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On keen Heegaard splittings

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Abstract.

In this paper, we introduce a new concept of *strongly keen* for Heegaard splittings, and show that, for any integers $n \geq 2$ and $g \geq 3$, there exists a strongly keen Heegaard splitting of genus g whose Hempel distance is n.

§1. Introduction

The *curve complex* $\mathcal{C}(S)$ of a compact surface S introduced by Harvey[4] has been used to prove many deep results in 3-dimentional topology. In particular, Hempel [5] defined the *Hempel distance* for a Heegaard splitting $V_1 \cup_S V_2$ by $d(S) = d_S(\mathcal{D}(V_1), \mathcal{D}(V_2)) = \min\{d_S(x, y) \mid x \in \mathcal{D}(V_1), y \in \mathcal{D}(V_2)\}$, where d_S is the simplicial distance of $\mathcal{C}(S)$ (for the definition, see Section 2), and $\mathcal{D}(V_i)$ is the disk complex of the handlebody V_i $(i = 1, 2)$. There have been many works on Hempel distance. For example, some authors showed that the existence of high distance Heegaard splittings (see [1, 3, 5], for example). Moreover, it is also shown that there exist Heegaard splittings of Hempel distance exactly n for various integers n (see [2, 6, 7, 11, 12], for example). Here we note that the pair (x, y) in the above definition that realizes $d(S)$ may not be unique. Hence it may be natural to settle: we say that a Heegaard splitting $V_1 \cup_S V_2$ is *keen* if its Hempel distance is realized by a unique pair of elements of $\mathcal{D}(V_1)$ and $\mathcal{D}(V_2)$. Namely, $V_1 \cup_S V_2$ is keen if it satisfies the following.

- If $d_S(a, b) = d_S(a', b') = d_S(\mathcal{D}(V_1), \mathcal{D}(V_2))$ for $a, a' \in \mathcal{D}(V_1)$ and $b, b' \in \mathcal{D}(V_2)$, then $a = a'$ and $b = b'$.

In Proposition 3.1, we give necessary conditions for a Heegaard splitting to be keen. We note that these show that Heegaard splittings given in

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[6, 7, 11] are not keen (Remark 3.2). We also note that Proposition 3.1 shows that every genus-2 Heegaard splitting with Hempel distance $n\,(\geq 1)$ is not keen.

By the way, for a keen Heegaard splitting $V_1 \cup_S V_2$, the geodesics joining the unique pair of elements of $\mathcal{D}(V_1)$ and $\mathcal{D}(V_2)$ may not be unique (see Remark 4.15). We say that a Heegaard splitting $V_1 \cup_S V_2$ is *strongly keen* if the geodesics joining the pair of elements of $D(V_1)$ and $D(V_2)$ are unique. The main result of this paper gives the existence of strongly keen Heegaard splitting with Hempel distance n for each $g \geq 3$ and $n \geq 2$ as follows.

Theorem 1.1. *For any integers $n \geq 2$ and $g \geq 3$, there exists a 3-manifold with a strongly keen genus-g Heegaard splitting of Hempel distance n.*

§2. Preliminaries

Let S be a compact connected orientable surface. A simple closed curve in S is *essential* if it does not bound a disk in S and is not parallel to a component of ∂S. An arc properly embedded in S is *essential* if it does not co-bound a disk in S together with an arc on ∂S.

Heegaard splittings

A connected 3-manifold C is a *compression-body* if there exists a closed (possibly empty) surface F and a 0-handle B such that C is obtained from $(F \times [0,1]) \cup B$ by adding 1-handles to $F \times \{1\} \cup \partial B$. The subsurface of ∂C corresponding to $F \times \{0\}$ is denoted by $\partial_- C$, and $\partial_+ C$ denotes the subsurface $\partial C \setminus \partial_- C$ of ∂C. A compression-body C is called a *handlebody* if $\partial_- C = \emptyset$.

Let M be a closed orientable 3-manifold. We say that $V_1 \cup_S V_2$ is a *Heegaard splitting* of M if V_1 and V_2 are handlebodies in M such that $V_1 \cup V_2 = M$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2 = S$. The genus of S is called the *genus* of the Heegaard splitting $V_1 \cup_S V_2$. Alternatively, given a Heegaard splitting $V_1 \cup_S V_2$ of M, we may regard that there is a homeomorphism $f : \partial V_2 \to \partial V_1$ such that M is obtained from V_1 and V_2 by identifying ∂V_1 and ∂V_2 via f. When we take this viewpoint, we will denote the Heegaard splitting by the expression $V_1 \cup_f V_2$.

Curve complexes

Let S be a compact connected orientable surface with genus g and p boundary components, where $3g + b > 4$. We call such surfaces *non-sporadic*. The *curve complex* $\mathcal{C}(S)$ is defined as follows: each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve on S, and a collection of $k + 1$ vertices forms a k-simplex of $\mathcal{C}(S)$ if they can be realized by mutually disjoint curves in S. The *arc-and-curve complex* $\mathcal{AC}(S)$ is defined similarly, as follows: each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential properly embedded arc or an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k-simplex of $\mathcal{AC}(S)$ if they can be realized by mutually disjoint arcs or simple closed curves in S. The symbol $\mathcal{C}^0(S)$ (resp. $\mathcal{AC}^0(S)$) denotes the 0-skeleton of $\mathcal{C}(S)$ (resp. $\mathcal{AC}(S)$). Throughout this paper, for a vertex $x \in \mathcal{C}^0(S)$ we often abuse notation and use x to represent (the isotopy class of) a geometric representative of x, and we assume that any pair of geometric representatives has minimal intersections.

For two vertices a, b of $\mathcal{C}(S)$, we define the *distance* $d_{\mathcal{C}(S)}(a, b)$ between a and b, which will be denoted by $d_S(a, b)$ in brief, as the minimal number of 1-simplexes of a simplicial path in $\mathcal{C}(S)$ joining a and b. For a subset A of $\mathcal{C}^0(S)$, we define $\mathrm{diam}_S(A) :=$ the diameter of A in $\mathcal{C}(S)$. Similarly, we can define $d_{\mathcal{AC}(S)}(a, b)$ for $a, b \in \mathcal{AC}^0(S)$ and $\mathrm{diam}_{\mathcal{AC}(S)}(A)$ for $A \subset \mathcal{AC}^0(S)$.

For a sequence $a_0, a_1, \ldots, a_n$ of vertices in $\mathcal{C}(S)$ with $a_i \cap a_{i+1} = \emptyset$ in S $(i = 0, 1, \ldots, n-1)$, we denote by $[a_0, a_1, \ldots, a_n]$ the path in $\mathcal{C}(S)$ with vertices $a_0, a_1, \ldots, a_n$ in this order. We say that a path $[a_0, a_1, \ldots, a_n]$ is a *geodesic* if $n = d_S(a_0, a_n)$.

Let C be a compression-body. A disk D properly embedded in C is *essential* if ∂D is an essential simple closed curve in $\partial_+ C$. Then the *disk complex* $\mathcal{D}(C)$ is the subset of $\mathcal{C}^0(\partial_+ C)$ consisting of the vertices with representatives bounding essential disks of C.

For a genus-$g(\geq 2)$ Heegaard splitting $V_1 \cup_S V_2$, the *Hempel distance* of $V_1 \cup_S V_2$ is defined by $d_S(\mathcal{D}(V_1), \mathcal{D}(V_2)) = \min\{d_S(x, y) \mid x \in \mathcal{D}(V_1), y \in \mathcal{D}(V_2)\}$.

Subsurface projection maps

For a set Y, let $\mathcal{P}(Y)$ denote the set consisting of the finite subsets of Y. Let S be a compact connected orientable surface, and let X be a subsurface of S. We suppose that both S and X are non-sporadic, and each component of ∂X is either contained in ∂S or essential in S. Let $\pi_A : \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(X))$ and $\pi_0 : \mathcal{P}(\mathcal{AC}^0(X)) \to \mathcal{P}(\mathcal{C}^0(X))$ be maps defined as follows: for $\alpha \in \mathcal{C}^0(S)$, take a representative of α so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_A(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,
- $\pi_0(\{\alpha_1, \ldots, \alpha_n\})$ is the union, for all $i = 1, \ldots, n$, of the set of all isotopy classes of the components of $\partial N(\alpha_i \cup \partial X)$ which are

essential in X, where $N(\alpha_i \cup \partial X)$ is a regular neighborhood of $\alpha_i \cup \partial X$ in X.

