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A characterization of Gorenstein planar graphs

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Abstract.

We graph-theoretically classify all Gorenstein planar graphs.

§1. Introduction

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Let Δ be a *simplicial complex* on the vertex set $\{1, \dots, n\}$ and I_Δ the *Stanley-Reisner* ideal of Δ in R. We say that Δ is *Cohen-Macaulay* (resp. *Gorenstein*) over k if so is R/I_Δ. One of the main problems is to characterize the Cohen-Macaulay and the Gorenstein property of Δ from the combinatorial data of Δ (see e.g. [13]). For example, Δ is Gorenstein whenever its geometric realization, denoted by $|\Delta|$, is isomorphic to a *sphere* according to [13, Corollary 5.2]. In general, these properties depend on not only the structure of Δ but also the characteristic of k (see [3, 12]).

In this paper we are concerned only with *flag complexes*. A simplicial complex is a flag complex if all of its minimal non-faces are two element sets. Flag complexes are closely related to graphs.

Let G be a simple graph with the vertex set $V(G) = \{1, \dots, n\}$ and the edge set $E(G)$. We associate to G a quadratic squarefree monomial ideal

$$I(G) = (x_i x_j \mid ij \in E(G)) \subset R,$$

which is called the *edge ideal* of G. An *independent set* in G is a set of vertices no two of which are adjacent to each other. The *independence complex* of G, denoted by $\Delta(G)$, is the set of independent sets of G. It is well-known that $I_{\Delta(G)} = I(G)$, so that $\Delta(G)$ is a flag complex. Conversely, any flag complex is the independence complex of some graph.

We say that G is Cohen-Macaulay (resp. Gorenstein) over k if so is $R/I(G)$; G is *well-covered* if very maximal independent set of G has the

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same size, that is $\alpha(G)$, the *independence number* of G. A well-covered graph G is said to be a member of the class W_2 if $G \setminus v$ is well-covered with $\alpha(G \setminus v) = \alpha(G)$ for every vertex v (see [11, 14]); where $G \setminus v$ stands for the induced subgraph of G on the vertex set $V(G) \setminus \{v\}$. At first sight, if G is Gorenstein then G is in W_2 (see [7, Lemma 3.1]). In general we cannot read off the Gorenstein property of a graph just from its structure since this property as usual depends on the characteristic of k (see [8, Proposition 3.1]), so now we focus on some classes of graphs such as (see [4, 6, 8]):

(1) A bipartite graph is Gorenstein if and only if it is consists of disjoint edges.
(2) A chordal graph is Gorenstein if and only if it is consists of disjoint edges.
(3) A triangle-free graph is Gorenstein if and only if it is a member of W_2.

The main theme of this paper is to characterize Gorenstein graphs among *planar graphs.* A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a planar graph. Because a graph is Gorenstein if and only if every its connected component is Gorenstein by Lemma 2.2, so that it suffices to characterize Gorentein planar connected graphs.

This work is done for triangle-free planar graphs (see [7, 8]). Namely, for $m \geqslant 1$, let G_m be the graph with the vertex set $\{1, 2, \ldots, 3m-1\}$ and the edge set

$$\{\{1,2\}, \{\{3s-1, 3s\}, \{3s, 3s+1\}, \{3s+1, 3s+2\}, \\ \{3s+2, 3s-2\}\}_{s=1,\ldots,m-1}, \{\{3t-3, 3t\}\}_{t=2,\ldots,m-1}\}.$$

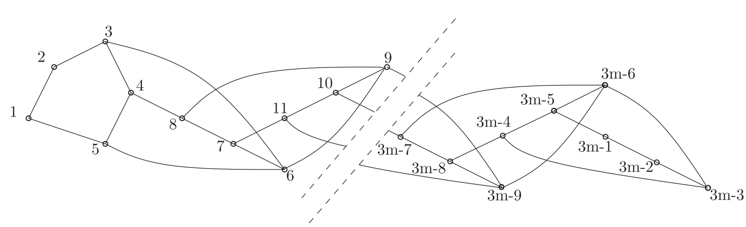

Fig. 1. The graph G_m

Then the family $\mathcal{G} = \{G_m\}_{m \geqslant 1}$ is exactly Gorenstein triangle-free planar connected graphs (see [8, Remark 4.5]). It is worth mentioning that this family can be visualized by a recursive construction due to Pinter [10]:

