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Algorithms for D-modules, integration, and generalized functions with applications to statistics

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Abstract.

This is an enlarged and revised version of the slides presented in a series of survey lectures given by the present author at MSJ SI 2015 in Osaka. The goal is to introduce an algorithm for computing a holonomic system of linear (ordinary or partial) differential equations for the integral of a holonomic function over the domain defined by polynomial inequalities. It applies to the cumulative function of a polynomial of several independent random variables with e.g., a normal distribution or a gamma distribution. Our method consists in Gröbner basis computation in the Weyl algebra, i.e., the ring of differential operators with polynomial coefficients. In the algorithm, generalized functions are inevitably involved even if the integrand is a usual function. Hence we need to make sure to what extent purely algebraic method of Gröbner basis applies to generalized functions which are based on real analysis.

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§1. Introduction: aim and an example from statistics

A univariate function is called holonomic if it satisfies a (non-trivial) linear ordinary differential equation. Special functions such as the hypergeometric function or the Bessel function are holonomic, as well as rational functions and their exponential and logarithm. As is well-known, the solutions of a linear ordinary differential equation constitute a finite dimensional vector space.

A D-module is a system of linear (partial or ordinary) differential equations with polynomial (or analytic function) coefficients. There is a special class of D-modules which are called holonomic, the solution spaces of which are finite dimensional vector spaces. This notion was introduced by Mikio Sato and J. Bernstein independently. Bernstein [2], [3] introduced a special class of linear partial differential equations with polynomial coefficients which was called the Bernstein class in [4]. On the other hand, Sato and his collaborators M. Kashiwara, T. Kawai [31] introduced the notion of a holonomic system, which was called at first a maximally overdetermined system, in the category of differential operators with analytic coefficients.

A holonomic function is a differentiable or a generalized function which is a solution of a holonomic system. For example, $\exp(f) = e^f$ is a holonomic function for any polynomial $f = f(x_1, \dots, x_n)$. In statistics, most of important probability density functions, such as those of the multivariate normal distribution and the gamma distribution are holonomic. Our aim is to find a holonomic system which is satisfied by the integral of a holonomic function over the domain defined by polynomial inequalities.

As an example, let us consider the integral

$$F(t) = \frac{1}{2\pi}\int_{D(t)} \exp\Bigl(-\frac{1}{2}(x^2+y^2)\Bigr)\,dxdy,\ \ D(t) = \{(x,y) \in \mathbb{R}^2 \mid xy \leq t\}.$$

It can be regarded as the cumulative distribution function of xy with (x, y) being a random vector with the two dimensional standard normal (Gaussian) distribution. Let us introduce the Heaviside function $Y(t)$ such that $Y(t) = 1$ for $t > 0$ and $Y(t) = 0$ for $t < 0$. (One does not need to mind the value at $t = 0$.) $Y(t)$ is discontinuous at $t = 0$ and its derivative $Y'(t)$ as a generalized function coincides with Dirac's delta function $\delta(t)$. As a generalized function, $\delta(t)$ vanishes outside of $t = 0$ and $t\delta(t) = 0$ holds everywhere in $\mathbb{R}$.

By using the Heaviside function, we rewrite $F(t)$ as

$$F(t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp\Bigl(-\frac{1}{2}(x^2+y^2)\Bigr)Y(t-xy)\,dxdy.$$

Differentiation under the integral sign yields

$$v(t) := F'(t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp\Big(-\frac{1}{2}(x^2+y^2)\Big)\delta(t-xy)\,dxdy.$$

The integrand $u(x,y,t) := \exp\Big(-\frac{1}{2}(x^2+y^2)\Big)\delta(t-xy)$ satisfies a holonomic system

$$(\partial_y + x\partial_t + y)u = (\partial_x + y\partial_t + x)u = (t-xy)u = 0$$

with $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, $\partial_t = \partial/\partial t$ as is easily checked. The integration algorithm for D-modules (see 5.2) outputs an answer

$$(t\partial_t^2 + \partial_t - t)v(t) = 0. \tag{1}$$

In fact, we have an equality

$$y\partial_t(\partial_y + x\partial_t + y) - y(\partial_x + y\partial_t + x) + (\partial_t^2 - 1)(t - xy) \\ = -\partial_x y + \partial_y y\partial_t + t\partial_t^2 + \partial_t - t$$

in the ring of differential operators. Since the differential operator on the left-hand side annihilates $u(x,y,t)$, we get

$$(t\partial_t^2 + \partial_t - t)v(t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}(t\partial_t^2 + \partial_t - t)u(x,y,t)\,dxdy \\ = \frac{1}{2\pi}\int_{\mathbb{R}^2}\partial_x(yu(x,y,t))\,dxdy - \frac{1}{2\pi}\int_{\mathbb{R}^2}\partial_y(y\partial_t u(x,y,t))\,dxdy = 0.$$

The integrals on the last line vanish since $yu(x,y,t)$ and $y\partial_t u(x,y,t)$ are 'rapidly decreasing' in x, y; this reasoning shall be made precise in 4.3.

It follows that $w(z) := v(-iz)$ satisfies the Bessel differential equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + z^2 w = 0.$$

Together with the property that $v(t) \to 0$ as $t \to \pm\infty$ and $v(-t) = v(t)$, this implies

$$v(t) = CH_0^{(1)}(i|t|) \qquad (t \neq 0)$$

with some constant C, where $H_0^{(1)}(z)$ is a Hankel function. This fact was observed, for example, by Wishart and Bartlett [36] as a special case. Note that $v(t)$ is discontinuous at $t = 0$ but is integrable and satisfies (1) in the sense of generalized functions on the whole real line $\mathbb{R}$.

It also follows from (1) that the characteristic function, i.e., the Fourier transform

$$\hat{v}(\tau) = \int_{-\infty}^{\infty} e^{it\tau} v(t)\,dt = \int_{\mathbb{R}^2} \exp\Big(i\tau xy - \frac{1}{2}(x^2+y^2)\Big)\,dxdy$$

satisfies a differential equation

$$(\tau^2+1)\frac{d}{d\tau}\hat{v}(\tau) + \tau\hat{v}(\tau) = 0.$$

Together with $\hat{v}(0) = 1$, this implies $\hat{v}(\tau) = (\tau^2+1)^{-1/2}$. Thus we get an alternative expression

$$v(t) = V_+(t+i0) + V_-(t-i0) = \lim_{\varepsilon\to+0}(V_+(t+i\varepsilon) + V_-(t-i\varepsilon))$$

as a hyperfunction of Mikio Sato ([30]) with

$$V_+(t+is) = \frac{1}{2\pi}\int_{-\infty}^{0} \frac{\exp(-i(t+is)\tau)}{\sqrt{\tau^2+1}}\,d\tau,$$
$$V_-(t+is) = \frac{1}{2\pi}\int_{0}^{\infty} \frac{\exp(-i(t+is)\tau)}{\sqrt{\tau^2+1}}\,d\tau,$$

where $V_+(t+is)$ and $V_-(t+is)$ are holomorphic functions of $t+is$ on the upper half plane $s>0$ and on the lower half plane $s<0$ respectively.

In general, for a holonomic function $u(x,y)$ with $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_d)$, let us consider the integral

$$v(y) = \int_{D(y)} u(x,y)\,dx_1\cdots dx_n,$$
$$D(y) = \{x\in\mathbb{R}^n \mid f_j(x,y)\geq 0 \quad (1\leq j\leq m)\}$$

with real polynomials $f_1,\dots,f_m$ in (x,y). We rewrite it as

$$v(y) = \int_{\mathbb{R}^n} u(x,y)Y(f_1(x,y))\cdots Y(f_m(x,y))\,dx_1\cdots dx_n$$

and apply the D-module theoretic integration algorithm to obtain a holonomic system for $v(y)$, assuming that the integrand and its derivatives are 'rapidly decreasing' with respect to the integration variables x. In the process, we also need an algorithm to compute a holonomic system for the product $uY(f_1)\cdots Y(f_m)$ as a generalized function. Then the D-module theory assures us that the obtained system of differential equations for $v(y)$ is holonomic.

Finally, let us remark that we cannot use differential operators with rational function coefficients since generalized functions are involved in the computation. For example, $x\partial_x Y(x) = 0$ does not imply $\partial_x Y(x) = \delta(x) = 0$; we cannot factor out x.

The organization of this article is as follows:

Section 2 is a hopefully concise exposition on the very beginning of the D-module theory; the central subject is holonomic D-modules. More advanced topics such as D-modules with regular singularities are not treated. The presentation is almost self-contained with some arguments and examples supplied in the next section after armed with Gröbner bases.

In Section 3, we introduce Gröbner bases over the ring of differential operators. One point is that we can compute Gröbner bases with respect to arbitrary monomial orders that are not necessarily well-orders, which will be needed in the integration algorithm. We also describe first applications of Gröbner bases to D-module theory: computation of the characteristic variety, and a proof of the equivalence of the two definitions of holonomicity introduced in the previous section.

In Section 4, we briefly review the theory of distributions in the sense of generalized functions from our viewpoint, with mention of the relation with statistical distributions. Especially, we introduce some classes of distributions which are adapted to our integration algorithm developed in the following sections.

Section 5 is a review on the integration of D-modules both from theoretical and algorithmic viewpoints; the material should be more or less standard by now.

In the first subsection of Section 6, we give some examples of integrals which correspond to random variables with respect to the multivariate standard normal distribution such as the example above. In a somewhat technical subsection 6.2, we introduce an algorithm to compute a holonomic system for the product of complex powers of polynomials and a holonomic function. This enables us to compute, in 6.3, a holonomic system for the integral of a holonomic function over the domain defined by arbitrary polynomial inequalities. Finally in 6.4, we treat the integral of a function with some auxiliary parameters which satisfies a holonomic difference-differential system.

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§2. Basics of D-module theory

We review the theory of D-modules, more precisely, of modules over the Weyl algebra, which was initiated by J. Bernstein [2], [3]. A standard reference is the first chapter of [4]. A D-module corresponds to a system of linear (ordinary or partial) differential equations with polynomial coefficients. The notion of holonomic modules, also called the Bernstein class of modules, and its characterizations are most essential. We remark that the notion of holonomic modules over the ring of differential operators with complex analytic coefficients was independently introduced by M. Sato, T. Kawai, and M. Kashiwara [31].

2.1. The ring of differential operators

Let $\mathbb{K}$ be an arbitrary field of characteristic zero. We denote by $\mathbb{K}[x] := \mathbb{K}[x_1, \ldots, x_n]$ the ring of polynomials in indeterminates $x = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{K}$. A *derivation* $\theta : \mathbb{K}[x] \to \mathbb{K}[x]$ is a $\mathbb{K}$-linear map that satisfies

$$\theta(fg) = \theta(f)g + f\theta(g) \qquad (\forall f, g \in \mathbb{K}[x]).$$

The set $\mathrm{Der}_{\mathbb{K}}\mathbb{K}[x]$ of the derivations constitutes a $\mathbb{K}[x]$-module. For $i = 1, \ldots, n$, define a derivation $\partial_i = \partial_{x_i}$ by the partial derivative

$$\partial_i \ : \ \mathbb{K}[x] \ni f \longmapsto \frac{\partial f}{\partial x_i} \in \mathbb{K}[x].$$

Then $\partial_1, \ldots, \partial_n$ are a $\mathbb{K}[x]$-basis of $\mathrm{Der}_{\mathbb{K}}\mathbb{K}[x]$. In fact, if $\theta \in \mathrm{Der}_{\mathbb{K}}\mathbb{K}[x]$, then it is easy to see that

$$\theta = \theta(x_1)\partial_1 + \cdots + \theta(x_n)\partial_n.$$

Let $\mathrm{End}_{\mathbb{K}}\mathbb{K}[x]$ be the $\mathbb{K}$-algebra consisting of the $\mathbb{K}$-linear endomorphisms of $\mathbb{K}[x]$. The ring D_n is defined to be the $\mathbb{K}$-subalgebra of $\mathrm{End}_{\mathbb{K}}\mathbb{K}[x]$ that is generated by $\mathbb{K}[x]$ and $\mathrm{Der}_{\mathbb{K}}\mathbb{K}[x]$, or equivalently, by $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_n$. This ring D_n is called the *ring of differential operators* in the variables $x = (x_1, \ldots, x_n)$ with polynomial coefficients, or, more simply, the n-th *Weyl algebra* over $\mathbb{K}$.

An element $a = a(x)$ of $\mathbb{K}[x]$ is regarded as an element of D_n as the multiplication operator $f \mapsto af$ for $f \in \mathbb{K}[x]$. With this identification, D_n contains $\mathbb{K}[x]$ as a subring. The ring D_n is a non-commutative $\mathbb{K}$-algebra. In fact, for $a \in \mathbb{K}[x]$ regarded as an element of D_n, the product in D_n satisfies

$$\partial_i a = a\partial_i + \partial_i(a) = a\partial_i + \frac{\partial a}{\partial x_i} \qquad (i = 1, \dots, n).$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $\mathbb{N} = \{0, 1, 2, \dots\}$, we use the notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\alpha = \partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then an element P of D_n is uniquely written in a finite sum

$$P = P(x, \partial) = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha,\beta} x^\alpha \partial^\beta = \sum_{\beta \in \mathbb{N}^n} a_\beta(x) \partial^\beta$$

with $a_{\alpha,\beta} \in \mathbb{K}$ and $a_\beta(x) = \sum_\alpha a_{\alpha,\beta} x^\alpha$, which is called the *normal form* of P. In fact, $a_\beta(x)$ are uniquely determined by the action of P on $\mathbb{K}[x]$ as follows: First we have $a_{(0,\dots,0)}(x) = P1$. Next, we have

$$a_{(1,0,\dots,0)}(x) = Px_1 - a_{(0,\dots,0)}(x) x_1,$$

and so on. Here we need the assumption that the characteristic of $\mathbb{K}$ is zero.

Introducing commutative indeterminates $\xi = (\xi_1, \dots, \xi_n)$ which corresponds to ∂, we associate with this P a polynomial

$$P(x, \xi) := \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha,\beta} x^\alpha \xi^\beta \ \in \mathbb{K}[x, \xi] = \mathbb{K}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$$

and call it the *total symbol* of P. Note that P must be in the normal form when ξ is substituted for ∂. By this correspondence, D_n is isomorphic to $\mathbb{K}[x, \xi]$ as a $\mathbb{K}$-vector space but not as a ring.

The product $R = PQ$ in D_n can be effectively computed by using the *Leibniz formula*

$$R(x, \xi) = \sum_{\nu \in \mathbb{N}^n} \frac{1}{\nu!} \left(\frac{\partial}{\partial \xi}\right)^\nu P(x, \xi) \cdot \left(\frac{\partial}{\partial x}\right)^\nu Q(x, \xi) \tag{2}$$

in terms of total symbols, where we use the notation $\nu! = \nu_1! \cdots \nu_n!$ for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$.

Example 2.1. Set $n = 1$ and write $x = x_1$ and $\partial = \partial_1$. Consider the product $R := \partial^m x^m$ with a non-negative integer m. Since the total symbols of ∂^m and x^m are ξ^m and x^m respectively, the Leibniz formula

(2) gives the total symbol $R(x,\xi)$ as

$$R(x,\xi) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left(\frac{\partial}{\partial \xi}\right)^{\nu} \xi^m \cdot \left(\frac{\partial}{\partial x}\right)^{\nu} x^m$$
$$= \sum_{\nu=0}^{m} \frac{1}{\nu!} \{m(m-1)\cdots(m-\nu+1)\}^2 \xi^{m-\nu} x^{m-\nu}.$$

This implies

$$\partial^m x^m = \sum_{\nu=0}^{m} \frac{1}{\nu!} \{m(m-1)\cdots(m-\nu+1)\}^2 x^{m-\nu} \partial^{m-\nu}.$$

Exercise 1. Show that an element $P = \sum_{\beta \in \mathbb{N}} a_\beta(x) \partial^\beta$ of D_n with $a_\beta(x) \in \mathbb{K}[x]$ defines the zero endomorphism of $\mathbb{K}[x]$ if and only if $a_\beta(x) = 0$ for any β.

Exercise 2. Prove the Leibniz formula (2).

Exercise 3. Set $n = 1$ and $x = x_1$, $\partial = \partial_1$. For a positive integer m, prove the formulae

$$x^m \partial^m = x\partial(x\partial - 1)\cdots(x\partial - m + 1), \quad \partial^m x^m = \partial x(\partial x + 1)\cdots(\partial x + m - 1).$$

2.2. The D-module formalism

Given $P_1, \ldots, P_r \in D_n$, let us consider a system of linear (partial or ordinary) differential equations

$$P_1 u = \cdots = P_r u = 0 \tag{3}$$

for an unknown function u. Let $I := D_n P_1 + \cdots + D_n P_r$ be the left ideal of D_n generated by $P_1, \ldots, P_r$. Then (3) is equivalent to

$$Pu = 0 \qquad (\forall P \in I).$$

Here we suppose that the unknown function u belongs to some 'function space' $\mathcal{F}$ which is a left D_n-module.

For $\mathcal{F}$ to be a left D_n-module, it is necessary that any function f belonging to $\mathcal{F}$ be infinitely differentiable and multiplication ah by an arbitrary polynomial $a \in \mathbb{K}[x]$ make sense. Here are examples of 'function spaces':

Example 2.2. By the definition, $\mathbb{K}[x]$ has a natural structure of left D_n-module since D_n is a subalgebra of $\mathrm{End}_{\mathbb{K}}\mathbb{K}[x]$. So $\mathbb{K}[x]$ has two structures: a subring of D_n and a left D_n-module. Hence for $f \in \mathbb{K}[x]$ and $P \in D_n$, Pf has two meanings:

- Pf as the product in D_n with f regarded as an element of the subring $\mathbb{K}[x]$ of D_n.
- Pf as the action of P on the element f of the left D_n-module $\mathbb{K}[x]$. In other words, we regard f as a function subject to derivations.

This might cause some confusion. In [29], the action of P on an element f of a left D_n-module is conspicuously denoted $P \bullet f$ for distinction. We shall denote, if needed, $Pf = P(f)$ to clarify the action of P on f, and $Pf = P \cdot f$ to emphasize that it is the product in D_n, following the traditional notation in D-module theory.

Example 2.3. The field $\mathbb{K}(x) = \mathbb{K}(x_1, \ldots, x_n)$ of rational functions has a natural structure of left D_n-module. For a point $a = (a_1, \ldots, a_n)$ of the affine space $\mathbb{K}^n$, the set $\mathbb{K}[x]_a$ of regular functions at a, i.e., the elements of $\mathbb{K}(x)$ whose denominators do not vanish at a, also has a natural structure of D_n-module. More generally, the localization $\mathbb{K}[x][S^{-1}]$ by a multiplicative subset S of $\mathbb{K}[x]$ is also a left D_n-module.

Example 2.4. Set $\mathbb{K} = \mathbb{C}$. Let $C^\infty(U)$ be the set of the complex-valued C^∞ functions on an open set U of the n-dimensional real Euclidean space $\mathbb{R}^n$. Then each ∂_i acts on $C^\infty(U)$ as differentiation and x_i as multiplication. This makes $C^\infty(U)$ a left D_n-module. Let $C_0^\infty(U)$ be the set of C^∞ functions on U with compact support. More precisely, $f \in C^\infty(U)$ belongs to $C_0^\infty(U)$ if and only if there is a compact subset K of U such that $f(x) = 0$ for any $x \in U \setminus K$. Then $C_0^\infty(U)$ is a left D_n-submodule of $C^\infty(U)$.

Other examples of such $\mathcal{F}$ with $\mathbb{K} = \mathbb{C}$ are the set $\mathcal{O}(\Omega)$ of holomorphic functions on an open subset Ω of $\mathbb{C}^n$, the set $\mathcal{D}'(U)$ of the Schwartz distributions on an open subset U of $\mathbb{R}^n$, and the set $S'(\mathbb{R}^n)$ of tempered distributions, which shall be introduced later, as well as the set $\mathcal{B}(U)$ of the hyperfunctions (of Mikio Sato) on an open subset U of $\mathbb{R}^n$.