We call the composition $\pi_0 \circ \pi_A$ the *subsurface projection* and denote it by π_X. We say that α *misses* X (resp. α *cuts* X) if $\alpha \cap X = \emptyset$ (resp. $\alpha \cap X \neq \emptyset$). The following lemma can be proved by using [10, Lemma 2.2].

Lemma 2.1. *Let $A \in \mathcal{P}(\mathcal{AC}^0(X))$ and $n \in \mathbb{N}$. If* $\operatorname{diam}_{\mathcal{AC}(X)}(A) \leq n$, *then* $\operatorname{diam}_X(\pi_0(A)) \leq 2n$.

The following lemma is proved by using the above lemma.

Lemma 2.2. ([6, Lemma 2.1]) *Let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $\mathcal{C}(S)$ such that every α_i cuts X. Then* $\operatorname{diam}_X(\pi_X(\alpha_0) \cup \pi_X(\alpha_n)) \leq 2n$.

Maps induced on curve complexes

Let Y, Z be non-sporadic surfaces. Suppose that there exists an embedding $\varphi : Y \to Z$ such that for each component l of ∂Y either $\varphi(l) \subset \partial Z$ or $\varphi(l)$ is essential in Z. Note that φ naturally induces maps $\mathcal{C}^0(Y) \to \mathcal{C}^0(Z)$ and $\mathcal{P}(\mathcal{C}^0(Y)) \to \mathcal{P}(\mathcal{C}^0(Z))$. Throughout this paper, under this setting, we abuse notation and use φ to denote these maps.

Let S be a non-sporadic closed surface.

Lemma 2.3. *Let X be a non-sporadic subsurface of S such that each component of ∂X is essential in S. Let $\alpha, \beta \in \mathcal{C}^0(S)$ such that α, β cut X. For any $k \in \mathbb{N}$, there exists a homeomorphism $h : S \to S$ such that $h|_{S \setminus X} = \mathrm{id}_{S \setminus X}$ and that* $\operatorname{diam}_X(\pi_X(\alpha) \cup \pi_X(h(\beta))) > k$.

Proof. Let γ be an element of $\pi_X(\beta)$. Take and fix a pseudo-Anosov homeomorphism $f : X \to X$ such that $f|_{\partial X} = \mathrm{id}_{\partial X}$. Then, by [9, Proposition 4.6], there is a positive integer n such that

$$d_X(\gamma, f^n(\gamma)) > k + \operatorname{diam}_X(\pi_X(\alpha) \cup \pi_X(\beta)).$$

Let $h : S \to S$ be the extension of f^n. Then

$$\begin{aligned}
& k + \operatorname{diam}_X(\pi_X(\alpha) \cup \pi_X(\beta)) \\
<\ & d_X(\gamma, h(\gamma)) \\
\leq\ & \operatorname{diam}_X(\pi_X(\beta) \cup \pi_X(\alpha)) + \operatorname{diam}_X(\pi_X(\alpha) \cup h(\pi_X(\beta))) \\
=\ & \operatorname{diam}_X(\pi_X(\beta) \cup \pi_X(\alpha)) + \operatorname{diam}_X(\pi_X(\alpha) \cup \pi_X(h(\beta)))
\end{aligned}$$

and hence we have $\operatorname{diam}_X(\pi_X(\alpha) \cup \pi_X(h(\beta))) > k$. Q.E.D.

The following two lemmas can be proved by using arguments in the proof of [6, Propositions 4.1, 4.4].

Lemma 2.4. *Let* $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ *and* $[\beta_0, \beta_1, \ldots, \beta_m]$ *be geodesics in* $\mathcal{C}(S)$*. Suppose that* α_n *and* β_0 *are non-separating on* S*, and let* $X = \mathrm{Cl}(S \setminus N(\alpha_n))$*. Let* $h : S \to S$ *be a homeomorphism such that*

- $h(\beta_0) = \alpha_n$*, and*
- $\mathrm{diam}_X(\pi_X(\alpha_0) \cup \pi_X(h(\beta_m))) > 2(n+m)$*.*

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n (= h(\beta_0)), h(\beta_1), \ldots, h(\beta_m)]$ *is a geodesic in* $\mathcal{C}(S)$*.*

Moreover, every geodesic connecting α_0 *and* $h(\beta_m)$ *passes through* α_n*. In fact, for any geodesic* $[\gamma_0, \gamma_1, \ldots, \gamma_{n+m}]$ *in* $\mathcal{C}(S)$ *such that* $\gamma_0 = \alpha_0$ *and* $\gamma_{n+m} = h(\beta_m)$*, we have* $\gamma_n = \alpha_n$*.*

Lemma 2.5. *Suppose that the genus of* S *is greater than* 2*. Let* $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ *and* $[\beta_0, \beta_1, \ldots, \beta_m]$ *be geodesics in* $\mathcal{C}(S)$*. Suppose that* $\alpha_{n-1} \cup \alpha_n$ *and* $\beta_0 \cup \beta_1$ *are non-separating on* S*, and let* $X = \mathrm{Cl}(S \setminus N(\alpha_{n-1} \cup \alpha_n))$*. Let* $h : S \to S$ *be a homeomorphism such that*

- $h(\beta_0) = \alpha_{n-1}$*,* $h(\beta_1) = \alpha_n$*, and*
- $\mathrm{diam}_X(\pi_X(\alpha_0) \cup \pi_X(h(\beta_m))) > 2(n+m-1)$*.*

Then $[\alpha_0, \alpha_1, \ldots, \alpha_{n-1} (= h(\beta_0)), \alpha_n (= h(\beta_1)), h(\beta_2), \ldots, h(\beta_m)]$ *is a geodesic in* $\mathcal{C}(S)$*.*

Moreover, every geodesic connecting α_0 *and* $h(\beta_m)$ *passes through* α_{n-1} *or* α_n*. In fact, for any geodesic* $[\gamma_0, \gamma_1, \ldots, \gamma_{n+m-1}]$ *in* $\mathcal{C}(S)$ *such that* $\gamma_0 = \alpha_0$ *and* $\gamma_{n+m-1} = h(\beta_m)$*, we have* $\gamma_{n-1} = \alpha_{n-1}$ *or* $\gamma_n = \alpha_n$*.*

Remark 2.6. Note that, in Lemmas 2.4 and 2.5, since S is closed and non-sporadic (that is, the genus of S is greater than 1) in Lemma 2.4 and the genus of S is greater than 2 in Lemma 2.5, the subsurfaces denoted by X are non-sporadic.

§3. Keen Heegaard splittings

Recall that a Heegaard splitting $V_1 \cup_S V_2$ is called *keen* if its Hempel distance is realized by a unique pair of elements of $\mathcal{D}(V_1)$ and $\mathcal{D}(V_2)$.

Proposition 3.1. *Let* $V_1 \cup_S V_2$ *be a genus-*$g(\geq 2)$ *Heegaard splitting with Hempel distance* $n(\geq 1)$*. Let* $[l_0, l_1, \ldots, l_n]$ *be a geodesic in* $\mathcal{C}(S)$ *such that* $l_0 \in \mathcal{D}(V_1)$ *and* $l_n \in \mathcal{D}(V_2)$*. If* $V_1 \cup_S V_2$ *is keen, then the following holds.*

(1) l_0 *and* l_n *are non-separating on* S*.*
(2) l_1 *and* l_{n-1} *are non-separating on* S*.*
(3) $l_0 \cup l_1$ *and* $l_{n-1} \cup l_n$ *are separating on* S*.*

Proof. (1) Assume on the contrary that either l_0 or l_n is separating on S. Without loss of generality, we may assume that l_0 is separating on S. Let D_0 be a disk properly embedded in V_1 such that $\partial D_0 = l_0$.