(1) Begin with $G_1 = K_2$ and $G_2 = C_5$;

(2) Given the graph G_m for $m \geqslant 2$, let x be a vertex of degree 2 in G_m and $N_G(x) = \{y, z\}$. Then construct the graph G_{m+1} with precisely three more points than G_m as follows. Let the three new points be a, b and c. Then join a to three points b, y, z; b to c and c to x (see Figure 2).

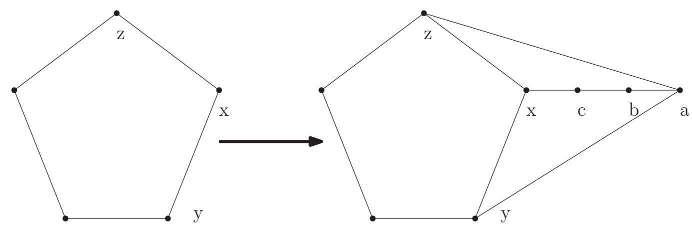

Fig. 2. The construction of G_3 from G_2

Moving away from triangle-free graphs, we get a new Gorenstein planar graph R_3 (the name due to [11]) as in Figure 3. The main result of the paper says that Gorenstein planar connected graphs are exactly the family $\mathcal{G}$ and R_3.

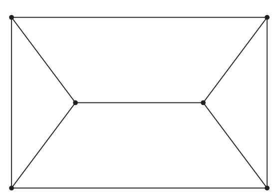

Fig. 3. The graph R_3

Theorem 3.7 *Let G be a planar connected graph without isolated vertices. Then, G is Gorenstein if and only if G is either G_m, where $m = \alpha(G)$, or R_3.*

The paper is organized as follows. In Section 1 we set up some basic terminology for simplicial complexes and graphs. In Section 2 we characterize Gorenstein planar graphs.

§2. Preliminaries

Let Δ be a simplicial complex on the vertex set $V = \{1, \dots, n\}$. Thus, Δ is a collection of subsets of V closed under taking subsets; that is, if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. Given a field k we now define the Stanley-Reisner ideal of Δ to be the square-free monomial ideal

$$I_\Delta = (x_{j_1} \cdots x_{j_i} \mid j_1 < \cdots < j_i \text{ and } \{j_1, \dots, j_i\} \notin \Delta)$$

in $R = k[x_1, \dots, x_n]$ and the *Stanley-Reisner* ring of Δ to be the quotient ring $k[\Delta] = R/I_\Delta$. We say that Δ is Cohen-Macaulay (resp. Gorenstein) if $k[\Delta]$ is Cohen-Macaulay (resp. Gorenstein). For $F \in \Delta$, we define the dimension of $F \in \Delta$ to be $\dim F = |F| - 1$ and the dimension of Δ to be $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. The link of F inside Δ is its subcomplex:

$$\mathrm{lk}_\Delta F = \{H \in \Delta \mid H \cup F \in \Delta \text{ and } H \cap F = \emptyset\}.$$

Let f_i be the number of i-dimensional faces of Δ. Then the *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$ of Δ is defined by

$$\widetilde{\chi}(\Delta) := \sum_{i=-1}^{d} (-1)^i f_i,$$

where $d = \dim(\Delta)$.

We call Δ is *pure* if all its facets have the same dimension, and *Eulerian* if it is pure and $\widetilde{\chi}(\mathrm{lk}_\Delta F) = (-1)^{\dim(\mathrm{lk}_\Delta F)}$ for all $F \in \Delta$.