Now for a left ideal I of D_n, consider the residue module $M := D_n/I$, which is a left D_n-module generated by the residue class $\overline{1}$ of $1 \in \mathbb{K}[x] \subset D_n$. Fix a left D_n-module $\mathcal{F}$ as your favorite function space. A map $\varphi : M \to \mathcal{F}$ is D_n-linear, or a D_n-homomorphism, if

$$\varphi(u + v) = \varphi(u) + \varphi(v), \quad \varphi(Pu) = P\varphi(u) \quad (\forall u, v \in M, \ \forall P \in D_n).$$

Let $\mathrm{Hom}_{D_n}(M, \mathcal{F})$ be the set of the D_n-homomorphisms of M to $\mathcal{F}$, which is a $\mathbb{K}$-vector space. Since M is generated by $\overline{1}$ as left D_n-module, $\varphi \in \mathrm{Hom}_{D_n}(M, \mathcal{F})$ is uniquely determined by $\varphi(\overline{1}) \in \mathcal{F}$. On the other hand, for φ to be well-defined, it is necessary and sufficient that $\varphi(\overline{1})$ be annihilated by I, i.e., $P\varphi(\overline{1}) = 0$ for any $P \in I$. In conclusion, we have a $\mathbb{K}$-isomorphism

$$\mathrm{Hom}_{D_n}(M, \mathcal{F}) \ni \varphi \overset{\sim}{\longmapsto} \varphi(\overline{1}) \in \{f \in \mathcal{F} \mid Pf = 0 \quad (\forall P \in I)\}.$$

For an element u of a left D_n-module $\mathcal{F}$, we define the annihilator of u in D_n to be the left ideal

$$\mathrm{Ann}_{D_n} u = \{P \in D_n \mid Pu = 0\}.$$

Then we have

$$I = \mathrm{Ann}_{D_n} \overline{1} = \{P \in D_n \mid P\overline{1} = 0 \in M\}$$

by the definition.

We started with a left ideal I of D_n generated by given $P_1, \dots, P_r \in D_n$ and considered a left D_n-module $M = D_n/I$. We can argue in the reverse order: Let M be a finitely generated left D_n-module and let $u_1, \dots, u_m \in M$ be generators of M, i.e., assume that for any $u \in M$, there exist $P_1, \dots, P_m \in D_n$ such that $u = P_1 u_1 + \cdots + P_m u_m$. Set

$$N := \{(P_1, \dots, P_m) \in (D_n)^m \mid P_1 u_1 + \cdots + P_m u_m = 0\},$$

which is a left D_n-submodule of the free module $(D_n)^m$.

Since D_n is a left (and right) Noetherian ring (this can be proved by using a Gröbner basis in D_n), N is also finitely generated over D_n. Hence there exist

$$Q_i = (Q_{i1}, \dots, Q_{im}) \in (D_n)^m \qquad (i = 1, \dots, r)$$

which generate N as left D_n-module. Then we have an exact sequence of left D_n-modules

$$(D_n)^r \xrightarrow{\psi} (D_n)^m \xrightarrow{\varphi} M \longrightarrow 0, \tag{4}$$

which is called a *presentation* of M. Here φ and ψ are homomorphisms of left D_n-modules defined by, for $P_i \in D_n$,

$$\varphi((P_1, \dots, P_m)) = P_1 u_1 + \cdots + P_m u_m,$$
$$\psi((P_1, \dots, P_r)) = \begin{pmatrix} P_1 & \cdots & P_r \end{pmatrix} \begin{pmatrix} Q_{11} & \cdots & Q_{1m} \\ \vdots & & \vdots \\ Q_{r1} & \cdots & Q_{rm} \end{pmatrix}$$

and $N = \ker \varphi = \mathrm{im}\, \psi$ holds.

From (4) we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{D_n}(M, \mathcal{F}) \xrightarrow{\varphi^*} \mathrm{Hom}_{D_n}((D_n)^m, \mathcal{F}) \xrightarrow{\psi^*} \mathrm{Hom}_{D_n}((D_n)^r, \mathcal{F}).$$

Since $\mathrm{Hom}_{D_n}((D_n)^m, \mathcal{F})$ is isomorphic to $\mathcal{F}^m$, this yields

$$0 \longrightarrow \mathrm{Hom}_{D_n}(M, \mathcal{F}) \xrightarrow{\varphi^*} \mathcal{F}^m \xrightarrow{\psi^*} \mathcal{F}^r.$$

Regarding the elements of $\mathcal{F}^m$ as column vectors, we have, for $h \in \mathrm{Hom}_{D_n}(M, \mathcal{F})$ and $f_1, \ldots, f_m \in \mathcal{F}$,

$$\varphi^*(h) = \begin{pmatrix} h(u_1) \\ \vdots \\ h(u_m) \end{pmatrix}, \quad \psi^*\left(\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}\right) = \begin{pmatrix} Q_{11} & \cdots & Q_{1m} \\ \vdots & & \vdots \\ Q_{r1} & \cdots & Q_{rm} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

Hence we have an isomorphism

$$\begin{aligned} &\mathrm{Hom}_{D_n}(M, \mathcal{F}) \\ &\quad \cong \mathrm{Ker}\,\psi^* = \left\{ {}^t(f_1, \ldots, f_m) \in \mathcal{F}^m \mid \sum_{j=1}^m Q_{ij} f_j = 0 \quad (i = 1, \ldots, r) \right\} \end{aligned}$$

as $\mathbb{K}$-vector space. Note that the generators $u_1, \ldots, u_m$ of M also satisfy the same equations

$$\sum_{j=1}^m Q_{ij} u_j = 0 \qquad (i = 1, \ldots, r)$$

in M. In this way, we can regard a finitely generated left D_n-module M as a system of linear differential equations for unknown functions in a function space which correspond to generators of M.

Example 2.5. Let us consider $\mathbb{K}[x]$ as a left D_n-module. Since D_n contains $\mathbb{K}[x]$ as a subring, $\mathbb{K}[x]$ is generated by 1 as a left D_n-module. For $P \in D_n$, there exist $Q_1, \ldots, Q_n \in D_n$ and $r(x) \in \mathbb{K}[x]$ such that

$$P = Q_1\partial_1 + \cdots + Q_n\partial_n + r(x).$$

Then the action of P on 1 is $P(1) = r(x)$, which vanishes if and only if $r(x) = 0$. This implies $\mathbb{K}[x] \cong D_n/(D_n\partial_1 + \cdots + D_n\partial_n)$ and a presentation of $\mathbb{K}[x]$ is given by

$$(D_n)^n \xrightarrow{\cdot {}^t(\partial_1, \ldots, \partial_n)} D_n \xrightarrow{\varphi} \mathbb{K}[x] \longrightarrow 0$$

with $\varphi(P) = P(1)$. In the same way we can show

$$\mathrm{Hom}_{D_n}(\mathbb{K}[x], \mathcal{F}) \cong \{f \in \mathcal{F} \mid \partial_1 f = \cdots = \partial_n f = 0\} = \mathbb{K}$$

for $\mathcal{F} = \mathbb{K}[x], \mathbb{K}(x), \mathbb{K}[x]_a$, or for $\mathcal{F} = C^\infty(U)$ with an open connected set U of $\mathbb{R}^n$ if $\mathbb{K} = \mathbb{C}$.

Exercise 4. Confirm the formulae above for $\varphi*$ and ψ^*.

Exercise 5. Construct a $\mathbb{C}$-isomorphism $\mathrm{Hom}_{D_n}(\mathbb{C}[x], C^\infty(U)) \cong \mathbb{C}$ for an open connected set U of $\mathbb{R}^n$, where D_n is the n-th Weyl algebra over $\mathbb{C}$. What happens if U is not connected?

2.3. Weight vector and filtration

A *weight vector* w for D_n is an integer vector

$$w = (w_1, \dots, w_n; w_{n+1}, \cdots, w_{2n}) \in \mathbb{Z}^{2n}$$

with the conditions $w_i + w_{n+i} \geq 0$ for $i = 1, \dots, n$, which are necessary in view of the commutation relation $\partial_i x_i = x_i \partial_i + 1$ in D_n. For a nonzero differential operator P of the form $P = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha,\beta} x^\alpha \partial^\beta$, we define its *$w$-order* to be

$$\mathrm{ord}_w(P) = \max\{\langle w, (\alpha, \beta)\rangle \mid a_{\alpha,\beta} \neq 0\}$$

with

$$\langle w, (\alpha, \beta)\rangle := w_1\alpha_1 + \cdots + w_n\alpha_n + w_{n+1}\beta_1 + \cdots + w_{2n}\beta_n.$$

We set $\mathrm{ord}_w(0) := -\infty$. A weight vector w induces the *w-filtration*

$$F_k^w(D_n) := \{P \in D_n \mid \mathrm{ord}_w(P) \leq k\} \qquad (k \in \mathbb{Z})$$

on the ring D_n. In general, for two $\mathbb{K}$-subspaces V, W of D_n, we denote by VW the $\mathbb{K}$-subspace of D_n spanned by products PQ with $P \in V$ and $Q \in W$.

The w-filtration satisfies the properties:

$$F_k^w(D_n) \subset F_{k+1}^w(D_n), \quad \bigcup_{k \in \mathbb{Z}} F_k^w(D_n) = D_n,$$

$$1 \in F_0^w(D_n), \quad F_k^w(D_n) F_l^w(D_n) \subset F_{k+l}^w(D_n), \quad \bigcap_{k \in \mathbb{Z}} F_k^w(D_n) = \{0\}.$$

The *w-graded ring* associated with this filtration is defined to be

$$\mathrm{gr}^w(D_n) := \bigoplus_{k \in \mathbb{Z}} \mathrm{gr}_k^w(D_n), \qquad \mathrm{gr}_k^w(D_n) := F_k^w(D_n)/F_{k-1}^w(D_n).$$

Let P be a nonzero element of D_n with $m := \mathrm{ord}_w(P)$. Then we denote by $\sigma^w(P)$ the residue class of P in $\mathrm{gr}^w_k(D_n) \subset \mathrm{gr}^w(D_n)$. We set $\sigma^w(0) = 0$. It is easy to see that $\sigma^w(PQ) = \sigma^w(P)\sigma^w(Q)$ holds for any $P, Q \in D_n$ by using the Leibniz formula.

If $w_i + w_{n+i} > 0$, then the w-order of $\partial_i x_i - x_i \partial_i = 1$ is zero while that of $x_i\partial_i$ is positive. Hence $\sigma^w(x_i)$ and $\sigma^w(\partial_i)$ commute in $\mathrm{gr}^w(D_n)$. In this case, we denote $\sigma^w(x_i)$ and $\sigma^w(\partial_i)$ simply by x_i and ξ_i regarding these as commutative indeterminates.

On the other hand, if $w_i + w_{n+i} = 0$, then we have

$$\sigma^w(\partial_i)\sigma^w(x_i) - \sigma^w(x_i)\sigma^w(\partial_i) = 1$$

in $\mathrm{gr}^w(D_n)$, the same commutation relation as that for x_i and ∂_i in D_n. Hence we will denote $\sigma^w(x_i)$ and $\sigma^w(\partial_i)$ by x_i and ∂_i for simplicity.

Lemma 2.6. *Assume that* $w_i \geq 0$ *holds for* $i = 1, \dots, 2n$, *or else* $|w_i| \leq 1$ *and* $w_i + w_{n+i} = 0$ *hold for* $i = 1, \dots, n$. *Then* $F^w_k(D_n)$ *is a finitely generated left (and right)* $F^w_0(D_n)$*-module for each integer* k.

Proof. First, suppose $w_i \geq 0$ for all i. Then for any positive integer k, $F^w_k(D_n)$ is generated over $F^w_0(D_n)$ by the finite set

$$\{x^\alpha \partial^\beta \mid \langle w, (\alpha, \beta)\rangle = k,\ \alpha_i = 0 \text{ if } w_i = 0,\ \beta_i = 0 \text{ if } w_{n+i} = 0\}.$$

Now suppose $|w_i| \leq 1$ and $w_i + w_{n+i} = 0$ for $i = 1, \dots, n$. We may assume $w_i \geq 0$ for $1 \leq i \leq n$ by exchanging x_i and ∂_i if necessary. Each element of D_n is expressed as a linear combination of a finite set of 'monomials' of the form $x^\alpha\partial^\beta$. If $\langle (w_1, \dots, w_n), \alpha\rangle > \langle (w_1, \dots, w_n), \beta\rangle$, then there exists $\gamma \in \mathbb{N}^n$ such that $\alpha - \gamma \in \mathbb{N}^n$ and

$$\langle (w_1, \dots, w_n), \alpha - \gamma\rangle = \langle (w_1, \dots, w_n), \beta\rangle.$$

Then the w-order of $x^\alpha\partial^\beta = x^\gamma x^{\alpha-\gamma}\partial^\beta$ is $\langle (w_1, \dots, w_n), \gamma\rangle \geq 0$ and $x^{\alpha-\gamma}\partial^\beta$ belongs to $F^w_0(D_n)$. Hence $F^w_k(D_n)$ is generated by a finite set

$$\{x^\gamma \mid \langle (w_1, \dots, w_n), \gamma\rangle = k,\ \gamma_i = 0 \text{ if } w_i = 0\}$$

over $F^w_0(D_n)$ if $k > 0$. Likewise, $F^w_k(D_n)$ is generated by a finite set

$$\{\partial^\gamma \mid \langle (w_{n+1}, \dots, w_{2n}), \gamma\rangle = k,\ \gamma_i = 0 \text{ if } w_{n+i} = 0\}$$

over $F^w_0(D_n)$ if $k < 0$ since D_n is spanned by $\partial^\beta x^\alpha$. Q.E.D.

Lemma 2.7. *Assume* $|w_i| \leq 1$ *for* $1 \leq i \leq 2n$. *Then for integers* j, k *one has* $F^w_j(D_n)F^w_k(D_n) = F^w_{j+k}(D_n)$ *if* $j \geq 0$, $k \geq 0$ *or else* $j \leq 0$, $k \leq 0$.

Proof. The statement is easily shown if each component of w is 1 or 0. We can argue componentwise. Assume $w_i = -1$, and consequently $w_{n+i} = 1$. Suppose the w-order of $x_i^{\alpha_i}\partial_i^{\beta_i}$ is $j+k$ with $j, k \geq 0$. This means $\beta_i - \alpha_i = j + k$ and consequently $k \leq \beta_i$. Then ∂_i^k belongs to $F_k(D_n)$ and $x_i^{\alpha_i}\partial_i^{\beta_i - k}$ to $F_j(D_n)$. The case $j, k \leq 0$ is similar. Q.E.D.

Note that the lemma above does not hold in general without the assumption on w. For example, if $n = 1$ and $w = (1; 2)$, then ∂_1 belongs to $F_2(D)$ but does not belong to $F_1(D_1)F_1(D_1)$.

The *Rees algebra* $R^w(D_n)$ associated with the w-filtration is defined by

$$R^w(D_n) := \bigoplus_{k\in\mathbb{Z}} F_k^w(D_n)T^k \ \subset\ D_n[T]$$

with an indeterminate T. We have isomorphisms

$$(5)\quad R^w(D_n)/(T-1)R^w(D_n) \cong D_n, \quad R^w(D_n)/TR^w(D_n) \cong \mathrm{gr}^w(D_n)$$

as $\mathbb{K}$-algebra. Note that D_n, $\mathrm{gr}^w(D_n)$, and $R^w(D_n)$ are left (and right) Noetherian rings. This can be proved by using Gröbner bases which will be introduced in the next section.

Let M be a left D_n-module. A family $\{F_k(M)\}_{k\in\mathbb{Z}}$ of $\mathbb{K}$-subspaces $F_k(M)$ of M is called a *w-filtration* on M if it satisfies

(1) $F_k(M) \subset F_{k+1}(M)$ for all $k \in \mathbb{Z}$,

(2) $\bigcup_{k\in\mathbb{Z}} F_k(M) = M$,

(3) $F_j^w(D_n)F_k(M) \subset F_{j+k}(M)$ for all $j, k \in \mathbb{Z}$.

For a w-filtration $\{F_k(M)\}$, let

$$\mathrm{gr}(M) := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k(M), \qquad \mathrm{gr}_k(M) := F_k(M)/F_{k-1}(M)$$

be the associated graded module, which is a left $\mathrm{gr}^w(D_n)$-module.

Definition 2.8. A w-filtration $\{F_k(M)\}$ of a left D_n-module M is called *good* if there exist a finite number of elements $u_i \in F_{k_i}(M)$ and $k_i \in \mathbb{Z}$ $(i = 1, \ldots, m)$ such that

$$F_k(M) = F_{k-k_1}^w(D_n)u_1 + \cdots + F_{k-k_m}^w(D_n)u_m \qquad (\forall k \in \mathbb{Z}).$$

It follows from the definition that a left D_n-module M has a good w-filtration if and only if M is finitely generated over D_n. The following lemma is also an immediate consequence of the definition:

Lemma 2.9. *Let $\{F_k(M)\}$ be a good w-filtration on a left D_n-module M. Let N be a left D_n-submodule of M. Define a w-filtration on M/N by*

$$F_k(M/N) := F_k(M)/(F_k(M) \cap N) \subset M/N.$$

Then $\{F_k(M/N)\}$ is a good w-filtration on M/N.

Lemma 2.10. *Let $\{F_k(M)\}$ and $\{F'_k(M)\}$ be w-filtrations on a left D_n-module M. Assume that $\{F_k(M)\}$ is good. Then there exists an integer l such that $F_k(M) \subset F'_{k+l}(M)$ for any $k \in \mathbb{Z}$.*

Proof. There exist $u_i \in F_{k_i}(M)$ such that

$$F_k(M) = F^w_{k-k_1}(D_n)u_1 + \cdots + F^w_{k-k_m}(D_n)u_m \qquad (\forall k \in \mathbb{Z}).$$

There exists an integer l such that each u_i belongs to $F'_l(M)$. Then we have

$$F_k(M) \subset F^w_{k-k_1}(D_n)F'_l(M) + \cdots + F^w_{k-k_m}(D_n)F'_l(M) \subset F'_{k-k_0+l}(M)$$

with $k_0 := \min\{k_1, \ldots, k_m\}$. Q.E.D.

Proposition 2.11. *Let $\{F_k(M)\}$ be a good w-filtration on a left D_n-module M. Then*

(1) *The associated graded module* $\mathrm{gr}(M)$ *is finitely generated over* $\mathrm{gr}^w(D_n)$*. In particular, each homogeneous component* $\mathrm{gr}_k(M)$ *is a finitely generated $\mathrm{gr}^w_0(D_n)$-module if w satisfies the assumption of Lemma 2.6.*

(2) *If $w_i \geq 0$ for all i, then $\{F_k(M)\}$ is bounded below*; *i.e., there exists $k_0 \in \mathbb{Z}$ such that $F_k(M) = \{0\}$ for any $k \leq k_0$.*

Proof. (1) By the assumption, there exist $u_1, \ldots, u_m \in M$ such that

$$F_k(M) = F^w_{k-k_1}(D_n)u_1 + \cdots + F^w_{k-k_m}(D_n)u_m \qquad (\forall k \in \mathbb{Z}). \tag{6}$$

Hence for any $u \in F_k(M) \setminus F_{k-1}(M)$, there exist $P_i \in F^w_{k-k_i}(D_n)$ such that

$$u = P_1u_1 + \cdots + P_mu_m.$$

Let $\overline{u}$ be the residue class of u in $\mathrm{gr}_k(M)$ and $\overline{u}_i$ be that of u_i in $\mathrm{gr}_{k_i}(M)$. Set $\overline{P}_i = \sigma^w(P_i)$ if $\mathrm{ord}_w(P_i) = k - k_i$, and $\overline{P}_i = 0$ otherwise. Then we have

$$\overline{u} = \overline{P}_1\overline{u}_1 + \cdots + \overline{P}_m\overline{u}_m$$

in $\mathrm{gr}(M)$. Hence $\mathrm{gr}(M)$ is generated by $\overline{u}_i$ $(1 \leq i \leq m)$ over $\mathrm{gr}^w(D_n)$.

(2) We have $F_k(M) = 0$ for $k < \min\{k_1, \ldots, k_m\}$ in view of (6) and $F_{-1}(D_n) = \{0\}$. Q.E.D.

Proposition 2.12. *Regard* $L = (D_n)^m$ *as a free left* D_n*-module. Fixing integers* $l_1, \dots, l_m$, *set*

$$F_k(L) := F^w_{k-l_1}(D_n)e_1 + \cdots + F^w_{k-l_m}(D_n)e_m \qquad (\forall k \in \mathbb{Z})$$

with $e_1 = (1, 0, \dots, 0)$, $\dots$, $e_m = (0, \dots, 0, 1) \in \mathbb{Z}^m$. *Let* N *be a left* D_n*-submodule of* L *and assume* $P_i = (P_{i1}, \dots, P_{im}) \in L$ $(i = 1, \dots, q)$ *generate* N *and, at the same time, their residue classes* $\overline{P}_1, \dots, \overline{P}_m$ *in* $\mathrm{gr}(L)$ *generate the graded submodule* $\mathrm{gr}(N)$ *of* $\mathrm{gr}(L)$, *which is associated with the induced filtration* $\{F_k(L) \cap N\}$. *Suppose* $P_i \in F_{k_i}(L) \setminus F_{k_i - 1}(L)$. *Under these conditions,*

$$F_k(L) \cap N = F^w_{k-k_1}(D_n)P_1 + \cdots + F^w_{k-k_m}(D_n)P_m$$

holds for any $k \in \mathbb{Z}$. *In particular,* $\{F_k(L) \cap N\}$ *is a good* w*-filtration on* N. *Moreover, if* $w_i \geq 0$ *for all* i, *the assumption that* P_i *generate* N *is not necessary.*

Proof. This is standard in the case $w_i \geq 0$ for all i, which will suffice for the application in the next subsection. In fact, for any element P of $F_k(L) \cap N$, there exist $Q'_i \in F^w_{k-k_i}(D_n)$ such that

$$P - \sum_{i=1}^{q} Q'_i P_i \ \in F_{k-1}(L) \cap N$$

by the assumption. Then we can conclude by induction on k since $F_k(L) = \{0\}$ for sufficiently small k. For the general case we need the completion with respect to the filtration; see the proof of Theorem 10.6 in [27] for details. Q.E.D.