Let $V_1^{(1)}$ be the component of $V_1 \setminus D_0$ that contains l_1, and let $V_1^{(2)}$ be the other component. It is easy to see that there is an essential disk D_0' properly embedded in $V_1^{(2)}$ such that $D_0' \cap D_0 = \emptyset$. Then $l_0' := \partial D_0'$ is also disjoint from l_1, and hence, $[l_0', l_1, \ldots, l_n]$ is a geodesic in $\mathcal{C}(S)$. Hence, we have $d_S(l_0', l_n) = d_S(\mathcal{D}(V_1), \mathcal{D}(V_2))$, where l_0' is an element of $\mathcal{D}(V_1)$ different from l_0, a contradiction.

(2) Assume on the contrary that either l_1 or l_{n-1}, say l_1, is separating on S. Let $S^{(1)}$ be the component of $S \setminus l_1$ that contains l_0. Since l_0 is non-separating on S by (1) and l_1 is separating on S, we can see that l_0 is non-separating on $S^{(1)}$. Then there exists an essential simple closed curve l^* on $S^{(1)}$ such that l^* intersects l_0 transversely in one point. Let D_0 be a disk properly embedded in V_1 such that $\partial D_0 = l_0$, and let D_0^+ and D_0^- be the components of $\mathrm{Cl}(\partial N(D_0) \setminus \partial V_1)$, where $N(D_0)$ is a regular neighborhood of D_0 in V_1. Take the subarc of l^* lying outside of the product region $N(D_0)$ between D_0^+ and D_0^-, and let D_0'' be the disk in V_1 obtained from $D_0^+ \cup D_0^-$ by adding a band along the subarc of l^*. Then $l_0'' := \partial D_0''$ is also disjoint from l_1, and hence, $[l_0'', l_1, \ldots, l_n]$ is a geodesic in $\mathcal{C}(S)$. Hence, we have $d_S(l_0'', l_n) = d_S(\mathcal{D}(V_1), \mathcal{D}(V_2))$, where l_0'' is an element of $\mathcal{D}(V_1)$ different from l_0, a contradiction.

(3) Assume on the contrary that either $l_0 \cup l_1$ or $l_{n-1} \cup l_n$, say $l_0 \cup l_1$, is non-separating on S. Then there exists an essential simple closed curve l^* on S such that l^* intersects l_0 transversely in one point and $l^* \cap l_1 = \emptyset$. We can lead to a contradiction by the arguments in (2). Q.E.D.

Remark 3.2. (1) By Proposition 3.1, we see that every genus-2 Heegaard splitting with Hempel distance $n\,(\geq 1)$ is not keen. In fact, if a genus-2 Heegaard splitting $V_1 \cup_S V_2$ is keen, and $[l_0, l_1, \ldots, l_n]$ is a path that realizes the Hempel distance, then by (1) and (2) of Proposition 3.1, we see that $l_0 \cup l_1$ cuts S into four punctured sphere, contradicting (3) of Proposition 3.1. Hence, if a genus-g Heegaard splitting with Hempel distance $n\,(\geq 1)$ is keen, then $g \geq 3$.

(2) Heegaard splittings given in [6, 7, 11] are not keen, since their Hempel distances are realized by pairs of separating elements.

§4. Proof of Theorem 1.1 when $n \geq 4$

Let n and g be integers with $n \geq 4$ and $g \geq 3$. Let S be a closed connected orientable surface of genus g. Let l_0 and l_1 be non-separating simple closed curves on S such that $l_0 \cap l_1 = \emptyset$, $l_0 \cup l_1$ is separating and l_0, l_1 are not parallel on S. Let $F_1 = \mathrm{Cl}(S \setminus N(l_1))$. Choose and fix an integer $k \in \{2, 3, \ldots, n-2\}$. Let $[l_1', l_2', \ldots, l_k']$ and $[l_1'', l_2'', \ldots, l_{n-k}'']$ be geodesics in $\mathcal{C}(S)$ such that l_1', l_k', l_1'' and l_{n-k}'' are non-separating on S. (For the existence of such geodesics, see [6] or the proof of Proposition 4.14 below for example.) By Lemma 2.3, there exist homeomorphisms $h_1 : S \to S$ and $h_2 : S \to S$ such that

- $h_1(l'_1) = l_1$,
- $h_2(l''_1) = l_1$,
- $\mathrm{diam}_{F_1}(\pi_{F_1}(l_0) \cup \pi_{F_1}(h_1(l'_k))) \geq 4n + 16$, and
- $\mathrm{diam}_{F_1}(\pi_{F_1}(l_0) \cup \pi_{F_1}(h_2(l''_{n-k}))) \geq 4n + 16$.

Note that $\pi_{F_1}(l_0) = \{l_0\}$ since $l_0 \cap l_1 = \emptyset$. By Lemma 2.4, $[l_0, l_1 (= h_1(l'_1)), h_1(l'_2), \ldots, h_1(l'_k)]$ and $[l_0, l_1(= h_2(l''_1)), h_2(l''_2), \ldots, h_2(l''_{n-k})]$ are geodesics in $\mathcal{C}(S)$. Let $F_k = \mathrm{Cl}(S \setminus N(h_1(l'_k)))$. By Lemma 2.3, there exists a homeomorphism $h_3 : S \to S$ such that

- $h_3(h_2(l''_{n-k})) = h_1(l'_k)$, and
- $\mathrm{diam}_{F_k}(\pi_{F_k}(l_0) \cup \pi_{F_k}(h_3(l_0))) > 2n$.

Let $l_i = h_1(l'_i)$ for $i \in \{2, \ldots, k\}$, $l_i = h_3(h_2(l''_{n-i}))$ for $i \in \{k+1, \ldots, n-1\}$, and $l_n = h_3(l_0)$. By Lemma 2.4, $[l_0, l_1, \ldots, l_n]$ is a geodesic in $\mathcal{C}(S)$. Moreover, by the construction of the geodesic, the following are satisfied.

(G1) l_0, l_1, l_{n-1} and l_n are non-separating on S,
(G2) $l_0 \cup l_1$ and $l_{n-1} \cup l_n$ are separating on S,
(G3) $\mathrm{diam}_{F_1}(\pi_{F_1}(l_0) \cup \pi_{F_1}(l_k)) \geq 4n + 16$,
(G4) $\mathrm{diam}_{F_{n-1}}(\pi_{F_{n-1}}(l_k) \cup \pi_{F_{n-1}}(l_n)) \geq 4n + 16$, and
(G5) $\mathrm{diam}_{F_k}(\pi_{F_k}(l_0) \cup \pi_{F_k}(l_n)) > 2n$,

where $F_{n-1} = \mathrm{Cl}(S \setminus N(l_{n-1}))$.

Let C_1 and C_2 be copies of the compression-body obtained by adding a 1-handle to $F \times [0, 1]$, where F is a closed connected orientable surface of genus $g - 1$. Let D_1 (resp. D_2) be the non-separating essential disk properly embedded in C_1 (resp. C_2) corresponding to the co-core of the 1-handle. We may assume that $\partial_+ C_1 = S$ and $\partial D_1 = l_0$. Choose a homeomorphism $f : \partial_+ C_2 \to \partial_+ C_1$ such that $f(\partial D_2) = l_n$.

Let H_1 and H_2 be copies of the handlebody of genus $g - 1$. In the remainder of this section, we identify ∂H_i and $\partial_- C_i$ $(i = 1, 2)$ so that we obtain a keen Heegaard splitting of genus g whose Hempel distance is n.

For each $i = 1, 2$, let $C'_i = \mathrm{Cl}(C_i \setminus N(D_i))$ and $X_i = \partial C'_i \cap \partial_+ C_i$. Note that C'_i is homeomorphic to $\partial_- C_i \times [0, 1]$. Let $\varphi_i : C'_i \to \partial_- C_i \times [0, 1]$ be a homeomorphism such that $\varphi_i(\partial C'_i \setminus \partial_- C_i) = \partial_- C_i \times \{1\}$ and $\varphi_i(\partial_- C_i) = \partial_- C_i \times \{0\}$, and let $\psi_i : \partial_- C_i \times \{1\} \to \partial_- C_i \times \{0\}$ be the natural homeomorphism. Let $P_i : X_i \to \partial_- C_i$ be the composition of the inclusion map $X_i \to \partial C'_i \setminus \partial_- C_i$ and the map $\left(\varphi_i|_{\partial_- C_i}\right)^{-1} \circ \psi_i \circ \left(\varphi_i|_{\partial C'_i \setminus \partial_- C_i}\right) : \partial C'_i \setminus \partial_- C_i \to \partial_- C_i$.