The restriction of Δ to a subset S of V is $\Delta_S := \{F \in \Delta \mid F \subseteq S\}$. The star of a vertex v in Δ is $\mathrm{st}_\Delta(v) := \{F \in \Delta \mid F \cup \{v\} \in \Delta\}$. Let $\mathrm{core}(V) := \{x \in V \mid \mathrm{st}_\Delta(x) \neq \Delta\}$, then the core of Δ is $\mathrm{core}(\Delta) := \Delta_{\mathrm{core}(V)}$. For a vertex v, $\Delta = \mathrm{st}_\Delta(v)$ means that Δ is a cone over v, hence $\Delta = \mathrm{core}(\Delta)$ means that Δ is not a cone.

Let Δ and Γ be simplicial complexes with disjoint vertex sets V and W, respectively. Define the join $\Delta * \Gamma$ to be the simplicial complex on the vertex set $V \cup W$ with faces $F \cup H$, where $F \in \Delta$ and $H \in \Gamma$. It follows that $\Delta = \mathrm{core}(\Delta) * \langle V \setminus \mathrm{core}(V)\rangle$; where for a finite set P, we denote $\langle P \rangle$ to be the simplex over P.

We then have a criterion for determining when a simplicial complex Δ is Gorenstein due to Stanley (see [13, Theorem 5.1 in Chapter II]).

Lemma 2.1. *Δ is Gorenstein if and only if and only if* $\mathrm{core}(\Delta)$ *is an Euler complex which is Cohen-Macaulay.*

Let G be a simple graph. For a subset S of $V(G)$, the neighborhood of S in G is

$$N_G(S) := \{v \in V(G) \setminus S \mid uv \in E(G) \text{ for some } u \in S\},$$

the close neighborhood of S is $N_G[S] := S \cup N_G(S)$. We denote by G_S the induced subgraph of G on the vertex set $V(G) \setminus N_G[S]$. If v is a vertex of G, we write $N_G(v)$ (resp. $N_G[v]$ and G_v) instead of $N_G(\{v\})$ (resp. $N_G[\{v\}]$ and $G_{\{v\}}$). The *degree* of v in G is $\deg_G(v) := |N_G(v)|$. If $\deg_G(v) = 0$, then v is called an *isolated vertex* of G.

For the independence complex of G, we have $\dim(\Delta(G)) = \alpha(G) - 1$. Clearly, G is well-covered if and only if $\Delta(G)$ is pure and $\Delta(G) = \operatorname{core}(\Delta(G))$ if and only if G has no isolated vertices.

Lemma 2.2. *Let G be a simple graph. Then, G is Gorenstein if and only every connected component of G is also Gorenstein.*

Proof. Let $G_1, \dots, G_s$ be connected components of G. Then,

$$\Delta(G) = \Delta(G_1) * \cdots * \Delta(G_s).$$

Hence, $\Delta(G)$ is Gorenstein if and only if so are $\Delta(G_1), \dots, \Delta(G_s)$ by [2], and the lemma follows. Q.E.D.

Lemma 2.3. *Let G be a Gorenstein graph without isolated vertices and let S be an independent subset of G. Then,*

(1) *If G_S is a Gorenstein graph without isolated vertices*;
(2) $\alpha(G_S) = \alpha(G) - |S|$

Proof. (1) Since G has no isolated vertices, $\Delta(G) = \operatorname{core}(\Delta(G))$. By Lemma 2.1, $\Delta(G)$ is Eulerian. Note that $\Delta(G_S) = \operatorname{lk}_{\Delta(G)}(S)$, so $\Delta(G_S)$ is Eulerian. In particular, $\widetilde{\chi}(\Delta(G_S)) \neq 0$, so $\Delta(G)$ is not a cone. It follows that G_S has no isolated vertices.

On the other hand, $\Delta(G_S)$ is Cohen-Macaulay by [5, Corollary 8.1.8]. Therefore, $\Delta(G_S)$ is Gorenstein by Lemma 2.1, i.e. G_S is Gorenstein.

(2) Because $\Delta(G)$ is pure, we have

$$\alpha(G_S) = \dim(\Delta(G_S)) + 1 = \dim(\Delta(G)) - |S| + 1 = \alpha(G) - |S|,$$

and the lemma follows. Q.E.D.

