Advanced Studies in Pure Mathematics 78, 2018
Singularities in Generic Geometry
pp. 365–381

Singularities of secant maps on closed plane curves

María Carmen Romero Fuster and
Luís Sanhermelando Rodríguez

Abstract.

We study the singularities of secant maps associated to pairs of plane curves providing their geometrical interpretation up to codimension 2. We show that for most pairs of closed plane curves the secant map is a stable map from the torus to the plane. We determine the isotopy type of the singular set of the secant map associated to pairs of convex closed curves in terms of their Whitney indices.

§1. Introduction

The local properties of secant maps associated to curves in 3-space were studied by J.W. Bruce [2], who proved that for generic pairs of curves these map is locally stable and thus may only have isolated crosscap points. We consider here the secant map associated to closed plane curves $\alpha, \beta : S^1 \to \mathbb{R}^2$ and analyze its singularities from the local and multi-local viewpoints, providing their geometrical characterization up to codimension 2 (Theorem 3.1). As a consequence of Thom's fundamental transversality lemma, we show that the secant map of a generic pair of closed plane curves is a stable map from the torus to the plane (Theorem 4.5). In the particular case $\alpha = \beta$, we see that for most rigid motions ϕ on the plane, the pair $(\alpha, \phi \cdot \alpha)$ is a generic couple of curves.

From a global viewpoint, we prove that the number of singular curves of the secant map of a generic pair of closed convex curves with respective Whitney indexes n and m, is exactly twice the maximum common divisor $\mu_{n,m}$ of n and m (Theorem 5.10). Moreover, all the singular curves are of type $(\frac{n}{\mu_{n,m}}, \frac{m}{\mu_{n,m}})$. As a consequence, we get that

Received January 26, 2016.
Revised January 28, 2017.
Key words and phrases. Singularities, stability, plane curves, Whitney index, isotopy invariants.
Work partially supported by DGCYT and FEDER grant no. MTM2015-64013-P.

given a convex curve α with Whitney index n, the secant map of pair $(\alpha, \phi \cdot \alpha)$ (where ϕ is any rigid motion such that $(\alpha, \phi \cdot \alpha)$ is a good pair of curves) has exactly n singular curves, all of them being toric curves of type $(1,1)$.

The general case, including non necessarily convex curves, will be treated in a forthcoming paper. We would like to thank J. Martínez Alfaro and R. Oset for helpful comments.

§2. Isotopy invariants of closed curves

It is a well known fact that a generic immersion of S^1 into the plane has normal crossings, in other words, a finite number of transverse double points. In fact, standard transversality arguments in jet spaces lead, as a consequence of Thom's Transversality Theorem ([7]), to the fact that the set of regular closed plane curves with normal crossings is open and dense in $C^\infty(S^1, \mathbb{R}^2)$ with the Whitney C^∞-topology. Analogous arguments show that for a curve α lying in an open and dense subset of $C^\infty(S^1, \mathbb{R}^2)$, the inflection points are isolated and do not coincide with the vertices of α. This means that the curvature of α and its derivative do not vanish at the same time. We shall denote by S the open and dense subset of $C^\infty(S^1, \mathbb{R}^2)$ made of closed plane curves with normal crossings and isolated inflection points that satisfy this condition.

An *isotopy* between two regular plane curves α and β is a smooth map

$$\begin{array}{rcl} F : S^1 \times [0,1] & \longrightarrow & \mathbb{R}^2 \\ (s,u) & \longmapsto & F_u(s) \end{array}$$

such that F_u is a regular plane curve for all $u \in [0,1]$ and $F_0 = \alpha$ and $F_1 = \beta$

We say that F is a *stable isotopy* provided F_u is a stable immersion, for all $u \in [0,1]$, that is, F_u is a closed regular plane curve with normal crossings.

The *Gauss map* of a regular curve $\alpha : S^1 \to \mathbb{R}^2$ is given by

$$\begin{array}{rcl} G_\alpha : S^1 & \longrightarrow & S^1 \\ s & \longmapsto & \dfrac{\alpha'(s)}{\|\alpha'(s)\|}. \end{array}$$

Denote by κ_α the curvature function of α. An inflection point s_0 of α is characterized by the condition $\kappa_\alpha(s_0) = 0$, or equivalently $\alpha''(s_0) = 0$.

So we have that s_0 is a singular point of G_α if and only if it is an inflection point of α.

The *Whitney index* of a closed curve α is defined as the degree of the map $G_\alpha : S^1 \to S^1$. We denote it by i_α. The following theorem tells us that this index is a complete isotopy invariant for the space of closed regular plane curves.

Theorem 1 (H. Whitney [10]). *Two closed regular plane curves are isotopic if and only if they have the same Whitney index.*

We recall that the first homology group of the torus $S^1 \times S^1$ is generated by two loops that we shall denote as γ_1 (parallel) and γ_2 (meridian) and we have that any loop γ in the torus is homotopic to $a\gamma_1 + b\gamma_2$, for convenient $a, b \in \mathbb{Z}$, we then say that γ is of type (a, b). Closed curves of type $(0, 0)$ are homotopically trivial. Two curves on the torus are isotopic if and only if they have the same type (a, b).

§3. Singularities of secant maps on plane curves

Given two plane curves $\alpha, \beta : S^1 \to \mathbb{R}^2$, the *secant map* between them is defined as:

$$\begin{array}{rcc} S_{\alpha,\beta} : S^1 \times S^1 & \longrightarrow & \mathbb{R}^2 \\ (s,t) & \longmapsto & (\alpha(s) - \beta(t)). \end{array}$$

Our purpose in this section is to analyze the singularities of secant maps associated to couples of curves (α, β) providing their geometrical interpretation. All the curves we consider along this paper will be immersed curves.

We shall say that a smooth map f from a surface X to the plane is $\mathcal{A}$-*stable* if any element lying in a small enough open neighbourhood of f in $C^\infty(X, \mathbb{R}^2)$ with the Whitney C^∞-topology is $\mathcal{A}$-equivalent to f. A well known theorem of Whitney states that the critical set of stable maps from a surface to the plane is composed of smooth curves made of fold points and isolated cusp points ([7], [11]). From the global viewpoint, we have that the image of the critical set (apparent contour) of a stable map from a closed surface to the plane is a collection of closed plane curves with transverse intersections and isolated singular points (simple cusps) which correspond to the cusps of the map.

