N}\hat{W}(\hat{Z}(\ ))\\
&=q^{1/2N}q^{(N-1/N)/2}\hat{Z}(\uparrow) - q^{-1/2N}q^{-(N-1/N)/2}\hat{Z}(\uparrow)\\
&=(q^{N/2}-q^{-N/2})\hat{Z}(\uparrow).
\end{aligned}
$$

Hence we have $\hat{W}(\hat{Z}(\bigcirc)) = [N]$. □

4 The modified Kontsevich invariant and Vassiliev invariants

4.1 Vassiliev invariants of framed knots

We denote framed knots with even[5] framings simply by knots in this section. Let $\mathcal{K}$ be the vector space freely spanned by knots over $\mathbb{C}$. A *singular knot* is an immersion of S^1 into S^3 whose singularities are transversal double points. We regard a singular knot as an element in $\mathcal{K}$ by linearly removing each singularity by the following relation



for example, see Figure 4.1. We define the subspace $\mathcal{K}_d$ of $\mathcal{K}$ by

$$\mathcal{K}_d = span\{\text{the singular knots with } d \text{ singular points}\}.$$

[5]The framing of a framed knot usually changes by even, by a crossing change. Hence we consider framed knots only with even framings. If we considered framed knots only with odd framings, we obtain the same results as in this section. This suggestion is due to Thang Le.