It is clear that l_1 represents an essential simple closed curve on X_1. Since l_1 is non-separating on S, $P_1(l_1)$ is an essential simple closed curve on $\partial_- C_1$. By [5, Theorem 2.7] and its proof (see also [1, Theorem 2.4]), there exists a homeomorphism $f_1 : \partial H_1 \to \partial_- C_1$ such that

$$d_{\partial_- C_1}(f_1(\mathcal{D}(H_1)), P_1(l_1)) \geq 2. \tag{1}$$

Let $V_1 = C_1 \cup_{f_1} H_1$, that is, V_1 is the manifold obtained from C_1 and H_1 by identifying $\partial_- C_1$ and ∂H_1 via f_1. Note that V_1 is a handlebody.

Claim 4.1. *l_1 intersects every element of $\mathcal{D}(V_1) \setminus \{l_0\}$.*

Proof. Assume on the contrary that there exists an element a of $\mathcal{D}(V_1) \setminus \{l_0\}$ such that $a \cap l_1 = \emptyset$. Let D_a be a disk in V_1 bounded by a, and recall that l_0 bounds the disk D_1 in C_1, and hence, in V_1 (see Fig. 1). We may assume that $|D_a \cap D_1| = |D_a \cap N(D_1)|$ and is minimal. By using innermost disk arguments, we see that $D_a \cap D_1$ has no loop components. Let Δ be a disk properly embedded in $C_1' \cup_{f_1} H_1$ defined as follows.

- If $D_a \cap D_1 = \emptyset$, let $\Delta = D_a$.
- If $D_a \cap D_1 \neq \emptyset$, let Δ be the closure of a component of $D_a \setminus N(D_1)$ that is outermost in D_a.

Since $a \cap l_1 = \emptyset$, the disk Δ is disjoint from l_1. Since l_0, l_1 are non-separating and $l_0 \cup l_1$ is separating on S by the condition (G2), and $a \neq l_0$, we see that Δ is essential in $C_1' \cup_{f_1} H_1$.

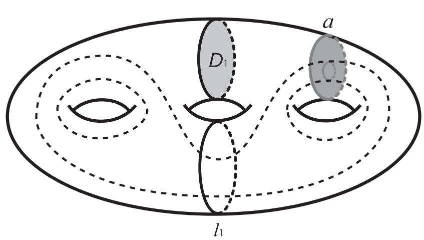

Fig. 1

Since C_1' is homeomorphic to $\partial_- C_1 \times [0, 1]$, we may assume that Δ is obtained by gluing a vertical annulus in C_1' and an essential disk Δ' in H_1 via f_1, after boundary compressions and isotopies toward $\partial_- C_1$ if necessary. This together with $\Delta \cap l_1 = \emptyset$ implies that $d_{\partial_- C_1}(f_1(\partial \Delta'), P_1(l_1)) \leq 1$. Since $f_1(\partial \Delta') \in f_1(\mathcal{D}(H_1))$, we have $d_{\partial_- C_1}(f_1(\mathcal{D}(H_1)), P_1(l_1)) \leq 1$, a contradiction to the inequality (1). Q.E.D.

Let $\pi_{F_1} = \pi_0 \circ \pi_A : \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(F_1)) \to \mathcal{P}(\mathcal{C}^0(F_1))$ be the subsurface projection introduced in Section 2. Recall that $\pi_{F_1}(l_0) = \{l_0\}$ since $l_0 \cap l_1 = \emptyset$.

Claim 4.2. *For any element $a \in \mathcal{D}(V_1)$, we have $\pi_{F_1}(a) \neq \emptyset$, and $\mathrm{diam}_{F_1}(l_0 \cup \pi_{F_1}(a)) \leq 4$.*

Proof. Note that, by Claim 4.1, we immediately have $\pi_{F_1}(a) \neq \emptyset$. If $a = l_0$ or $a \cap l_0 = \emptyset$, that is, $d_S(l_0, a) \leq 1$, then we have $\mathrm{diam}_{F_1}(l_0 \cup \pi_{F_1}(a)) \leq 2$ by Lemma 2.2. Hence, we suppose that $a \neq l_0$ and $a \cap l_0 \neq \emptyset$ in the following.

Let D_a be a disk in V_1 bounded by a, and recall that l_0 bounds the disk D_1 in V_1. Here, we may assume that $|a \cap l_1| = |a \cap N(l_1)|$ and is minimal. We may also assume that $|D_a \cap D_1| = |D_a \cap N(D_1)|$ and is minimal. Let Δ be the closure of a component of $D_a \setminus N(D_1)$ that is outermost in D_a. If $\Delta \cap l_1 = \emptyset$, then we can lead to a contradiction by arguments in the proof of Claim 4.1. Hence, $\Delta \cap l_1 \neq \emptyset$. Since $l_0 \cup l_1$ is separating on S by the condition (G2), there exists a component γ of $\mathrm{Cl}(\partial\Delta \setminus (N(D_1) \cup N(l_1)))$ such that $\partial\gamma \subset \partial N(l_1)$. It is clear that γ is an essential arc on F_1. Note that γ is disjoint from l_0, that is, $d_{\mathcal{AC}(F_1)}(l_0, \gamma) = 1$, since $l_0 \cap \Delta = \emptyset$ and γ is a subarc of $\partial\Delta$. Since $\gamma \in \pi_A(a)$, we have $d_{\mathcal{AC}(F_1)}(l_0, \pi_A(a)) \leq d_{\mathcal{AC}(F_1)}(l_0, \gamma) = 1$. Hence,

$$\begin{aligned} \mathrm{diam}_{\mathcal{AC}(F_1)}(l_0 \cup \pi_A(a)) &\leq d_{\mathcal{AC}(F_1)}(l_0, \pi_A(a)) + \mathrm{diam}_{\mathcal{AC}(F_1)}(\pi_A(a)) \\ &\leq 1 + 1 = 2. \end{aligned}$$

By Lemma 2.1, we have $\mathrm{diam}_{F_1}(l_0 \cup \pi_{F_1}(a)) \leq 4$. Q.E.D.

Lemma 4.3. *$d_S(\mathcal{D}(V_1), l_n) = n$.*

Proof. Since $l_0 \in \mathcal{D}(V_1)$, we have $d_S(\mathcal{D}(V_1), l_n) \leq n$. To prove $d_S(\mathcal{D}(V_1), l_n) = n$, assume on the contrary that $d_S(\mathcal{D}(V_1), l_n) < n$. Then there exists a geodesic $[m_0, m_1, \ldots, m_p]$ in $\mathcal{C}(S)$ such that $p < n$, $m_0 \in \mathcal{D}(V_1)$ and $m_p = l_n$.

Claim 4.4. *$m_i = l_1$ for some $i \in \{0, 1, \ldots, p\}$.*

Proof. Assume on the contrary that $m_i \neq l_1$ for every $i \in \{0, 1, \ldots, p\}$. Namely, every m_i cuts F_1. By Lemma 2.2, we have

$$\mathrm{diam}_{F_1}(\pi_{F_1}(m_0) \cup \pi_{F_1}(m_p)) \leq 2p. \tag{2}$$

Similarly, we have

$$\mathrm{diam}_{F_1}(\pi_{F_1}(l_n) \cup \pi_{F_1}(l_k)) \leq 2(n-k). \tag{3}$$

By the triangle inequality, we have

$$\begin{aligned}\operatorname{diam}_{F_1}(\pi_{F_1}(l_0)\cup\pi_{F_1}(l_k)) \quad\leq\quad & \operatorname{diam}_{F_1}(\pi_{F_1}(l_0)\cup\pi_{F_1}(m_0))\\ & + \operatorname{diam}_{F_1}(\pi_{F_1}(m_0)\cup\pi_{F_1}(m_p))\\ & + \operatorname{diam}_{F_1}(\pi_{F_1}(l_n)\cup\pi_{F_1}(l_k)).\end{aligned} \tag{4}$$

By the inequalities (2), (3), (4) and Claim 4.2, we obtain

$$\begin{aligned}\operatorname{diam}_{F_1}(\pi_{F_1}(l_0)\cup\pi_{F_1}(l_k)) \quad & \leq \quad 4+2p+2(n-k)\\ & < \quad 4+2n+2n,\end{aligned} \tag{5}$$

which contradicts the condition (G3). Q.E.D.