Given a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^2$, we denote its singular set by Σf and its apparent contour $f(\Sigma f)$ by $C(f)$. The codimension 1 germs from the plane to the plane were first studied by Gaffney and Ruas (unpublished work) and subsequently by other authors [1, 3, 4, 6, 12]. A complete classification of the simple germs from the plane to the plane, including all the germs of codimensions one and two, was obtained by J. Rieger [9]. The following list contains all these germs up to codimension 2. Observe that all of them have corank at most one.

singularity	**normal form**	**$\mathcal{A}$-codimension**
regular	(x, y)	0
fold	(x, y^2)	0
cusp	$(x, y^3 + xy)$	0
lips and beaks ($4_2^{\pm}$)	$(x, y^3 \pm x^2y)$	1
goose (4_3)	$(x, y^3 + x^3y)$	2
swallowtail (5)	$(x, y^4 + xy)$	1
butterfly ($6^{\pm}$)	$(x, xy + y^5 \pm y^7)$	2
gulls (11_5)	$(x, xy^2 + y^4 + y^5)$	2

It is well known that the set of stable maps from a closed surface M to the plane is a residual subset of $C^\infty(M, S^1)$.

Given a germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of corank one, by appropriate changes of variables at the source and the target, we can write it as:

$$\begin{array}{rcl} f : (\mathbb{R}^2, 0) & \longrightarrow & (\mathbb{R}^2, 0) \\ (x, y) & \longmapsto & (x, f_2(x, y)) \end{array}$$

where $j^\infty f_2(x, y) = \sum_{i,j=1}^{\infty} c_{i,j} x^i y^j$.

The following tables provide the characterization of each singularity in terms of the coefficients $c_{i,j}$.

singularity	**normal form**	**conditions**	**frequency**
regular	(x, y)	$c_{0,1} \neq 0$	open dense subset
fold	(x, y^2)	$c_{0,1} = 0$ $c_{0,2} \neq 0$	smooth curves
cusp	$(x, y^3 + xy)$	$c_{0,1} = 0$ $c_{0,2} = 0$ $c_{1,1} \neq 0$ $c_{0,3} \neq 0$	isolated points
swallowtail	$(x, y^4 + xy)$	$c_{0,1} = 0$ $c_{0,2} = 0$ $c_{0,3} = 0$ $c_{1,1} \neq 0$ $c_{0,4} \neq 0$	isolated points in 1-parameter families

singularity	normal form	conditions	frequency
lips beaks	$(x, y^3 + x^2y)$ $(x, y^3 - x^2y)$	$c_{0,1} = 0$ $c_{0,2} = 0$ $c_{1,1} = 0$ $c_{0,3} \neq 0$	isolated points in 1-parameter families
lips beaks		$3c_{2,1}c_{0,3} > c_{1,2}^2$ $3c_{2,1}c_{0,3} < c_{1,2}^2$	
butterfly ($6\pm$)		$c_{0,1} = c_{0,2} = 0$ $c_{0,3} = c_{0,4} = 0$ $c_{1,1}, c_{0,5} \neq 0$	isolated points in 2-parameter families
6^+ 6^-	$(x, xy + y^5 + y^7)$ $(x, xy + y^5 - y^7)$	$c_{0,7} > 0$ $c_{0,7} < 0$	
goose (4_3)	$(x, y^3 + x^3y)$	$c_{0,1} = c_{0,2} = 0$ $c_{1,1} = c_{2,1} = 0$ $c_{0,3}, c_{3,1} \neq 0$	isolated points in 2-parameter families
gulls (11_5)	$(x, xy^2 + y^4 + y^5)$	$c_{0,1} = c_{0,2} = 0$ $c_{1,1} = c_{0,3} = 0$ $c_{1,2}, c_{0,4}, c_{0,5} \neq 0$	isolated points in 2-parameter families

The list of multi-germs from the plane to the plane up to codimension one was first determined by Chíncaro ([5]). A complete list of multi-germs up to codimension 2, obtained by F. Aicardi and T. Ohmoto [8], can be seen in the next table.

singularity	normal form	conditions	frequency
transversal cross of 2 folds	$(x, y^2; \bar{x}, \bar{y}^2)$	$x = \bar{x}$ $y^2 = \bar{y}^2$ $c_{0,1} = 0$ $c_{0,2} \neq 0$ $\bar{c}_{0,1} = 0$ $\bar{c}_{0,2} \neq 0$	isolated points
cross of fold and cusp	$(x, y^2; \bar{x}, \bar{y}^3 + \bar{x}\bar{y})$	$x = \bar{x}$ $y^2 = \bar{y}^3 + \bar{x}\bar{y}$ $c_{0,1} = 0$ $c_{0,2} \neq 0$ $\bar{c}_{0,1} = 0$ $\bar{c}_{0,2} = 0$ $\bar{c}_{1,1} \neq 0$ $\bar{c}_{0,3} \neq 0$	isolated points in 1-parameter families

singularity	**normal form**	**conditions**	**frequency**
cross of 3 folds mutually transversal	$(x, y^2; \bar{x}, \bar{y}^2; \bar{\bar{x}}, \bar{\bar{y}}^2)$	$x = \bar{x} = \bar{\bar{x}}$ $y^2 = \bar{y}^2 = \bar{\bar{y}}^2$ $c_{0,1} = 0$ $c_{0,2} \neq 0$ $\bar{c}_{0,1} = 0$ $\bar{c}_{0,2} \neq 0$ $\bar{\bar{c}}_{0,1} = 0$ $\bar{\bar{c}}_{0,2} \neq 0$	isolated points in 1-parameter families

We analyze now the geometrical meaning of the singularities of the above tables in the particular case of secant maps. Given a couple of plane curves (α, β), consider the Jacobian map of $S_{\alpha,\beta}$:

$$JS_{\alpha,\beta}(s,t) = \begin{pmatrix} \alpha_1'(s) & -\beta_1'(t) \\ \alpha_2'(s) & -\beta_2'(t) \end{pmatrix},$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s))$ and $\beta(s) = (\beta_1(s), \beta_2(s))$. Since both curves are regular, it follows that rank $JS_{\alpha,\beta}(s,t) \geq 1$, for all $(s,t) \in S^1 \times S^1$. Moreover, we have that $(s,t) \in S^1 \times S^1$ is a singular point of $S_{\alpha,\beta}$ if and only if the tangent vectors $\alpha'(s)$ and $\beta'(t)$ are parallel (we denote this as $\alpha'(s)||\beta'(t)$).