The following is an analogue of the Artin-Rees lemma in commutative algebra:

Proposition 2.13. *Let* M *be a finitely generated left* D_n*-module and* $\{F_k(M)\}$ *be a good* w*-filtration on* M. *Let* N *be a left* D_n*-submodule of* M. *Then the induced filtration* $\{N \cap F_k(M)\}$ *on* N *is good.*

Proof. By the assumption, there exist $u_1, \dots, u_m \in M$ such that

$$F_k(M) = F^w_{k-k_1}(D_n)u_1 + \cdots + F^w_{k-k_m}(D_n)u_m \qquad (\forall k \in \mathbb{Z}).$$

Set $L = (D_n)^m$ and define a D_n-homomorphism $\varphi : L \to M$ by

$$\varphi(A_1, \dots, A_m) = A_1 u_1 + \cdots + A_m u_m \qquad (A_i \in D_n).$$

Define a w-filtration on L by

$$F_k(L) = \{(A_1, \dots, A_m) \in L \mid A_i \in F^w_{k-k_i}(D_n) \quad (1 \leq i \leq m)\}.$$

Then $\varphi(F_k(L)) = F_k(M)$ holds for any $k \in \mathbb{Z}$ by the construction. Now $N' := \varphi^{-1}(N)$ is a left D_n-submodule of L and finitely generated since D_n is Noetherian. Hence Proposition 2.12 (or else Theorem 3.14) assures the existence of $Q_1, \dots, Q_p \in N'$ and $l_1 \dots, l_p \in \mathbb{Z}$ such that

$$F_k(L) \cap N' = F^w_{k-l_1}(D_n)Q_1 + \cdots + F^w_{k-l_p}(D_n)Q_p \qquad (\forall k \in \mathbb{Z}).$$

Set $L' = (D_n)^p$ and define a D_n-homomorphism $\psi : L' \to N'$ by

$$\psi(B_1, \dots, B_p) = B_1Q_1 + \cdots + B_pQ_p \qquad (B_1, \dots, B_p \in D_n).$$

Define a w-filtration on L' by

$$F_k(L') = \{(B_1, \dots, B_p) \in L' \mid B_i \in F^w_{k-l_i}(D_n) \quad (1 \le i \le p)\}.$$

Then we have

$$(\varphi \circ \psi)(F_k(L')) = \varphi(F_k(L) \cap N') = F_k(M) \cap N.$$

In fact, if u belongs to $F_k(M) \cap N$, then there exists $Q \in F_k(L)$ such that $u = \varphi(Q)$ and consequently Q belongs to $F_k(L) \cap N'$. Thus $\{F_k(M) \cap N\}$ is a good w-filtration. Q.E.D.

Exercise 6. Let $w \in \mathbb{Z}^{2n}$ be a weight vector for D_n and set

$$d = \min\{w_i + w_{n+i} \mid 1 \le i \le n\}.$$

Suppose $P \in F^w_k(D_n)$ and $Q \in F^w_l(D_n)$ and show that the commutator $[P, Q] := PQ - QP$ belongs to $F^w_{k+l-d}(D_n)$.

Exercise 7. Set $n = 1$, $w = (-1; 1)$, and $M = D_1/I$ with the left ideal $I = D_1(x_1^2\partial_1 - 1)$ of D_1. Set

$$F_k(M) = F^w_k(D_1)/(F^w_k(D_1) \cap I) \qquad (k \in \mathbb{Z}).$$

(1) Show that $\{F_k(M)\}$ is a good w-filtration on M.
(2) Show that $F_k(M) = M$ for any $k \in \mathbb{Z}$ and that $\mathrm{gr}(M) = \{0\}$.

Exercise 8. Set $n = 1$ and regard $\mathbb{K}[x]$ as a left D_1-module. Define $F_k(\mathbb{K}[x]) = \{f \in \mathbb{K}[x] \mid \deg f \le 2k\}$ for $k \in \mathbb{Z}$. Then prove the following:
(1) $\{F_k(\mathbb{K}[x])\}$ is a $(1; 1)$-filtration on $\mathbb{K}[x]$.
(2) The associated graded module $\mathrm{gr}(\mathbb{K}[x])$ is not finitely generated over $\mathrm{gr}^{(1;1)}(D_1)$.
(3) $\{F_k(\mathbb{K}[x])\}$ is not a good $(1; 1)$-filtration, but it is a good $(2; 1)$-filtration.

Exercise 9. Prove the $\mathbb{K}$-algebra isomorphisms (5).

2.4. Holonomic D-module and characteristic variety

Following J. Bernstein [2], [3], let us define the notion of holonomic system by using the weight vector $(\mathbf{1};\mathbf{1}) = (1,\dots,1;1,\dots,1) \in \mathbb{Z}^{2n}$.

Let M be a finitely generated left D_n-module and $\{F_k(M)\}$ a good $(\mathbf{1};\mathbf{1})$-filtration on M. Note that $\mathrm{gr}^{(\mathbf{1};\mathbf{1})}(D_n)$ is isomorphic to the polynomial ring $\mathbb{K}[x,\xi]$ as a graded ring in which indeterminates $x_1, \dots, x_n$, $\xi_1, \dots, \xi_n$ are all of order one. By Proposition 2.11, $\mathrm{gr}(M)$ is a finitely generated graded $\mathbb{K}[x,\xi]$-module and each $\mathrm{gr}_k(M)$ is a finite dimensional $\mathbb{K}$-vector space. Moreover, $\mathrm{gr}^{(\mathbf{1};\mathbf{1})}(D_n)$ is generated by $\mathrm{gr}_1^{(\mathbf{1};\mathbf{1})}(D_n)$ as $\mathbb{K}$-algebra.

In this situation, it is well-known in commutative algebra (see e.g., [5], [8]) that there exist a (Hilbert) polynomial $H(k) = \sum_{j=0}^{d} c_j k^j \in \mathbb{Q}[k]$ and an integer k_0 such that

$$H(k) = \sum_{j \le k} \dim_{\mathbb{K}} \mathrm{gr}_j(M) = \dim_{\mathbb{K}} F_k(M) \qquad (\forall k \ge k_0)$$

and that $d!c_d$ is a positive integer.

Proposition 2.14. *The leading term $c_d k^d$ of $H(k)$ does not depend on the choice of a good $(\mathbf{1};\mathbf{1})$-filtration $\{F_k(M)\}$. Hence it is an invariant of a finitely generated left D_n-module M. The degree d of $H(k)$ is called the* dimension *of M and denoted* $\dim M$*. The* multiplicity *of M is defined to be the positive integer $d!c_d$ and denoted* $\mathrm{mult}\, M$*.*

Proof. Let $\{F_k(M)\}$ and $\{F'_k(M)\}$ be two good $(\mathbf{1};\mathbf{1})$-filtrations on M. There exist polynomials $H(k)$, $G(k)$, and an integer k_0 such that

$$\dim_{\mathbb{K}} F_k(M) = H(k), \quad \dim_{\mathbb{K}} F'_k(M) = G(x) \qquad (\forall k \ge k_0).$$

On the other hand, by Lemma 2.10, there exists a non-negative integer k_1 such that

$$F'_{k-k_1}(M) \subset F_k(M) \subset F'_{k+k_1}(M) \qquad (\forall k \in \mathbb{Z}).$$

Hence we have $G(k-k_1) \le H(k) \le G(k+k_1)$ for any $k \ge k_0$. This implies that the leading terms of $H(k)$ and of $G(k)$ coincide. Q.E.D.

Example 2.15. Since

$$\dim_{\mathbb{K}} F_k^{(\mathbf{1};\mathbf{1})}(D_n) = \binom{2n+k}{2n} = \frac{1}{(2n)!}k^{2n} + (\text{lower degree terms in } k),$$

the dimension of D_n as a left D_n module equals $2n$, and the multiplicity is one.

Theorem 2.16 (Bernstein's inequality)**.** *If M is a finitely generated nonzero left D_n-module, then* $\dim M$ *is greater than or equal to n.*

Proof. We follow the argument in §30 of [12], which is based on a lemma by A. Joseph. Let $\{F_k(M)\}$ be a good $(\mathbf{1};\mathbf{1})$-filtration on M. We may assume $F_0(M) \neq \{0\}$ and $F_{-1}(M) = \{0\}$ by shifting k if necessary. Let us define a $\mathbb{K}$-homomorphism

$$\Psi_k : F_k^{(\mathbf{1};\mathbf{1})}(D_n) \ni P \longmapsto \Psi_k(P) \in \mathrm{Hom}_{\mathbb{K}}(F_k(M), F_{2k}(M)),$$

where $\Psi_k(P)$ denotes the natural $\mathbb{K}$-homomorphism $P : F_k(M) \to F_{2k}(M)$. Let us show that Ψ_k is injective by induction on k. First, Ψ_0 is injective since $F_0^{(\mathbf{1};\mathbf{1})}(D_n) = \mathbb{K}$. Now assume Ψ_j is injective if $j \leq k-1$. Let P be a nonzero element of $F_k^{(\mathbf{1};\mathbf{1})}(D_n)$. We may assume $P \notin \mathbb{K}$ since $\Psi_k(P) \neq 0$ otherwise. Then, we have $[P, \partial_i] \neq 0$ or $[P, x_i] \neq 0$ with some i. In fact, $[P, \partial_i] = P\partial_i - \partial_i P$ vanishes if and only if P does not contain x_i, and $[P, x_i]$ vanishes if and only if P does not contain ∂_i.

First assume $[P, \partial_i] \neq 0$. Since $[P, \partial_i]$ belongs to $F_{k-1}^{(\mathbf{1};\mathbf{1})}(D_n)$, there exists an element u of $F_{k-1}(M)$ such that $[P, \partial_i]u \neq 0$ by the induction hypothesis. Hence either $P\partial_i u \neq 0$ or $Pu \neq 0$ holds. Since u and $\partial_i u$ belong to $F_k(M)$, this shows $\Psi_k(P) \neq 0$.

The case $[P, x_i] \neq 0$ can be treated similarly with ∂_i replaced by x_i in the argument above. Thus we have proved that Ψ_k is injective for $k \geq 0$. From this we obtain

$$\begin{aligned}\dim_{\mathbb{K}} F_k^{(\mathbf{1};\mathbf{1})}(D_n) &\leq \dim_{\mathbb{K}} \mathrm{Hom}_{\mathbb{K}}(F_k(M), F_{2k}(M)) \\ &= (\dim_{\mathbb{K}} F_k(M))(\dim_{\mathbb{K}} F_{2k}(M)).\end{aligned}$$

There exists a polynomial $H(k)$ such that $H(k) = \dim_{\mathbb{K}} F_k(M)$ for sufficiently large k. Thus we have

$$H(k)H(2k) \geq \dim_{\mathbb{K}} F_k^{(\mathbf{1};\mathbf{1})}(D_n) = \binom{2n+k}{2n} \qquad (\forall k \gg 0).$$

Comparing the degrees in k, we get $2\deg H(k) \geq 2n$, consequently $\dim M = \deg H(k) \geq n$. Q.E.D.

Definition 2.17. A finitely generated left D_n-module M is called *holonomic* or a *holonomic system* if $\dim M \leq n$, that is if $\dim M = n$ or else $M = 0$.

Example 2.18. Let us show that $\mathbb{K}[x]$ is a holonomic left D_n-module. It is easy to see that

$$F_k(\mathbb{K}[x]) = \{f \in \mathbb{K}[x] \mid \deg f \leq k\} \qquad (k \in \mathbb{Z})$$

constitute a $(\mathbf{1};\mathbf{1})$-filtration on $\mathbb{K}[x]$. Moreover, it is a good filtration since

$$F_k(\mathbb{K}[x]) = \sum_{|\alpha|\le k} \mathbb{K}x^\alpha = F_k^{(\mathbf{1};\mathbf{1})}(D_n)1.$$

It follows that $\dim \mathbb{K}[x] = n$ since

$$\dim_{\mathbb{K}} F_k(\mathbb{K}[x]) = \binom{n+k}{n} = \frac{1}{n!}k^n + (\text{lower order terms in } k).$$

Proposition 2.19. *Let*

$$0 \longrightarrow N \xrightarrow{\varphi} M \xrightarrow{\psi} L \longrightarrow 0$$

be an exact sequence of finitely generated left D_n-modules. Then

(1) *M is holonomic if and only if both N and L are holonomic.*
(2) *If M is holonomic, then* $\operatorname{mult} M = \operatorname{mult} N + \operatorname{mult} L$ *holds, where we define the multiplicity of the zero module to be zero.*

Proof. Let $\{F_k(M)\}$ be a good $(\mathbf{1};\mathbf{1})$-filtration on M and set

$$F_k(N) := \varphi^{-1}(F_k(M)), \qquad F_k(L) := \psi(F_k(M)).$$

Then $\{F_k(N)\}$ is a good filtration on N by Proposition 2.13 and $\{F_k(L)\}$ is a good filtration on L by Lemma 2.9. Hence the assertions follow from

$$\dim_{\mathbb{K}} F_k(M) = \dim_{\mathbb{K}} F_k(N) + \dim_{\mathbb{K}} F_k(L).$$

Q.E.D.

Let us recall another characterization of a holonomic system by using the weight vector $w = (\mathbf{0};\mathbf{1}) = (0,\dots,0;1,\dots,1)$. Let M be a finitely generated left D_n-module and $\{F_k(M)\}$ be a good $(\mathbf{0};\mathbf{1})$-filtration on M. Then $\operatorname{gr}(M)$ is a finitely generated $\mathbb{K}[x,\xi]$-module. Let us denote by $\overline{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$. In general, for a finitely generated $\mathbb{K}[x,\xi]$-module M', its *support* is the algebraic set of $\overline{\mathbb{K}}^{2n}$ defined by

$$\operatorname{Supp} M' := \{(a,b) \in \overline{\mathbb{K}}^n \times \overline{\mathbb{K}}^n \mid M'_{(a,b)} := \overline{\mathbb{K}}[x,\xi]_{(a,b)} \otimes_{\mathbb{K}[x,\xi]} M' \neq 0\},$$

where $\overline{\mathbb{K}}[x,\xi]_{(a,b)}$ denotes the localization of $\overline{\mathbb{K}}[x,\xi]$ at (a,b), i.e., the localization at the maximal ideal corresponding to the point (a,b).

Proposition 2.20. *The support of* $\operatorname{gr}(M)$ *does not depend on the choice of a good $(\mathbf{0};\mathbf{1})$-filtration $\{F_k(M)\}$ on M.*

Proof. We follow the argument of Kashiwara [14]. Let $\{F_k(M)\}$ and $\{F'_k(M)\}$ be good $(\mathbf{0};\mathbf{1})$-filtrations on M and $\mathrm{gr}(M)$ and $\mathrm{gr}'(M)$ be the associated graded modules respectively. Then by Lemma 2.10 we may assume that there exists an integer $k_1 \geq 0$ such that

$$F_{k-k_1}(M) \subset F'_k(M) \subset F_k(M) \qquad (\forall k \in \mathbb{Z})$$

by shifting the index of $F'_k(M)$ if necessary.

Let us argue by induction on k_1. The case $k_1 = 0$ is trivial. Suppose $k_1 = 1$ and consider the following two exact sequences

$$0 \longrightarrow F'_k(M)/F_{k-1}(M) \to F_k(M)/F_{k-1}(M) \to F_k(M)/F'_k(M) \to 0,$$
$$0 \to F_{k-1}(M)/F'_{k-1}(M) \to F'_k(M)/F'_{k-1}(M) \to F'_k(M)/F_{k-1}(M) \to 0.$$

It follows that

$$\mathrm{Supp}\,\mathrm{gr}(M) = \mathrm{Supp} \bigoplus_{k\in\mathbb{Z}} F'_k(M)/F_{k-1}(M) \cup \mathrm{Supp} \bigoplus_{k\in\mathbb{Z}} F_k(M)/F'_k(M),$$
$$\mathrm{Supp}\,\mathrm{gr}'(M) = \mathrm{Supp} \bigoplus_{k\in\mathbb{Z}} F'_k(M)/F_{k-1}(M) \cup \mathrm{Supp} \bigoplus_{k\in\mathbb{Z}} F_{k-1}(M)/F'_{k-1}(M)$$

since $\overline{\mathbb{K}}[x,\xi]_{(a,b)}$ is a flat module over $\mathbb{K}[x,\xi]$. Hence $\mathrm{Supp}\,\mathrm{gr}(M)$ and $\mathrm{Supp}\,\mathrm{gr}'(M)$ coincide.

Now suppose $k_1 \geq 2$ and set

$$F''_k(M) = F_{k-1}(M) + F'_k(M) \qquad (k \in \mathbb{Z}).$$

Let $\mathrm{gr}''(M)$ be the graded $\mathbb{K}[x,\xi]$-module associated with the good filtration $\{F''_k(M)\}$. It follows from the definition

$$F_{k-1}(M) \subset F''_k(M) \subset F_k(M), \quad F''_{k-k_1+1}(M) \subset F'_k(M) \subset F''_k(M)$$

for any $k \in \mathbb{Z}$. By the induction hypothesis, we have

$$\mathrm{Supp}\,\mathrm{gr}(M) = \mathrm{Supp}\,\mathrm{gr}''(M) = \mathrm{Supp}\,\mathrm{gr}'(M).$$

Q.E.D.

Definition 2.21. Let M be a finitely generated left D_n-module and $\{F_k(M)\}$ be a good $(\mathbf{0};\mathbf{1})$-filtration on M. Then the *characteristic variety* $\mathrm{Char}(M)$ of M is defined to be the support $\mathrm{Supp}\,\mathrm{gr}(M)$ of the graded module $\mathrm{gr}(M)$ associated with $\{F_k(M)\}$.

Since $\mathrm{gr}(M)$ is a graded $\mathbb{K}[x,\xi]$-module with $x_1,\dots,x_n$ of order zero, and $\xi_1,\dots,\xi_n$ of order one, $\mathrm{Char}(M)$ is a homogeneous set with respect to ξ; i.e, if (a,b) belongs to $\mathrm{Char}(M)$, then so does (a,cb) for any $c\in\overline{\mathbb{K}}$.

The following theorem is proved, e.g., in Chapter 3 of [4] by using a homological method based on Auslander-Buchsbaum-Serre theorem (cf. [5], [8]). We will give a more elementary proof in 3.4.

Theorem 2.22. *Let M be a finitely generated left D_n-module. Then the dimension* $\dim M$ *defined through the* $(\mathbf{1};\mathbf{1})$*-filtration coincides with the Krull dimension* (*not as a graded module*) *of the* $\mathbb{K}[x,\xi]$*-module* $\mathrm{gr}(M)$ *associated with a good* $(\mathbf{0};\mathbf{1})$*-filtration on* M*.*

Especially, M is holonomic if and only if the dimension of the characteristic variety is n or else $M=0$. More strongly, it is known that the dimension of each irreducible component of the characteristic variety is of dimension $\geq n$. This fact was first proved by Sato-Kawai-Kashiwara [31] in the analytic category, and by Gabber [9] in a purely algebraic setting. See [33] for extension to general weight vectors.

Example 2.23. Let us regard $\mathbb{K}[x]$ as a left D_n-module. Then

$$F_k(\mathbb{K}[x]) = F_k^{(\mathbf{0};\mathbf{1})}(D_n)1 \qquad (k\in\mathbb{Z})$$

constitute a good $(\mathbf{0};\mathbf{1})$-filtration on $\mathbb{K}[x]$. It is easy to see that $F_k(\mathbb{K}[x]) = \mathbb{K}[x]$ if $k\geq 0$, and $F_k(\mathbb{K}[x])=\{0\}$ if $k\leq -1$. Hence the associated graded module is

$$\mathrm{gr}(\mathbb{K}[x]) = \bigoplus_{k\in\mathbb{Z}} F_k(\mathbb{K}[x])/F_{k-1}(\mathbb{K}[x]) = \mathbb{K}[x].$$

As a $\mathbb{K}[x,\xi]$-module, $\mathbb{K}[x]$ is isomorphic to $\mathbb{K}[x,\xi]/\langle\xi_1,\dots,\xi_n\rangle$, where $\langle\xi_1,\dots,\xi_n\rangle$ denotes the ideal of $\mathbb{K}[x,\xi]$ generated by $\xi_1,\dots,\xi_n$. Hence we get

$$\mathrm{Char}(M) = \{(x,\xi)\in\overline{\mathbb{K}}^{2n} \mid \xi_1=\cdots=\xi_n=0\}.$$

Exercise 10. Set $M=D_n/D_n\partial_1^m$ with a positive integer m and the coefficient field $\mathbb{K}=\mathbb{C}$.