In the sequence we also need a criterion on deciding whether a given graph is planar. A *minor* of a graph G is any graph obtainable from G by means of a sequence of vertex and edge deletions and edge contractions. Alternatively, consider a partition $(V_0, V_1, \dots, V_m)$ of $V(G)$ such that

$G[V_i]$ is connected, $1 \leqslant i \leqslant m$, and let H be the graph obtained from G by deleting V_0 and shrinking each induced subgraph $G[V_i]$, $1 \leqslant i \leqslant m$, to a single vertex. Then any spanning subgraph of H is a minor of G. A minor which is isomorphic to K_5 or $K_{3,3}$ is called a Kuratowski minor. Then, Wagner's theorem (see [1, Theorem 10.3]) says that: *G is planar if and only if G has no Kuratowski minor.*

§3. A Characterization of Planar Gorenstein graphs

In this section we characterize Gorenstein planar graphs. A graph is called *trivial* if it has only one vertex. We always use the symbol G to indicate a simple graph without isolated vertices. We first consider graphs with small independence numbers.

Lemma 3.1. *Let G be a graph with $\alpha(G) = 1$. Then, G is Gorenstein if and only if G is an edge.*

Proof. Since $\alpha(G) = 1$, G is a complete graph. Note that G is not trivial by our assumption, $|V(G)| \geqslant 2$. By [6, Corollary 2.2], we conclude that G is Gorenstein if and only if it is an edge. Q.E.D.

Lemma 3.2. *R_3 is Gorenstein.*

Proof. Since the geometric realization of $\Delta(R_3)$ is isomorphic to the cycle C_6, the lemma now follows from [13, Theorem 5.1 in Chapter II]. Q.E.D.

Lemma 3.3. *Let G be a planar graph with $\alpha(G) = 2$. Then G is Gorenstein if and only if G is two disjoint edges, or a pentagon, or R_3.*

Proof. Let $n := |V(G)|$. For each vertex v of G, G_v is a Gorenstein graph without isolated vertices by Lemma 2.2. Since, $\alpha(G_v) = \alpha(G) - 1 = 1$, G_v is just one edge by Lemma 3.1. It follows that $\deg_G(v) = n - 3$ for any $v \in V(G)$. Let G^c denote the complement of G. Then, $\deg_{G^c}(v) = 2$ for all $v \in V(G^c)$, and hence G^c is an n-cycle. Since $\alpha(G) = 2$, we have $n \geqslant 4$. We now consider the following cases depending on n:

Case 1: $n = 4$. Then, G is two disjoint edges.

Case 2: $n = 5$. Then, G is a pentagon.

Case 3: $n = 6$. Then, G is just R_3.

Case 4: $n \geqslant 7$. Assume that $V(G) = \{1, 2, \ldots, n\}$. If we delete all vertices $\{8, \ldots, n\}$ from G, and then contract the edge $\{2, 6\}$, then we obtain a graph that has a minor $K_{3,3}$. It follows that G is not planar by Wagner's theorem, a contradiction.

In summary, G is Gorenstein if and only if G is two disjoint edges, or a pentagon, or R_3. Clearly, these graphs are Gorenstein. Q.E.D.

We call a graph G is *locally triangle-free* if G_v is triangle-free for every vertex v of G. The next result plays a key role in characterizing Gorenstein graphs with higher independence numbers (see [9, Theorem 4.1]).

Lemma 3.4. *Let G be a locally triangle-free graph. Then, G is Gorenstein if and only if G is a triangle-free graph in W_2 except for* 4 *graphs*: P_{10}, P_{12}, Q_9 *and* Q_{12} (see Figure 4)*.*