We can write $S_{\alpha,\beta}$ in the standard normal form

$$\begin{array}{rcl} f : (\mathbb{R}^2, 0) & \longrightarrow & (\mathbb{R}^2, 0) \\ (x,y) & \longmapsto & (x, f_2(x,y)) \end{array}$$

where $f_2(x,y)$, x and y are functions of s and t. We shall work locally at a singular point (s,t) that by simplicity of notation we shall consider to be $(0,0)$. Since it is a singular point of $S_{\alpha,\beta}$, we have that $\alpha'(s)||\beta'(t)$, therefore it is possible to make a change of variables in $\mathbb{R}^2$ such that $\alpha(s) = (s, a(s))$ and $\beta(t) = (t, b(t))$ for convenient functions a and b defined in a neighbourhood of $(0,0)$. Then we have,

$$\begin{array}{rcl} S_{\alpha,\beta} : (\mathbb{R}^2, 0) & \longrightarrow & (\mathbb{R}^2, 0) \\ (s,t) & \longmapsto & (s-t, a(s) - b(t)). \end{array}$$

By applying the change of variables $x = s - t, y = s + t$ we get

$$\bar{S}_{\alpha,\beta}(x,y) = (x, f_2(x,y))$$

with $f_2(x,y) = a(\frac{x+y}{2}) - b(\frac{y-x}{2})$. If we write the Taylor series of the functions a and b at the origin as $j^\infty a(0) = \sum_{i=1}^{\infty} a_i s^i$ and $j^\infty b(0) = \sum_{j=1}^{\infty} b_j t^j$, we obtain

$$j^\infty f_2(x,y) = \left(a_1\left(\tfrac{x+y}{2}\right) + a_2\left(\tfrac{x+y}{2}\right)^2 + \cdots\right) - \left(b_1\left(\tfrac{y-x}{2}\right) + b_2\left(\tfrac{y-x}{2}\right)^2 + \cdots\right)$$
$$= \tfrac{a_1+b_1}{2}x + \tfrac{a_1-b_1}{2}y + \tfrac{a_2-b_2}{4}x^2 + 2\tfrac{a_2+b_2}{4}xy + \tfrac{a_2-b_2}{4}y^2 + \cdots = \textstyle\sum_{i,j=1}^\infty c_{i,j}x^i y^j,$$

where $c_{i,j} = \begin{pmatrix} i+j \\ i \end{pmatrix} \frac{a_{i+j}+(-1)^{i+1}b_{i+j}}{2^{i+j}}$.

We can now write the values of the coefficients $c_{i,j}$ in the above tables in terms of the coefficients a_i and b_j. Here we observe that $c_{i,j} = 0$ if and only if $a_{i+j} = (-1)^i b_{i+j}$.

Lemma 2. *The goose* (4_3) *cannot occur as a singularity of secant maps.*

Proof. This follows immediately from the above table by observing that in the particular case of a secant map we have that $c_{2,1} = 0$ if and only if $a_3 = b_3$ if and only if $c_{0,3} = 0$. Q.E.D.

Now, by analyzing the coefficients $c_{i,j}$ in terms of the a_i and b_j, we obtain the following geometrical interpretation for the singularities of a secant map up to codimension 2:

singularity	**normal form**	a_i **and** b_j	**geometrical meaning**
regular	(x,y)	$a_1 \neq b_1$	$\alpha'(s) \nparallel \beta'(t)$
fold	(x,y^2)	$a_1 = b_1$ $a_2 \neq b_2$	$\alpha'(s) \parallel \beta'(t)$ $\kappa_\alpha(s) \neq \kappa_\beta(t)$
cusp	(x,y^3+xy)	$a_1 = b_1$ $a_2 = b_2 \neq 0$ $a_3 \neq b_3$	$\alpha'(s) \parallel \beta'(t)$ $\kappa_\alpha(s) = \kappa_\beta(t) \neq 0$ $\kappa'_\alpha(s) \neq \kappa'_\beta(t)$
swallowtail	(x,y^4+xy)	$a_1 = b_1$ $a_2 = b_2 \neq 0$ $a_3 = b_3 \neq 0$ $a_4 \neq b_4$	$\alpha'(s) \parallel \beta'(t)$ $\kappa_\alpha(s) = \kappa_\beta(t) \neq 0$ $\kappa'_\alpha(s) = \kappa'_\beta(t) \neq 0$ $\kappa''_\alpha(s) \neq \kappa''_\beta(t)$
    lips beaks	 (x,y^3+x^2y) (x,y^3-x^2y)	$a_1 = b_1$ $a_2 = b_2 = 0$ $a_3 \neq b_3$ $a_3 \neq 0, b_3 \neq 0$ $a_3 > b_3$ $a_3 < b_3$	$\alpha'(s) \parallel \beta'(t)$ $\kappa_\alpha(s) = 0 = \kappa_\beta(t)$ $\kappa'_\alpha(s) \neq \kappa'_\beta(t)$ $\kappa'_\alpha(s) \neq 0, \kappa'_\beta(t) \neq 0$ $\kappa'_\alpha(s) > \kappa'_\beta(t)$ $\kappa'_\alpha(s) < \kappa'_\beta(t)$