(1) Give a presentation of the graded module $\mathrm{gr}(M)$ associated with the good $(\mathbf{1};\mathbf{1})$-filtration

$$F_k(M) = F_k^{(\mathbf{1};\mathbf{1})}(D_n)/(F_k^{(\mathbf{1};\mathbf{1})}(D_n)\cap D_n\partial_1^m)$$

and compute $\dim M$.

(2) Give a presentation of the graded module $\mathrm{gr}(M)$ associated with the good $(\mathbf{0};\mathbf{1})$-filtration

$$F_k(M) = F_k^{(\mathbf{0};\mathbf{1})}(D_n)/(F_k^{(\mathbf{0};\mathbf{1})}(D_n) \cap D_n\partial_1^m)$$

and compute $\mathrm{Char}M$.

§3. Gröbner bases in the ring of differential operators

In this section, we quickly review the theory of Gröbner bases over the Weyl algebra. In D-module theory, one often needs a Gröbner basis with respect to a monomial order which is not a well-ordering; for this we need homogenization technique. A good reference is the first chapter of [29]. See also [23], [27].

3.1. Definitions and basic properties

Recall that $\xi = (\xi_1, \ldots, \xi_n)$ are the commutative variables corresponding to derivations $\partial_i = \partial_{x_i}$ $(i = 1, \ldots, n)$. Let

$$M(x, \xi) = \{x^\alpha \xi^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$$

be the set of the monomials in $\mathbb{K}[x, \xi]$. A total order $\prec$ on $M(x, \xi)$ is called a *monomial order for* D_n if it satisfies

(1) $u \prec v \quad \Rightarrow \quad uw \prec vw \quad (\forall u, v, w \in M(x, \xi))$,

(2) $1 \prec x_i\xi_i$ for any $i = 1, \ldots, n$.

A monomial order $\prec$ is called a *term order* if

(3) $1 \prec x^\alpha\xi^\beta$ for any $(\alpha, \beta) \in \mathbb{N}^{2n} \setminus \{(\mathbf{0}, \mathbf{0})\}$.

This is equivalent to the condition that the monomial order $\prec$ be a well-ordering.

Now fix a monomial order $\prec$. For a nonzero element

$$P = \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha \partial^\beta \qquad (a_{\alpha,\beta} \in \mathbb{K})$$

of D_n, its initial monomial $\mathrm{in}_\prec(P)$ is defined to be the maximum nonzero monomial

$$\mathrm{in}_\prec(P) = \max{}_\prec\{x^\alpha\xi^\beta \mid a_{\alpha\beta} \neq 0\}$$

of $P(x, \xi)$ with respect to $\prec$. Note that $\mathrm{in}_\prec(P)$ belongs to $\mathbb{K}[x, \xi]$ instead of D_n so that monomial ideals make sense.

By using the Leibniz formula and the conditions (1) and (2), we can verify that $\mathrm{in}_\prec(PQ) = \mathrm{in}_\prec(P)\mathrm{in}_\prec(Q) = \mathrm{in}_\prec(QP)$ holds in $\mathbb{K}[x, \xi]$ for nonzero $P, Q \in D_n$.

Definition 3.1. Let I be a left ideal of D_n. A finite subset G of $I \setminus \{0\}$ is called a *Gröbner basis* of I with respect to a monomial order $\prec$ if

(1) G generates I as a left ideal;
(2) $\mathrm{in}_\prec(G) := \{\mathrm{in}_\prec(P) \mid P \in G\}$ generates the monomial ideal $\mathrm{in}_\prec(I)$ in $\mathbb{K}[x, \xi]$ which is generated by the set $\{\mathrm{in}_\prec(P) \mid P \in I,\ P \neq 0\}$.

First, let us recall

Lemma 3.2 (Dickson)**.** *Every monomial ideal* (*i.e.*, *an ideal generated by monomials*) *of* $\mathbb{K}[x, \xi]$ *is finitely generated.*

See e.g., 2.4 of [6] for the proof.

Proposition 3.3. *For any left ideal I of D_n, and any monomial order $\prec$, there exists a Gröbner basis G of I with respect to $\prec$. In particular, D_n is left Noetherian.*

Proof. Let G be a finite generating set of I. Since $\mathrm{in}_\prec(I)$ is a monomial ideal of $\mathbb{K}[x, \xi]$, there exists a finite set G' of I such that $\{\mathrm{in}_\prec(P) \mid P \in G'\}$ generates $\mathrm{in}_\prec(I)$ by Lemma 3.2. Then $G \cup G'$ is a Gröbner basis of I with respect to $\prec$. Q.E.D.

For a term order, we can compute a Gröbner basis of I by using division and Buchberger's criterion applied to D_n.

Now let $w \in \mathbb{Z}^{2n}$ be a weight vector for D_n (see 2.3). A monomial order $\prec$ on $M(x, \xi)$ is *adapted* to w if

$$x^\alpha \xi^\beta \prec x^{\alpha'} \xi^{\beta'} \ \Rightarrow\ \langle w, (\alpha, \beta)\rangle \leq \langle w, (\alpha', \beta')\rangle.$$

There exists a term order that is adapted to w if and only if $w_i \geq 0$ for any $i = 1, \ldots, n$.

For an arbitrary monomial order $\prec$ for D_n, define another monomial order $\prec_w$ by

$$\begin{aligned} x^\alpha \xi^\beta \prec_w x^{\alpha'} \xi^{\beta'} \ \Leftrightarrow\ & \langle w, (\alpha, \beta)\rangle < \langle w, (\alpha', \beta')\rangle \\ & \text{or } (\langle w, (\alpha, \beta)\rangle = \langle w, (\alpha', \beta')\rangle \text{ and } x^\alpha \xi^\beta \prec x^{\alpha'} \xi^{\beta'}). \end{aligned}$$

Then $\prec_w$ is adapted to w.

Recall that the residue class in $\mathrm{gr}^w_k(D_n)$ of $P \in F_k(D_n) \setminus F_{k-1}(D_n)$ is denoted by $\sigma^w(P)$ (it is denoted by $\mathrm{in}_w(P)$ in [29]). For a nonzero element P of $\mathrm{gr}^w(D_n)$ and a monomial order $\prec$ for D_n, the initial monomial $\mathrm{in}_\prec(P)$ is defined as a monomial in $\mathbb{K}[x, \xi]$.

The following is an immediate consequence of the definitions.

Lemma 3.4. *If a monomial order $\prec$ for D_n is adapted to a weight vector w for D_n, then one has $\mathrm{in}_\prec(\sigma^w(P)) = \mathrm{in}_\prec(P)$ for any nonzero element P of D_n.*

The following proposition enables us to look at w-filtrations of left D_n-modules from a computational viewpoint. Note that the weight vector w may have negative components and hence the monomial order $\prec_w$ may not be a term order.

Proposition 3.5. *Let w be a weight vector for D_n, $\prec$ be a term order, and I be a left ideal of D_n. Let G be a Gröbner basis of I with respect to $\prec_w$. Then $\mathrm{gr}^w(G) := \{\sigma^w(P) \mid P \in G\}$ generates over $\mathrm{gr}^w(D_n)$ the graded ideal*

$$\mathrm{gr}^w(I) := \bigoplus_{k\in\mathbb{Z}} (I \cap F_k^w(D_n))/(I \cap F_{k-1}^w(D_n))$$

associated with the induced filtration $\{F_k^w(D_n) \cap I\}$ on I.

Proof. Set $G = \{P_1, \ldots, P_r\}$. We denote by $\langle\sigma^w(G)\rangle$ the left ideal of $\mathrm{gr}^w(D_n)$ generated by $\sigma^w(P_1), \ldots, \sigma^w(P_r)$. Let P be a nonzero element of I. Let m be the w-order of P. We have only to show that $\sigma^w(P)$ belongs to $\langle\sigma^w(G)\rangle$.

By the assumption, the monomial $\mathrm{in}_\prec(\sigma^w(P)) = \mathrm{in}_{\prec_w}(P)$ belongs to the monomial ideal $\langle\mathrm{in}_{\prec_w}(G)\rangle$ generated by $\mathrm{in}_{\prec_w}(G)$. Hence there exist $Q_1 \in D_n$ whose total symbol is a monomial, and $i_1 \in \{1, \ldots, r\}$ such that

$$\mathrm{in}_\prec(\sigma^w(P)) = \mathrm{in}_{\prec_w}(Q_1)\mathrm{in}_{\prec_w}(P_{i_1}) = \mathrm{in}_{\prec_w}(Q_1 P_{i_1}).$$

In particular, the w-order of $R_1 := P - Q_1 P_{i_1}$ is $\leq m$. If $\mathrm{ord}_w(R_1) < m$, then $\sigma^w(P) = \sigma^w(Q_1)\sigma^w(P_{i_1})$ belongs to $\langle\mathrm{gr}^w(G)\rangle$ and we are done.

Assume $\mathrm{ord}_w(R_1) = m$. Then we have

$$\sigma^w(R_1) = \sigma^w(P) - \sigma^w(Q_1)\sigma^w(P_{i_1}), \qquad \mathrm{in}_{\prec_w}(R_1) \prec \mathrm{in}_{\prec_w}(P)$$

and R_1 belongs to I. Since the order $\prec_w$ restricted to $\{(\alpha, \beta) \in \mathbb{N}^{2n} \mid \langle w, (\alpha, \beta)\rangle = m\}$ coincides with $\prec$, which is a well-order, this process terminates and we obtain finite number of operators $Q_1, \ldots, Q_l$ and $i_1, \ldots, i_l \in \{1, \ldots, r\}$ such that

$$R_l := P - \sum_{j=1}^{l} Q_j P_{i_j} \in F_{m-1}^w(D_n), \qquad \mathrm{ord}_w(Q_j) + \mathrm{ord}_w(P_{i_j}) = m$$

for $1 \leq j \leq l$. This implies $\sigma^w(P) = \sum_{j=1}^{l} \sigma^w(Q_j)\sigma^w(P_{i_j})$ belongs to $\langle\sigma^w(G)\rangle$. Q.E.D.

Definition 3.6. Let I be a left ideal of D_n. A finite subset G of $I \setminus \{0\}$ is called a *w-involutive basis* of I if the following two conditions hold:

(1) G generates I over D_n.
(2) $\sigma^w(G) := \{\sigma^w(P) \mid P \in G\}$ generates $\mathrm{gr}^w(I)$ over $\mathrm{gr}^w(D_n)$.

Theorem 3.7. *Let I be a left ideal of D_n and set $M = D_n/I$. Let $\prec$ be a term order for D_n and w a weight vector for D_n. Suppose that $G = \{P_1, \dots, P_m\}$ is a Gröbner basis of I with respect to $\prec_w$ and set $k_i = \mathrm{ord}_w(P_i)$. Then*

(1) *G is a w-involutive basis of I.*
(2) *Let $\varphi(P)$ be the residue class of $P \in D_n$ in M and $\psi : (D_n)^m \to D_n$ be the D_n-homomorphism defined by*

$$\psi(A_1, \dots, A_m) = A_1 P_1 + \cdots + A_m P_m \qquad (A_1, \dots, A_m \in D_n).$$

Then the exact sequence

$$(D_n)^m \xrightarrow{\psi} D_n \xrightarrow{\varphi} M \longrightarrow 0$$

induces, for each $k \in \mathbb{Z}$, the exact sequence

$$\bigoplus_{i=1}^{m} F^w_{k-k_i}(D_n) \xrightarrow{\psi_k} F^w_k(D_n) \xrightarrow{\varphi_k} F_k(M) \longrightarrow 0$$

with $F_k(M) = F^w_k(D_n)/(F^w_k(D_n) \cap I)$.

Proof. (1) is an immediate consequence of Proposition 3.5.

(2) It follows from the definition that φ_k is surjective. Applying Proposition 2.12 to I and G, we have

$$F_k(I) := F^w_k(D_n) \cap I = F^w_{k-k_1}(D_n)P_1 + \cdots + F^w_{k-k_m}(D_n)P_m \qquad (\forall k \in \mathbb{Z}).$$

This completes the proof since $\ker \varphi_k = I \cap F^w_k(D_n) = F_k(I)$. Q.E.D.

We can dispense with Proposition 2.12 if G is obtained by the homogenization introduced in the next subsection (see Theorem 3.14).

Exercise 11. Set $I = D_n\partial_1 + \cdots + D_n\partial_n$ and w be an arbitrary weight vector for w. Show that $G := \{\partial_1, \dots, \partial_n\}$ is a w-involutive basis of I.

Exercise 12. Set $n = 2$ and $P_1 = \partial_1$, $P_2 = \partial_1^2 + \partial_2$. Let I be the left ideal of D_2 generated by P_1 and P_2. Let $w = (w_1, w_2; w_3, w_4)$ be a weight vector for D_2. Show that $\{P_1, P_2\}$ is a w-involutive basis of I if and only if $2w_3 < w_4$.

3.2. Homogenization trick

For a monomial order $\prec$ in which 1 is not the smallest element, the division algorithm cannot be performed directly. To bypass this difficulty, we introduce the $(\mathbf{1};\mathbf{1})$-homogenized ring. First, recall the Rees algebra

$$R^{(\mathbf{1};\mathbf{1})}(D_n) = \bigoplus_{k\in\mathbb{Z}} F_k^{(\mathbf{1};\mathbf{1})}(D_n)T^k$$

of D_n with respect to the $(\mathbf{1};\mathbf{1})$-filtration.

Let $D_n^{(h)}$ be the $\mathbb{K}$-vector space with the basis $\{x^\alpha\partial^\beta h^k \mid \alpha,\beta\in\mathbb{N}^n, k\in\mathbb{N}\}$, where h is a new indeterminate. Define a $\mathbb{K}$-isomorphism $\Psi : R^{(\mathbf{1};\mathbf{1})}(D_n) \to D_n^{(h)}$ by

$$\Psi(x^\alpha\partial^\beta T^k) = x^\alpha\partial^\beta h^{k-|\alpha|-|\beta|}.$$

Note that $x^\alpha\partial^\beta T^k \in R^{(\mathbf{1};\mathbf{1})}(D_n)$ means $|\alpha|+|\beta|\le k$.

We can make $D_n^{(h)}$ a graded $\mathbb{K}$-algebra by using the graded $\mathbb{K}$-algebra structure of $R^{(\mathbf{1};\mathbf{1})}(D_n)$ via Ψ. Let us call this $D^{(h)}$ the *homogenized Weyl algebra*, which was introduced, in connection with Gröbner bases, by Takayama and Assi-Castro-Granger [1] independently. In fact, $D^{(h)}$ was implemented by Takayama in his computer algebra system Kan [34] as early as 1994.

The image of $F_k^{(\mathbf{1};\mathbf{1})}(D_n)T^k$ by Ψ consists of the elements of $D_n^{(h)}$ which are homogeneous of degree k in x,∂,h. For an element P of D_n, we set

$$P^{(h)} := \Psi(PT^k) \quad \text{with } k := \mathrm{ord}_{(\mathbf{1};\mathbf{1})}P,$$

which is called the ($(\mathbf{1};\mathbf{1})$-) homogenization of P. For example, since $\partial_i x_j T^2 = (x_i\partial_j + \delta_{ij})T^2$ holds in $R^{(\mathbf{1};\mathbf{1})}(D_n)$, we have

$$\partial_i x_j = \Psi(\partial_i x_j T^2) = \Psi(x_i\partial_j T^2) + \delta_{ij}\Psi(T^2) = x_i\partial_j + \delta_{ij}h^2$$

in $D_n^{(h)}$. Conversely, the dehomogenization (substituting 1 for h) $D_n^{(h)} \ni P \mapsto P|_{h=1} \in D_n$ defines a ring homomorphism so that $(P^{(h)})|_{h=1} = P$ holds for $P\in D_n$.

For elements P,Q of $D_n^{(h)}$, let $P(x,\xi,h)$ and $Q(x,\xi,h)$ be their total symbols defined in a similar manner as in D_n. Then the total symbol of $R := PQ$ is given by

$$R(x,\xi,h) = \sum_{\nu\in\mathbb{N}^n} \frac{h^{2\nu}}{\nu!}\left(\frac{\partial}{\partial\xi}\right)^\nu P(x,\xi,h)\cdot\left(\frac{\partial}{\partial x}\right)^\nu Q(x,\xi,h).$$

Definition 3.8. An order $\prec$ on $M(x,\xi,h) = \{x^\alpha\xi^\beta h^j \mid \alpha,\beta\in\mathbb{N}^n, j\in\mathbb{N}\}$ is called a *monomial order for* $D_n^{(h)}$ if

(1) $u \prec v \quad \Rightarrow \quad uw \prec vw \quad (\forall u, v, w \in M(x, \xi, h))$
(2) $h^2 \prec x_i \xi_i$ for any $i = 1, \dots, n$.

A monomial order $\prec$ is called a *term order* if $1 \preceq x^\alpha \xi^\beta h^j$ for any $\alpha, \beta \in \mathbb{N}$ and $j \in \mathbb{N}$.

Definition 3.9. Let $P = \sum_{\alpha,\beta,j} c_{\alpha,\beta,j} x^\alpha \partial^\beta h^j$ be a nonzero element of $D_n^{(h)}$ and $\prec$ a monomial order for $D_n^{(h)}$. Then the *initial monomial* $\mathrm{in}_\prec(P)$ of P is the monomial $x^{\alpha_0} \xi^{\beta_0} h^{j_0}$ such that

$$(\alpha_0, \beta_0, j_0) = \max{}_\prec \{(\alpha, \beta, j) \in \mathbb{N}^{2n+1} \mid c_{\alpha,\beta,j} \neq 0\}.$$

The leading coefficient $\mathrm{LC}_\prec(P)$ and the leading term $\mathrm{LT}_\prec(P)$ are defined to be c_{α_0,β_0,j_0} and $c_{\alpha_0,\beta_0,j_0} x^{\alpha_0} \partial^{\beta_0} h^{j_0}$ respectively. Note that $\mathrm{LT}_\prec(P)$ belongs to $D_n^{(h)}$ while $\mathrm{in}_\prec(P)$ belongs to $\mathbb{K}[x, \xi, h]$.

Definition 3.10. Let J be a left ideal of $D_n^{(h)}$. A finite subset G of $J \setminus \{0\}$ is called a *Gröbner basis* of J with respect to a monomial order $\prec$ if

(1) G generates J as a left ideal;
(2) $\mathrm{in}_\prec(G) := \{\mathrm{in}_\prec(P) \mid P \in G\}$ generates the monomial ideal $\mathrm{in}_\prec(J)$ in $\mathbb{K}[x, \xi, h]$ which is generated by the set $\{\mathrm{in}_\prec(P) \mid P \in J,\ P \neq 0\}$.

Proposition 3.11 (division). *Let $G = \{P_1, \dots, P_m\}$ be a finite set of nonzero elements of $D_n^{(h)}$ and $\prec$ be a term order for $D_n^{(h)}$. Then for any $P \in D_n^{(h)}$, there exist $Q_1, \dots, Q_m, R \in D_n^{(h)}$ such that*

$$P = Q_1 P_1 + \cdots + Q_m P_m + R, \qquad \mathrm{in}_\prec(Q_j P_j) \preceq \mathrm{in}_\prec(P) \text{ if } Q_j \neq 0$$

and that $\mathrm{in}_\prec(R)$ is not divisible by $\mathrm{in}_\prec(P_j)$ for $1 \leq j \leq m$. Moreover, if G is a Gröbner basis of the left ideal J generated by G with respect to $\prec$, then $R = 0$ if and only if P belongs to J.

Proof. The existence of Q_j and R can be proved by induction in the well-order $\prec$, in the same way as in the polynomial ring. Suppose that G is a Gröbner basis with respect to $\prec$ and that P belongs to J. If $R \neq 0$, then $\mathrm{in}_\prec(R)$ must be divisible by $\mathrm{in}_\prec(P_j)$ for some j since R belongs to J. This contradicts the assumption. Hence we have $R = 0$. Q.E.D.

