Rectifying developable surfaces of framed base curves and framed helices

Shun'ichi Honda

Abstract.

We study the rectifying developable surface of a framed base curve and a framed helix in the Euclidean space. A framed base curve is a smooth curve with a moving frame which may have singular points. By using the curvature of a framed base curve, we investigate the rectifying developable surface and a framed helix. Moreover, we introduce two new invariants of a framed base curve, which characterize singularities of the rectifying developable surface and a framed helix.

§1. Introduction

There are several articles concerning singularities of the tangent developable surface and the focal developable surface of a space curve with singular points ([5, 6, 7, 8]). In [6, 7, 8] Ishikawa investigated relationships between singularities of the tangent developable surface and the type (a_1, a_2, a_3) of a space curve. In [5] the author and Takahashi introduced relationships between singularities of the focal developable surface of a framed base curve and invariants, that is, the curvature of a framed curve. On the other hand, Izumiya, Katsumi and Yamasaki introduced the rectifying developable surface of a regular space curve in [9]. They showed relationships between singularities of the rectifying developable surface of a regular space curve and geometric invariants of the curve which are deeply related to the order of contact with a helix. A regular space curve γ is always a geodesic of its rectifying developable surface. In this sense, the rectifying developable surface is an interesting subject.

In this paper we consider the rectifying developable surface of a space curve with singular points, and a helix which may have singular points. In order to define these notions, we apply the theory of framed base curves under a certain condition. A framed base curve is a smooth space curve with a moving frame which may have singular points, see [4] and Appendix A. In Section 3, we define the rectifying developable surface of a framed base curve under a certain condition. That is a natural generalization of the rectifying developable surface of a regular space curve in [9]. By using two new invariants, we give basic properties of the rectifying developable surface (cf. Proposition 3.2 and Theorem 3.3). Moreover, we define a framed helix and consider relationships between the rectifying developable surface of a framed base curve and a framed helix in Section 4. In Section 5, we introduce the notion of support functions of a framed base curve. By using this function, we give relationships between singularities of the rectifying developable surface and invariants of a framed base curve. The proof of Theorem 3.3 is given in Section 5. We give some examples of the rectifying developable surface of a framed base curve and a framed helix in Section 6.

All maps and manifolds considered here are differential of class C^∞.

§2.   Basic notions

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the canonical inner product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$, where $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $\|\boldsymbol{a}\| = \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle}$. We define the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ by

$$
\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{vmatrix} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},
$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the canonical basis on $\mathbb{R}^3$.

We quickly review some basic concepts on classical differential geometry of regular space curves in $\mathbb{R}^3$. Let I be an interval. Suppose that $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ is a regular space curve with linearly independent condition, that is, $\dot{\boldsymbol{\gamma}}(t)$ and $\ddot{\boldsymbol{\gamma}}(t)$ are linearly independent for all $t \in I$, where $\dot{\boldsymbol{\gamma}}(t) = (d\boldsymbol{\gamma}/dt)(t)$ and $\ddot{\boldsymbol{\gamma}}(t) = (d^2\boldsymbol{\gamma}/dt^2)(t)$. Then we have an orthonormal frame

$$
\{T(t), N(t), B(t)\} = \left\{ \frac{\dot{\boldsymbol{\gamma}}(t)}{\|\dot{\boldsymbol{\gamma}}(t)\|}, \frac{(\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)) \times \dot{\boldsymbol{\gamma}}(t)}{\|(\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)) \times \dot{\boldsymbol{\gamma}}(t)\|}, \frac{\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)}{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|} \right\}
$$

along $\boldsymbol{\gamma}(t)$, which is called the *Frenet frame* along $\boldsymbol{\gamma}(t)$. Then we have the following Frenet-Serret formula:

$$
\begin{pmatrix} \dot{T}(t) \\ \dot{N}(t) \\ \dot{B}(t) \end{pmatrix} = \begin{pmatrix} 0 & \|\dot{\boldsymbol{\gamma}}(t)\|\kappa(t) & 0 \\ -\|\dot{\boldsymbol{\gamma}}(t)\|\kappa(t) & 0 & \|\dot{\boldsymbol{\gamma}}(t)\|\tau(t) \\ 0 & -\|\dot{\boldsymbol{\gamma}}(t)\|\tau(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix},
$$

where

$$
\kappa(t) = \frac{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|}{\|\dot{\boldsymbol{\gamma}}(t)\|^3}, \ \tau(t) = \frac{\det(\dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t), \dddot{\boldsymbol{\gamma}}(t))}{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|^2},
$$

where $\dddot{\boldsymbol{\gamma}}(t) = (d^3\boldsymbol{\gamma}/dt^3)(t)$. We call $\kappa(t)$ a *curvature* and $\tau(t)$ a *torsion* of $\boldsymbol{\gamma}(t)$. Note that the curvature $\kappa(t)$ and the torsion $\tau(t)$ are independent of a choice of parametrization. For any regular space curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, we define a vector $D(t) = \tau(t)T(t) + \kappa(t)B(t)$ and we call it a *Darboux vector* along $\boldsymbol{\gamma}(t)$ (cf. [9, 11]). Since $\kappa(t) > 0$, we also define a spherical Darboux vector $\overline{D} : I \to S^2$ by

$$
\overline{D}(t) = \frac{\tau(t)T(t) + \kappa(t)B(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}.
$$

and the rectifying developable surface $RD_{\boldsymbol{\gamma}} : I \times \mathbb{R} \to \mathbb{R}^3$ of $\boldsymbol{\gamma}(t)$ by

$$
RD_{\boldsymbol{\gamma}}(t, u) = \boldsymbol{\gamma}(t) + u\overline{D}(t) = \boldsymbol{\gamma}(t) + u\frac{\tau(t)T(t) + \kappa(t)B(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}.
$$

By a direct calculation, we have

$$
\frac{\partial RD_{\boldsymbol{\gamma}}}{\partial t}(t, u) \times \frac{\partial RD_{\boldsymbol{\gamma}}}{\partial u}(t, u)
$$
$$
= -\left(\frac{\kappa(t)\|\dot{\boldsymbol{\gamma}}(t)\|}{\sqrt{\kappa^2(t) + \tau^2(t)}} + u\frac{\kappa(t)\dot{\tau}(t) - \dot{\kappa}(t)\tau(t)}{\kappa^2(t) + \tau^2(t)} \right) N(t).
$$

Therefore, $(t_0, u_0) \in I \times \mathbb{R}$ is a singular point of $RD_{\boldsymbol{\gamma}}$ if and only if

$$
\frac{\kappa(t_0)\|\dot{\boldsymbol{\gamma}}(t_0)\|}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}} + u_0\frac{\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)}{\kappa^2(t_0) + \tau^2(t_0)} = 0.
$$

If (t_0, u_0) is a singular point of $RD_{\boldsymbol{\gamma}}$, then we have $u_0 \neq 0$, that is, $RD_{\boldsymbol{\gamma}}$ has no singular value on the base curve $\boldsymbol{\gamma}(t)$. Izumiya, Katsumi and Yamasaki investigated the rectifying developable surfaces of a regular space curve in [9].

In this paper we do not assume that $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ is a regular curve with linearly independent condition, so that $\boldsymbol{\gamma}$ may have singular points.

If γ has a singular point, we can not construct the Frenet frame along $\gamma(t)$. However, we can define the Frenet type frame along $\gamma(t)$ under a certain condition.

Definition 2.1. We say that $\gamma : I \to \mathbb{R}^3$ is a *Frenet type framed base curve* if there exist a regular spherical curve $\mathcal{T} : I \to S^2$ and a smooth function $\alpha : I \to \mathbb{R}$ such that $\dot{\gamma}(t) = \alpha(t)\mathcal{T}(t)$ for all $t \in I$. Then we call $\mathcal{T}(t)$ a *unit tangent vector* and $\alpha(t)$ a *speed function* of $\gamma(t)$.