singularity	**normal form**	a_i **and** b_j	**geometrical meaning**
butterfly	$(x, xy + y^5 \pm y^7)$	$a_1 = b_1$	$\alpha'(s) \parallel \beta'(t)$
		$a_2 = b_2 \neq 0$	$\kappa_\alpha(s) = \kappa_\beta(t) \neq 0$
		$a_3 = b_3$	$\kappa'_\alpha(s) = \kappa'_\beta(t)$
		$a_4 = b_4$	$\kappa''_\alpha(s) = \kappa''_\beta(t)$
		$a_5 \neq b_5$	$\kappa'''_\alpha(s) \neq \kappa'''_\beta(t)$
6^+		$a_7 > b_7$	$\kappa'''''_\alpha(s) > \kappa'''''_\beta(t)$
6^-		$a_7 < b_7$	$\kappa'''''_\alpha(s) < \kappa'''''_\beta(t)$
gulls	$(x, xy^2 + y^4 + y^5)$	$a_1 = b_1$	$\alpha'(s) \parallel \beta'(t)$
		$a_2 = b_2 = 0$	$\kappa_\alpha(s) = \kappa_\beta(t) = 0$
		$a_3 = b_3 \neq 0$	$\kappa'_\alpha(s) = \kappa'_\beta(t) \neq 0$
		$a_4 \neq b_4$	$\kappa''_\alpha(s) \neq \kappa''_\beta(t)$

We observe that (s, t) is a *cusp point* of the secant map $S_{\alpha,\beta}$ if and only if the translation T_{12} of vector $\beta(t) - \alpha(s)$ takes the tangent line and osculating circle of α at s to the tangent line and osculating circle of β at t. In other words, the curves $T_{12} \cdot \alpha$ and β have a contact of order 2 at the point $p = T_{12} \cdot \alpha(s) = \beta(t)$.

For the codimension 1 phenomena we have:

a) (s, t) is a *swallowtail point* of $S_{\alpha,\beta}$ if and only if the curves $T_{12} \cdot \alpha$ and β have a contact of order 3 at the point $p = T_{12} \cdot \alpha(s) = \beta(t)$ and this is not an inflection point.

b) (s, t) is a *lips* (*or beaks*) *point* of $S_{\alpha,\beta}$ if and only if the curves $T_{12} \cdot \alpha$ and β have a contact of order 2 at the point $p = T_{12} \circ \alpha(s) = \beta(t)$, which is an inflection point of both curves.

By applying these arguments to all the germs of codimension lesser or equal to 2 and we can state:

Theorem 3. *The following table provides the geometrical interpretation of all the possible singularities*, *up to codimension* 2, *of the secant maps associated to a couple of plane curves* α *and* β, *where* T_{12} *denotes the translation of the plane given by the vector* $\alpha(s) - \beta(t)$:

singularity type	***$\mathcal{A}$-codimensión***	***geometrical interpretation***
regular	*0*	$\alpha'(s) \nparallel \beta'(t)$
fold	*0*	$\alpha'(s) \parallel \beta'(t)$
cusp	*0*	$\alpha'(s) \parallel \beta'(t)$ $\kappa_\alpha(s) = \kappa_\beta(t)$
lips or beaks	*1*	$\alpha'(s) \parallel \beta'(t)$ $\kappa_\alpha(s) = \kappa_\beta(t) = 0$
swallowtail	*1*	$T_{12} \circ \alpha$ *and* β *have contact of order 3*
butterfly	*2*	$T_{12} \circ \alpha$ *and* β *have contact of order 4*
gulls	*2*	$T_{12} \circ \alpha$ *and* β *have contact of order 3 and vanishing curvature*

§4. Stability of secant maps

Given two plane curves $\alpha, \beta : S^1 \longrightarrow \mathbb{R}^2$, consider their secant map

$$\begin{array}{rcl} S_{\alpha,\beta} : S^1 \times S^1 & \longrightarrow & \mathbb{R}^2 \\ (s,t) & \longmapsto & (\alpha(s) - \beta(t)). \end{array}$$

As above, we can take a convenient reparametrization and a change of variables to take this map to the form $\bar{S}_{\alpha,\beta}(x,y) = (x, f_2(x,y))$, where $f_2(x,y) = a(\frac{x+y}{2}) - b(\frac{y-x}{2})$.

We recall the following transversality result due to R. Thom (see [7])

Lemma 4. (***Fundamental transversality lemma***) *Given differentiable manifolds X, B, Y, let W be a submanifold of Y and $j : B \longrightarrow C^\infty(X,Y)$ a (non necessarily continuous) map. Suppose that $\Phi : X \times B \longrightarrow Y$ is a smooth map, such that $\Phi(x,b) = j(b)(x)$ and $\Phi \pitchfork W$. Then the subset $\{b \in B | j(b) \pitchfork W\}$ is dense in B.*

We use this lemma in order to prove the following:

Proposition 5. *Given two plane curves $\alpha, \beta : S^1 \to \mathbb{R}^2$, for almost all (in the sense of an open and dense subset of) Euclidean motions $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, the secant map of the pair $(\alpha, \phi \circ \beta)$ is a locally stable map (that is, it may only have fold curves and isolated cusp singularities) from the torus to the plane.*

Proof. Consider the subset $J^3(S^1,\mathbb{R}^2)\times J^3(S^1,\mathbb{R}^2)$ of pairs of 3-jets of closed plane curves in which we denote the natural local coordinates as $(s,\ t,\ r^1_{0,0},\ r^2_{0,0},\ r^1_{1,0},\ r^2_{1,0},\ r^1_{0,1},\ r^2_{0,1},\ r^1_{2,0},\ r^2_{2,0},\ r^1_{1,1},\ r^2_{1,1},\ r^1_{0,2},\ r^2_{0,2},\ r^1_{3,0},\ r^2_{3,0},\ r^1_{2,1},\ r^2_{2,1},\ r^1_{1,2},\ r^2_{1,2},\ r^1_{0,3},\ r^2_{0,3})$. We take in $J^3(S^1,\mathbb{R}^2)\times J^3(S^1,\mathbb{R}^2)$ the variety

$$T=\{j^3f\in J^3(S^1\times S^1,\mathbb{R}^2)\mid r^i_{k,j}=0 \text{ if } k\neq 0\wedge j\neq 0\}$$

made of 3-jets of maps with vanishing crossed derivatives. Observe that any 3-jet of $J^3(S^1\times S^1,\mathbb{R}^2)$ originated by a linear combination of two closed curves belongs to T. Denote by $O(2)$ the group of all plane rotations and define the map,

$$\begin{array}{rcl}\lambda : O(2) & \longrightarrow & C^\infty(J^3(S^1,\mathbb{R}^2)\times J^3(S^1,\mathbb{R}^2),T)\\ \varphi & \longmapsto & \lambda(\varphi)=S_\varphi\end{array}$$

where

$$\begin{array}{rcl}S_\varphi : J^3(S^1,\mathbb{R}^2)\times J^3(S^1,\mathbb{R}^2) & \longrightarrow & T\\ (j^3\alpha(0),j^3\beta(0)) & \longmapsto & j^3((\varphi\circ\alpha)-\beta)(0).\end{array}$$