Definition 3.12. Let $\prec$ be a term order for $D_n^{(h)}$. For nonzero $P, Q \in D_n^{(h)}$, write $\mathrm{LT}_\prec(P) = a x^\alpha \partial^\beta h^j$ and $\mathrm{LT}_\prec(Q) = b x^{\alpha'} \partial^{\beta'} h^k$ with $a, b \in \mathbb{K} \setminus \{0\}$. Set

$$\begin{aligned}(\alpha'', \beta'', l) &= (\alpha, \beta, j) \wedge (\alpha', \beta', k) \\ &:= (\min\{\alpha_1, \alpha'_1\}, \dots, \min\{\beta_n, \beta'_n\}, \min\{j, k\}).\end{aligned}$$

Then the *S-pair* of P and Q with respect to $\prec$ is an element of $D_n^{(h)}$ defined by

$$\mathrm{sp}_\prec(P,Q) = bx^{\alpha'-\alpha''}\partial^{\beta'-\beta''}h^{k-l}P - ax^{\alpha-\alpha''}\partial^{\beta-\beta''}h^{j-l}Q.$$

Theorem 3.13 (Buchberger's criterion in $D_n^{(h)}$). *Let J be a left ideal of $D_n^{(h)}$ and $\prec$ be a term order for $D_n^{(h)}$. Let $G = \{P_1,\dots,P_m\}$ be a finite subset of $J \setminus \{0\}$ which generates J. Then the following two conditions are equivalent:*

(1) *G is a Gröbner basis of J with respect to $\prec$.*

(2) *If $\mathrm{sp}_\prec(P_i,P_j) \neq 0$ for $1 \le i < j \le m$, then there exist $Q_{ijk} \in D_n^{(h)}$ such that*

$$\mathrm{sp}_\prec(P_i,P_j) = Q_{ij1}P_1 + \cdots + Q_{ijm}P_m,$$
$$\mathrm{in}_\prec(Q_{ijk}P_k) \preceq \mathrm{in}_\prec(\mathrm{sp}_\prec(P_i,P_j)) \text{ if } Q_{ijk} \neq 0 \quad (1 \le k \le m).$$

Proof. We see that (1) implies (2) by division. Assume (2) and let P be a nonzero element of I. We have only to show that $\mathrm{in}_\prec(P)$ belongs to the monomial ideal $\langle \mathrm{in}_\prec(G)\rangle$. Let us consider the expression of the form

$$P = Q_1P_1 + \cdots + Q_mP_m \qquad (Q_1,\dots,Q_m \in D_n^{(h)}). \tag{7}$$

Since $\prec$ is a well-order, we may assume that this expression is minimal in the sense that $a := \max\{\mathrm{in}_\prec(Q_iP_i) \mid Q_i \neq 0\}$ is minimum among such expressions. Then $\mathrm{in}_\prec(P) \preceq a$ holds. If $\mathrm{in}_\prec(P) = a$, then we are done since a belongs to $\langle \mathrm{in}_\prec(G)\rangle$.

Suppose $\mathrm{in}_\prec(P) \prec a$. Let A be the set of $i \in \{1,\dots,m\}$ such that $\mathrm{in}_\prec(P_iQ_i) = a$. We may assume $A = \{1,\dots,l\}$. We may also assume that the leading coefficients of P_i are all one. Set

$$c_k = \mathrm{LC}_\prec(Q_k), \quad S_k := c_k^{-1}\mathrm{LT}_\prec(Q_k), \quad Q'_k = Q_k - \mathrm{LT}_\prec(Q_k) = Q_k - c_kS_k$$

for $1 \le k \le l$. Then we have

$$P = \sum_{k=1}^{l} c_kS_kP_k + \sum_{k=1}^{l} Q'_kP_k + \sum_{k=l+1}^{m} Q_kP_k \tag{8}$$

with the property that $\mathrm{in}_\prec(Q'_kP_k) \prec a$ if $1 \le k \le l$ and $Q'_k \neq 0$, and $\mathrm{in}_\prec(Q_kP_k) \prec a$ if $l < k \le m$ and $Q_k \neq 0$. This implies that the initial monomial of $\sum_{k=1}^{l} c_kS_kP_k$ is smaller than a in $\prec$. The first term can be rewritten as

$$\begin{aligned}(9)\quad &\sum_{k=1}^{l} c_k S_k P_k \\ &= \sum_{k=1}^{l-1}(c_1+\cdots+c_k)(S_kP_k - S_{k+1}P_{k+1}) + (c_1+\cdots+c_l)S_lP_l.\end{aligned}$$

Let $\mathrm{sp}_{\prec}(P_i, P_j)$ be given by $S_{ji}P_i - S_{ij}P_j$. There exist monomials u_k such that

$$\mathrm{in}_{\prec}(S_k)\mathrm{in}_{\prec}(P_k) = \mathrm{in}_{\prec}(S_{k+1})\mathrm{in}_{\prec}(P_{k+1}) = u_k \mathrm{LCM}(\mathrm{in}_{\prec}(P_k), \mathrm{in}_{\prec}(P_{k+1}))$$

and

$$\mathrm{in}_{\prec}(S_k) = u_k S_{k+1,k}(x,\xi,h), \quad \mathrm{in}_{\prec}(S_{k+1}) = u_k S_{k,k+1}(x,\xi,h)$$

for $1 \le k \le l-1$. Take $U_k \in D_n^{(h)}$ whose total symbol is u_k and set

$$A_k := S_k - U_k S_{k+1,k}, \quad B_k := S_{k+1} - U_k S_{k,k+1}.$$

Then we have

$$S_kP_k - S_{k+1}P_{k+1} = U_k(S_{k+1,k}P_k - S_{k,k+1}P_{k+1}) + A_kP_k - B_kP_{k+1}.$$

Combined with (9), this yields

$$\begin{aligned}\sum_{k=1}^{l} c_k S_k P_k &= \sum_{k=1}^{l-1}(c_1+\cdots+c_k)U_k \mathrm{sp}_{\prec}(P_k, P_{k+1}) + (c_1+\cdots+c_l)S_lP_l \\ &\quad + \sum_{k=1}^{l-1}(c_1+\cdots+c_k)(A_kP_k - B_kP_{k+1}) \\ &= \sum_{j=1}^{m}\sum_{k=1}^{l-1}(c_1+\cdots+c_k)U_kQ_{k,k+1,j}P_j + (c_1+\cdots+c_l)S_lP_l \\ &\quad + \sum_{k=1}^{l-1}(c_1+\cdots+c_k)(A_kP_k - B_kP_{k+1}).\end{aligned}$$

Here the initial monomials of $U_kQ_{k,k+1,j}P_j$, A_kP_k, and B_kP_{k+1} are smaller than a, as well as the initial monomial of $\sum_{k=1}^{l} c_kS_kP_k$, while the initial monomial of S_lP_l is a. It follows that $c_1+\cdots+c_l = 0$ and hence

$$\sum_{k=1}^{l} c_k S_k P_k = \sum_{j=1}^{m}\sum_{k=1}^{l-1}(c_1 + \cdots + c_k) U_k Q_{k,k+1,j} P_j + \sum_{k=1}^{l-1}(c_1 + \cdots + c_k)(A_k P_k - B_k P_{k+1}).$$

Substituting this for the first term of the right-hand side of (8) gives an expression of P which contradicts the minimality of (7). Hence we must have $\mathrm{in}_{\prec}(P) = a$. This completes the proof. Q.E.D.

This criterion assures that the Buchberger algorithm applies to $D_n^{(h)}$. Note also that this criterion and the proof works in D_n if $\prec$ is a term order for D_n.

Now let $\prec$ be an arbitrary monomial order for D_n. We define a monomial order $\prec_h$ on $M(x, \xi, h)$ by

$$x^\alpha \xi^\beta h^j \prec_h x^{\alpha'} \xi^{\beta'} h^k \;\Leftrightarrow\; |\alpha| + |\beta| + j < |\alpha'| + |\beta'| + k$$
$$\text{or } (|\alpha| + |\beta| + j = |\alpha'| + |\beta'| + k \text{ and } x^\alpha \xi^\beta \prec x^{\alpha'} \xi^{\beta'}).$$

Then $\prec_h$ is a term order for $D_n^{(h)}$. Hence the division and the Buchberger algorithm works with $\prec_h$ in $D_n^{(h)}$. If P is a nonzero homogeneous element of $D_n^{(h)}$, then $\mathrm{in}_{\prec}(P|_{h=1}) = \mathrm{in}_{\prec_h}(P)|_{h=1}$ holds.

Theorem 3.14. *Let I be the left ideal of D_n generated by nonzero elements $P_1, \ldots, P_r$ of I, and $\prec$ be an arbitrary monomial order for D_n. Let J be a left ideal of $D_n^{(h)}$ generated by $P_1^{(h)}, \ldots, P_r^{(h)}$ and $\{Q'_1, \ldots, Q'_m\}$ be a Gröbner basis of J with respect to $\prec_h$, which can be computed by Buchberger's algorithm.*

Set $Q_i := Q'_i|_{h=1}$. Then $\{Q_1, \ldots, Q_m\}$ is a Gröbner basis of I with respect to $\prec$. Moreover, for any nonzero element P of I, there exist $U_1, \ldots, U_m \in D_n$ such that

$$P = U_1 Q_1 + \cdots + U_m Q_m, \qquad \mathrm{in}_{\prec}(U_i Q_i) \preceq \mathrm{in}_{\prec}(P) \text{ if } U_i \neq 0.$$

In particular, if $\prec$ is adapted to w, then $Q_1, \ldots, Q_m$ are a w-involutive basis of I; more precisely, one has

$$I \cap F_k^w(D_n) = F_{k-k_1}^w(D_n) Q_1 + \cdots + F_{k-k_m}^w(D_n) Q_m \qquad (\forall k \in \mathbb{Z})$$

with $k_i := \mathrm{ord}_w(Q_i)$.

Proof. Let P be a nonzero element of I. Then there exist $A_1, \dots,$ $A_r \in D_n$ such that $P = A_1 P_1 + \cdots + A_r P_r$. Homogenizing the both sides of this equation, we obtain

$$h^l P^{(h)} = h^{l_1} A_1^{(h)} P_1^{(h)} + \cdots + h^{l_m} A_m^{(h)} P_m^{(h)}$$

with some $l, l_1, \dots, l_m \in \mathbb{N}$. Hence $h^l P^{(h)}$ belongs to J. Since $Q'_1, \dots,$ Q'_m are a Gröbner basis of J, division algorithm in $D_n^{(h)}$ produces an expression

$$h^l P^{(h)} = \sum_{j=1}^{m} B_j Q'_j$$

with some homogeneous elements B_j of $D_n^{(h)}$ such that $\mathrm{in}_{\prec_h}(B_j Q'_j) \preceq_h$ $\mathrm{in}_{\prec_h}(h^l P^{(h)})$ if $B_j \neq 0$. Dehomogenization yields

$$P = \sum_{j=1}^{m} B_j|_{h=1} Q_j, \qquad \mathrm{in}_{\prec}(B_j|_{h=1} Q_j) \preceq \mathrm{in}_{\prec}(P). \tag{10}$$

In particular, $\mathrm{in}_{\prec}(P)$ is divisible by $\mathrm{in}_{\prec}(Q_j)$ for some j. Hence $Q_1, \dots,$ Q_m are a Gröbner basis of I with respect to $\prec$. The last statement of the theorem also follows from (10) if $\prec$ is adapted to w. Q.E.D.

Example 3.15. Set $n = 2$, $w = (0, 1; 0, -1)$, and $P_1 = x_1 - x_2^2$, $P_2 = 2x_2\partial_1 + \partial_2$. Fix a term order $\prec$ for D_2 which is adapted to the weight vector $(1, 1; 1, 1)$ such that $x_1\xi_1 \succ x_2\xi_2$. Let us compute a Gröbner basis of $I := D_2P_1 + D_2P_2$ with respect to the monomial order$\prec_w$. Homogenization gives

$$P_1^{(h)} = x_1 h - \underline{x_2^2}, \qquad P_2^{(h)} = \underline{2x_2\partial_1} + \partial_2 h$$

with the leading terms with respect to $\prec_{wh}$, which is the term order for $D_2^{(h)}$ defined by $\prec_w$, being underlined. Their S-pair is

$$P'_3 := \mathrm{sp}_{\prec_w}(P_1^{(h)}, P_2^{(h)}) = 2\partial_1 P_1^{(h)} + x_2 P_2^{(h)} = \underline{2x_1\partial_1 h} + x_2\partial_2 h + 2h^3.$$

By using the Buchberger criterion, we can check that $P_1^{(h)}$, $P_2^{(h)}$, P'_3 are a Gröbner basis of the left ideal $J := D_2^{(h)} P_1^{(h)} + D_2^{(h)} P_2^{(h)}$ of $D_2^{(h)}$ with respect to $\prec_{wh}$. Hence P_1, P_2, and $P_3 := P'_3|_{h=1} = 2x_1\partial_1 + x_2\partial_2 + 2$ are a Gröbner basis of I with respect to $\prec_w$.

The notion and the algorithm of Gröbner basis can be extended to submodules of free modules over D_n or $D_n^{(h)}$ of finite rank.

Exercise 13. In the example above, confirm that $P_1^{(h)}$, $P_2^{(h)}$, P'_3 are a Gröbner basis of J with respect to $\prec_{wh}$.

3.3. Computation of the characteristic variety and the singular locus

Let I be a left ideal of D_n and consider the left D_n-module $M = D_n/I$. As was seen in 2.4, the most fundamental invariants of M are the dimension and the characteristic variety. Now let us deduce a more concrete description of the characteristic variety. Let P be a nonzero differential operator written in the form

$$P = P(x, \partial) = \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha,\beta} x^\alpha \partial^\beta \qquad (a_{\alpha,\beta} \in \mathbb{K})$$

and set $m := \mathrm{ord}_{(\mathbf{0};\mathbf{1})}(P)$. Then the *principal symbol* of P is the polynomial defined by

$$\sigma(P)(x, \xi) = \sum_{|\beta|=m} \sum_{\alpha} a_{\alpha,\beta} x^\alpha \xi^\beta.$$

It can be identified with the residue class of P in $\mathrm{gr}^{(\mathbf{0};\mathbf{1})}(D_n) \cong \mathbb{K}[x, \xi]$. Note that $\sigma(P)(x, \xi)$ is homogeneous with respect to ξ.

In general, let w be a weight vector for D_n satisfying $w_i \geq 0$ for $i = 1, \ldots, 2n$, and $w_i + w_{n+i} > 0$ for $i = 1, \ldots, n$. Let $\prec$ be an arbitrary term order for D_n. Then the Buchberger algorithm applied to I with the term order $\prec_w$ yields a Gröbner basis $G = \{P_1, \ldots, P_m\}$ of I with respect to $\prec_w$. Proposition 3.5 assures that G is a w-involutive basis of I; that is, $\sigma^w(G)$ generates the graded ideal $\mathrm{gr}^w(I)$ associated with the filtration $\{F_k^w(D_n) \cap I\}$ on I.

Let $\mathrm{gr}^w(M)$ be the graded module associated with the good w-filtration $F_k^w(M) := F_k^w(D_n)/(F_k^w(D_n) \cap I)$. Then there exists a graded exact sequence

$$0 \longrightarrow \mathrm{gr}^w(I) \longrightarrow \mathrm{gr}^w(D_n) \longrightarrow \mathrm{gr}^w(M) \longrightarrow 0.$$

Note that $\mathrm{gr}^w(D_n)$ can be regarded as $\mathbb{K}[x, \xi]$ by the assumption on w. Hence one has an isomorphism

$$\mathrm{gr}^w(M) \cong \mathbb{K}[x, \xi]/\mathrm{gr}^w(I) \cong \mathbb{K}[x, \xi]/(\mathbb{K}[x, \xi]\sigma^w(P_1) + \cdots + \mathbb{K}[x, \xi]\sigma^w(P_m))$$

as $\mathbb{K}[x, \xi]$-module. In particular, setting $w = (\mathbf{0}; \mathbf{1})$, we obtain

$$\mathrm{Char}(M) = \{(x, \xi) \in \overline{\mathbb{K}}^{2n} \mid \sigma(P_1)(x, \xi) = \cdots = \sigma(P_m)(x, \xi) = 0\}.$$

Let $\pi : \overline{\mathbb{K}}^{2n} \ni (x, \xi) \mapsto x \in \overline{\mathbb{K}}^n$ be the projection. Then the *singular locus* of M is defined by

$$\mathrm{Sing}(M) := \pi(\mathrm{Char}(M) \setminus (\overline{\mathbb{K}}^n \times \{0\})).$$

It is an algebraic set of $\overline{\mathbb{K}}^n$ since $\mathrm{gr}(M)$ is homogeneous with respect to ξ. In particular, if M is holonomic, then $\mathrm{Sing}(M)$ is an algebraic set of codimension ≥ 1, or an empty set, since $\mathrm{Char}(M)$ is homogeneous with respect to ξ.

The set $\mathrm{Char}(M) \setminus (\{0\} \times \overline{\mathbb{K}}^n)$ can be regarded as the subset of $\overline{\mathbb{K}}^n \times \mathbb{P}^{n-1}(\overline{\mathbb{K}})$, where $\mathbb{P}^{n-1}(\overline{\mathbb{K}})$ is the $(n-1)$-dimensional projective space over $\overline{\mathbb{K}}$. Thus the problem of finding $\mathrm{Sing}(M)$ from $\mathrm{Char}(M)$ is completely solved by what is called the projective elimination theory, as is described in Chapter 8 of [6] in detail with a complete proof.

Proposition 3.16. *Assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero and set $M = D_n/I$ with a left ideal I of D_n. Let $f_1(x,\xi), \dots, f_m(x,\xi)$ be polynomials homogeneous in ξ which generate $\mathrm{gr}^{(\mathbf{0};\mathbf{1})}(I)$. Let J_i be the ideal of $\mathbb{K}[x,\xi]$ generated by $f_1, \dots, f_m$ with the variable ξ_i replaced by 1. Set $I_i = J_i \cap \mathbb{K}[x]$. Then* $\mathrm{Sing}(M)$ *is the algebraic subset of $\mathbb{K}^n$ defined as the zeros of the ideal $I_1 \cap \dots \cap I_n$.*

Thus we can compute $\mathrm{Sing}(M)$ from $\mathrm{Char}(M)$ by using appropriate Gröbner bases in $\mathbb{K}[x,\xi]$; this fact was pointed out in [21], where it is also noticed that the characteristic variety as is defined here coincides with the analytic definition using the differential operators with analytic coefficients. Even if M is generated by more than one elements over D_n, we can compute $\mathrm{Char}(M)$ by using a Gröbner basis for a submodule of the free module (see [21] for details).

Example 3.17. Let

$$P = a_m(x)\partial^m + a_{m-1}(x)\partial^{m-1} + \dots + a_0(x) \qquad (a_i(x) \in \mathbb{K}[x],\ a_m(x) \neq 0)$$

be a linear ordinary differential operator of order $m \geq 1$ with $x = x_1$ and $\partial = \partial_1$. Set $M = D_1/D_1P$. Then we have

$$\begin{aligned}\mathrm{Char}(M) &= \{(x,\xi) \in \overline{\mathbb{K}}^2 \mid \sigma(P)(x,\xi) = a_m(x)\xi^m = 0\} \\ &= \{(x,\xi) \in \overline{\mathbb{K}}^2 \mid a_m(x) = 0\} \cup \{(x,0) \mid x \in \overline{\mathbb{K}}\}.\end{aligned}$$

Hence M is holonomic and $\mathrm{Sing}(M) = \{x \in \mathbb{K} \mid a_m(x) = 0\}$, a point of which is called a *singular point* of P.

Example 3.18. Let f be an arbitrary nonzero polynomial of $x = (x_1, \dots, x_n)$. For each $i = 1, \dots, n$, $\partial_i f = f\partial_i + f_i$ annihilates the rational function $1/f$, with $f_i := \partial f/\partial x_i$. Set $M = D_n/I$ with

$$I := D_n\partial_1 f + \dots + D_n\partial_n f.$$

This is a 'naive' D-module for the rational function $1/f$.

For example, set $n = 2$ and $f(x) = x_1^3 - x_2^2$ which has a cusp singularity at the origin. We can check that $\partial_1 f$ and $\partial_2 f$ are a $(\mathbf{0};\mathbf{1})$-involutive basis of I. This gives

$$\begin{aligned}\mathrm{Char}(M) &= \{(x,\xi) \in \overline{\mathbb{K}}^4 \mid \xi_1 f(x_1,x_2) = \xi_2 f(x_1,x_2) = 0\} \\ &= \{(x,\xi) \mid x_1^3 - x_2^2\} \cup \{(x,\xi) \mid \xi = 0\}, \\ \mathrm{Sing}(M) &= \{(x_1,x_2) \mid x_1^3 - x_2^2 = 0\}.\end{aligned}$$

Hence the dimension of $\mathrm{Char}(M)$ is 3, consequently M is not holonomic. In fact, I is much smaller than $\mathrm{Ann}_{D_2}(1/f)$, which is generated by $3x_1^2\partial_2 + 2x_2\partial_1$ and $2x_1\partial_1 + 3x_2\partial_2 + 6$. There is an algorithm to compute $\mathrm{Ann}_{D_n}(1/f)$ for an arbitrary polynomial f and $D_n/\mathrm{Ann}_{D_n}(1/f)$ is always holonomic (see [22], [29]).