Clearly, t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$. We define a *unit principal normal vector* $\mathcal{N}(t) = \dot{\mathcal{T}}(t)/\|\dot{\mathcal{T}}(t)\|$ and a *unit binormal vector* $\mathcal{B}(t) = \mathcal{T}(t) \times \mathcal{N}(t)$ of $\gamma(t)$. Then we have an orthonormal frame $\{\mathcal{T}(t), \mathcal{N}(t), \mathcal{B}(t)\}$ along $\gamma(t)$, which is called the *Frenet type frame* along $\gamma(t)$. Then we have the following Frenet-Serret type formula:

$$\begin{pmatrix} \dot{\mathcal{T}}(t) \\ \dot{\mathcal{N}}(t) \\ \dot{\mathcal{B}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \mathcal{T}(t) \\ \mathcal{N}(t) \\ \mathcal{B}(t) \end{pmatrix},$$

where

$$\kappa(t) = \|\dot{\mathcal{T}}(t)\|, \quad \tau(t) = \frac{\det(\mathcal{T}(t), \dot{\mathcal{T}}(t), \ddot{\mathcal{T}}(t))}{\|\dot{\mathcal{T}}(t)\|^2}.$$

We call $\kappa(t)$ a *curvature* and $\tau(t)$ a *torsion* of γ. Note that the curvature $\kappa(t)$ and the torsion $\tau(t)$ are depend on a choice of parametrization.

We define a vector $\mathcal{D}(t)$ along $\gamma(t)$ by

$$\mathcal{D}(t) = \tau(t)\mathcal{T}(t) + \kappa(t)\mathcal{B}(t),$$

which is called a *Darboux type vector* along $\gamma(t)$. By using the Darboux type vector, the Frenet-Serret type formula is rewritten as follows:

$$\begin{cases} \dot{\mathcal{T}}(t) = \mathcal{D}(t) \times \mathcal{T}(t), \\ \dot{\mathcal{N}}(t) = \mathcal{D}(t) \times \mathcal{N}(t), \\ \dot{\mathcal{B}}(t) = \mathcal{D}(t) \times \mathcal{B}(t). \end{cases}$$

Thus the Darboux type vector plays an important role for the study of framed base curves. Since $\kappa(t) > 0$, we can define a *spherical Darboux-type vector* by

$$\overline{\mathcal{D}}(t) = \frac{\tau(t)\mathcal{T}(t) + \kappa(t)\mathcal{B}(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}.$$

Remark 2.2. Since $\mathcal{T}(t)$ is a regular curve, we uniquely obtain the unit principal normal vector $\mathcal{N}(t)$ and the unit binormal vector $\mathcal{N}(t)$.

Therefore, $\kappa(t)$, $\tau(t)$ and $\overline{\mathcal{D}}(t)$ is uniquely determined with respect to $\mathcal{T}(t)$. On the other hand, we can easily check that $\overline{\mathcal{D}}(t)$ is a spherical dual of $\mathcal{N}(t)$ (cf. [12]).

Example 2.3. Above objects are natural generalizations of corresponding notions for regular space curves. In fact, let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a regular space curve with linearly independent condition. If we take $\mathcal{T}(t) = T(t)$ and $\alpha(t) = \|\dot{\gamma}(t)\|$, then $\mathcal{N}(t) = N(t)$, $\mathcal{B}(t) = B(t)$ and the curvature $\kappa(t)$ (respectively, $\tau(t)$) as a framed base curve coincides with the curvature $\kappa(t)$ (respectively, the torsion $\tau(t)$) in the sense of classical differential geometry. Therefore, we have $\mathcal{D}(t) = D(t)$ and $\overline{\mathcal{D}}(t) = \overline{D}(t)$.

We can easily check that $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ is a framed base curve (cf. [4] and Appendix A). More precisely, $(\boldsymbol{\gamma}, \mathcal{N}, \mathcal{B}) : I \to \mathbb{R}^3 \times \Delta \subset \mathbb{R}^3 \times S^2 \times S^2$ is a framed curve with the curvature of the framed curve $(\tau(t), -\kappa(t), 0, \alpha(t))$. This is the reason why we call $\boldsymbol{\gamma}$ the Frenet type framed base curve. In [4], the author and Takahashi have shown the existence and the uniqueness for framed curves. Since Remark 2.2, the speed function $\alpha(t)$, the curvature $\kappa(t)$ and the torsion $\tau(t)$ are invariants of the pair $(\boldsymbol{\gamma}, \mathcal{T})$.

§3. Rectifying developable surfaces

In this section, we consider the rectifying developable surface of a Frenet type framed base curve. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a Frenet type framed base curve with unit tangent vector $\mathcal{T}(t)$.

Definition 3.1. We define a map $\mathcal{RD}_{\boldsymbol{\gamma}} : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\mathcal{RD}_{\boldsymbol{\gamma}}(t, u) = \boldsymbol{\gamma}(t) + u\overline{\mathcal{D}}(t) = \boldsymbol{\gamma}(t) + u\frac{\tau(t)\mathcal{T}(t) + \kappa(t)\mathcal{B}(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}.$$

We call $\mathcal{RD}_{\boldsymbol{\gamma}}$ the *rectifying developable surface* of Frenet type framed curve $\boldsymbol{\gamma}$.

Since Example 2.3, Definition 3.1 is a natural generalization of the rectifying developable surface of a regular space curve in [9]. The rectifying developable surface $\mathcal{RD}_{\boldsymbol{\gamma}}(t, u)$ is a ruled surface and we have

$$\dot{\overline{\mathcal{D}}}(t) = \left(\frac{\kappa(t)\dot{\tau}(t) - \dot{\kappa}(t)\tau(t)}{\kappa^2(t) + \tau^2(t)} \right) \frac{\kappa(t)\mathcal{T}(t) - \tau(t)\mathcal{B}(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}},$$

so that we have

$$\det\left(\dot{\gamma}, \overline{\mathcal{D}}, \dot{\overline{\mathcal{D}}}\right) = \det\left(\alpha\mathcal{T}, \frac{\tau\mathcal{T} + \kappa\mathcal{B}}{\sqrt{\kappa^2 + \tau^2}}, \left(\frac{\kappa\dot{\tau} - \dot{\kappa}\tau}{\kappa^2 + \tau^2}\right) \frac{\kappa\mathcal{T} - \tau\mathcal{B}}{\sqrt{\kappa^2 + \tau^2}} \right)$$
$$= 0$$

for all $t \in I$. This means that $\mathcal{RD}_\gamma$ is a developable surface (cf. [10]). Moreover, we introduce two invariants $\delta(t)$, $\sigma(t)$ as follows:

$$\delta(t) = \frac{\kappa(t)\dot{\tau}(t) - \dot{\kappa}(t)\tau(t)}{\kappa^2(t) + \tau^2(t)},$$

$$\sigma(t) = \frac{\alpha(t)\tau(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}} - \frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)}{\delta(t)\sqrt{\kappa^2(t) + \tau^2(t)}}\right), \quad \text{(when } \delta(t) \neq 0\text{).}$$

We remark that $\delta(t)$ corresponds to $(d/dt)(\tau/\kappa)(t)$ which is investigated in [9]. By a direct calculation, $\delta(t) = 0$ if and only if $\dot{\overline{\mathcal{D}}}(t) = \mathbf{0}$. We can also calculate that

$$\frac{\partial \mathcal{RD}_\gamma}{\partial t}(t, u) \times \frac{\partial \mathcal{RD}_\gamma}{\partial u}(t, u) = -\left(\frac{\alpha(t)\kappa(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}} + u\delta(t)\right)\mathcal{N}(t).$$

Therefore, $(t_0, u_0) \in I \times \mathbb{R}$ is a singular point of $\mathcal{RD}_\gamma$ if and only if

$$\frac{\alpha(t_0)\kappa(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}} + u_0 \frac{\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)}{\kappa^2(t_0) + \tau^2(t_0)} = 0.$$

Since $\dot{\mathcal{N}}(t) \neq \mathbf{0}$ for all $t \in I$, $\mathcal{RD}_\gamma$ is a wave front (cf. [1, 2]).