It is easy to see that the map

$$\begin{array}{rcl}\Lambda : J^3(S^1,\mathbb{R}^2)\times J^3(S^1,\mathbb{R}^2)\times O(2) & \longrightarrow & T\\ (j^3\alpha(0),j^3\beta(0),\varphi) & \longmapsto & \lambda(\varphi)(j^3\alpha(0),j^3\beta(0))\end{array}$$

is a submersion and hence transversal to any submanifold of T.

We can now take $X=J^3(S^1,\mathbb{R}^2)\times J^3(S^1,\mathbb{R}^2)$, $B=O(2)$, $Y=T$, $j=\lambda$ and $\Phi=\Lambda$ and apply the above Thom's fundamental transversality to the two following submanifolds:

$$W_1=\left\{f\in T\mid \begin{array}{c} r^1_{1,0}r^2_{0,1}=r^2_{1,0}r^1_{0,1}\\ r^1_{2,0}r^2_{0,2}=r^2_{2,0}r^1_{0,2}\\ r^1_{3,0}r^2_{0,3}=r^2_{3,0}r^1_{0,3}\end{array}\right\},$$

$$W_2=\left\{f\in T\mid \begin{array}{c} r^1_{1,0}r^2_{0,1}=r^2_{1,0}r^1_{0,1}\\ r^1_{1,0}r^2_{2,0}=r^1_{2,0}r^2_{1,0}\\ r^1_{0,1}r^2_{0,2}=r^1_{0,2}r^2_{0,1}\end{array}\right\}.$$

These can be respectively considered as the subsets of 3-jets of couples of curves whose secant map has a corank one singularity of swallowtail type or worse and those having a corank two singularity of lips/beaks type or worse. As a consequence of Thom's fundamental transversality lemma, we obtain two dense subsets of rotations ϕ for which the pair $(\phi\circ\alpha,\beta)$ respectively avoids such singularities. Moreover, we can also take each one of these as an open subset, for we observe that if any one of the above requirements holds for a couple (α, β), due to the compactness of the domain, standard topological arguments warranty that they will also hold for any small enough rotation of β. Then the intersection of these two open dense subsets provides the required open and dense subset. Q.E.D.

Lemma 6. *The image of the derivative of $S_{\alpha,\beta}$ at a corank one point (s,t) is a vector parallel to $\alpha'(s)$ (and hence to $\beta'(t)$).*

Proof. Consider parametrisations α and β given by $\alpha(s) = (s, \alpha_2(s))$ and $\beta(t) = (t, \beta_2(t))$, with $j^\infty \alpha_2(0) = \sum_{i=1}^{\infty} a_i s^i$ and $j^\infty \beta_2(0) = \sum_{j=1}^{\infty} b_j t^j$. Then we can write $S_{\alpha,\beta}(s,t) = (s - t, \alpha_2(s) - \beta_2(t))$ and the differential of $S_{\alpha,\beta}$ is given by

$$DS_{\alpha,\beta} = \begin{pmatrix} 1 & -1 \\ \alpha_2'(s) & -\beta_2'(t) \end{pmatrix}.$$

The singular set $\Sigma(S_{\alpha,\beta})$ is a 1-manifold made of the points satisfying the condition $\beta_2'(t) - \alpha_2'(s) = 0$. Consider the orthogonal $(-\beta_2''(t), -\alpha_2''(s))$ to the tangent vector to $\Sigma(S_{\alpha,\beta})$ at (s,t) and take the differential of $S_{\alpha,\beta}$

$$S_{\alpha,\beta*}(-\beta_2''(t), -\alpha_2''(s)) = \begin{pmatrix} -\beta_2''(t) - \alpha_2''(s) \\ -\alpha_2'(s)\beta_2''(t) - \beta_2'(t)\alpha_2''(s) \end{pmatrix}.$$

If (s,t) is a singular point, we have $\beta_2'(t) = \alpha_2'(s)$. So $S_{\alpha,\beta*}(-\beta_2''(t), -\alpha_2''(s)) = (-\beta_2''(t) - \alpha_2''(s))(1, \alpha_2'(s))$. On the other hand, $-\beta_2''(t) - \alpha_2''(s) \neq 0$ because the point is a corank one singularity, therefore we obtain the required result. Q.E.D.

Proposition 7. *Given two plane curves $\alpha, \beta : S^1 \to \mathbb{R}^2$, for almost all Euclidean motion $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ (in the sense of an open and dense subset of motions), the secant map of the pair $(\alpha, \phi \circ \beta)$ is a globally stable map from the torus to the plane.*

Proof. We just need to add appropriate multilocal conditions to the open and dense subset of rotations obtained in the above proposition. Essentially, we need to ensure that the secant map will avoid the following phenomena:

a) Tangencial crossing of 2 folds.
b) Crossing of a fold and a cusp.
c) Crossing of 3 folds.