By Claim 4.4, we have $d_S(m_i, m_p) = d_S(l_1, l_n)$. Since $[m_0, m_1, \ldots, m_p]$ and $[l_0, l_1, \ldots, l_n]$ are geodesics, $d_S(m_i, m_p) = p-i$ and $d_S(l_1, l_n) = n-1 > p-1$. Hence, $p-i > p-1$, which implies $i = 0$, that is, $m_0 = l_1$. This contradicts Claim 4.1. Hence, we have $d_S(\mathcal{D}(V_1), l_n) = n$. Q.E.D.

Note that $f^{-1}(l_{n-1})$ represents an essential simple closed curve on X_2. Since $f^{-1}(l_{n-1})$ is non-separating on $\partial_+ C_2$ by the condition (G1), $P_2(f^{-1}(l_{n-1}))$ is an essential simple closed curve on $\partial_- C_2$. By [5, Theorem 2.7] and its proof (see also [1, Theorem 2.4]), there exists a homeomorphism $f_2 : \partial H_2 \to \partial_- C_2$ such that

$$d_{\partial_- C_2}(f_2(\mathcal{D}(H_2)), P_2(f^{-1}(l_{n-1}))) \geq 2. \tag{6}$$

Let $V_2 = C_2 \cup_{f_2} H_2$. Then $V_1 \cup_f V_2$ is a genus-g Heegaard splitting.

Claims 4.5, 4.6 and Lemma 4.7 below can be proved by the arguments similar to those for Claims 4.1, 4.2 and Lemma 4.3, respectively.

Claim 4.5. *l_{n-1} intersects every element of $f(\mathcal{D}(V_2)) \setminus \{l_n\}$.*

Claim 4.6. *For any element $a \in f(\mathcal{D}(V_2))$, we have $\pi_{F_{n-1}}(a) \neq \emptyset$, and $\operatorname{diam}_{F_{n-1}}(l_n \cup \pi_{F_{n-1}}(a)) \leq 4$.*

Lemma 4.7. *$d_S(f(\mathcal{D}(V_2)), l_0) = n$.*

Claim 4.8. *(1) $\operatorname{diam}_{F_1}(\pi_{F_1}(f(\mathcal{D}(V_2)))) \leq 12$.*
(2) $\operatorname{diam}_{F_{n-1}}(\pi_{F_{n-1}}(\mathcal{D}(V_1))) \leq 12$.

Proof. By Lemma 4.3, we have $d_S(\mathcal{D}(V_1), l_{n-1}) = n-1 \geq 3$. Hence, by [8, Theorem 1], $\operatorname{diam}_{F_{n-1}}(\pi_{F_{n-1}}(\mathcal{D}(V_1))) \leq 12$. Similarly, we have $\operatorname{diam}_{F_1}(\pi_{F_1}(f(\mathcal{D}(V_2)))) \leq 12$ by Lemma 4.7 and [8]. Q.E.D.

Lemma 4.9. *$d_S(\mathcal{D}(V_1), f(\mathcal{D}(V_2))) = n$. Namely, the Hempel distance of the Heegaard splitting $V_1 \cup_f V_2$ is n.*

Proof. Since $l_0 \in \mathcal{D}(V_1)$ and $l_n \in f(\mathcal{D}(V_2))$, we have

$$d_S(\mathcal{D}(V_1), f(\mathcal{D}(V_2))) \leq n.$$

Let $[m_0, m_1, \ldots, m_p]$ be a geodesic in $\mathcal{C}(S)$ such that $m_0 \in \mathcal{D}(V_1)$, $m_p \in f(\mathcal{D}(V_2))$ and $p \leq n$.

Claim 4.10. *$m_i = l_1$ for some $i \in \{0, 1, \ldots, p\}$.*

Proof. Assume on the contrary that $m_i \neq l_1$ for every $i \in \{0, 1, \ldots, p\}$. Namely, every m_i cuts F_1. By Lemma 2.2, we have

$$\mathrm{diam}_{F_1}(\pi_{F_1}(m_0) \cup \pi_{F_1}(m_p)) \leq 2p. \tag{7}$$

Recall that $k \in \{2, 3, \ldots, n-2\}$. Similarly, we have

$$\mathrm{diam}_{F_1}(\pi_{F_1}(l_n) \cup \pi_{F_1}(l_k)) \leq 2(n-k). \tag{8}$$

By the triangle inequality, we have

$$\begin{aligned} \mathrm{diam}_{F_1}(\pi_{F_1}(l_0) \cup \pi_{F_1}(l_k)) \leq\ & \mathrm{diam}_{F_1}(\pi_{F_1}(l_0) \cup \pi_{F_1}(m_0)) \\ & + \mathrm{diam}_{F_1}(\pi_{F_1}(m_0) \cup \pi_{F_1}(m_p)) \\ & + \mathrm{diam}_{F_1}(\pi_{F_1}(m_p) \cup \pi_{F_1}(l_n)) \\ & + \mathrm{diam}_{F_1}(\pi_{F_1}(l_n) \cup \pi_{F_1}(l_k)). \end{aligned} \tag{9}$$

By the inequalities (7), (8), (9) together with Claims 4.2 and 4.8, we obtain

$$\begin{aligned} \mathrm{diam}_{F_1}(\pi_{F_1}(l_0) \cup \pi_{F_1}(l_k)) & \leq 4 + 2p + 12 + 2(n-k) \\ & < 4 + 2n + 12 + 2n, \end{aligned} \tag{10}$$

which contradicts the condition (G3). Q.E.D.

The following claim can be proved similarly.

Claim 4.11. *$m_j = l_{n-1}$ for some $j \in \{0, 1, \ldots, p\}$.*

Note that $l_1 \notin \mathcal{D}(V_1)$ by Claim 4.1. Note also that $l_1 \notin f(\mathcal{D}(V_2))$ since, otherwise, we have $d_S(f(\mathcal{D}(V_2)), l_0) \leq d_S(l_1, l_0) = 1$, which contradicts Lemma 4.7. Since $m_0 \in \mathcal{D}(V_1)$ and $m_p \in f(\mathcal{D}(V_2))$ by the assumption, we have $m_i (= l_1) \neq m_0$ and $m_i (= l_1) \neq m_p$, which implies $1 \leq i \leq p-1$. Similarly, we have $1 \leq j \leq p-1$. Hence, we have

$$|i - j| \leq (p-1) - 1 = p - 2. \tag{11}$$

On the other hand, by Claims 4.10 and 4.11, we have

$$|i - j| = d_S(m_i, m_j) = d_S(l_1, l_{n-1}) = n - 2,$$

which together with the inequality (11) implies $p = n$. Hence,

$$d_S(\mathcal{D}(V_1), f(\mathcal{D}(V_2))) = n.$$

Q.E.D.

Lemma 4.12. *The Heegaard splitting $V_1 \cup_f V_2$ is keen.*

Proof. Let $[m_0, m_1, \ldots, m_n]$ be a geodesic in $\mathcal{C}(S)$ such that $m_0 \in \mathcal{D}(V_1)$ and $m_n \in f(\mathcal{D}(V_2))$. By the proof of Lemma 4.9, we have $m_1 = l_1$ and $m_{n-1} = l_{n-1}$. By Claims 4.1 and 4.5, we have $m_0 = l_0$ and $m_n = l_n$. Q.E.D.

In Claim 4.13 and Proposition 4.14, we show that the existence of strongly keen Heegaard splitting.