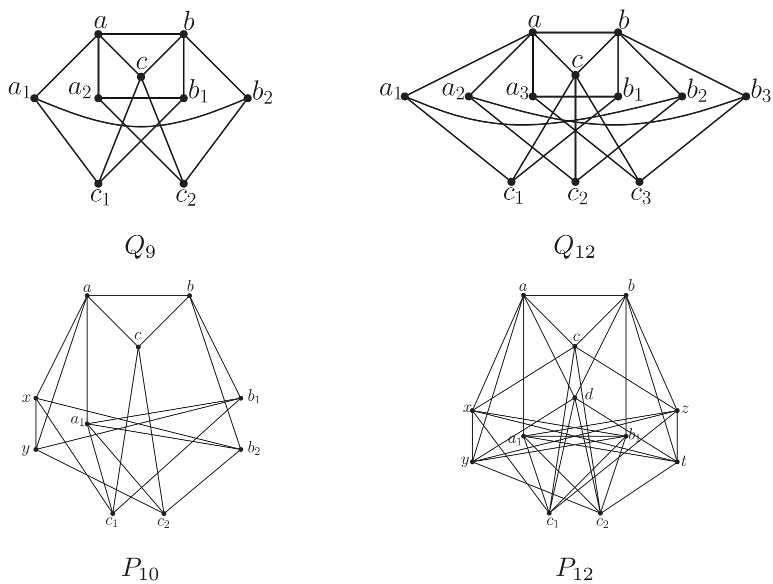

Fig. 4. Four graphs Q_9, Q_{12}, P_{10} and P_{12}

By using this result, we next characterize Gorenstein locally triangle-free planar graph. Namely,

Lemma 3.5. *Let G be a Gorenstein planar connected graph. If G is locally triangle-free, then G is either a triangle-free graph or R_3.*

Proof. If $\alpha(G) \leqslant 2$, the lemma follows from Lemmas 3.1 and 3.3.

Assume that $\alpha(G) \geqslant 3$. By Lemma 3.4, G is a triangle-free graph or one of P_{10}, P_{12}, Q_9 or Q_{12}. Therefore, it remains to verify that P_{10}, P_{12}, Q_9 and Q_{12} are not planar. In order to do that, we use Wagner's theorem.

For the graph Q_9, by shrinking a_1 to c_1, b_2 to c_2, a to a_2 and b to b_1 we see that P_{10} has a minor K_5. Hence, Q_9 is not planar.

For the graph Q_{12}, by shrinking a_1, a_2 and a_3 to a; and b_1, b_2 and b_3 to b we see that Q_{12} has a minor $K_{3,3}$. Hence, Q_{12} is not planar.

For the graph P_{10}, by shrinking a and b to c, b_1 to b_2, x and y to c_1, we see that P_{10} has a minor K_5. Hence, P_{10} is not planar.

Finally, for the graph P_{12}, first deleting two vertices a_1 and b_1; and then shrinking a and b to c, x to y, and z to t, we see that P_{12} has a minor $K_{3,3}$. Hence, P_{12} is not planar, and the lemma follows. Q.E.D.

Lemma 3.6. *If G is a Gorenstein planar connected graph, then it is locally triangle-free.*

Proof. We prove by induction on $\alpha(G)$. If $\alpha(G) \leqslant 2$, the lemma follows from Lemmas 3.1 and 3.3.

If $\alpha(G) \geqslant 3$. Assume on contrary that G is not locally triangle-free, so that G_x would have a triangle for some vertex x of G. We first prove the following claims.

Claim 1: Every connected component of G_v is either a triangle-free non-trivial graph or R_3 for any vertex v of G.

Indeed, by Lemma 2.3 we have G_v is Gorenstein graph with $\alpha(G_v) = \alpha(G) - 1$ which has no isolated vertices. Note that G_v is planar as well. By the induction hypothesis, G_v is locally triangle-free. Thus, every component of G_v is a triangle-free graph or R_3 by Lemmas 2.2 and 3.5, as claimed.

Let $G_1, \ldots, G_m$ be connected components of G_x. By Claim 1, each G_i is either a Gorenstein non-trivial triangle-free graph or R_3. Since G_x has a triangle, we may assume that G_1 is R_3. Let $X := N_G(x)$.

Claim 2: There is a vertex of G_1 which is not adjacent to any vertex in X.