Proposition 3.2. *Let* $\gamma : I \to \mathbb{R}^3$ *be a Frenet type framed base curve with* $\mathcal{T}(t)$. *Then we have the following*:

(A)   *The following are equivalent*:
    (1) $\mathcal{RD}_\gamma$ *is a cylinder,*
    (2) $\delta(t) = 0$ *for all* $t \in I$.
(B)   *If* $\delta(t) \neq 0$ *for all* $t \in I$, *then the following are equivalent*:
    (3) $\mathcal{RD}_\gamma$ *is a conical surface,*
    (4) $\sigma(t) = 0$ *for all* $t \in I$.

Proof.   (A) By definition, $\mathcal{RD}_\gamma$ is a cylinder if and only if $\overline{\mathcal{D}}(t)$ is a constant. Since

$$\dot{\overline{\mathcal{D}}}(t) = \delta(t)\frac{\kappa(t)\mathcal{T}(t) - \tau(t)\mathcal{B}(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}},$$

$\overline{\mathcal{D}}(t)$ is a constant if and only if $\delta(t) = 0$ for all $t \in I$.
    (B) We consider the striction curve $\boldsymbol{\sigma}(t)$ defined by

$$\boldsymbol{\sigma}(t) = \boldsymbol{\gamma}(t) - \frac{\langle\dot{\boldsymbol{\gamma}}(t), \dot{\overline{\mathcal{D}}}(t)\rangle}{\langle\dot{\overline{\mathcal{D}}}(t), \dot{\overline{\mathcal{D}}}(t)\rangle}\overline{\mathcal{D}}(t) = \boldsymbol{\gamma}(t) - \frac{\alpha(t)\kappa(t)}{\delta(t)\sqrt{\kappa^2(t) + \tau^2(t)}}\overline{\mathcal{D}}(t).$$

Then (B)-(3) is equivalent to the condition $\dot{\boldsymbol{\sigma}}(t) = 0$ for all $t \in I$. We can calculate that

$$
\begin{aligned}
\dot{\boldsymbol{\sigma}} &= \dot{\boldsymbol{\gamma}} - \frac{d}{dt}\left(\frac{\alpha\kappa}{\delta\sqrt{\kappa^2 + \tau^2}}\right)\overline{\mathcal{D}} - \frac{\alpha\kappa}{\delta\sqrt{\kappa^2 + \tau^2}}\dot{\overline{\mathcal{D}}} \\
&= \alpha\mathcal{T} - \frac{d}{dt}\left(\frac{\alpha\kappa}{\delta\sqrt{\kappa^2 + \tau^2}}\right)\overline{\mathcal{D}} - \frac{\alpha\kappa}{\sqrt{\kappa^2 + \tau^2}}\frac{\kappa\mathcal{T} - \tau\mathcal{B}}{\sqrt{\kappa^2 + \tau^2}} \\
&= \left(\frac{\alpha\tau}{\sqrt{\kappa^2 + \tau^2}} - \frac{d}{dt}\left(\frac{\alpha\kappa}{\delta\sqrt{\kappa^2 + \tau^2}}\right)\right)\frac{\tau\mathcal{T} + \kappa\mathcal{B}}{\sqrt{\kappa^2 + \tau^2}} \\
&= \sigma\overline{\mathcal{D}}.
\end{aligned}
$$

It follows that (B)-(3) and (B)-(4) are equivalent. $\qquad\square$

We give relationships between singularities of the rectifying developable surface of a framed base curve and such invariants.

Theorem 3.3. *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $\mathcal{T}(t)$. Then we have the following*:

(1) (t_0, u_0) *is a regular point of $\mathcal{RD}_\gamma$ if and only if*

$$
\frac{\alpha(t_0)\kappa(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}} + u_0\delta(t_0) \neq 0.
$$

(2)   *Suppose that (t_0, u_0) is a singular point of $\mathcal{RD}_\gamma$, then the rectifying developable surface $\mathcal{RD}_\gamma$ is locally diffeomorphic to the cuspidal edge **ce** at (t_0, u_0) if*
    (i) $\delta(t_0) \neq 0$, $\sigma(t_0) \neq 0$ *and*

$$
u_0 = -\frac{\alpha(t_0)\kappa(t_0)}{\delta(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)}},
$$

or
    (ii) $\delta(t_0) = \alpha(t_0) = 0$, $\dot{\delta}(t_0) \neq 0$ *and*

$$
u_0 \neq -\dot{\alpha}(t_0)\kappa(t_0)\frac{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}{\kappa(t_0)\ddot{\tau}(t_0) - \ddot{\kappa}(t_0)\tau(t_0)},
$$

or
    (iii) $\delta(t_0) = \alpha(t_0) = 0$ *and* $\dot{\alpha}(t_0) \neq 0$.

(3)   *Suppose that (t_0, u_0) is a singular point of $\mathcal{RD}_\gamma$, then the rectifying developable surface $\mathcal{RD}_\gamma$ is locally diffeomorphic to the swallowtail **sw** at (t_0, u_0) if $\delta(t_0) \neq 0$, $\sigma(t_0) = 0$, $\dot{\sigma}(t_0) \neq 0$ and*

$$
u_0 = -\frac{\alpha(t_0)\kappa(t_0)}{\delta(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.
$$

Here, $\boldsymbol{ce} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0); \ (u, v) \mapsto (u, v^2, v^3)$ is the cuspidal edge, $\boldsymbol{sw} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0); \ (u, v) \mapsto (3u^4 + u^2 v, 4u^3 + 2uv, v)$ is the swallowtail.

Remark 3.4. Suppose that t_0 is a singular point of $\boldsymbol{\gamma}$. Then $(t_0, 0)$ is a singular point of $\mathcal{RD}_{\boldsymbol{\gamma}}$.

Remark 3.5. $\boldsymbol{\gamma}$ is always a geodesic of the rectifying developable surface $\mathcal{RD}_{\boldsymbol{\gamma}}(t)$, away from singular points of $\boldsymbol{\gamma}$ (cf. [9]). Therefore, $\boldsymbol{\gamma}$ is the geodesic which formed to stride over singular points of $\mathcal{RD}_{\boldsymbol{\gamma}}$.

§4.   Framed helices

In this section, we define a helix which may have singular points. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $\mathcal{T}(t)$.

Definition 4.1. We say that $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ is a *framed helix* if there exist constants $\boldsymbol{v} \in S^2$ and $C \in \mathbb{R}$ such that $\mathcal{T}(t) \cdot \boldsymbol{v} = C$ for all $t \in I$.

Definition 4.1 means that the tangent line of $\boldsymbol{\gamma}$ makes a constant angle with a fixed direction. In this sense, a framed helix is a natural generalization of a regular helix. The invariant $\delta(t)$ characterize a framed helix. In fact, we can prove the following Proposition.