Claim 4.13. *In the above construction, if the following conditions are satisfied, then the Heegaard splitting constructed from the geodesic $[l_0, l_1, \ldots, l_n]$ is strongly keen.*

- *The geodesic $[l'_1, l'_2, \ldots, l'_k]$ (resp. $[l''_1, l''_2, \ldots, l''_{n-k}]$) is the unique geodesic from l'_1 to l'_k (resp. l''_1 to l''_{n-k}).*

Proof. By the proof of Lemma 4.12, $m_i = l_i$ holds for $i = 0$, 1, $n-1$ and n. Moreover, by the condition (G5) and Lemma 2.4, we have $m_k = l_k$. Hence, if the geodesics $[l'_1, l'_2, \ldots, l'_k]$ (resp. $[l''_1, l''_2, \ldots, l''_{n-k}]$) is the unique geodesic connecting l'_1 and l'_k (resp. l''_1 and l''_{n-k}), then we obtain the desired result. Q.E.D.

Hence the next proposition completes the proof of Theorem 1.1.

Proposition 4.14. *Let S be a closed, non-sporadic surface. For each p, there exists a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_p]$ in $\mathcal{C}(S)$ such that each α_i ($i = 0, 1, \ldots, p$) is non-separating on S and $[\alpha_0, \alpha_1, \ldots, \alpha_p]$ is the unique geodesic connecting α_0 and α_p.*

Proof. Let α_0 and α_1 be non-separating simple closed curve on S such that $\alpha_0 \cap \alpha_1 = \emptyset$, and let $X_1 = \mathrm{Cl}(S \setminus N(\alpha_1))$. Let α'_2 be a non-separating simple closed curve on S disjoint from α_1. By Lemma 2.3, there exists a homeomorphism $g_1 : S \to S$ such that $g_1(\alpha_1) = \alpha_1$ and $\mathrm{diam}_{X_1}(\pi_{X_1}(\alpha_0) \cup \pi_{X_1}(g_1(\alpha'_2))) > 4$. Let $\alpha_2 = g_1(\alpha'_2)$. By Lemma 2.4, $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic in $\mathcal{C}(S)$. Moreover, by Lemma 2.4, $[\alpha_0, \alpha_1, \alpha_2]$ is the unique geodesic connecting α_0 and α_2.

For any positive integer p, we repeat this process to construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_p]$ inductively as follows. Suppose we have constructed a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_i]$ for $i < p$ such that

- α_i is non-separating on S, and

- $[\alpha_0, \alpha_1, \ldots, \alpha_i]$ is the unique geodesic connecting α_0 and α_i.

Let $X_i = \mathrm{Cl}(S \backslash N(\alpha_i))$. Let α'_{i+1} be a non-separating simple closed curve on S disjoint from α_i. By Lemma 2.3, there exists a homeomorphism $g_i : S \to S$ such that $g_i(\alpha_i) = \alpha_i$ and $\mathrm{diam}_{X_i}(\pi_{X_i}(\alpha_0) \cup \pi_{X_i}(g_i(\alpha'_{i+1}))) > 2(i+1)$. Let $\alpha_{i+1} = g_i(\alpha'_{i+1})$. By Lemma 2.4, $[\alpha_0, \alpha_1, \ldots, \alpha_{i+1}]$ is a geodesic in $\mathcal{C}(S)$. Moreover, every geodesic connecting α_0 and α_{i+1} passes through α_i. Since $[\alpha_0, \alpha_1, \ldots, \alpha_i]$ is the unique geodesic connecting α_0 and α_i, we have that $[\alpha_0, \alpha_1, \ldots, \alpha_{i+1}]$ is the unique geodesic connecting α_0 and α_{i+1}. Hence, we obtain a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_p]$ such that every α_i $(i = 0, 1, \ldots, p)$ is non-separating on S and $[\alpha_0, \alpha_1, \ldots, \alpha_p]$ is the unique geodesic connecting α_0 and α_p. Q.E.D.

Remark 4.15. There exists a keen Heegaard splitting which is not strongly keen. For example, in the construction at the beginning of this section, let $k = 3$ and take l'_1 and l'_3 such that l'_1 and l'_3 intersect transversely in one point. Let l'_2 be an essential simple closed curve disjoint from $l'_1 \cup l'_3$. Then $[l'_1, l'_2, l'_3]$ is a geodesic in $\mathcal{C}(S)$. We can apply the arguments up to Lemma 4.12 to obtain a geodesic $[l_0, l_1, \ldots, l_n]$ and obtain a keen Heegaard splitting $V_1 \cup_f V_2$. Since l_1 and l_3 intersect transversely in one point, there exists an essential simple closed curve l^*_2 that is disjoint from $l_1 \cup l_3$ and different from l_2. Then $[l_0, l_1, l^*_2, l_3, \ldots, l_n]$ is a geodesic realizing the Hempel distance of $V_1 \cup_f V_2$ which is different from $[l_0, l_1, l_2, l_3, \ldots, l_n]$. Hence, $V_1 \cup_f V_2$ is not strongly keen.

§5. Proof of Theorem 1.1 when $n = 2$

Let $n = 2$ and g be an integer with $g \geq 3$. Let S be a closed connected orientable surface of genus g. Let l_0 and l_1 be non-separating simple closed curves on S such that $l_0 \cup l_1$ is separating on S and l_0, l_1 are not parallel on S. By Lemma 2.3, there exists a homeomorphism $h : S \to S$ such that $h(l_1) = l_1$ and

$$d_{F_1}(l_0, h(l_0)) > 12,$$

where $F_1 = \mathrm{Cl}(S \setminus N(l_1))$. Let $l_2 = h(l_0)$. By Lemma 2.4, $[l_0, l_1, l_2]$ is a geodesic in $\mathcal{C}(S)$.

Let C_1 and C_2 be copies of the compression-body obtained by adding a 1-handle to $F \times [0, 1]$, where F is a closed connected orientable surface of genus $g - 1$. Let D_1 and D_2 be the non-separating essential disk properly embedded in C_1 and C_2 corresponding to the co-cores of the 1-handles, respectively. We may assume that $\partial_+ C_1 = S$ and $\partial D_1 = l_0$. Choose a homeomorphism $f : \partial_+ C_2 \to \partial_+ C_1$ such that $f(\partial D_2) = l_2$.

Let H_i, C_i', X_i, P_i $(i = 1, 2)$ be as in Section 4. Note that l_1 is non-separating on S, and hence, $P_1(l_1)$ and $P_2(f^{-1}(l_1))$ are essential simple closed curves on $\partial_- C_1$ and $\partial_- C_2$, respectively. By [5, Theorem 2.7] and its proof (see also [1, Theorem 2.4]), there exist homeomorphisms $f_1 : \partial H_1 \to \partial_- C_1$ and $f_2 : \partial H_2 \to \partial_- C_2$ such that $d_{\partial_- C_1}(f_1(\mathcal{D}(H_1)), P_1(l_1)) \geq 2$ and $d_{\partial_- C_2}(f_2(\mathcal{D}(H_2)), P_2(f^{-1}(l_1))) \geq 2$, respectively. Let $V_i = C_i \cup_{f_i} H_i$ $(i = 1, 2)$. Then, $V_1 \cup_f V_2$ is a genus-g Heegaard splitting. By the arguments similar to those for Claims 4.1, 4.2, 4.5 and 4.6, we obtain the following.

Claim 5.1. (1) *l_1 intersects every element of $\mathcal{D}(V_1) \setminus \{l_0\}$ and every element of $f(\mathcal{D}(V_2)) \setminus \{l_2\}$.*

(2) *For any element $a \in \mathcal{D}(V_1)$, we have $\pi_{F_1}(a) \neq \emptyset$, and* $\mathrm{diam}_{F_1}(l_0 \cup \pi_{F_1}(a)) \leq 4$.

(3) *For any element $a \in f(\mathcal{D}(V_2))$, we have $\pi_{F_1}(a) \neq \emptyset$, and* $\mathrm{diam}_{F_1}(l_2 \cup \pi_{F_1}(a)) \leq 4$.