Indeed, assume on contrary that every vertex of G_1 is adjacent to some vertex in X. By deleting $G_2, \ldots, G_m$, and shrinking X into x, we obtain a graph G' as in Figure 5. By shrinking two vertices inside the rectangle, we see that G' has a minor $K_{3,3}$, so G is not planar, and the claim follows.

Because G is connected, there is some vertex in X which is adjacent to some vertex of G_1. Together with By Claim 2, we can partition X into two nonempty subsets $X = Y \cup Z$ such that every vertex in Y is adjacent to some vertex of G_1; but every vertex in Z is not adjacent to any vertex of G_1.

Claim 3: $m = 1$.

Indeed, assume that $m \geqslant 2$. Let v be a vertex of G_i for some $i = 2, \ldots, m$. If v is not adjacent to any vertex in Y, then there would be a connected component of G_v that contains G_1 and the vertex x.

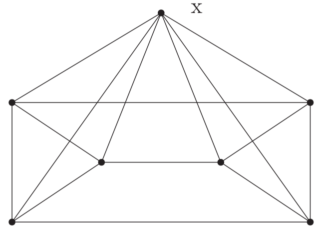

Fig. 5. The graph G'

This is impossible because every connected component of G_v is either a triangle-free graph or R_3 by Claim 1. Hence, every vertex of G_i for every $i = 2, \ldots, m$ is adjacent to all vertices in Y.

By Claim 2, there is a vertex of G_1, say a, which is not adjacent to any vertex in X. Then, G_a has a connected component, say H, that contains $N_G[x]$ and $G_2, \ldots, G_m$. Since G_2 is non-trivial by Claim 1 and every vertex of G_2 is adjacent to all vertices in Y, H must contain triangles, so H is isomorphic to R_3 by Claim 1. Hence, we may assume that $E(H) = \{xy, xz, xw, yz, zu, wu, yv, uv, wv\}$, so that G_2 is just the edge uv. In this case, $Y = \{w\}$ and $Z = \{y, z\}$. But then the graph G_z has a component that contains $G_1 = R_3$ and w. This contradicts Claim 1. Therefore, $m = 1$, as claimed.

We now return to prove the lemma. By Claim 3, we have $G_x = G_1$. Since G_1 is R_3, $\alpha(G) = \alpha(R_3) + 1 = 3$. Assume that $E(G_1) = \{ab, bc, ca, de, ef, fd, ad, be, cf\}$ and a is not adjacent to any vertex in X. Fix any vertex $z \in Z$. Then, the set $S := \{a, z\}$ is an independent set of G, and then G_S is Gorenstein with $\alpha(G_S) = \alpha(G) - |S| = 1$ by Lemma 2.3. Thus, G_S is an edge by Lemma 3.1, and thus it must be the edge ef. Consequently, $N_G[x] = N_G[z]$. Fix any another vertex y of Y. Then, (xyz) is a triangle of G because of $N_G[x] = N_G[z]$.

In the other hand, since $\alpha(G_a) = \alpha(G) - 1 = 2$ and G_a contains the triangle (xyz), G_a must be R_3 by Claim 1. Note also that x, z are vertices of G_a and

$$N_{G_a}[x] = N_G[x] = N_G[z] = N_{G_a}[z].$$

This is impossible since neighborhoods of two distinct vertices of R_3 are different. Therefore, G is locally triangle-free, and the proof is complete. Q.E.D.

We are in position to prove the main result of this paper.

Theorem 3.7. *Let G be a planar connected graph without isolated vertices. Then, G is Gorenstein if and only if G is either G_m, where $m = \alpha(G)$, or R_3.*

Proof. Assume that G is Gorenstein. If $\alpha(G) \leqslant 2$, then the theorem follows from Lemmas 3.1 and 3.3. If $\alpha(G) \geqslant 3$, then G is a triangle-free graph by Lemma 3.6, and then it is a member of $\mathcal{G}$ by [8, Remark 4.5]. Thus, $G = G_m$ for some $m \geqslant 1$. Since $\alpha(G_m) = m$ by [7, Lemma 3.7], we have $m = \alpha(G)$.

The converse implication follows from [8, Remark 4.5] together with Lemma 3.2. Q.E.D.

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