The arguments run in a similar way as those in Proposition 5. For the condition a) we apply lemma 4 in the following context:

$X = {}_2J^3(S^1, \mathbb{R}^2) \times {}_2J^3(S^1, \mathbb{R}^2)$ (${}_2J^3(S^1, \mathbb{R}^2)$ being the space of couples of 3-jets with different sources), $B = O(2)$, $j = \lambda$, $\Phi = \Lambda$, where

$$Y = T_2 = \{{}_2j^3 f \in {}_2J^3(S^1 \times S^1, \mathbb{R}^2) \mid \begin{array}{l} r^i_{a,b} = 0 \text{ if } a \neq 0 \wedge b \neq 0 \\ \bar{r}^i_{a,b} = 0 \text{ if } a \neq 0 \wedge b \neq 0\} \end{array}$$

and

$$W = W_3 = \left\{ f \in T_2 \mid \begin{array}{l} r^1_{0,0} = \bar{r}^1_{0,0} \wedge r^2_{0,0} = \bar{r}^2_{0,0} \\ r^1_{1,0} r^2_{0,1} = r^2_{1,0} r^1_{0,1} \\ \bar{r}^1_{1,0} \bar{r}^2_{0,1} = \bar{r}^2_{1,0} \bar{r}^1_{0,1} \\ r^1_{1,0} \bar{r}^2_{0,1} = \bar{r}^2_{1,0} r^1_{0,1} \end{array} \right\}$$

is the subset of multijets of curves having a tangencial crossing of two folds. Now, as a consequence of the Fundamental Transversality Lemma we can ensure the existence of a residual subset of rotations of the curve β, such that the secant map of the new pair does not meet W_3.

The condition b) is treated through an analogous argument where X, B and Y are as above and

$$W = W_4 = \left\{ f \in T_2 \mid \begin{array}{l} r^1_{0,0} = \bar{r}^1_{0,0} \wedge r^2_{0,0} = \bar{r}^2_{0,0} \\ r^1_{1,0} r^2_{0,1} = r^2_{1,0} r^1_{0,1} \\ \bar{r}^1_{1,0} \bar{r}^2_{0,1} = \bar{r}^2_{1,0} \bar{r}^1_{0,1} \\ \bar{r}^1_{2,0} \bar{r}^2_{0,2} = \bar{r}^2_{2,0} \bar{r}^1_{0,2} \end{array} \right\}.$$

In case c), we also take X, B are as above and put

$$Y = T_3 = \left\{ {}_3j^3 f \in {}_3J^3(S^1 \times S^1, \mathbb{R}^2) \mid \begin{array}{ll} r^i_{a,b} = 0 & \text{if } a \neq 0 \wedge b \neq 0 \\ \bar{r}^i_{a,b} = 0 & \text{if } a \neq 0 \wedge b \neq 0 \\ \bar{\bar{r}}^i_{a,b} = 0 & \text{if } a \neq 0 \wedge b \neq 0 \end{array} \right\}$$

and

$$W = W_5 = \left\{ f \in T_3 \mid \begin{array}{l} r^1_{0,0} = \bar{r}^1_{0,0} = \bar{\bar{r}}^1_{0,0} \wedge r^2_{0,0} = \bar{r}^2_{0,0} = \bar{\bar{r}}^2_{0,0} \\ r^1_{1,0} r^2_{0,1} = r^2_{1,0} r^1_{0,1} \\ \bar{r}^1_{1,0} \bar{r}^2_{0,1} = \bar{r}^2_{1,0} \bar{r}^1_{0,1} \\ \bar{\bar{r}}^1_{1,0} \bar{\bar{r}}^2_{0,1} = \bar{\bar{r}}^2_{1,0} \bar{\bar{r}}^1_{0,1} \end{array} \right\}.$$

Finally, arguing as in the previous proposition, we show the existence of open and dense subsets of rotations whose corresponding multi-jets do not cut respectively W_3, W_4 and W_5. By taking the intersections of these subsets we obtain the required open dense subset. Q.E.D.

Theorem 8. *There is an open and dense subset of couples of closed plane curves in $\mathbb{R}^2$ with the Whitney C^∞-topology, whose corresponding secant map is a (globally) stable map from the torus to the plane.*

Proof. As a consequence of the Proposition 7 we can approach any pair of closed plane curves (α, β) by a sequence of pairs $\{(\alpha_n, \beta_n)\}_{n=1}^{\infty}$ whose corresponding secant map is globally stable. From this we can conclude the density. On the other hand, to see the openness of this set, we observe that given a pair of curves (α, β), for which $S_{\alpha,\beta}$ is a globally stable map, any other couple $(\bar{\alpha}, \bar{\beta})$ which is near enough to (α, β) in the C^∞-Whitney topology (and thus in the C^3-topology) also satisfies the requirements to be globally stable (for their 3-jets will avoid the subsets W_1, W_2, W_3, W_4 and W_5 constructed in the above propositions).
Q.E.D.

Remark 9. *Observe that analogous arguments can be applied when $\alpha = \beta$ in order to show that for almost all Euclidean motions $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, the secant map of the pair $(\alpha, \phi \circ \alpha)$ is a (globally) stable map from the torus to the plane.*

§5. Global viewpoint

Definition 5.1. *Given two closed plane curves α, β we say that (α, β) is a stable pair of curves if the secant map $S_{\alpha,\beta}$ is A-stable.*

The singular set of the secant map of a stable pair of closed plane curves is a finite set of disjoint closed regular curves in the torus. Our aim in this section is to obtain global relations between the isotopy type of the closed plane curves α and β and those of the singular curves of $S_{\alpha,\beta}$ as closed curves in the torus. We denote by $\{\Sigma_i\}_{i=1}^m$ the set of singular curves of $S_{\alpha,\beta}$. Since each Σ_i is a closed regular curve in $S^1 \times S^1$, we can choose a continuous regular parametrization $\sigma_i : S^1 \to S^1 \times S^1$, such that $\sigma_i(S^1) = \Sigma_i$. Consider now the two natural projections $\Pi_s, \Pi_t : S^1 \times S^1 \longrightarrow S^1$, respectively given by $\Pi_s(s,t) = s$ and $\Pi_t(s,t) = t$ and denote by G_α and G_β the respective Gauss maps of the closed plane curves α and β. Then it is easy to see that $G_\alpha \circ \Pi_s \circ \sigma_i = G_\beta \circ \Pi_t \circ \sigma_i$. We will denote this map as G_{σ_i} and refer to it as the *Gauss map of the toric curve σ_i*.

Recall that the singularities of the Gauss map G_α (resp. G_β) are the inflection points of the plane curve α (resp. β). Then we have the following characterization for the singularities of the maps G_{σ_i}.