Exercise 14. In the example above with $n = 2$ and $f = x_1^3 - x_2^2$

(1) Verify that $\partial_1 f$ and $\partial_2 f$ are a $(\mathbf{0};\mathbf{1})$-involutive basis of the ideal I which they generate.
(2) Verify that $P_1 := 3x_1^2\partial_2 + 2x_2\partial_1$ and $P_2 := 2x_1\partial_1 + 3x_2\partial_2 + 6$ annihilate $1/f$.
(3) Find a $(\mathbf{0};\mathbf{1})$-involutive basis of $J := D_2P_1 + D_2P_2$ and verify that D_2/J is holonomic.
(4) Find the singular locus of D_2/J.

Exercise 15. Find the characteristic variety and the singular locus of the left D_n-module $M = D_n/(D_nx_1 + \cdots + D_nx_n)$.

Exercise 16. Let $\mathbb{K} = \mathbb{C}$ and let $f(x) \in \mathbb{C}[x] = \mathbb{C}[x_1,\ldots,x_n]$. Consider a C^∞ function $e^{f(x)}$ on $\mathbb{R}^n$. Set $f_i = \partial_i(f)$ and $M := D_n/I$ with

$$I := D_n(\partial_1 - f_1) + \cdots + D_n(\partial_n - f_n).$$

(1) Show that $I = \mathrm{Ann}_{D_n} e^{f(x)} := \{P \in D_n \mid Pe^{f(x)} = 0\}$.
(2) Show that $\mathrm{Hom}_{D_n}(M, C^\infty(\mathbb{R}^n)) \cong \mathbb{C}e^{f(x)}$.
(3) Find the characteristic variety and the singular locus of M.

3.4. Equivalence of two definitions of holonomicity

The purpose of this subsection is to prove Theorem 2.22 by using only basic tools in commutative algebra and Gröbner basis.

Definition 3.19 ([17])**.** A map φ from $\mathbb{N}$ to $\{t \in \mathbb{R} \mid t \geq 0\}$ is said to be of *polynomial growth* if there exists $\nu \in \mathbb{R}$ such that $\varphi(n) \leq n^\nu$ for $n \gg 0$. Then we define the degree of φ by

$$\deg\varphi = \inf\{\nu \mid \varphi(n) \leq n^\nu \text{ for } n \gg 0\}.$$

We set $\deg\varphi = \infty$ if φ is not of polynomial growth.

Definition 3.20 ([5])**.** A function $\varphi : \mathbb{Z} \to \mathbb{R}$ is called a *quasi-polynomial* (of period r) if there exist a positive integer r and polynomials H_i $(i = 0, 1, \ldots, r-1)$ such that

$$\varphi(jr+i) = H_i(jr+i) \qquad (\forall j \in \mathbb{Z},\, 0 \le \forall i \le r-1).$$

Then one has $\deg \varphi = \max_{0 \le i \le r-1} \deg H_i$.

Let w be a weight vector such that $w_i > 0$ for $1 \le i \le 2n$. In this case we call w positive and denote it by $w > 0$.

Proposition 3.21 ([33], Theorem 5.2)**.** *Let $w \in \mathbb{N}^{2n}$ be a positive weight vector for D_n. Let $\{F_k(M)\}$ be a good w-filtration on a left D_n-module M. Then the degree of the function*

$$\varphi(k) = \dim_{\mathbb{K}} F_k(M)$$

of $k \ge 0$ does not depend either on w or on the choice of a good w-filtration $\{F_k(M)\}$.

Proof. For a fixed w, the fact that $\deg \varphi$ does not depend on the choice of a good w-filtration can be proved by using Lemma 2.10.

Let $u_1, \ldots, u_m$ be a set of generators of M. Define a good w-filtration and a good $(\mathbf{1};\mathbf{1})$-filtration on M by

$$F_k^w(M) := F_k^w(D_n)u_1 + \cdots + F_k^w(D_n)u_m,$$
$$F_k^{(\mathbf{1};\mathbf{1})}(M) := F_k^{(\mathbf{1};\mathbf{1})}(D_n)u_1 + \cdots + F_k^{(\mathbf{1};\mathbf{1})}(D_n)u_m$$

respectively and $\mathrm{gr}^w(M)$ and $\mathrm{gr}^{(\mathbf{1};\mathbf{1})}(M)$ be associated graded modules. Set $\psi(k) = \dim_{\mathbb{K}} F_k^{(\mathbf{1};\mathbf{1})}(M)$. Then there exists a polynomial G such that $\psi(k) = G(k)$ for all $k \gg 0$. On the other hand, since there exists $P(T) \in \mathbb{Z}[T, T^{-1}]$ such that

$$\sum_{k \in \mathbb{Z}} \dim_{\mathbb{K}} \mathrm{gr}_k^w(M)\, T^k = \frac{P(T)}{\prod_{i=1}^n (1 - T^{w_i})(1 - T^{w_{n+i}})},$$

there exists a quasi-polynomial H of some period r such that $\varphi(k) = H(k)$ for all $k \gg 0$ (see e.g., Proposition 4.4.1 and Theorem 4.4.3 of [5]). Let d be an integer not smaller than $\max\{w_i \mid 1 \le i \le 2n\}$. Then the inclusion

$$F_k^w(D_n) \subset F_k^{(\mathbf{1};\mathbf{1})}(D_n) \subset F_{dk}^w(D_n) \qquad (\forall k \ge 0)$$

yields

$$F_k^w(M) \subset F_k^{(\mathbf{1};\mathbf{1})}(M) \subset F_{dk}^w(M) \qquad (\forall k \ge 0),$$

and hence

$$\varphi(k) \le \psi(k) \le \varphi(dk).$$

This implies $\deg \varphi = \deg \psi$. Q.E.D.

Proposition 3.22. *Let $w \in \mathbb{N}^{2n}$ be a positive weight vector and $M' = \oplus_j M'_j$ be a finitely generated graded module over $\mathbb{K}[x,\xi]$ in which x_i is of order w_i and ξ_i is of order w_{n+i}. Then the degree of the function*

$$\varphi(k) = \sum_{j \le k} \dim_{\mathbb{K}} M'_j$$

coincides with the Krull dimension of M' as a (not graded) $\mathbb{K}[x,\xi]$-module.

Proof. Let $u'_1, \ldots, u'_m$ be w-homogeneous generators of M' with $u'_i \in M'_{k_i}$. Then we have

$$M'_k = \mathbb{K}[x,\xi]^w_{k-k_1} u'_1 + \cdots + \mathbb{K}[x,\xi]^w_{k-k_m} u'_m,$$
$$\mathbb{K}[x,\xi]^w_k := \sum_{\langle w,(\alpha,\beta)\rangle = k} \mathbb{K} x^\alpha \xi^\beta.$$

Forgetting the w-graded structure of M', set

$$F_k(M') = F^{(\mathbf{1};\mathbf{1})}_{k-k_1}(\mathbb{K}[x,\xi]) u'_1 + \cdots + F^{(\mathbf{1};\mathbf{1})}_{k-k_m}(\mathbb{K}[x,\xi]) u'_m,$$
$$F^{(\mathbf{1};\mathbf{1})}_k(\mathbb{K}[x,\xi]) := \sum_{|\alpha|+|\beta| \le k} \mathbb{K} x^\alpha \xi^\beta.$$

Let $\mathrm{gr}(M') = \oplus_k F_k(M')/F_{k-1}(M')$ be the associated graded module. It is well-known in commutative algebra (e.g., Corollary 13.6 of [8]) that the Krull dimension of M' coincides with the degree of the function $\psi(k) := \dim_{\mathbb{K}} F_k(M')$, which equals a polynomial for $k \gg 0$. (One can dispense with the Krull dimension by adopting the degree of $\psi(k)$ as the definition of the dimension as in Chapter 9 of [6].) On the other hand, we can show that $\varphi(k)$ is a quasi-polynomial for $k \gg 0$ and its degree coincides with that of ψ by the same argument over $\mathbb{K}[x,\xi]$ instead of D_n as the proof of Proposition 3.21. Q.E.D.

Now let us prove Theorem 2.22. If M is generated by $u_1, \ldots, u_m$, then it is easy to see that

$$\dim M = \max_{1 \le i \le m} \dim D_n u_i, \qquad \mathrm{Char}(M) = \bigcup_{1 \le i \le m} \mathrm{Char}(D_n u_i)$$

hold. Hence we may assume that M is generated by a single element, and consequently that $M = D_n/I$ with a left ideal I of D_n.

Let $\prec$ be a term order for D_n adapted to the weight vector $(\mathbf{1};\mathbf{1})$. Let $G = \{P_1, \dots, P_m\}$ be a Gröbner basis of I with respect to the term order $\prec_{(\mathbf{0};\mathbf{1})}$. There exist $Q_{ijk} \in D_n$ such that

$$\mathrm{sp}_{\prec_{(\mathbf{0};\mathbf{1})}}(P_i, P_j) = \sum_{k=1}^{m} Q_{ijk} P_k,$$
$$\mathrm{in}_{\prec_{(\mathbf{0};\mathbf{1})}}(Q_{ijk}P_k) \preceq_{(\mathbf{0};\mathbf{1})} \mathrm{in}_{\prec_{(\mathbf{0};\mathbf{1})}}(\mathrm{sp}_{\prec_{(\mathbf{0};\mathbf{1})}}(P_i, P_j)) \quad \text{if } Q_{ijk} \neq 0.$$

Set $w = (1, \dots, 1; d, \dots, d) = (d-1)(\mathbf{0};\mathbf{1}) + (\mathbf{1};\mathbf{1})$. If we take d large enough, then the initial monomials of P_i and Q_{ijk} with respect to $\prec_{(\mathbf{0};\mathbf{1})}$ stay unchanged with $\prec_{(\mathbf{0};\mathbf{1})}$ replaced by $\prec_w$ since $\prec$ is adapted to $(\mathbf{1};\mathbf{1})$. Hence G is also a Gröbner basis of I with respect to $\prec_w$ in view of the Buchberger criterion in D_n.

Let $\mathrm{gr}^{(\mathbf{0};\mathbf{1})}(M)$ and $\mathrm{gr}^w(M)$ be the graded modules associated with filtrations

$$F_k^{(\mathbf{0};\mathbf{1})}(M) = F_k^{(\mathbf{0};\mathbf{1})}(D_n)/(F_k^{(\mathbf{0};\mathbf{1})}(D_n) \cap I),$$
$$F_k^w(M) = F_k^w(D_n)/(F_k^w(D_n) \cap I) \qquad (k \in \mathbb{Z})$$

respectively. Then from the argument above and Proposition 3.5 we have

$$\mathrm{gr}^{(\mathbf{0};\mathbf{1})}(M) = \mathbb{K}[x,\xi]/\langle \sigma^{(\mathbf{0};\mathbf{1})}(G)\rangle, \qquad \mathrm{gr}^w(M) = \mathbb{K}[x,\xi]/\langle \sigma^w(G)\rangle$$

and

$$\mathrm{in}_\prec(\sigma^{(\mathbf{0};\mathbf{1})}(P_i)) = \mathrm{in}_{\prec_{(\mathbf{0};\mathbf{1})}}(P_i) = \mathrm{in}_{\prec_w}(P_i) = \mathrm{in}_\prec(\sigma^w(P_i)) \quad (1 \le i \le m).$$

Since the Hilbert polynomial of $\mathbb{K}[x,\xi]/\langle \sigma^w(G)\rangle$ coincides with that of $\mathbb{K}[x,\xi]/\langle \mathrm{in}_\prec(\sigma^w(G))\rangle$ (see Chapter 9 of [6]), it follows that $\mathrm{gr}^{(\mathbf{0};\mathbf{1})}(M)$ and $\mathrm{gr}^w(M)$ have the same Krull dimension. Set

$$F_k^{(\mathbf{1};\mathbf{1})}(M) = F_k^{(\mathbf{1};\mathbf{1})}(D_n)/(F_k^{(\mathbf{1};\mathbf{1})}(D_n) \cap I).$$

By Propositions 3.21 and 3.22, the Krull dimension of $\mathrm{gr}^w(M)$ is

$$\deg\Bigl(\sum_{j \le k} \dim_{\mathbb{K}} \mathrm{gr}_j^w(M)\Bigr) = \deg \dim_{\mathbb{K}} F_k^w(M) = \deg \dim_{\mathbb{K}} F_k^{(\mathbf{1};\mathbf{1})}(M),$$

where deg denotes the degree as a function of k. It is also equal to $\dim M$ by the definition. This completes the proof of Theorem 2.22.

§4. Distributions as generalized functions

We briefly review the theory of distributions (generalized functions) introduced by L. Schwartz [32]. See also [10] for plenty of interesting examples. The main purpose here is to introduce some classes of distributions which adapt nicely to the integration algorithm based on differentiation under the integral sign. The terminology coined by Schwartz has an origin in probability theory. So we also consider probability and cumulative distribution functions associated with the multivariate normal distribution or the gamma distribution as integrals of generalized functions.

4.1. Definitions and basic properties

Definition 4.1. Let $C_0^\infty(U)$ be the set of the complex-valued C^∞ functions on an open set U of $\mathbb{R}^n$ with compact support. Here the support of a C^∞ function f on U is defined to be the closure in U of the set $\{x \in U \mid f(x) \neq 0\}$ and denoted by $\operatorname{supp} u$. A *distribution* u on U is a linear mapping

$$u : C_0^\infty(U) \ni \varphi \longmapsto \langle u, \varphi \rangle \in \mathbb{C}$$

such that $\lim_{j\to\infty} \langle u, \varphi_j \rangle = 0$ holds for a sequence $\{\varphi_j\}$ of $C_0^\infty(U)$ if there is a compact set $K \subset U$ such that $\operatorname{supp} \varphi_j \subset K$ for any j and

$$\lim_{j\to\infty} \sup_{x\in U} |\partial^\alpha \varphi_j(x)| = 0 \quad \text{for any } \alpha \in \mathbb{N}^n,$$

where $x = (x_1, \dots, x_n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ with $\partial_j = \partial/\partial x_j$. If this is the case, we say that $\{\varphi_j\}$ converges to 0 in $C_0^\infty(U)$, which makes $C_0^\infty(U)$ a topological vector space that is also denoted by $\mathcal{D}(U)$. The set of the distributions on U is denoted by $\mathcal{D}'(U)$.

A Lebesgue measurable function $u(x)$ defined on an open set U of $\mathbb{R}^n$ is called locally integrable on U if it is integrable on any compact subset of U. We can regard a locally integrable function $u(x)$ on U as a distribution on U through the pairing

$$\langle u, \varphi \rangle = \int_U u(x)\varphi(x)\,dx \qquad (\forall \varphi \in C_0^\infty(U)).$$

Identifying two locally integrable functions which are equal to each other almost everywhere in U (i.e., outside a set of measure 0), we can regard the set $L^1_{\mathrm{loc}}(U)$ of the locally integrable functions on U as a subspace of $\mathcal{D}'(U)$.

Let u be a distribution on U. The derivative $\partial_k u$ of u with respect to x_k is defined by

$$\langle \partial_k u, \varphi \rangle = -\langle u, \partial_k \varphi \rangle \quad \text{for any } \varphi \in C_0^\infty(U).$$

For a C^∞ function a on U, the product au is defined by

$$\langle au, \varphi \rangle = \langle u, a\varphi \rangle \quad \text{for any } \varphi \in C_0^\infty(U).$$

In particular, by these actions of the derivations and the polynomial multiplications, $\mathcal{D}'(U)$ has a natural structure of left D_n-module.

Example 4.2. Set $n = 1$. The Heaviside function $Y(x)$ is the measurable function on $\mathbb{R}$ such that $Y(x) = 1$ for $x > 0$ and $Y(x) = 0$ for $x < 0$. The Dirac delta function $\delta(x)$ is a distribution on $\mathbb{R}$ defined by

$$\langle \delta(x), \varphi \rangle = \varphi(0) \qquad (\forall \varphi \in C_0^\infty(\mathbb{R})).$$

The derivative $Y'(x)$ of $Y(x)$ as a distribution coincides with $\delta(x)$ since

$$\langle Y'(x), \varphi(x) \rangle = -\langle Y(x), \varphi'(x) \rangle = -\int_0^\infty \varphi'(x)\, dx = \varphi(0) = \langle \delta(x), \varphi \rangle$$

holds for any $\varphi \in C_0^\infty(\mathbb{R})$. The derivative $\delta'(x)$ of $\delta(x)$ is defined by

$$\langle \delta'(x), \varphi(x) \rangle = -\langle \delta(x), \varphi'(x) \rangle = -\varphi'(0).$$

In the same way, the k-th derivative $\delta^{(k)}(x) \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\langle \delta^{(k)}(x), \varphi(x) \rangle = (-1)^k \langle \delta(x), \varphi^{(k)}(x) \rangle = (-1)^k \varphi^{(k)}(0).$$

Example 4.3. The n-dimensional delta function $\delta(x)$ is the distribution defined by $\langle \delta(x), \varphi \rangle = \varphi(0)$ for any $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Let $u \in \mathcal{D}'(U)$ with an open set U of $\mathbb{R}^n$. Let V be an open subset of U. Then there exists a natural inclusion $C_0^\infty(V) \subset C_0^\infty(U)$. The restriction $v := u|_V$ of u to V is defined by

$$\langle v, \varphi \rangle = \langle u, \varphi \rangle \qquad (\forall \varphi \in C_0^\infty(V)).$$

Then $U \longmapsto \mathcal{D}'(U)$, where U are open sets of $\mathbb{R}^n$, constitutes a sheaf on $\mathbb{R}^n$. For $u \in \mathcal{D}'(U)$, the *support* $\operatorname{supp} u$ is defined to be the smallest closed set Z in U such that $u|_{U \setminus Z} = 0$, i.e., $\langle u, \varphi \rangle = 0$ for any $\varphi \in C_0^\infty(U \setminus Z)$. For example, with $x = x_1$ we have $\operatorname{supp} \delta(x) = \{0\}$ and $\operatorname{supp} Y(x) = \{x \in \mathbb{R} \mid x \geq 0\}$.

The set of the distributions on U whose supports are compact sets of U is denoted by $\mathcal{E}'(U)$. ($\mathcal{E}'(U)$ means the dual space of $\mathcal{E}(U) = C^\infty(U)$.)

Let u belong to $\mathcal{E}'(U)$ and $K := \operatorname{supp} u$ be its support. Then for an arbitrary $\varphi \in C^\infty(U)$, the pairing

$$\langle u, \varphi(x) \rangle = \langle u, \chi(x)\varphi(x) \rangle$$

is well-defined with a cut-off function $\chi \in C_0^\infty(U)$ such that $\chi(x) = 1$ on an open set $V \subset U$ such that $K \subset V$. This pairing does not depend on the choice of χ. In fact, assume $\tilde{\chi} \in C_0^\infty(U)$ satisfies the same condition. Then since

$$\operatorname{supp}(\chi - \tilde{\chi}) \cap \operatorname{supp} u = \emptyset,$$

$\langle u, \chi\varphi \rangle = \langle u, \tilde{\chi}\varphi \rangle$ holds for any $\varphi \in C^\infty(U)$.

Definition 4.4. A C^∞ function φ on $\mathbb{R}^n$ is called a *rapidly decreasing function* if $|P\varphi|$ is bounded on $\mathbb{R}^n$ for any differential operator $P \in D_n$. The set of the rapidly decreasing functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$; it contains $C_0^\infty(\mathbb{R}^n)$ as a subspace.

Definition 4.5. A *tempered distribution* u on $\mathbb{R}^n$ is a $\mathbb{C}$-linear mapping

$$u : \mathcal{S}(\mathbb{R}^n) \ni \varphi \longmapsto \langle u, \varphi \rangle \in \mathbb{C}$$

such that $\lim_{j\to\infty} \langle u, \varphi_j \rangle = 0$ holds for any sequence $\{\varphi_j\}$ of $\mathcal{S}(\mathbb{R}^n)$ which satisfies the condition

$$\lim_{j\to\infty} \sup_{x\in\mathbb{R}^n} |P\varphi_j(x)| = 0 \quad \text{for any } P \in D_n.$$

The sequence $\{\varphi_j\}$ with this condition is said to converge to 0 in $\mathcal{S}(\mathbb{R}^n)$, which makes $\mathcal{S}(\mathbb{R}^n)$ a topological vector space. The set of the tempered distributions on $\mathbb{R}^n$ is denoted by $\mathcal{S}'(\mathbb{R}^n)$, which can be regarded as a subspace of $\mathcal{D}'(\mathbb{R}^n)$ since $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$. Moreover, $\mathcal{E}'(\mathbb{R}^n)$ is a subspace of $\mathcal{S}'(\mathbb{R}^n)$. Any $P \in D_n$ defines a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$ and of $\mathcal{S}'(\mathbb{R}^n)$. Hence $\mathcal{S}'(\mathbb{R}^n)$ is a left D_n-submodule of $\mathcal{D}'(\mathbb{R}^n)$.