Proposition 4.2. *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $\mathcal{T}(t)$. Then the following are equivalent:*
(1)   *$\boldsymbol{\gamma}$ is a framed helix.*
(2)   *$\delta(t) = 0$ for all $t \in I$.*

Proof. Suppose that $\boldsymbol{\gamma}(t)$ is a framed helix. Here we put $\boldsymbol{v} = a(t)\mathcal{T}(t) + b(t)\mathcal{N}(t) + c(t)\mathcal{B}(t)$, where $a(t)$, $b(t)$ and $c(t)$ are smooth functions. By the assumption,

$$(1) \qquad\qquad \langle \boldsymbol{v}, \mathcal{T}(t) \rangle = a(t) = C.$$

Moreover, taking the derivative of the both sides of the formula (1), we have

$$-b(t)\kappa(t)\mathcal{T}(t) + (C\kappa(t) + \dot{b}(t) - c(t)\tau(t))\mathcal{N}(t) + (\dot{c}(t) + b(t)\tau(t))\,\mathcal{B}(t) = \boldsymbol{0}.$$

Then we have $b(t) = 0$, $c(t) = C_1$ and $C = C_1(\tau(t)/\kappa(t))$, where C_1 is a constant. On the other hand, since

$$1 = \|\boldsymbol{v}\|^2 = C_1^2 \left(\frac{\tau^2(t)}{\kappa^2(t)} + 1 \right),$$

$C_1 \neq 0$. Thus, $C/C_1 = \tau(t)/\kappa(t)$. We remark that $\delta(t) = 0$ if and only if $(d/dt)(\tau/\kappa)(t) = 0$. Hence, we have $\delta(t) = 0$ for all $t \in I$.

Conversely, suppose that $\delta(t) = 0$ for all $t \in I$. We put a constant vector $\boldsymbol{v} = (\tau(t)/\kappa(t))\mathcal{T}(t) + \mathcal{B}(t)$ and $\overline{\boldsymbol{v}} = \boldsymbol{v}/\|\boldsymbol{v}\|$. Then

$$\langle \overline{\boldsymbol{v}}, \mathcal{T}(t) \rangle = \frac{\frac{\tau(t)}{\kappa(t)}}{\sqrt{\frac{\tau^2(t)}{\kappa^2(t)} + 1}},$$

that is, $\langle \overline{\boldsymbol{v}}, \mathcal{T}(t) \rangle$ is a constant. This means that $\boldsymbol{\gamma}(t)$ is a framed helix. $\qquad \square$

Corollary 4.3. *Let* $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ *be a Frenet type framed base curve with* $\mathcal{T}(t)$. *Then the following are equivalent:*
(1) $\mathcal{RD}_{\boldsymbol{\gamma}}$ *is a cylinder,*
(2) $\delta(t) = 0$ *for all* $t \in I$,
(3) $\boldsymbol{\gamma}$ *is a framed helix.*

We recall the notion of the contact between framed curves, see [4]. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3 \times \Delta; t \mapsto (\boldsymbol{\gamma}(t), \boldsymbol{\nu}_1(t), \boldsymbol{\nu}_2(t))$ and $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}}_1, \widetilde{\boldsymbol{\nu}}_2) : \widetilde{I} \to \mathbb{R}^3 \times \Delta; u \mapsto (\widetilde{\boldsymbol{\gamma}}(u), \widetilde{\boldsymbol{\nu}}_1(u), \widetilde{\boldsymbol{\nu}}_2(u))$ be framed curves, respectively. Let k be a natural number. We denote the curvatures $\mathcal{F}(t) = (\ell(t), m(t), n(t), \alpha(t))$ and $\widetilde{\mathcal{F}}(u) = (\widetilde{\ell}(u), \widetilde{m}(u), \widetilde{n}(u), \widetilde{\alpha}(u))$ for convenience. We say that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ *and* $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}}_1, \widetilde{\boldsymbol{\nu}}_2)$ *have* k-*th order contact at* $t = t_0$, $u = u_0$ if

$$\frac{d^i}{dt^i}(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)(t_0) = \frac{d^i}{du^i}(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}}_1, \widetilde{\boldsymbol{\nu}}_2)(u_0),$$

$$\frac{d^k}{dt^k}(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)(t_0) \neq \frac{d^k}{du^k}(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}_1}, \widetilde{\boldsymbol{\nu}}_2)(u_0)$$

for $i = 0, 1, ..., k - 1$. Moreover, we say that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ *and* $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}_1}, \widetilde{\boldsymbol{\nu}_2})$ *have at least* k-*th order contact at* $t = t_0$, $u = u_0$ if

$$\frac{d^i}{dt^i}(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)(t_0) = \frac{d^i}{du^i}(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}_1}, \widetilde{\boldsymbol{\nu}_2})(u_0),$$

for $i = 0, 1, ..., k - 1$.

In general, we may assume that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ and $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}}_1, \widetilde{\boldsymbol{\nu}}_2)$ have at least first order contact at any point $t = t_0$, $u = u_0$, up to congruence as framed curves.

Theorem 4.4. ([4]) *If* $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ *and* $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}}_1, \widetilde{\boldsymbol{\nu}}_2)$ *have at least* $(k + 1)$-*th order contact at* $t = t_0$, $u = u_0$ *then*

$$(2) \qquad\qquad \frac{d^i}{dt^i}\mathcal{F}(t_0) = \frac{d^i}{du^i}\widetilde{\mathcal{F}}(u_0),$$

for $i = 0, 1, ..., k-1$. Conversely, if the condition (2) hold, then (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least $(k + 1)$-th order contact at $t = t_0$, $u = u_0$, up to congruence as framed curves.

By Theorem 4.4, we can show the following propositions:

Proposition 4.5. *If $(\gamma, \mathcal{N}, \mathcal{B})(t_0)$ and $(\widetilde{\gamma}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{B}})(u_0)$ have at least $(k + 2)$-th order contact as framed curves, then $\delta^{(p)}(t_0) = \widetilde{\delta}^{(p)}(u_0)$ (for $0 \le p \le k - 1$), where*

$$\delta^{(p)}(t_0) = (d^p\delta/dt^p)(t_0) \text{ and } \widetilde{\delta}^{(p)}(u_0) = (d^p\widetilde{\delta}/du^p)(u_0).$$

Proposition 4.6. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $\mathcal{T}(t)$. Then there exists a framed curve $(\widetilde{\gamma}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{B}}) : I \to \mathbb{R}^3 \times \Delta$ such that $\widetilde{\gamma}(t)$ is a framed helix, and $(\gamma, \mathcal{N}, \mathcal{B})$ and $(\widetilde{\gamma}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{B}})$ have at least second order contact as framed curves at a point $t_0 \in I$.*

Proof. Choose any fixed value $t = t_0$ of the parameter. We consider a new curvature as a framed curve

$$(\widetilde{\tau}(t), -\widetilde{\kappa}(t), 0, \widetilde{\alpha}(t)) = ((\tau(t_0)/\kappa(t_0))\kappa(t), -\kappa(t), 0, \alpha(t)).$$

Since the existence and the uniqueness of framed curves, there exists a framed curve $(\widetilde{\gamma}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{B}})$ with $(\widetilde{\tau}(t), -\widetilde{\kappa}(t), 0, \widetilde{\alpha}(t))$. Moreover, by Theorem 4.4 and an appropriate Euclid transformation, $(\gamma, \mathcal{N}, \mathcal{B})$ and $(\widetilde{\gamma}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{B}})$ have at least second order contact as framed curves at $t_0 \in I$. On the other hand, by a direct calculation, we have $\widetilde{\delta}(t) = 0$ for all $t \in I$. Thus, $\widetilde{\gamma}(t)$ is a framed helix. $\square$

§5. Support functions

For a Frenet type framed base curve $\gamma : I \to \mathbb{R}^3$, we define a function $G : I \times \mathbb{R}^3 \to \mathbb{R}$ by $G(t, \boldsymbol{x}) = \langle \boldsymbol{x} - \gamma(t), \mathcal{N}(t) \rangle$. We call G a *support function* of γ with respect to the unit principal normal vector $\mathcal{N}(t)$. We denote that $g_{\boldsymbol{x}_0}(t) = G(t, \boldsymbol{x}_0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}^3$. Then we have the following proposition.