Lemma 10. *The map G_{σ_i} has a singularity at a point $(\alpha(s_0), \beta(t_0)) \in \Sigma_i$ if and only if either s_0 is a singular point of G_α, or t_0 is a singular point of G_β.*

Proof. Clearly, since $G_{\sigma_i} = G_\alpha \circ \Pi_s \circ \sigma_i = G_\beta \circ \Pi_t \circ \sigma_i$, the singular points of G_α and G_β lead to singular points of G_{σ_i}. On the other hand, since the map σ_i has no singular points, we have that a singularity of G_{σ_i} must be either a singular point of G_α or a singular point of Π_s. In case it is not a singular point of G_α, then we have a singular point for Π_s, but this implies that it is not a singular point of Π_t, from which we get that it must be a singular point of G_β. Q.E.D.

Definition 5.2. *We define the tangency function of a pair* (α, β) *as*

$$\begin{array}{rcl} B_{\alpha,\beta} : S^1 \times S^1 & \longrightarrow & \mathbb{R} \\ (s,t) & \longmapsto & det(\alpha'(s), \beta'(t)). \end{array}$$

Lemma 11. *The singular points of the secant map* $S_{\alpha,\beta}$ *are the zeros of* $B_{\alpha,\beta}$*.*

Proof. We have that $(s,t) \in B^{-1}_{\alpha,\beta}(0) \Leftrightarrow det(\alpha'(s), \beta'(t)) = 0 \Leftrightarrow \alpha'(s) \parallel \beta'(t) \Leftrightarrow (s,t) \in \sum(S_{\alpha,\beta})$. Q.E.D.

Remark 12. *Observe that the above lemma implies that* $B^{-1}_{\alpha,\beta}(0)$ *is a non necessarily connected regular closed curve in the torus.*

Proposition 13. *There is an open and dense subset of stable pairs* (α, β)*, for which* 0 *is a regular value of the tangency function* $B_{\alpha,\beta}$*.*

Proof. We have that 0 fails to be a regular value if there exist (s,t) such that $B_{\alpha,\beta}(s,t) = 0$ and $\frac{\partial B_{\alpha,\beta}}{\partial s}(s,t) = \frac{\partial B_{\alpha,\beta}}{\partial t}(s,t) = 0$. But this implies that $det(\alpha'(s), \beta'(t)) = 0 = det(\alpha''(s), \beta'(t)) = det(\alpha'(s), \beta''(t))$, which implies that rank $\{\alpha'(s), \beta'(t), \alpha''(s), \beta''(t)\} = 1$. This amounts to say that the curves α and β have an inflection point respectively s and t and their tangent vectors are parallel. Clearly, a small rotation of one of the curves will avoid this situation. Q.E.D.

Definition 5.3. *We say that a stable pair closed plane curves* (α, β) *is a good pair if* 0 *is a regular value of its tangency function* $B_{\alpha,\beta}$*.*

Lemma 14. *Given a good pair of closed plane curves* (α, β)*,*

a) (s,t) *is a singular point of* $\Pi_t|_{B^{-1}_{\alpha,\beta}(0)}$ *if and only if* $\alpha(s)$ *is an inflection point of* α *(= singular point of* G_α*).*

b) (s,t) *is a singular point of* $\Pi_s|_{B^{-1}_{\alpha,\beta}(0)}$ *if and only if* $\beta(t)$ *is an inflection point of* β *(= singular point of* G_β*).*

Proof. The singularities of the projections $\Pi_s, \Pi_t : S^1 \times S^1 \to S^1$ on $B^{-1}_{\alpha,\beta}(0)$ are given by the zeros of the partial derivatives of $B_{\alpha,\beta}$. Then we have,

$$s \in \Sigma\Pi_t|_{B^{-1}_{\alpha,\beta}(0)} \Leftrightarrow \frac{\partial B_{\alpha,\beta}}{\partial s}(s,t) = 0.$$

But this means that $\frac{\partial}{\partial s}(det(\alpha'(s), \beta'(t))) = det(\alpha''(s), \beta'(t)) = 0$. Now, $(s,t) \in B^{-1}_{\alpha,\beta}(0)$, so we have that rank $\{\alpha'(s), \beta'(t), \alpha''(s)\} = 1$. Therefore $\alpha(s)$ is an inflection point of α (i.e. a singular point of G_α). Analogously, (s,t) is a singular point of $\Pi_s|_{B^{-1}_{\alpha,\beta}(0)} \Leftrightarrow \beta(t)$ is an inflection point of β (i.e. a singular point of G_β). Q.E.D.

Proposition 15. *The existence of homotopically trivial connected components in the singular set of the secant map $S_{\alpha,\beta}$ implies that both curves are non convex.*

Proof. The restriction of each one of the projections Π_s and Π_t to any homotopically trivial curve in the torus has at least two singular points. Then, as a consequence of Lemma 14 we get that each one of the curves α and β must have at least two inflection points. Q.E.D.

Given two natural numbers n and m, we shall denote by $\mu_{n,m}$ the maximum common divisor of n and m.

Theorem 16. *Given a good pair of closed convex plane curves (α, β) with respective Whitney indices $n, m > 0$, we have that $S_{\alpha,\beta}$ has exactly $2\mu_{n,m}$ singular curves, all of them being toric curves of type $(\frac{n}{\mu_{n,m}}, \frac{m}{\mu_{n,m}})$.*

Proof. We can choose the same orientation in both curves. From the convexity of the curves, we get that each tangent direction occurs exactly $2n$ times in α and $2m$ times in β (for G_α and G_β are both regular maps). Given $v \in S^1$, denote $I_v = \{s \in S^1 : \alpha'(s)||v\}$ and $J_v = \{s \in S^1 : \beta'(s)||v\}$. Once we fix a point in the curve $\alpha(S^1)$ we can label the points of I_v and J_v according to the natural order given by the orientation of α and β respectively. So we can write $I_v = \{s_i\}_{i=1}^{2n}$ and $J_v = \{t_j\}_{j=1}^{2m}$. We observe that there is a unique singular curve of $S_{\alpha,\beta}$ passing through each pair $(s_i, t_j) \in I_v \times J_v$. Moreover, the projections Π_s and Π_t are strictly monotone functions in s and t (for the maxima and minima of Π_s and Π_t correspond respectively to the inflection points of α and β and these curves are assumed to be convex). Therefore, we can assume that the Gauss map G_{σ_i} is strictly increasing on each singular curve σ_i. These curves connect the point (s_i, t_j) to the point (s_{i+1}, t_{j+1}), for $i = 2, \cdots, n-1$ and $j = 2, \cdots, n-1$; moreover, for each j, the point (s_{2n}, t_j) is connected to (s_1, t_{j+1}).