Let $f(x) = f(x_1, \dots, x_n)$ be a complex-valued locally integrable function on $\mathbb{R}^n$ of polynomial growth; i.e., assume that there exists a constant $C > 0$ and a non-negative integer m such that

$$|f(x)| \le C(1 + |x|^2)^m = C(1 + x_1^2 + \cdots + x_n^2)^m \qquad (\forall x \in \mathbb{R}^n).$$

Then $f(x)$ defines a tempered distribution through the pairing

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx \qquad (\forall \varphi \in \mathcal{S}(\mathbb{R}^n)).$$

Example 4.6. Let $f_1, \dots, f_p$ be polynomials in $x = (x_1, \dots, x_n)$ with real coefficients. Let $\lambda_1, \dots, \lambda_p$ be complex numbers such that $\mathrm{Re}\,\lambda_j \geq 0$ $(j = 1, \dots, p)$. Set

$$((f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p})(x) = \begin{cases} f_1(x)^{\lambda_1} \cdots f_p(x)^{\lambda_p} & \text{if } f_1(x) > 0, \dots, f_p(x) > 0 \\ 0 & \text{otherwise} \end{cases},$$

where we use the convention that $t^\lambda = \exp(\lambda \log t)$ with real $\log t$ for $t > 0$ and $\lambda \in \mathbb{C}$. Then it is easy to see that $(f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p}$ is locally integrable and of polynomial growth, thus can be regarded as a tempered distribution. In particular, $(f_1)_+^0 \cdots (f_p)_+^0$ coincides with $Y(f_1) \cdots Y(f_p)$.

Theorem 4.7 (Sato-Kawai-Kashiwara)**.** *Let M be a finitely generated left D_n-module. For a point a of $\mathbb{R}^n$, let us denote by $\mathcal{D}'_a$ the stalk of the sheaf $\mathcal{D}'$ at a, which is the inductive limit $\mathcal{D}'_a = \varinjlim \mathcal{D}'(U)$ where U are open neighborhoods of a.*

(1) *If M is holonomic, then $\mathrm{Hom}_{D_n}(M, \mathcal{D}'_a)$ is a finite dimensional vector space over $\mathbb{C}$ for any $a \in \mathbb{R}^n$.*

(2) *Let U be an open set of $\mathbb{R}^n$. Then any distribution solution of M is real analytic on $U' := U \setminus \mathrm{Sing}(M)$; i.e., the natural $\mathbb{C}$-linear map*

$$\mathrm{Hom}_{D_n}(M, \mathcal{A}(U')) \longrightarrow \mathrm{Hom}_{D_n}(M, \mathcal{D}'(U'))$$

is an isomorphism, where $\mathcal{A}(U')$ denotes the set of complex-valued real analytic functions on U'.

In fact, this theorem holds in a weaker assumption that M is an analytic D-module and with $\mathcal{D}'(U)$ being replaced by the set $\mathcal{B}(U)$ of hyperfunctions. Under this weaker assumption, the statement (1) is due to Kashiwara (see Theorem 5.1.7 of [14] for a more refined formulation). The statement (2) was first noticed by M. Sato with the introduction of the theory of microfunctions developed together with Kawai and Kashiwara ([31]).

Example 4.8. We have $\mathrm{Hom}_{D_n}(\mathbb{C}[x], \mathcal{D}'(\mathbb{R}^n)) \cong \mathbb{C}$. In fact, $\mathbb{C}[x] = D_n/(D_n\partial_1 + \cdots + D_n\partial_n)$ and we can prove that if $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfies $\partial_1 u = \cdots = \partial_n u = 0$, then u is a constant function (see Exercise 19). Since $\mathrm{Sing}(\mathbb{C}[x]) = \emptyset$, u is real analytic on $\mathbb{R}^n$.

Example 4.9. Set $M := D_n/(D_n x_1 + \cdots + D_n x_n)$ with $\mathbb{K} = \mathbb{C}$. Then $\mathrm{Hom}_{D_n}(M, \mathcal{D}'(\mathbb{R}^n))$ is one dimensional and spanned by the n-dimensional delta function $\delta(x)$. Since $\mathrm{Sing}(M) = \{0\}$, u is real analytic (zero in fact) on $\mathbb{R}^n \setminus \{0\}$.

Exercise 17. Let u be a distribution on $\mathbb{R}^n$ satisfying $x_1 u = \cdots = x_n u = 0$.

(1) Show that the support of u is contained in $\{0\}$ and hence u belongs to $\mathcal{E}'(\mathbb{R}^n)$.

(2) Prove that u is a constant multiple of the n-dimensional delta function $\delta(x)$. Use the fact that for $\varphi \in C^\infty(\mathbb{R}^n)$, there exist $\varphi_i \in C^\infty(\mathbb{R}^n)$ $(i = 1, \ldots, n)$ such that

$$\varphi(x) = \varphi(0) + x_1\varphi_1(x) + \cdots + x_n\varphi_n(x).$$

Exercise 18. Set $n = 1$ and write $x = x_1$, $\partial = \partial_1$. Define a $\mathbb{C}$-linear map $u : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ by

$$\langle u, \varphi \rangle = \lim_{\varepsilon \to +0} \int_\varepsilon^\infty \frac{\varphi(x) - \varphi(-x)}{x}\, dx \qquad (\varphi \in \mathcal{S}(\mathbb{R})).$$

(1) Show that u belongs to $\mathcal{S}'(\mathbb{R})$.

(2) Show that $xu = 1$, and hence $\partial x u = 0$ holds.

4.2. Product of distributions

The product of two distributions cannot be defined in general. There are some cases where the product is well-defined:

(1) Let U be an open set of $\mathbb{R}^n$ and V an open set of $\mathbb{R}^m$. For $u_1 \in \mathcal{D}'(U)$ and $u_2 \in \mathcal{D}'(V)$, their tensor product $u_1 \otimes u_2$, which is also denoted by $u_1(x)u_2(y)$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$, is defined as the unique distribution on $U \times V$ such that

$$\langle u_1 \otimes u_2,\, \varphi_1(x)\varphi_2(y) \rangle = \langle u_1,\, \varphi_1 \rangle \langle u_2,\, \varphi_2 \rangle$$

holds for any $\varphi_1 \in C_0^\infty(U)$ and $\varphi_2 \in C_0^\infty(V)$. Then

$$\langle u_1 \otimes u_2,\, \varphi(x, y) \rangle = \langle u_1,\, \langle u_2,\, \varphi(x, y) \rangle_y \rangle$$

holds for $\varphi(x, y) \in C_0^\infty(U \times V)$, where $\langle\rangle_y$ denotes the pairing of $\mathcal{D}'(V)$ and $C_0^\infty(V)$ with x fixed. See Chapter 4 of [32] for details. If $u_1 \in \mathcal{S}'(\mathbb{R}^n)$ and $u_2 \in \mathcal{S}'(\mathbb{R}^m)$, then $u_1 \otimes u_2$ belongs to $\mathcal{S}'(\mathbb{R}^{n+m})$.

(2) Let U be an open set of $\mathbb{R}^n$. For $u_1 \in C^\infty(U)$ and $u_2 \in \mathcal{D}'(U)$, the product $u = u_1 u_2$ is well-defined as an element of $\mathcal{D}'(U)$ and the Leibniz rule $\partial_i(u_1 u_2) = (\partial_i u_1)u_2 + u_1(\partial_1 u_2)$ holds for $i = 1, \ldots, n$. In fact, the pairing

$$\langle u_1 u_2, \varphi \rangle = \langle u_2, u_1 \varphi \rangle$$

is well-defined for $\varphi \in C_0^\infty(U)$, and we have

$$\begin{aligned}\langle \partial_i(u_1u_2),\, \varphi\rangle &= -\langle u_1u_2,\, \partial_i\varphi\rangle = -\langle u_2,\, u_1\partial_i\varphi\rangle \\ &= -\langle u_2,\, \partial_i(u_1\varphi) - (\partial_i u_1)\varphi\rangle \\ &= \langle \partial_i u_2,\, u_1\varphi\rangle + \langle u_2,\, (\partial_i u_1)\varphi\rangle \\ &= \langle u_1(\partial_i u_2) + (\partial_i u_1)u_2,\, \varphi\rangle.\end{aligned}$$

(3) Let u_1 belong to $\mathcal{S}(\mathbb{R}^n)$ and u_2 to $\mathcal{S}'(\mathbb{R}^n)$. Then u_1u_2 is well-defined as an element of $\mathcal{S}'(\mathbb{R}^n)$ and the Leibniz rule holds.

(4) Let u_1 and u_2 be measurable functions on U. Let p, q be positive real numbers or infinity such that $1/p + 1/q = 1$. If u_1 is locally L^p and u_2 is locally L^q (i.e., $|u_1|^p$ and $|u_2|^q$ are locally integrable if $p, q > 1$; the measure of the set $\{x \in U \mid |u_1(x)| > a\}$ is zero for some $a \in \mathbb{R}$ in case $p = \infty$), then the product $u = u_1u_2$ is well-defined as a locally integrable function. But the Leibniz rule does not make sense; in fact, the product $(\partial_1 u_1)u_2$ cannot be defined in general.

However, for example in one variable $x = x_1$, the product $\delta(x)^2$ or $Y(x)\delta(x)$ cannot be defined as distributions. If $u(x)$ is locally integrable, then $Y(x)u(x)$ is also a locally integrable function. But $\delta(x)u(x)$ cannot be defined in general. In particular, the Leibniz rule

$$\partial_x(Y(x)u(x)) = Y(x)u'(x) + \delta(x)u(x)$$

does not make sense in general because the products on the right-hand side cannot be defined unless u is C^∞ while the left-hand side is well-defined as distribution.

Proposition 4.10. *Let $f(x)$ be a real-valued C^∞ function on an open subset U of $\mathbb{R}^n$. If $u(t)$ belongs to $\mathcal{D}'(\mathbb{R})$ and $v(x)$ belongs to $\mathcal{D}'(U)$, then $u(t-f(x))v(x)$ is well-defined as an element of $\mathcal{D}'(\mathbb{R}\times U)$ and the Leibniz formulae*

$$\begin{aligned}&\frac{\partial}{\partial t}(u(t-f(x))v(x)) = u'(t-f(x))v(x),\\ &\frac{\partial}{\partial x_i}(u(t-f(x))v(x)) = u(t-f(x))\frac{\partial}{\partial x_i}v(x) - \frac{\partial f}{\partial x_i}u'(t-f(x))v(x)\end{aligned}$$

hold for $i = 1, \dots, n$ with $u'(t)$ being the derivative of $u(t)$. Moreover, if $f(x)$ is a polynomial, $u(t)$ belongs to $\mathcal{S}'(\mathbb{R})$ and $v(x)$ belongs to $\mathcal{S}'(\mathbb{R}^n)$, then $u(t-f(x))v(x)$ belongs to $\mathcal{S}'(\mathbb{R}^{n+1})$.

Proof. For $\varphi(t,x) \in C_0^\infty(\mathbb{R}\times U)$, set

$$\langle u(t-f(x))v(x),\, \varphi(t,x)\rangle = \langle u(t)v(x),\, \varphi(t+f(x),x)\rangle.$$

Since $\varphi(t+f(x),x)$ belongs to $C_0^\infty(\mathbb{R}\times U)$, this defines an element of $\mathcal{D}'(\mathbb{R}\times U)$. For $\varphi \in C_0^\infty(\mathbb{R}\times U)$, we have

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}(u(t-f(x))v(x)),\ \varphi(t,x)\right\rangle = -\left\langle u(t-f(x))v(x),\ \frac{\partial \varphi}{\partial t}(t,x)\right\rangle\\
&= -\left\langle u(t)v(x),\ \frac{\partial \varphi}{\partial t}(t+f(x),x)\right\rangle\\
&= \langle u'(t)v(x),\ \varphi(t+f(x),x)\rangle = \langle u'(t-f(x))v(x),\ \varphi(t,x)\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial x_i}(u(t-f(x))v(x)),\ \varphi(t,x)\right\rangle = -\left\langle u(t-f(x))v(x),\ \frac{\partial \varphi}{\partial x_i}(t,x)\right\rangle\\
&= -\left\langle u(t)v(x),\ \frac{\partial \varphi}{\partial x_i}(t+f(x),x)\right\rangle\\
&= -\left\langle u(t)v(x),\ \frac{\partial}{\partial x_i}\varphi(t+f(x),x) - \frac{\partial f}{\partial x_i}\frac{\partial \varphi}{\partial t}(t+f(x),x)\right\rangle\\
&= \langle u(t)\partial_{x_i}v(x),\ \varphi(t+f(x),x)\rangle - \left\langle u'(t)\frac{\partial f}{\partial x_i}v(x),\ \varphi(t+f(x),x)\right\rangle\\
&= \langle u(t-f(x))\partial_{x_i}v(x),\ \varphi(t,x)\rangle - \left\langle u'(t-f(x))\frac{\partial f}{\partial x_i}v(x),\ \varphi(t,x)\right\rangle.
\end{aligned}$$

If $\varphi(t,x)$ belongs to $\mathcal{S}(\mathbb{R}^{n+1})$ and $f(x)$ is a polynomial, then $\varphi(t+f(x),x)$ also belongs to $\mathcal{S}(\mathbb{R}^{n+1})$, as is seen by the inequality $t^2 \le 2((t+f(x))^2 + f(x)^2)$. This implies the last assertion. Q.E.D.

Example 4.11. Let $f(x)$ be a polynomial in $x=(x_1,\dots,x_n)$ with real coefficients and $v(x)$ be an element of $\mathcal{S}'(\mathbb{R}^n)$. Then $Y(t-f(x))v(x)$ and $\delta^{(k)}(t-f(x))v(x)$ with a non-negative integer k are well-defined as elements of $\mathcal{S}'(\mathbb{R}^{n+1})$ and we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(Y(t-f(x))v(x)) = \delta(t-f(x))v(x),\\
&\frac{\partial}{\partial t}(\delta^{(k)}(t-f(x))v(x)) = \delta^{(k+1)}(t-f(x))v(x).
\end{aligned}$$

Exercise 19. Let u be a distribution on $\mathbb{R}^n$ satisfying $\partial_1 u = 0$.

(1) Show that $\langle u,\varphi\rangle = 0$ if $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\int_{-\infty}^{\infty}\varphi(t,x_2,\dots,x_n)\,dt = 0.$$

(2) Show that there exists a distribution v on $\mathbb{R}^{n-1}$ such that $u = 1\otimes v$.

4.3. Integrals of distributions

Let us consider distributions in variables (x, y) with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_d)$. We regard x as the integration variables and y as parameters. Let $\varpi : \mathbb{R}^{n+d} \ni (x, y) \mapsto y \in \mathbb{R}^d$ be the projection. Let U be an open set of $\mathbb{R}^d$ and let u be a distribution defined on $\varpi^{-1}(U) = \mathbb{R}^n \times U$.

We would like to define the integral

$$\int_{\mathbb{R}^n} u(x, y)\, dx = \int_{\mathbb{R}^n} u(x, y)\, dx_1 \cdots dx_n$$

along the fibers of ϖ (i.e., with respect to x) as a distribution on U. However, we need some 'tameness' of u with respect to x for this integral to be well-defined. Let us introduce the following two sufficient conditions:

(1) Let u be a distribution on $\varpi^{-1}(U)$ such that $\varpi : \operatorname{supp} u \to \mathbb{R}^d$ is proper, i.e., for any compact set K of U, $\varpi^{-1}(K) \cap \operatorname{supp} u$ is compact.

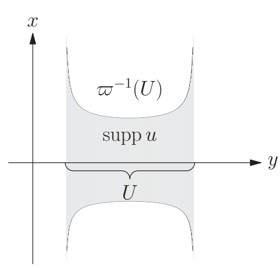

Fig. 1. An example of the support of $u \in \mathcal{E}'\mathcal{D}'(\mathbb{R}_x \times U)$

Let us denote by $\mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)$ the set of such distributions, which constitutes a left D_{n+d}-submodule of $\mathcal{D}'(\mathbb{R}^n \times U)$. The integral of $u \in \mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)$ with respect to x is defined by

$$\left\langle \int_{\mathbb{R}^n} u(x, y)\, dx,\ \varphi(y) \right\rangle = \langle u(x, y), 1(x)\varphi(y) \rangle \qquad (\forall \varphi(y) \in C_0^\infty(U)),$$

where $1(x)$ denotes the constant function with value 1. This integral belongs to $\mathcal{D}'(U)$. More precisely, the pairing above is defined as follows:

Choose $\chi(x, y) \in C^\infty(\varpi^{-1}(U))$ such that $\chi(x, y) = 1$ on an open set W of $\varpi^{-1}(U)$ which contains $\operatorname{supp} u$ and that $\varpi : \operatorname{supp} \chi \to U$ is proper, by using a partition of unity. Then we define

$$\langle u(x, y),\ 1(x)\varphi(y) \rangle := \langle u(x, y),\ \chi(x, y)\varphi(y) \rangle.$$

The right-hand side does not depend on such $\chi(x,y)$ since $\operatorname{supp}(1-\chi)\cap \operatorname{supp} u = \emptyset$.

(2) Let $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ be the subspace of $\mathcal{S}'(\mathbb{R}^{n+d})$ consisting of distributions of the form

(11)
$$u(x,y) = \sum_{j=1}^{m} u_j(x)v_j(x,y) \qquad (m \in \mathbb{N},\ u_j \in \mathcal{S}(\mathbb{R}^n),\ v_j \in \mathcal{S}'(\mathbb{R}^{n+d})).$$

We also denote $\mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}^d_y)$ to clarify the variables. Then $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ is a left D_{n+d}-submodule of $\mathcal{S}'(\mathbb{R}^{n+d})$. The integral of $u(x,y)$ is naturally defined as an element of $S'(\mathbb{R}^d)$ by the pairing

$$\left\langle \int_{\mathbb{R}^n} u(x,y)\,dx,\ \varphi(y) \right\rangle = \sum_{j=1}^{m} \langle v_j(x,y),\ u_j(x)\varphi(y)\rangle \qquad (\forall \varphi \in \mathcal{S}(\mathbb{R}^d)).$$

This integral does not depend on the choice of expression (11). In fact, assume $u(x,y)=0$ in (11) and take $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x)=1$ if $|x| \le 1$. Then for an arbitrary constant $r>0$, we have an equality

$$0 = \left\langle \sum_{j=1}^{m} u_j(x)v_j(x,y),\ \chi\Big(\frac{x}{r}\Big)\varphi(y) \right\rangle = \sum_{j=1}^{m} \left\langle v_j(x,y),\ \chi\Big(\frac{x}{r}\Big)u_j(x)\varphi(y) \right\rangle$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since $\chi(x/r)u_j(x)\varphi(y)$ converges to $u_j(x)\varphi(y)$ in $\mathcal{S}(\mathbb{R}^{n+d})$ as $r \to \infty$, we get

$$\sum_{j=1}^{m} \langle v_j(x,y),\ u_j(x)\varphi(y)\rangle = 0.$$

Proposition 4.12 (differentiation under the integral sign)**.** *Let $u(x,y)$ belong to $\mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)$ with an open subset U of $\mathbb{R}^d$, or else to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then*

$$P(y,\partial_y)\int_{\mathbb{R}^n} u(x,y)\,dx = \int_{\mathbb{R}^n} P(y,\partial_y)u(x,y)\,dx$$

holds for any $P = P(y,\partial_y) \in D_d$.

Proof. Let $u(x,y) = \sum_{j=1}^m u_j(x)v_j(x,y)$ with $u_j \in \mathcal{S}(\mathbb{R}^n)$ and $v_j \in \mathcal{S}'(\mathbb{R}^{n+d})$. Then for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\left\langle \partial_{y_i} \int_{\mathbb{R}^n} u(x,y)\,dx,\ \varphi(y) \right\rangle = -\left\langle \int_{\mathbb{R}^n} u(x,y)\,dx,\ \partial_{y_i}\varphi(y) \right\rangle$$
$$= -\sum_{j=1}^m \langle v_j(x,y),\ u_j(x)\partial_{y_i}\varphi(y)\rangle = -\sum_{j=1}^m \langle v_j(x,y),\ \partial_{y_i}(u_j(x)\varphi(y))\rangle$$
$$= \sum_{j=1}^m \langle \partial_{y_i} v_j(x,y),\ u_j(x)\varphi(y)\rangle = \left\langle \int_{\mathbb{R}^n} \partial_{y_i} u(x,y)\,dx,\ \varphi(y) \right\rangle$$

and

$$\left\langle y_i \int_{\mathbb{R}^n} u(x,y)\,dx,\ \varphi(y) \right\rangle = \sum_{j=1}^m \langle v_j(x,y),\ y_i u_j(x)\varphi(y)\rangle$$
$$= \sum_{j=1}^m \langle y_i v_j(x,y),\ u_j(x)\varphi(y)\rangle = \left\langle \int_{\mathbb{R}^n} y_i u(x,y)\,dx,\ \varphi(y) \right\rangle.$$

The case $u \in \mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)$ can be proved in the same manner. Q.E.D.