Proposition 5.1. *For a support function $g_{\boldsymbol{x}_0}(t) = \langle \boldsymbol{x}_0 - \gamma(t), \mathcal{N}(t) \rangle$, we have the following*:

(1) $g_{\boldsymbol{x}_0}(t_0) = 0$ *if and only if there exist $u, v \in \mathbb{R}$ such that* $\boldsymbol{x}_0 - \gamma(t_0) = u\mathcal{T}(t_0) + v\mathcal{B}(t_0)$.

(2) $g_{\boldsymbol{x}_0}(t_0) = \dot{g}_{\boldsymbol{x}_0}(t_0) = 0$ *if and only if there exists $u \in \mathbb{R}$ such that*

$$\boldsymbol{x}_0 - \gamma(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) + \kappa(t_0)\mathcal{B}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.$$

(A) *Suppose that $\delta(t_0) \neq 0$. Then we have the following:*

    (3) $g_{\boldsymbol{x}_0}(t_0) = \dot{g}_{\boldsymbol{x}_0}(t_0) = \ddot{g}_{\boldsymbol{x}_0}(t_0) = 0$ *if and only if*

$$(*) \ \boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = -\frac{\alpha(t_0)\kappa(t_0)}{\delta(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)}} \frac{\tau(t_0)\mathcal{T}(t_0) + \kappa(t_0)\mathcal{B}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.$$

    (4) $g_{\boldsymbol{x}_0}(t_0) = \dot{g}_{\boldsymbol{x}_0}(t_0) = \ddot{g}_{\boldsymbol{x}_0}(t_0) = g_{\boldsymbol{x}_0}^{(3)}(t_0) = 0$ *if and only if* $\sigma(t_0) = 0$ *and* $(*)$.

    (5) $g_{\boldsymbol{x}_0}(t_0) = \dot{g}_{\boldsymbol{x}_0}(t_0) = \ddot{g}_{\boldsymbol{x}_0}(t_0) = g_{\boldsymbol{x}_0}^{(3)}(t_0) = g_{\boldsymbol{x}_0}^{(4)}(t_0) = 0$ *if and only if* $\sigma(t_0) = 0$, $\dot{\sigma}(t_0) = 0$ *and* $(*)$.

(B) *Suppose that $\delta(t_0) = 0$. Then we have the following:*

    (6) $g_{\boldsymbol{x}_0}(t_0) = \dot{g}_{\boldsymbol{x}_0}(t_0) = \ddot{g}_{\boldsymbol{x}_0}(t_0) = 0$ *if and only if $\alpha(t_0) = 0$ and there exists $u \in \mathbb{R}$ such that*

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) + \kappa(t_0)\mathcal{B}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.$$

    (7) $g_{\boldsymbol{x}_0}(t_0) = \dot{g}_{\boldsymbol{x}_0}(t_0) = \ddot{g}_{\boldsymbol{x}_0}(t_0) = g_{\boldsymbol{x}_0}^{(3)}(t_0) = 0$ *if and only if one of the following conditions holds:*

       (a) $\dot{\delta}(t_0) \neq 0$, $\alpha(t_0) = 0$ *and*

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = -\dot{\alpha}(t_0)\kappa(t_0)\frac{\tau(t_0)\mathcal{T}(t_0) + \kappa(t_0)\mathcal{B}(t_0)}{\kappa(t_0)\ddot{\tau}(t_0) - \ddot{\kappa}(t_0)\tau(t_0)}.$$

       (b) $\dot{\delta}(t_0) = 0$, $\alpha(t_0) = \dot{\alpha}(t_0) = 0$ *and there exists $u \in \mathbb{R}$ such that*

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) + \kappa(t_0)\mathcal{B}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.$$

Proof. Since $g_{\boldsymbol{x}_0}(t) = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}(t), \mathcal{N}(t) \rangle$, we have the following calculations:

    (α) $g_{\boldsymbol{x}_0} = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, \mathcal{N} \rangle$,

    (β) $\dot{g}_{\boldsymbol{x}_0} = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, -\kappa\mathcal{T} + \tau\mathcal{B} \rangle$,

    (γ) $\ddot{g}_{\boldsymbol{x}_0} = \alpha\kappa + \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, -\dot{\kappa}\mathcal{T} - (\kappa^2 + \tau^2)\mathcal{N} + \dot{\tau}\mathcal{B} \rangle$,

    (δ) $g_{\boldsymbol{x}_0}^{(3)} = 2\alpha\dot{\kappa} + \dot{\alpha}\kappa + \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, (\kappa(\kappa^2 + \tau^2) - \ddot{\kappa})\mathcal{T}$
$$- 3(\kappa\dot{\kappa} + \tau\dot{\tau})\mathcal{N} + (-\tau(\kappa^2 + \tau^2) + \ddot{\tau})\mathcal{B} \rangle,$$

    (ϵ) $g_{\boldsymbol{x}_0}^{(4)} = \ddot{\alpha}\kappa + 3\dot{\alpha}\dot{\kappa} + 3\alpha\ddot{\kappa} - \alpha\kappa(\kappa^2 + \tau^2)$
$$+ \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, (\dot{\kappa}(6\kappa^2 + \tau^2) + 5\kappa\tau\dot{\tau} - \dddot{\kappa})\mathcal{T}$$
$$+ ((\kappa^2 + \tau^2)^2 - 4(\kappa\ddot{\kappa} + \tau\ddot{\tau}) - 3(\dot{\kappa}^2 + \dot{\tau}^2))\mathcal{N}$$
$$+ (-\dot{\tau}(\kappa^2 + 6\tau^2) - 5\kappa\dot{\kappa}\tau + \dddot{\tau})\mathcal{B} \rangle.$$

By definition and (α), (1) follows.

By (β), $g_{\boldsymbol{x}_0}(t_0) = \dot{g}_{\boldsymbol{x}_0}(t_0) = 0$ if and only if there exist $u, v \in \mathbb{R}$ such that $\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = u\mathcal{T}(t_0) + v\mathcal{B}(t_0)$ and $-\kappa(t_0)u + \tau(t_0)v = 0$. Since $\kappa(t_0) > 0$, we have

$$u = v\frac{\tau(t_0)}{\kappa(t_0)},$$

so that there exists $w \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = w\frac{\tau(t_0)\mathcal{T}(t_0) + \kappa(t_0)\mathcal{B}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.$$

Therefore (2) holds.

By (γ), $g_{\boldsymbol{x}_0}(t_0) = \dot{g}_{\boldsymbol{x}_0}(t_0) = \ddot{g}_{\boldsymbol{x}_0}(t_0) = 0$ if and only if there exists $u \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) + \kappa(t_0)\mathcal{B}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}$$

and

$$\alpha(t_0)\kappa(t_0) + u\frac{\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}} = 0.$$

It follows that

$$\frac{\alpha(t_0)\kappa(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}} + u\frac{\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)}{\kappa^2(t_0) + \tau^2(t_0)} = 0.$$

Thus,

$$\delta(t_0) = \frac{\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)}{\kappa^2(t_0) + \tau^2(t_0)} \neq 0 \text{ and } u = -\frac{\alpha(t_0)\kappa(t_0)}{\delta(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)}},$$

or $\delta(t_0) = 0$ and $\alpha(t_0) = 0$. This completes the proof of (A)-(3), and (B)-(6).