From a global viewpoint, we can decompose the torus into $4nm$ rectangles with vertices at the points $\{(s_i, t_j)\}_{i,j=1}^{2n,2m}$. Since the curves are convex, it follows from Proposition 15 that there are no homotopically trivial singular curves. Then the singular set is made by a union of arcs (without self-intersections) joining the opposite vertices in each one of the rectangles (i.e., arcs joining the point (s_i, t_j) with (s_{i+1}, t_{j+1})

for $i = 1, \cdots, 2n, j = 1, \cdots, 2m$ and (s_{2n}, t_j) with (s_1, t_{j+1}) for $j = 1, \cdots, 2m)$. Such curves can be seen, up to isotopy, as the union of all the principal diagonals of the small rectangles. We now prove that these are closed curves of type $(\frac{n}{\mu_{n,m}}, \frac{m}{\mu_{n,m}})$. From the topological viewpoint, we can substitute this grid by another one at which all the rectangles have the same size $= \frac{2\pi}{2n} \times \frac{2\pi}{2m}$. In this last case, the slope of each diagonal curve is given by $\sigma = \frac{m}{n} = \frac{m'}{n'}$, where $n' = \frac{n}{\mu_{m,n}}$ and $m' = \frac{m}{\mu_{m,n}}$. This implies that, up to identification of the torus with the plane rectangle $[0, 2\pi] \times [0, 2\pi]$, each one of these lines is given by the equation $y = \frac{m'}{n'}(x - x_0)$, where x_0 is the initial point of the considered line on the x-axis. Now, we observe that the increment of x along each vertical turn must be given by $\Delta x = 2\pi \frac{m'}{n'}$, so this curve needs to make m' turns in the vertical sense order to reach its initial point again. Analogously, it follows that it needs to make n' turns in the horizontal sense. Suppose now that there is some $m'_1 < m'$ such that $x_0 + m'_1 2\pi \frac{n'}{m'} = x_0 + 2\pi k$. Then we would have that $\frac{m'_1 - n'}{m'} = k$. But this implies that $\frac{m'}{n'} = \frac{k}{m'_1}$ which contradicts the assumption that $\mu_{n,m}$ is the maximum common divisor of m and n. Therefore, we can conclude that the number of vertical turns of each one of these curves is $m' = \frac{m}{\mu_{m,n}}$ and analogously, we get that the number of horizontal turns of each curve is $n' = \frac{n}{\mu_{m,n}}$. It now follows that the total number of curves must be $\mu_{2n,2m} = 2\mu_{n,m}$ as required. Q.E.D.

Finally, we call the attention on the following particular cases:

Corollary 16.1. *Suppose that (α, β) is a good pair of closed convex plane curves, where α is a standard circle and β is a convex plane curve with Whitney index n, then the secant map $S_{\alpha,\beta}$ has exactly 2 singular curves, both of them of type $(1, n)$.*

Corollary 16.2. *Given a convex closed plane curve α with Whitney index $n > 0$, for any rigid motion $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, such that the pair $(\alpha, \phi \cdot \alpha)$ is a good pair of curves, the secant map $S_{\alpha,\phi\cdot\alpha}$ has exactly $2n$ singular curves, all of them of type $(1, 1)$.*

Remark 17. *As a consequence of this last result we have that the topological type of the secant map of the pair $S_{\alpha,\phi\cdot\alpha}$ does not depend on the choice of the rigid motion ϕ, as far as $(\alpha, \phi \cdot \alpha)$ be a good pair of curves.*

References

[1] V.I. Arnol'd, Singularities of systems of rays. Russian Math. Surveys **38**, 1983, pp. 87–176.

[2] J.W. Bruce, Generic Space Curves and Secants. Proc. Royal Soc. Edimburgh A **98**, 1984, pp. 281–289.

[3] J.W. Bruce, Projections and reflections of generic surfaces in $\mathbb{R}^3$. Math. Scand. **54**, 1984, pp. 262–278.

[4] J.W. Bruce and P.J. Giblin, Outlines and their duals, Proc. London Math. Soc. (3) **50**, 1985, pp. 552–570.

[5] E. Chíncaro, Bifurcations of Whitney Maps. Doctoral Thesis, Univ. Fed. Minas Gerais (Brazil), 1978.

[6] T. Gaffney, The structure of $T\mathcal{A}(f)$, classification and an application to differential geometry. Singularities, Part 1 (Arcata, Calif., 1981), pp. 409–427, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983.

[7] M. Golubitsky and V. Guillemin, Stable Mappings and Their Singularities. GTM **14**, Springer-Verlag, New York, 1973.

[8] T. Ohmoto and F. Aicardi, First order local invariants of apparent contours. Toplogy **45**, 2006, pp. 27–45.

[9] J.H. Rieger, Families of maps from the plane to the plane. J. London Math. Soc. (2) **36**, 1987, pp. 351–369.

[10] H. Whitney, On regular closed curves in the plane. Compositio Math. **4**, 1937, pp. 276–284.

[11] H. Whitney, On singularities of mappings of Euclidean spaces I. Mappings of the plane into the plane. Ann. of Math. **62**, 1955, pp. 374–410.

[12] T. Yoshida, Y. Kabata and T. Ohmoto, Bifurcation of plane-to-plane mapgerms of corank 2. Q. J. Math. **66** (2015), no. 1, pp. 369–391.

Departament de Matemàtiques, Facultat de Matemàtiques, Universitat de València, Spain

E-mail address: `carmen.romero@uv.es`

E-mail address: `luis.sanhermelando@uv.es`