Lemma 5.2. *$V_1 \cup_f V_2$ is a strongly keen Heegaard splitting whose Hempel distance is* 2*.*

Proof. Since $l_0 \in \mathcal{D}(V_1)$ and $l_2 \in f(\mathcal{D}(V_2))$, we have

$$d_S(\mathcal{D}(V_1), f(\mathcal{D}(V_2))) \leq 2.$$

Let $[m_0, m_1, m_2]$ be a geodesic in $\mathcal{C}(S)$ such that $m_0 \in \mathcal{D}(V_1)$ and $m_2 \in f(\mathcal{D}(V_2))$. (Possibly, $m_1 \in \mathcal{D}(V_1)$ or $m_1 \in f(\mathcal{D}(V_2))$.) By Claim 5.1 (1), both m_0 and m_2 cut F_1. If m_1 also cuts F_1, then we have $\mathrm{diam}_{F_1}(\pi_{F_1}(m_0) \cup \pi_{F_1}(m_2)) \leq 4$ by Lemma 2.2, which together with Claim 5.1 (2) and (3) implies that

$$\begin{aligned} d_{F_1}(l_0, l_2) &\leq \mathrm{diam}_{F_1}(l_0 \cup \pi_{F_1}(m_0)) + \mathrm{diam}_{F_1}(\pi_{F_1}(m_0) \cup \pi_{F_1}(m_2)) \\ &\quad + \mathrm{diam}_{F_1}(\pi_{F_1}(m_2) \cup l_2) \\ &\leq 4 + 4 + 4 = 12. \end{aligned}$$

This contradicts the fact that $d_{F_1}(l_0, l_2) > 12$. Hence, m_1 misses F_1, that is, $m_1 = l_1$. By Claim 5.1 (1), we have $m_0 = l_0$ and $m_2 = l_2$, and we obtain the desired result. Q.E.D.

§6. Proof of Theorem 1.1 when $n = 3$

Let $n = 3$ and g be an integer with $g \geq 3$. Let S be a closed connected orientable surface of genus g. Let l_0 and l_1 be non-separating simple closed curves on S such that $l_0 \cup l_1$ is separating on S and l_0, l_1 are not parallel on S. Let l_2' be a simple closed curve on S such that

$l_2' \cap l_1 = \emptyset$ and $l_1 \cup l_2'$ is non-separating on S. By Lemma 2.3, there exists a homeomorphism $h_1 : S \to S$ such that $h_1(l_1) = l_1$ and

$$d_{F_1}(l_0, h_1(l_2')) > 8,$$

where $F_1 = \mathrm{Cl}(S \setminus N(l_1))$. Let $l_2 = h_1(l_2')$. By Lemma 2.4, $[l_0, l_1, l_2]$ is a geodesic in $\mathcal{C}(S)$. Note that there exists a homeomorphism $h_2 : S \to S$ such that $h_2(l_1) = l_2$ and $h_2(l_2) = l_1$, since l_1 and l_2 are non-separating on S. Let $l_3' = h_2(l_0)$. Note that $[l_1, l_2, l_3']$ is a geodesic in $\mathcal{C}(S)$.

Let $S' = \mathrm{Cl}(S \setminus N(l_1 \cup l_2))$. Let $\pi_{S'} = \pi_0 \circ \pi_A : \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(S')) \to \mathcal{P}(\mathcal{C}^0(S'))$ be the subsurface projection introduced in Section 2.

Claim 6.1. *There exists a homeomorphism $h : S \to S$ such that $h(l_1) = l_1$, $h(l_2) = l_2$ and $\mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(h(l_3'))) > 14$.*

Proof. Let γ be the closure of a component of $l_3' \setminus l_1$. Since $l_3' \cap l_2 = \emptyset$, we have $\gamma \in \pi_A(l_3')$, and hence, $\pi_0(\gamma) \in \pi_0(\pi_A(l_3')) = \pi_{S'}(l_3')$. Note that $\pi_0(\gamma)$ consists of a single simple closed curve or two disjoint simple closed curves on S'. By Lemma 2.3, there exists a homeomorphism $h : S \to S$ such that $h(l_1) = l_1$, $h(l_2) = l_2$ and $d_{S'}(\pi_{S'}(l_0), h(\pi_0(\gamma))) > 14$. This inequality, together with the fact that $h(\pi_0(\gamma)) \in h(\pi_{S'}(l_3'))$, implies

$$\begin{aligned} \mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(h(l_3'))) &= \mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup h(\pi_{S'}(l_3'))) \\ &\geq d_{S'}(\pi_{S'}(l_0), h(\pi_0(\gamma))) \\ &> 14. \end{aligned}$$

Q.E.D.

Let $l_3 = h(l_3')$. By Lemma 2.5, $[l_0, l_1, l_2, l_3]$ is a geodesic in $\mathcal{C}(S)$. Note that the following hold.

- $d_{F_1}(l_0, l_2) > 8$.
- $d_{F_2}(l_1, l_3) > 8$, where $F_2 = \mathrm{Cl}(S \setminus N(l_2))$, since $d_{F_1}(l_0, l_2) > 8$ and the homeomorphism $h \circ h_2$ sends l_0, l_1, l_2 to l_3, l_2, l_1, respectively.
- $\mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(l_3)) > 14$.

Let C_1 and C_2 be copies of the compression-body obtained by adding a 1-handle to $F \times [0, 1]$, where F is a closed connected orientable surface of genus $g - 1$. Let D_1 and D_2 be the non-separating essential disk properly embedded in C_1 and C_2 corresponding to the co-cores of the 1-handles, respectively. We may assume that $\partial_+ C_1 = S$ and $\partial D_1 = l_0$. Choose a homeomorphism $f : \partial_+ C_2 \to \partial_+ C_1$ such that $f(\partial D_2) = l_3$.

Let H_i, C_i', X_i, P_i $(i = 1, 2)$ be as in Section 4. Note that l_1 and l_2 are non-separating on S and not isotopic to l_0 or l_3. Hence, $P_1(l_1)$ and

$P_2(f^{-1}(l_2))$ are essential simple closed curves on $\partial_- C_1$ and $\partial_- C_2$, respectively. By [5, Theorem 2.7] and its proof (see also [1, Theorem 2.4]), there exist homeomorphisms $f_1 : \partial H_1 \to \partial_- C_1$ and $f_2 : \partial H_2 \to \partial_- C_2$ such that $d_{\partial_- C_1}(f_1(\mathcal{D}(H_1)), P_1(l_1)) \geq 2$ and $d_{\partial_- C_2}(f_2(\mathcal{D}(H_2)), P_2(f^{-1}(l_2))) \geq 2$, respectively. Let $V_i = C_i \cup_{f_i} H_i$ $(i = 1, 2)$. Then, $V_1 \cup_f V_2$ is a genus-g Heegaard splitting. By the arguments similar to those for Claims 4.1, 4.2, 4.5 and 4.6, we obtain the following.

Claim 6.2. (1) *l_1 intersects every element of $\mathcal{D}(V_1) \setminus \{l_0\}$, and l_2 intersects every element of $f(\mathcal{D}(V_2)) \setminus \{l_3\}$.*

(2) *For any element $a \in \mathcal{D}(V_1)$, we have $\pi_{F_1}(a) \neq \emptyset$, and* $\mathrm{diam}_{F_1}(l_0 \cup \pi_{F_1}(a)) \leq 4$.

(3) *For any element $a \in f(\mathcal{D}(V_2))$, we have $\pi_{F_2}(a) \neq \emptyset$, and* $\mathrm{diam}_{F_2}(l_3 \cup \pi_{F_2}(a)) \leq 4$.

Lemma 6.3. (1) *For any element $a \in \mathcal{D}(V_1)$, we have $\pi_{S'}(l_0) \neq \emptyset$, $\pi_{S'}(a) \neq \emptyset$, and* $\mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(a)) \leq 4$.

(2) *For any element $a \in f(\mathcal{D}(V_2))$, we have $\pi_{S'}(l_3) \neq \emptyset$, $\pi_{S'}(a) \neq \emptyset$, and* $\mathrm{diam}_{S'}(\pi_{S'}(l_3) \cup \pi_{S'}(a)) \leq 4$.