Suppose that $\delta(t_0) \neq 0$. By (δ),

$$g_{\boldsymbol{x}_0}(t_0) = \dot{g}_{\boldsymbol{x}_0}(t_0) = \ddot{g}_{\boldsymbol{x}_0}(t_0) = g_{\boldsymbol{x}_0}^{(3)}(t_0) = 0$$

if and only if

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = -\frac{\alpha(t_0)\kappa(t_0)}{\delta(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}\frac{\tau(t_0)\mathcal{T}(t_0) + \kappa(t_0)\mathcal{B}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}$$

and

$$2\alpha(t_0)\dot{\kappa}(t_0) + \dot{\alpha}(t_0)\kappa(t_0) - \frac{\alpha(t_0)\kappa(t_0)}{\delta(t_0)}\left(\frac{\kappa(t_0)\ddot{\tau}(t_0) - \ddot{\kappa}(t_0)\tau(t_0)}{\kappa^2(t_0) + \tau^2(t_0)}\right) = 0.$$

We can rewrite $\sigma(t)$:

$$\sigma = -\sqrt{\kappa^2 + \tau^2}\left(2\alpha\dot{\kappa} + \dot{\alpha}\kappa - \frac{\alpha\kappa}{\delta}\left(\frac{\kappa\ddot{\tau} - \ddot{\kappa}\tau}{\kappa^2 + \tau^2}\right)\right).$$

Therefore, (A)-(3) holds. By the similar arguments to the above, we have (A)-(5).

Suppose that $\delta(t_0) = 0$. Then by (δ), $g_{\boldsymbol{x}_0}(t_0) = \dot{g_{\boldsymbol{x}_0}}(t_0) = \ddot{g_{\boldsymbol{x}_0}}(t_0) = g_{\boldsymbol{x}_0}^{(3)}(t_0) = 0$ if and only if $\alpha(t_0) = 0$, there exists $u \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) + \kappa(t_0)\mathcal{B}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}$$

and

$$\dot{\alpha}(t_0)\kappa(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)} + u\left(\kappa(t_0)\ddot{\tau}(t_0) - \ddot{\kappa}(t_0)\tau(t_0)\right) = 0.$$

It follows that

$$\kappa(t_0)\ddot{\tau}(t_0) - \ddot{\kappa}(t_0)\tau(t_0) \neq 0 \text{ and } u = -\dot{\alpha}(t_0)\kappa(t_0)\frac{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}{\kappa(t_0)\ddot{\tau}(t_0) - \ddot{\kappa}(t_0)\tau(t_0)},$$

or

$$\kappa(t_0)\ddot{\tau}(t_0) - \ddot{\kappa}(t_0)\tau(t_0) = 0 \text{ and } \dot{\alpha}(t_0) = 0.$$

Therefore, we have (B)-(7)-(a) and (B)-(7)-(b). This completes the proof. $\qquad\square$

In order to prove Theorem 3.3, we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [3]. Let $\mathbb{R}^r$ be the r-dimensional Euclidean space with coordinates $(x_1, x_2, ..., x_r)$ and $F : (\mathbb{R} \times \mathbb{R}^r, (t_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(t) = F(t, \boldsymbol{x}_0)$. We say that f has the A_k-*singularity* at t_0 if $f^{(p)}(t_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(t_0) \neq 0$. Let F be an unfolding of f and f has the A_k-singularity $(k \geq 1)$ at t_0. We write the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at t_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(t, \boldsymbol{x}_0))(t_0) = \sum_{j=0}^{k-1}\alpha_{ji}(t - t_0)^j$ for $i = 1, ..., r$. Then F is called an $\mathcal{R}$-*versal unfolding* if the $k \times r$ matrix of coefficients $(\alpha_{ji})_{j=0,...,k-1;i=1,...,r}$ has rank k $(k \leq r)$. We introduce an important set concerning the unfoldings relative to the above notions. The *discriminant set* of F is defined to be

$$D_F = \left\{\boldsymbol{x} \in \mathbb{R}^r \mid \text{there exists } s \text{ such that } F = \frac{\partial F}{\partial t} = 0 \text{ at } (s, \boldsymbol{x})\right\}.$$

Then we have the following classification (cf. [3]).

Theorem 5.2. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (t, \boldsymbol{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of f which has the A_k-singularity at t_0. Suppose that F is an $\mathcal{R}$-versal unfolding.*

(1) *If $k = 2$, then D_F is locally diffeomorphic to the cuspidal edge $\boldsymbol{ce}$.*
(2) *If $k = 3$, then D_F is locally diffeomorphic to the swallowtail $\boldsymbol{sw}$.*

For the proof of Theorem 3.3, we have the following proposition.

Proposition 5.3. *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $\mathcal{T}(t)$. If $g_{\boldsymbol{x}_0}$ has the A_k-singularity $(k = 2, 3)$ at t_0, then G is an $\mathcal{R}$-versal unfolding of $g_{\boldsymbol{x}_0}$. Here, we assume that $\delta(t_0) \neq 0$ for $k = 3$.*

Proof. We write that $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $\mathcal{N}(t) = (n_1(t), n_2(t), n_3(t))$. Then we have

$$G(t, \boldsymbol{x}) = n_1(t)(x_1 - \gamma_1(t)) + n_2(t)(x_2 - \gamma_2(t)) + n_3(t)(x_3 - \gamma_3(t)),$$

so that

$$\frac{\partial G}{\partial x_i}(t, \boldsymbol{x}) = n_i(t), \quad (i = 1, 2, 3).$$

Therefore the 2-jet is

$$j^2 \frac{\partial G}{\partial x_i}(t_0, \boldsymbol{x}_0) = n_i(t_0) + \dot{n}_i(t_0)(t - t_0) + \frac{1}{2}\ddot{n}_i(t_0)(t - t_0)^2.$$

We consider the following matrix:

$$A = \begin{pmatrix} n_1(t_0) & n_2(t_0) & n_3(t_0) \\ \dot{n}_1(t_0) & \dot{n}_2(t_0) & \dot{n}_3(t_0) \\ \ddot{n}_1(t_0) & \ddot{n}_2(t_0) & \ddot{n}_3(t_0) \end{pmatrix} = \begin{pmatrix} \mathcal{N}(t_0) \\ \dot{\mathcal{N}}(t_0) \\ \ddot{\mathcal{N}}(t_0) \end{pmatrix}.$$

By the Frenet-Serret type formula, we have

$$\dot{\mathcal{N}}(t_0) = -\kappa(t_0)\mathcal{T}(t_0) + \tau(t_0)\mathcal{B}(t_0),$$
$$\ddot{\mathcal{N}}(t_0) = -\dot{\kappa}(t_0)\mathcal{T}(t_0) - (\kappa^2(t_0) + \tau^2(t_0))\mathcal{N}(t_0) + \dot{\tau}(t_0)\mathcal{B}(t_0).$$

Since $\{\mathcal{T}(t_0), \mathcal{N}(t_0), \mathcal{B}(t_0)\}$ is an orthonormal basis of $\mathbb{R}^3$, the rank of

$$A = \begin{pmatrix} \mathcal{N}(t_0) \\ -\kappa(t_0)\mathcal{T}(t_0) + \tau(t_0)\mathcal{B}(t_0) \\ -\dot{\kappa}(t_0)\mathcal{T}(t_0) - (\kappa^2(t_0) + \tau^2(t_0))\mathcal{N}(t_0) + \dot{\tau}(t_0)\mathcal{B}(t_0) \end{pmatrix}$$

is equal to the rank of

$$\begin{pmatrix} 0 & 1 & 0 \\ -\kappa(t_0) & 0 & \tau(t_0) \\ -\dot{\kappa}(t_0) & -(\kappa^2(t_0) + \tau^2(t_0)) & \dot{\tau}(t_0) \end{pmatrix}.$$

Therefore, rank $A = 3$ if and only if

$$0 \neq \kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0).$$

The last condition is equivalent to the condition $\delta(t_0) \neq 0$. Moreover, the rank of

$$\begin{pmatrix} \mathcal{N}(t_0) \\ \dot{\mathcal{N}}(t_0) \end{pmatrix} = \begin{pmatrix} \mathcal{N}(t_0) \\ -\kappa(t_0)\mathcal{T}(t_0) + \tau(t_0)\mathcal{B}(t_0) \end{pmatrix}$$

is always two.

If $g_{\boldsymbol{x}_0}$ has the A_k-singularity $(k = 2, 3)$ at t_0, then G is the $\mathcal{R}$-versal unfolding of $g_{\boldsymbol{x}_0}$. This completes the proof. □

Proof of Theorem 3.3. By a straightforward calculation, we have

$$\frac{\partial \mathcal{RD}_{\boldsymbol{\gamma}}}{\partial t}(t, u) \times \frac{\partial \mathcal{RD}_{\boldsymbol{\gamma}}}{\partial u}(t, u) = -\left(\frac{\alpha(t)\kappa(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}} + u\delta(t) \right) \mathcal{N}(t).$$

Therefore, (t_0, u_0) is a regular point of $\mathcal{RD}_{\boldsymbol{\gamma}}$ if and only if

$$\frac{\alpha(t_0)\kappa(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}} + u_0\delta(t_0) \neq 0.$$

This completes the proof of (1).

By Proposition 5.1-(2), the discriminant set $\mathcal{D}_G$ of the support function G of $\boldsymbol{\gamma}$ with respect to $\mathcal{N}(t)$ is the image of the rectifying developable surface of $\boldsymbol{\gamma}$.

Suppose that $\delta(t_0) \neq 0$. It follows from Proposition 5.1-(A)-(3), (4) and (5) that $g_{\boldsymbol{x}_0}$ has the A_2-type singularity (respectively, the A_3-type singularity) at $t = t_0$ if and only if

$$(\text{**}) \quad u_0 = -\frac{\alpha(t_0)\kappa(t_0)}{\delta(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}$$

and $\sigma(t_0) \neq 0$ (respectively, (**), $\sigma(t_0) = 0$ and $\dot{\sigma}(t_0) \neq 0$). By Theorem 5.2 and Proposition 5.3, we have (2)-(i) and (3).

Suppose that $\delta(t_0) = 0$. It follows from Proposition 5.1 (B)-(6) and (7) that $g_{\boldsymbol{x}_0}$ has the A_3-type singularity if and only if $\alpha(t_0) = 0$, $\dot{\alpha}(t_0) \neq 0$ and $\dot{\delta}(t_0) = 0$, or $\alpha(t_0) = 0$, $\dot{\delta}(t_0) \neq 0$ and

$$\dot{\alpha}(t_0)\kappa(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)} + u_0(\kappa(t_0)\ddot{\tau}(t_0) - \ddot{\kappa}(t_0)\tau(t_0)) \neq 0.$$

By Theorem 5.2 and Proposition 5.3, we have the assertion (2)-(β). This completes the proof. □

§6.   Examples

We give examples to understand the phenomena for rectifying developable surfaces of framed base curves and framed helices.

Example 6.1 (The astroid)**.** The *astroid* $\boldsymbol{\gamma} : [0, 2\pi) \to \mathbb{R}^3$ is defined by $\boldsymbol{\gamma}(t) = (\cos^3 t, \sin^3 t, \cos 2t)$. See Fig.1. Then

$$\mathcal{T}(t) = \frac{1}{5} (-3\cos t, 3\sin t, -4)$$

gives the unit tangent vector and $\alpha(t) = 5\cos t \sin t$ is a speed function. By a direct calculation, we have

$$\mathcal{N}(t) = (\sin t, \cos t, 0), \quad \mathcal{B}(t) = \frac{1}{5} (4\cos t, -4\sin t, -3),$$

$\kappa(t) = 3/5$ and $\tau(t) = 4/5$. Since $\delta(t) \equiv 0$ and Corollary 4.3, $\boldsymbol{\gamma}$ is a framed helix. For the astroid $\boldsymbol{\gamma}$, the rectifying developable surface is given by $\mathcal{RD}_{\boldsymbol{\gamma}}(t, u) = (\cos^3 t, \sin^3 t, -u + \cos 2t)$. By Theorem 3.3 (2)-(iii), we have the cuspidal edge singularities at $t = 0, \pi/2, \pi, 3\pi/2$ (Fig.2).
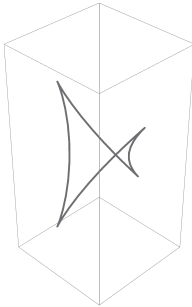

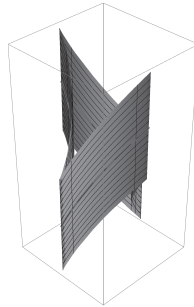Fig. 1. $\boldsymbol{\gamma}$ of Example 6.1

Fig. 2. $\boldsymbol{\gamma}$ and $\mathcal{RD}_{\boldsymbol{\gamma}}$
of Example 6.1

Example 6.2 (The spherical nephroid (cf. [13]))**.** The *spherical nephroid* $\boldsymbol{\gamma} : [0, 2\pi) \to S^2 \subset \mathbb{R}^3$ is defined by

$$\boldsymbol{\gamma}(t) = \left(\frac{3}{4}\cos t - \frac{1}{4}\cos 3t, \frac{3}{4}\sin t - \frac{1}{4}\sin 3t, \frac{\sqrt{3}}{2}\cos t \right).$$

See Fig.3. Then

$$\mathcal{T}(t) = \frac{1}{2} \left(\sqrt{3}\cos 2t, \sqrt{3}\sin 2t, -1 \right)$$

gives the unit tangent vector and $\alpha(t) = \sqrt{3}\sin(t)$ is a speed function. By a direct calculation, we have

$$\mathcal{N}(t) = (-\sin 2t, \cos 2t, 0)\,,\ \ \mathcal{B}(t) = \frac{1}{2}\left(\cos 2t, \sin 2t, \sqrt{3}\,\right),$$

$\kappa(t) = \sqrt{3}$ and $\tau(t) = -1$. Since $\delta(t) \equiv 0$ and Corollary 4.3, $\boldsymbol{\gamma}$ is a framed helix. For the spherical nephroid, the rectifying developable surface is given by

$$\mathcal{RD}_{\boldsymbol{\gamma}}(t, u) = \left(\frac{3}{4}\cos t - \frac{1}{4}\cos 3t, \frac{3}{4}\sin t - \frac{1}{4}\sin 3t, u + \frac{\sqrt{3}}{2}\cos t\right).$$

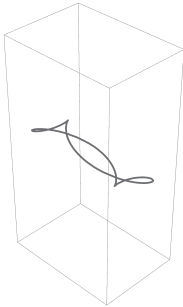By Theorem 3.3 (2)-(iii), we have the cuspidal edge singularities at $t = 0, \pi$ (Fig.4).