Proof. We give a proof for (1) only, since (2) can be proved similarly. Suppose that $\pi_{S'}(l_0) = \emptyset$ (resp. $\pi_{S'}(a) = \emptyset$). This means that for each component γ of $l_0 \cap S'$ (resp. $a \cap S'$), each component of $S' \setminus \gamma$ is an annulus. This shows that S' is a sphere with three boundary components, a contradiction. If $a = l_0$ or $a \cap l_0 = \emptyset$, then we have $\mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(a)) \leq 2$ by Lemma 2.2. Hence, we suppose that $a \neq l_0$ and $a \cap l_0 \neq \emptyset$ in the following.

Let D_a be a disk in V_1 bounded by a, and recall l_0 bounds the disk D_1 in V_1. We may assume that $|D_a \cap D_1|$ is minimal. Let Δ be the closure of a component of $D_a \setminus D_1$ that is outermost in D_a. Let $D_1^{(1)}$ and $D_1^{(2)}$ be the components of $D_1 \setminus \Delta$. By the minimality of $|D_a \cap D_1|$, the disks $D_1^{(1)} \cup \Delta$ and $D_1^{(2)} \cup \Delta$ are essential in V_1.

Claim 6.4. *$D_1^{(1)} \cup \Delta$ or $D_1^{(2)} \cup \Delta$, say $D_1^{(1)} \cup \Delta$, is not isotopic to D_1 in V_1.*

Proof. Let m_1 and m_2 be the two simple closed curves obtained from $l_0 (= \partial D_1)$ by a band move along $\Delta \cap \partial V_1$. Suppose both $D_1^{(1)} \cup \Delta$ and $D_1^{(2)} \cup \Delta$ are isotopic to D_1 in V_1. This implies that m_1 and m_2 are parallel in ∂V_1, and hence, they co-bound an annulus, say A, in S. Further, by slight isotopy, we may suppose that $l_0 \cap (m_1 \cup m_2) = \emptyset$. Note that l_0 is retrieved from $m_1 \cup m_2$ by a band move along an arc α such that $|\alpha \cap (\Delta \cap \partial V_1)| = 1$. Since l_0 is essential, $(\mathrm{int}\alpha) \cap A = \emptyset$. This shows that l_0 cuts off a punctured torus from ∂V_1, which contradicts the assumption that l_0 is non-separating on ∂V_1. Q.E.D.

Hence, by Claim 6.2 (1), l_1 intersects $D_1^{(1)} \cup \Delta$. Since $l_1 \cap D_1 = \emptyset$, l_1 intersects $\partial\Delta \setminus D_1$. Since $l_0 \cup l_1$ is separating on S, there is a subarc γ of $\partial\Delta \setminus D_1$ such that $\partial\gamma \subset l_1$. Let γ' be the closure of a component of $\gamma \setminus N(l_1 \cup l_2)$. Then γ' is an element of $\pi_A(a)\,(\subset \mathcal{AC}^0(S'))$. Hence, we have

$$\mathrm{diam}_{\mathcal{AC}(S')}(\gamma' \cup \pi_A(a)) \leq 1.$$

On the other hand, since γ' is disjoint from l_0, we have

$$\mathrm{diam}_{\mathcal{AC}(S')}(\pi_A(l_0) \cup \gamma') \leq 1.$$

By the triangle inequality, we have

$$\begin{aligned} \mathrm{diam}_{\mathcal{AC}(S')}(\pi_A(l_0) \cup \pi_A(a)) &\leq \mathrm{diam}_{\mathcal{AC}(S')}(\pi_A(l_0) \cup \gamma') \\ &\quad + \mathrm{diam}_{\mathcal{AC}(S')}(\gamma' \cup \pi_A(a)) \\ &\leq 1 + 1 = 2. \end{aligned}$$

By Lemma 2.1, we have $\mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(a)) \leq 4$. This completes the proof of Lemma 6.3 (1). Q.E.D.

Lemma 6.5. *$V_1 \cup_f V_2$ is a strongly keen Heegaard splitting whose Hempel distance is* 3.

Proof. Since $l_0 \in \mathcal{D}(V_1)$ and $l_3 \in f(\mathcal{D}(V_2))$, we have

$$d_S(\mathcal{D}(V_1), f(\mathcal{D}(V_2))) \leq 3.$$

Let $[m_0, \ldots, m_p]$ be a geodesic in $\mathcal{C}(S)$ such that $m_0 \in \mathcal{D}(V_1)$, $m_p \in f(\mathcal{D}(V_2))$ and $p \leq 3$.

Claim 6.6. *$m_i = l_1$ or $m_i = l_2$ for some $i \in \{0, \ldots, p\}$.*

Proof. Assume on the contrary that $m_i \neq l_1$ and $m_i \neq l_2$ for every $i \in \{0, \ldots, p\}$. Namely, every m_i cuts S'. By Lemma 2.2, we have

$$\mathrm{diam}_{S'}(\pi_{S'}(m_0) \cup \pi_{S'}(m_p)) \leq 2p \leq 6. \tag{12}$$

By the triangle inequality, we have

$$\begin{aligned} \mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(l_3)) &\leq \mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(m_0)) \\ &\quad + \mathrm{diam}_{S'}(\pi_{S'}(m_0) \cup \pi_{S'}(m_p)) \\ &\quad + \mathrm{diam}_{S'}(\pi_{S'}(m_p) \cup \pi_{S'}(l_3)). \end{aligned} \tag{13}$$

By the inequalities (12), (13) together with Lemma 6.3, we obtain

$$\mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(l_3)) \leq 4 + 6 + 4 = 14,$$

which contradicts the inequality $\mathrm{diam}_{S'}(\pi_{S'}(l_0) \cup \pi_{S'}(l_3)) > 14$ (see Claim 6.1). Q.E.D.

Assume that $m_i = l_1$ for some $i \in \{0, \ldots, p\}$. (The case where $m_i = l_2$ for some $i \in \{0, \ldots, p\}$ can be treated similarly.) By Claim 6.2 (1), we have $l_1 \notin \mathcal{D}(V_1)$ and $l_1 \notin f(\mathcal{D}(V_2))$, which imply $i \neq 0$ and $i \neq p$, respectively. Hence, we have $1 \leq i \leq p-1$ and $2 \leq p (\leq 3)$. If $p = 2$, then $m_1 = l_1$ and $m_2 \in f(\mathcal{D}(V_2))$, and hence

$$\begin{aligned} d_{F_2}(l_1, l_3) &= d_{F_2}(m_1, l_3) \\ &\leq \mathrm{diam}_{F_2}(m_1 \cup \pi_{F_2}(m_2)) + \mathrm{diam}_{F_2}(\pi_{F_2}(m_2) \cup l_3) \\ &\leq 2 + 4 = 6, \end{aligned} \tag{14}$$

which contradicts the inequality $d_{F_2}(l_1, l_3) > 8$. Hence, $p = 3$, and this implies that the Hempel distance of $V_1 \cup_f V_2$ is 3. Moreover, we have $i = 1$ (that is, $m_1 = l_1$) since, if $i = 2$, then $[l_0, l_1 (= m_2), m_3]$ is a path of length 2 from $\mathcal{D}(V_1)$ to $f(\mathcal{D}(V_2))$, a contradiction.

To prove $m_2 = l_2$, assume on the contrary that $m_2 \neq l_2$. Then m_2, as well as $m_1 (= l_1)$ and m_3, cuts F_2. By Lemma 2.2 and Claim 6.2 (3),

$$\begin{aligned} d_{F_2}(l_1, l_3) &= d_{F_2}(m_1, l_3) \\ &\leq \mathrm{diam}_{F_2}(m_1 \cup \pi_{F_2}(m_3)) + \mathrm{diam}_{F_2}(\pi_{F_2}(m_3) \cup l_3) \\ &\leq 4 + 4 = 8, \end{aligned} \tag{15}$$

which contradicts the inequality $d_{F_2}(l_1, l_3) > 8$. Hence, $m_2 = l_2$.

By Claim 6.2 (1), we have $m_0 = l_0$ and $m_3 = l_3$. Hence, $[l_0, l_1, l_2, l_3]$ is the unique geodesic realizing the Hempel distance. Q.E.D.

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