Fig. 3. $\boldsymbol{\gamma}$ of Example 6.2
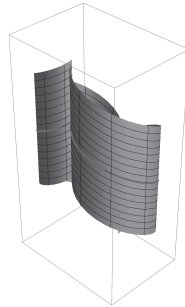
Fig. 4. $\boldsymbol{\gamma}$ and $\mathcal{RD}_{\boldsymbol{\gamma}}$ of Example 6.2

Example 6.3 ((2, 3, 5)-type). Let $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^3$ be

$$\boldsymbol{\gamma}(t) = \left(\frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{5}t^5\right).$$

See Fig.5. We say that $\boldsymbol{\gamma}$ is of type $(2, 3, 5)$. Then

$$\mathcal{T}(t) = \frac{1}{\sqrt{1 + t^2 + t^6}}\left(1, t, t^3\right)$$

gives the unit tangent vector and $\alpha(t) = t\sqrt{1 + t^2 + t^6}$ is a speed function. By a direct calculation, we have

$$\kappa(t) = \frac{\sqrt{1 + 9t^4 + 4t^6}}{1 + t^2 + t^6}, \ \ \tau(t) = \frac{6t\sqrt{1 + t^2 + t^6}}{1 + 9t^4 + 4t^6}.$$

Since $\delta(0) = 6$, $\sigma(0) = 1/6$ and $\alpha(0) = 0$ and Theorem 3.3 (2)-(i), the rectifying developable surface $\mathcal{RD}(t, u)$ is locally diffeomorphic to the cuspidal edge $\boldsymbol{ce}$ at $(0, 0)$ (Fig.6).
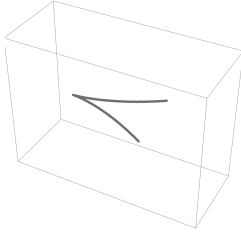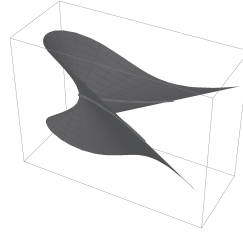

Fig. 5. $\boldsymbol{\gamma}$ of Example 6.3

Fig. 6. $\boldsymbol{\gamma}$ and $\mathcal{RD}_{\boldsymbol{\gamma}}$
of Example 6.3

§Appendix A.   Framed base curves in the Euclidean space

We define a set

$$\Delta = \{(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle \boldsymbol{\nu}_1, \boldsymbol{\nu}_1 \rangle = \langle \boldsymbol{\nu}_2, \boldsymbol{\nu}_2 \rangle = 1, \langle \boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \rangle = 0\}$$
$$= \{(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) \in S^2 \times S^2 \mid \langle \boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \rangle = 0\}.$$

Definition A.1. We say that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3 \times \Delta \subset \mathbb{R}^3 \times S^2 \times S^2$ is a *framed curve* if $\langle \dot{\boldsymbol{\gamma}}(t), \boldsymbol{\nu}_1(t) \rangle = 0$ and $\langle \dot{\boldsymbol{\gamma}}(t), \boldsymbol{\nu}_2(t) \rangle = 0$ for all $t \in I$. We also say that $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ is a *framed base curve* if there exists $(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \Delta$ such that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ is a framed curve.

Then we have the Frenet-Serret type formula of the framed curve $\boldsymbol{\gamma}$. We define $\boldsymbol{\mu}(t) = \boldsymbol{\nu}_1(t) \times \boldsymbol{\nu}_2(t)$ and call $\{\boldsymbol{\nu}_1(t), \boldsymbol{\nu}_2(t), \boldsymbol{\mu}(t)\}$ a *moving frame along the framed base curve* $\gamma(t)$. By standard arguments, we have the Frenet-Serret type formulae as follows:

Proposition A.2. ([4]) *Let* $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ *be a framed curve. Then we have*

$$\begin{pmatrix} \dot{\boldsymbol{\nu}}_1(t) \\ \dot{\boldsymbol{\nu}}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu}_1(t) \\ \boldsymbol{\nu}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\ell(t) = \langle \dot{\boldsymbol{\nu}}_1(t), \boldsymbol{\nu}_2(t) \rangle$, $m(t) = \langle \dot{\boldsymbol{\nu}}_1(t), \boldsymbol{\mu}(t) \rangle$ *and* $n(t) = \langle \dot{\boldsymbol{\nu}}_2(t), \boldsymbol{\mu}(t) \rangle$. *Moreover, there exists a smooth function* $\alpha(t)$ *such that* $\dot{\boldsymbol{\gamma}}(t) = \alpha(t)\boldsymbol{\mu}(t)$.

The quadruplet (ℓ, m, n, α) is an important invariant of a framed curve. We call (ℓ, m, n, α) the *curvature* of the framed curve. Note that t_0 is a singular point of $\boldsymbol{\gamma}$ if and only if $\alpha(t_0) = 0$.

Definition A.3. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ and $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}_1}, \widetilde{\boldsymbol{\nu}_2}) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. We say that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ and $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}_1}, \widetilde{\boldsymbol{\nu}_2})$ are *congruent* as framed curves if there exists a rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde{\boldsymbol{\gamma}}(t) = A(\boldsymbol{\gamma}(t)) + \boldsymbol{a}$, $\widetilde{\boldsymbol{\nu}_1}(t) = A(\boldsymbol{\nu}_1(t))$ and $\widetilde{\boldsymbol{\nu}_2}(t) = A(\boldsymbol{\nu}_2(t))$ for all $t \in I$.

We have shown the existence and the uniqueness for framed curves similarly to the case of regular space curves in [4].

Theorem A.4 (The Existence Theorem, [4]). *Let* $(\ell, m, n, \alpha) : I \to \mathbb{R}^4$ *be a smooth mapping. There exists a framed curve* $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ *whose curvature is* (ℓ, m, n, α).

Theorem A.5 (The Uniqueness Theorem, [4]). *Let* $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ *and* $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}_1}, \widetilde{\boldsymbol{\nu}_2}) : I \to \mathbb{R}^3 \times \Delta$ *be framed curves with curvatures* (ℓ, m, n, α) *and* $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$. *If* $(\ell, m, n, \alpha) = (\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$, *then* $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ *and* $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{\nu}_1}, \widetilde{\boldsymbol{\nu}_2})$ *are congruent as framed curves.*

References

[1] V. I. Arnol'd, *Singularities of Caustics and Wave fronts*, *Mathematics and its Applications* **62**, Kluwer Academic Publishers, 1990.

[2] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps Vol. I*, Birkhäuser, 1986.

[3] J. W. Bruce and P. J. Giblin, *Curves and singularities. A geometrical introduction to singularity theory. Second edition*, Cambridge University Press, Cambridge, 1992.

[4] S. Honda and M. Takahashi, Framed curves in the Euclidean space, *Adv. Geom.* **16** (2016), 265–276.

[5] S. Honda and M. Takahashi, Evolutes and focal surfaces of framed immersions in the Euclidean space, to appear in *Proc. Roy. Soc.* Edinburgh Sect. A.

[6] G. Ishikawa, Determinacy of envelope of the osculating hyperplanes to a curve, *Bull. London Math. Soc.* **25** (1993), 603–610.

[7] G. Ishikawa, Developable of a curve and its determinacy relative to the osculation-type, *Quart. J. Math. Oxford Ser.* (2) **46** (1995), 437–451.

[8] G. Ishikawa, Topological classification of the tangent developable of space curves, *J. London Math. Soc.* **62** (2000), 583–598.

[9] S. Izumiya, H. Katsumi and T. Yamasaki, The rectifying developable and the spherical Darboux image of a space curve, *Banach Center Publications* **50** (1999), 137–149.

[10] S. Izumiya and N. Takeuchi, Geometry of ruled surfaces, *Applicable Math. in the Golden Age* (2003), 305–338.

[11] J. Koenderink, *Solid Shape*, MIT Press, 1990.

[12] I. R. Porteous, Some remarks on duality in S^3, *Banach Center Publications*
     50 (1999), 217–226.
[13] M. Takahashi, Legendre curves in the unit spherical bundle over the unit
     sphere and evolutes, *Contemp. Math.* **675** (2016), 337–355.

Department of Mathematics, Hokkaido University,
Sapporo 060-0810, Japan.
E-mail address: `s-honda@math.sci.hokudai.ac.jp`