Example 4.13. Let $\delta(t)$ be the univariate delta function. We have $\int_{\mathbb{R}} \delta(t)\,dt = 1$ since

$$\int_{\mathbb{R}} \delta(t) = \langle \delta(t), 1(t)\rangle = 1.$$

More generally, let $f(t,x) = f(t,x_1,\dots,x_n)$ be a C^∞ function on $\mathbb{R}\times U$ with an open set U of $\mathbb{R}^n$. Since $\operatorname{supp} f(t,x)\delta(t) \subset \{(t,x) \mid t = 0\}$, $f(t,x)\delta(t)$ belongs to $\mathcal{E}'\mathcal{D}'(\mathbb{R}\times U)$. Note that $f(t,x)\delta(t) = f(0,x)\delta(t)$ holds since there exists $g \in C^\infty(\mathbb{R}\times U)$ such that $f(t,x) - f(0,x) = tg(t,x)$. Hence we get

$$\int_{\mathbb{R}} f(t,x)\delta(t)\,dt = \int_{\mathbb{R}} f(0,x)\delta(t)\,dt = f(0,x)\int_{\mathbb{R}} \delta(t)\,dt = f(0,x).$$

Example 4.14. Set $x = (x_1,\dots,x_n)$ and let a be an arbitrary positive constant. Let $f(x)$ be a real polynomial in x. Then $\exp(-a|x|^2)Y(t-f(x))$ belongs to $\mathcal{S}\mathcal{S}'(\mathbb{R}^n_x \times \mathbb{R}_t)$. Hence the integral

$$F(t) = \int_{\mathbb{R}^n} \exp(-a|x|^2)Y(t-f(x))\,dx$$

is well-defined as an element of $\mathcal{S}'(\mathbb{R}_t)$. Up to a constant multiple, $F(t)$ is the cumulative distribution function of $f(x)$ with x being the random vector with an n-dimensional normal (Gaussian) distribution. By Proposition 4.12 the derivative $F'(t)$ is given by the integral

$$F'(t) = \int_{\mathbb{R}^n} \exp(-a|x|^2)\delta(t - f(x))\, dx$$

as an element of $\mathcal{S}'(\mathbb{R})$.

Example 4.15. Let $f(x)$ be a real polynomial in $x = (x_1, \ldots, x_n)$, and $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers. Let us consider the integral

$$F(t) = \int_{\mathbb{R}^n} e^{-b_1x_1-\cdots-b_nx_n} Y(t - f(x))(x_1)_+^{a_1-1} \cdots (x_n)_+^{a_n-1}\, dx,$$

which can be regarded, up to a constant multiple depending on a_i, b_i, as the cumulative distribution function of $f(x)$ with x being the random vector with a multi-dimensional gamma distribution. Let $\chi(t)$ be a C^∞ function on $\mathbb{R}$ such that $\chi(t) = 1$ for $t \geq -1$ and $\chi(t) = 0$ for $t \leq -2$. Then we have

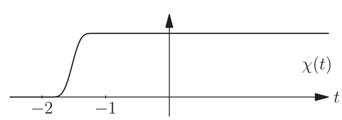

Fig. 2. The graph of $\chi(t)$ in Example 4.15

$$e^{-b_1x_1-\cdots-b_nx_n}(x_1)_+^{a_1-1} \cdots (x_n)_+^{a_n-1}$$
$$= e^{-b_1x_1-\cdots-b_nx_n}\chi(x_1)\cdots\chi(x_n)(x_1)_+^{a_1-1} \cdots (x_n)_+^{a_n-1}$$

and $e^{-b_1x_1-\cdots-b_nx_n}\chi(x_1)\cdots\chi(x_n)$ belongs to $\mathcal{S}(\mathbb{R}^n_x)$. Hence the integrand belongs to $\mathcal{S}\mathcal{S}'(\mathbb{R}^n_x \times \mathbb{R}_t)$ and consequently $F(t)$ is well-defined as an element of $\mathcal{S}'(\mathbb{R})$. Its derivative is given by

$$F'(t) = \int_{\mathbb{R}^n} e^{-b_1x_1-\cdots-b_nx_n}\delta(t - f(x))(x_1)_+^{a_1-1} \cdots (x_n)_+^{a_n-1}\, dx.$$

Example 4.16. Set $x = (x_1, \ldots, x_n)$ and let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers. For a real polynomial $f(x)$, set

$$F(t)$$
$$= \int_{\mathbb{R}^n} (x_1)_+^{a_1-1} \cdots (x_n)_+^{a_n-1}(1-x_1)_+^{b_1-1} \cdots (1-x_n)_+^{b_n-1} Y(t-f(x))\, dx.$$

The integrand is integrable and belongs to $\mathcal{E}'\mathcal{D}'(\mathbb{R}^n_x \times \mathbb{R}_t)$ since its support is contained in the n-cube $[0,1] \times \cdots \times [0,1]$. Hence $F(t)$ and its derivatives are well-defined as elements of $\mathcal{D}'(\mathbb{R})$. (In fact, $F(t)$ belongs to $\mathcal{S}'(\mathbb{R})$ since the integrand belongs also to $\mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}_t)$ multiplied by a suitable cut-off function.) Up to a constant multiple, $F(t)$ is the cumulative distribution function of $f(x)$ with x regarded as the random vector with a multivariate beta distribution.

The following proposition will play a crucial role in the integration algorithm for holonomic distributions, which will be introduced in the following sections.

Proposition 4.17. *Let $u(x,y)$ belong to $\mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)$ with an open subset U of $\mathbb{R}^d$, or else to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then one has*

$$\int_{\mathbb{R}^n} \partial_{x_i} u(x,y)\, dx = 0 \qquad (i = 1, \dots, n).$$

Proof. First, let us assume that $u(x,y)$ belongs to $\mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)$. Let $\chi(x,y)$ be an element of $C^\infty(\mathbb{R}^n \times U)$ which takes the value 1 on an open subset of $\mathbb{R}^n \times U$ containing $\operatorname{supp} u$ such that the projection map $\operatorname{supp} \chi \to U$ is proper. Then we have, by the definition of the integral,

$$\begin{aligned}\left\langle \int_{\mathbb{R}^n} \partial_{x_i} u(x,y)\, dx,\ \varphi(y) \right\rangle &= \langle \partial_{x_i} u(x,y),\ \chi(x,y)\varphi(y)\rangle \\ &= -\langle u(x,y),\ \partial_{x_i}\chi(x,y)\varphi(y)\rangle = 0\end{aligned}$$

for any $\varphi \in C_0^\infty(U)$ since $\partial_{x_i}\chi(x,y)$ vanishes on an open set containing $\operatorname{supp} u$.

Next, let us assume that $u(x,y)$ belongs to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. We may assume, without loss of generality, that $u(x,y) = v(x)w(x,y)$ with $v \in \mathcal{S}(\mathbb{R}^n)$ and $w \in \mathcal{S}'(\mathbb{R}^{n+d})$. Then it follows from the definition of the integral that

$$\begin{aligned}&\left\langle \int_{\mathbb{R}^n} \partial_{x_i}(v(x)w(x,y))\, dx,\ \varphi(y) \right\rangle \\ &= \left\langle \int_{\mathbb{R}^n} (\partial_{x_i} v(x)) w(x,y)\, dx,\ \varphi(y) \right\rangle + \left\langle \int_{\mathbb{R}^n} v(x)(\partial_{x_i} w(x,y))\, dx,\ \varphi(y) \right\rangle \\ &= \langle w(x,y),\, \partial_{x_i} v(x)\varphi(y)\rangle + \langle \partial_{x_i} w(x,y),\, v(x)\varphi(y)\rangle \\ &= \langle w(x,y),\, \partial_{x_i} v(x)\varphi(y)\rangle - \langle w(x,y),\, \partial_{x_i}(v(x)\varphi(y))\rangle = 0\end{aligned}$$

holds for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Q.E.D.

4.4. Holonomic distributions

We assume $\mathbb{K} = \mathbb{C}$. A distribution $u(x) \in \mathcal{D}(U)$, with an open set U of $\mathbb{R}^n$, is called *holonomic* if there exist a finite set of differential operators $P_1, \dots, P_m$ which annihilate u, i.e., $P_i u = 0$ holds in $\mathcal{D}'(U)$ for $i = 1, \dots, m$ such that the left D_n-module $D_n/(D_n P_1 + \cdots + D_n P_m)$ is holonomic. In other words, $u(x)$ is holonomic if and only if $D_n/\mathrm{Ann}_{D_n} u$ is a holonomic D_n-module, where

$$\mathrm{Ann}_{D_n} u = \{P \in D_n \mid Pu = 0\}$$

is the annihilator (ideal) of u.

For example, the univariate delta function $\delta(x)$ and the Heaviside function $Y(x)$ are holonomic since $x\delta(x) = x\partial_x Y(x) = 0$.

Proposition 4.18. *If elements u and v of $\mathcal{D}'(U)$ are holonomic, then $C_1 u + C_2 v$ and Pu are also holonomic for any $C_1, C_2 \in \mathbb{C}$ and $P \in D_n$.*

Proof. Set $I = \mathrm{Ann}_{D_n} u$ and $J = \mathrm{Ann}_{D_n} v$. Then D_n/I and D_n/J are holonomic. Since the annihilator of $C_1 u + C_2 v$ contains $I \cap J$, we have only to show that $D_n/(I \cap J)$ is holonomic. The left D_n-homomorphism of D_n to $(D_n)^2$ which sends $Q \in D_n$ to $(Q, -Q)$ induces an injective homomorphism

$$D_n/(I \cap J) \longrightarrow (D_n/I) \oplus (D_n/J).$$

This implies that $D_n/(I \cap J)$ is holonomic since $(D_n/I) \oplus (D_n/J)$ is holonomic.

The left ideal $I : P = \{Q \in D_n \mid QP \in I\}$ coincides with $\mathrm{Ann}_{D_n} Pu$. The left D_n-endomorphism of D_n which sends $Q \in D_n$ to QP induces an injective homomorphism $D_n/(I : P) \to D_n/I$. Hence $D_n/(I : P)$ is holonomic. Q.E.D.

Definition 4.19. Let $f(x)$ be a real polynomial in $x = (x_1, \dots, x_n)$. Set $\partial_x = (\partial_1, \dots, \partial_n)$ with $\partial_i = \partial/\partial x_i$. Let t be a single variable and set $\partial_t = \partial/\partial t$. For $P = P(x, \partial_x) \in D_n$, define $\tau(P, f) \in D_{n+1}$ by

$$\tau(P, f) = P(x, \partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t)$$

with $f_i = \partial f/\partial x_i$. This substitution is well-defined since $x_1, \dots, x_n$ and $\partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t$ satisfy the same commutation relations as $x_1, \dots, x_n$ and $\partial_1, \dots, \partial_n$.

Proposition 4.20. *Let $f(x)$ be a real polynomial in $x = (x_1, \dots, x_n)$ and suppose that $v \in \mathcal{D}'(U)$ with an open set U of $\mathbb{R}^n$ is holonomic. Then $Y(t - f(x))v(x)$ is holonomic. More concretely, let I be a left ideal of*

D_n which is contained in $\mathrm{Ann}_{D_n} v$ such that D_n/I is holonomic. Let J be the left ideal of D_{n+1} which is generated by $\{\tau(P,f) \mid P \in I\}$. Then left ideals

$$J_0 := J + D_{n+1}(t - f(x))\partial_t, \qquad J_1 := J + D_{n+1}(t - f(x))$$

of D_{n+1} annihilate $Y(t-f(x))v(x)$ and $\delta(t-f(x))v(x)$ respectively and both D_{n+1}/J_0 and D_{n+1}/J_1 are holonomic.

Proof. We have

$$(\partial_i + f_i\partial_t)(Y(t-f(x))v(x)) = Y(t-f(x))\partial_i v(x),$$
$$(\partial_i + f_i\partial_t)(\delta(t-f(x))v(x)) = \delta(t-f(x))\partial_i v(x)$$

for $i = 1, \dots, n$ by Proposition 4.10. Hence

$$\tau(P,f)(Y(t-f(x))v(x)) = Y(t-f(x))Pv(x),$$
$$\tau(P,f)(\delta(t-f(x))v(x)) = \delta(t-f(x))Pv(x)$$

hold for any $P \in D_n$. It follows that J_0 and J_1 annihilate $Y(t-f(x))v(x)$ and $\delta(t-f(x))v(x)$ respectively.

Let us show that D_{n+1}/J_0 is holonomic. Since D_n/I is holonomic, its characteristic variety $\mathrm{Char}(D_n/I)$ is an n-dimensional algebraic set of $\mathbb{C}^{2n}$. By the definition, we have

$$\begin{aligned}
&\mathrm{Char}(D_{n+1}/J_0)\\
&\subset \Big\{(x,t,\xi,\tau) \in \mathbb{C}^{2(n+1)} \mid \sigma(P)(x, \xi_1 + f_1\tau, \dots, \xi_n + f_n\tau) = 0\\
&\qquad (\forall P \in I),\ (t - f(x))\tau = 0\Big\}\\
&= \Big\{(x,t,\xi,\tau) \mid (x, \xi_1 + f_1\tau, \dots, \xi_n + f_n\tau) \in \mathrm{Char}(D_n/I),\ t = f(x)\Big\}\\
&\quad \cup \Big\{(x,t,\xi,\tau) \mid (x, \xi_1, \dots, \xi_n) \in \mathrm{Char}(D_n/I),\ \tau = 0\Big\}.
\end{aligned}$$

Since the last two sets are in one-to-one correspondence with the set $\mathrm{Char}(D_n/I) \times \mathbb{C}$, the dimension of $\mathrm{Char}(D_{n+1}/J)$ is $n+1$, which implies that D_{n+1}/J_0 is a holonomic module. Similarly, D_{n+1}/J_1 is also holonomic. Q.E.D.

Example 4.21. Let $f(x)$ be a real polynomial in $x = (x_1, \dots, x_n)$ and $a_1, \dots, a_n$ be positive real numbers. Set

$$u(x,t) = \exp(-a_1x_1^2 - \cdots - a_nx_n^2)\delta(t - f(x)).$$

Then $u = u(x,t)$ satisfies a holonomic system

$$(t - f(x))u = (\partial_1 + f_1\partial_t + 2a_1x_1)u = \cdots = (\partial_n + f_n\partial_t + 2a_nx_n)u = 0.$$

Lemma 4.22. *Let a be a positive real number. Then the univariate locally integrable function t_+^{a-1} in t satisfies $(t\partial_t - a + 1)t_+^{a-1}$ in $\mathcal{S}'(\mathbb{R})$.*

Proof. Let $\varphi(t)$ belong to $\mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned}\langle t\partial_t t_+^{a-1}, \varphi(t)\rangle &= -\langle t_+^{a-1}, \partial_t(t\varphi(t))\rangle = -\langle t_+^{a-1}, \varphi(t)\rangle - \langle t_+^{a-1}, t\varphi'(t)\rangle \\ &= -\int_0^\infty t^{a-1}\varphi(t)\,dt - \int_0^\infty t^a \varphi'(t)\,dt \\ &= -\int_0^\infty t^{a-1}\varphi(t)\,dt + a\int_0^\infty t^{a-1}\varphi(t)\,dt \\ &= \langle (a-1)t_+^{a-1}, \varphi(t)\rangle\end{aligned}$$

by integration by parts. Q.E.D.

Example 4.23. Let $f(x)$ be a real polynomial in $x = (x_1, \dots, x_n)$ and $a_1, \dots, a_n, b_1, \dots, b_n$ be positive real numbers. Set

$$u(x,t) = \exp(-b_1x_1 - \cdots - b_nx_n)\delta(t - f(x))(x_1)_+^{a_1-1}\cdots(x_n)_+^{a_n-1}.$$

Then $u = u(x,t)$ satisfies a holonomic system

$$(t - f(x))u = (x_i(\partial_i + f_i\partial_t + b_i) - a_i + 1)u = 0 \quad (i = 1, \dots, n).$$

Exercise 20. Set $n = 1$, $x = x_1$, $\partial = \partial_1$ and $M := D_1/D_1x\partial$. Show that $\mathrm{Hom}_{D_1}(M, \mathcal{D}'(\mathbb{R}))$ is two dimensional and spanned by $Y(x)$ and $Y(-x)$.

Exercise 21. Let $\lambda_1, \dots, \lambda_n$ be complex numbers such that $\mathrm{Re}\,\lambda_i > -1$ for $i = 1, \dots, n$. Set $f(x) = (x_1)_+^{\lambda_1}\cdots(x_n)_+^{\lambda_n}$.

(1) Show that $f(x)$ is locally integrable on $\mathbb{R}^n$ and belongs to $\mathcal{S}'(\mathbb{R}^n)$.

(2) Show that $f(x)$ satisfies linear differential equations

$$(x_1\partial_1 - \lambda_1)f(x) = \cdots = (x_n\partial_n - \lambda_n)f(x) = 0.$$

(3) Find the characteristic variety and the singular locus of the left D_n-module

$$M := D_n/(D_n(x_1\partial_1 - \lambda_1) + \cdots + D_n(x_n\partial_n - \lambda_n)).$$

4.5. Distributions with smooth parameters

Let Ω be an open set of $\mathbb{R}^p$. Introducing parameters $a = (a_1, \dots, a_p)$, let us define the space $\mathcal{ES}(\Omega \times \mathbb{R}^n)$ of rapidly decreasing functions with smooth parameters as the set of $\varphi(a,x) \in C^\infty(\Omega \times \mathbb{R}^n_x)$ such that

$$\sup_{K\times\mathbb{R}^n} |x^\alpha \partial_x^\beta \partial_a^\gamma \varphi(a,x)| < \infty$$

for any $\alpha, \beta \in \mathbb{N}^n$, $\gamma \in \mathbb{N}^p$, and any compact subset K of Ω. It is easy to see that $\mathcal{ES}(\Omega \times \mathbb{R}^n)$ is a left D_{p+n}-submodule of $C^\infty(\Omega \times \mathbb{R}^n)$. For example, e^{-ax^2} belongs to $\mathcal{ES}(\Omega \times \mathbb{R})$ with $\Omega = \{a \in \mathbb{R} \mid a > 0\}$.

Let $\mathcal{ESS}'(\Omega \times \mathbb{R}^n \times \mathbb{R}^d)$ be the set of $u \in \mathcal{D}'(\Omega \times \mathbb{R}^n_x \times \mathbb{R}^d_y)$ which can be written as

$$u(a,x,y) = \sum_{j=1}^{m} u_j(a,x)v_j(x,y)$$

with $u_j \in \mathcal{ES}(\Omega \times \mathbb{R}^n)$, $v_j \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$, and $m \in \mathbb{N}$. Then the integral $\int_{\mathbb{R}^n} u(a,x,y)\,dx$ of u with respect to x is defined by

$$\left\langle \int_{\mathbb{R}^n} u(a,x,y)\,dx,\ \varphi(a,y) \right\rangle = \sum_{j=1}^{m} \int_{\Omega} \langle v_j(x,y),\, u_j(a,x)\varphi(a,y)\rangle_{(x,y)}\,da$$

as an element of $\mathcal{D}'(\Omega \times \mathbb{R}^d)$ for $\varphi \in C_0^\infty(\Omega \times \mathbb{R}^d)$, where $\langle , \rangle_{(x,y)}$ denotes the pairing of $\mathcal{D}'(\mathbb{R}^n_x \times \mathbb{R}^d_y)$ and $C_0^\infty(\mathbb{R}^n_x \times \mathbb{R}^d_y)$ with a fixed, which is C^∞ with respect to a. The well-definedness of the integal can be proved by using a suitable cut-off function in the same way as in the case without parameters. For example, $e^{-ax^2}\delta(y - x^3)$ belongs to $\mathcal{ESS}'(\Omega \times \mathbb{R}_x \times \mathbb{R}_y)$ with $\Omega = \{a \in \mathbb{R} \mid a > 0\}$.

We can also consider the case $d = 0$, which we denote $\mathcal{ESS}'(\Omega \times \mathbb{R}^n)$. For example, $e^{ax-x^3}Y(x) = e^{ax-x^3}\chi(x)Y(x)$ belongs to $\mathcal{ESS}'(\mathbb{R}_a \times \mathbb{R}_x)$, where $\chi(x)$ is the same cut-off function as in Example 4.15. If $u(a,x)$ belongs to $\mathcal{ESS}'(\Omega \times \mathbb{R}^n)$, then $\int_{\mathbb{R}^n} u(a,x)\,dx$ belongs to $C^\infty(\Omega)$.

Proposition 4.24. *Let Ω be an open set of $\mathbb{R}^p_a$ and let u belong to $\mathcal{ESS}'(\Omega \times \mathbb{R}^n_x \times \mathbb{R}^d_y)$.*

(1) $P \int_{\mathbb{R}^n} u(a,x,y)\,dx = \int_{\mathbb{R}^n} Pu(a,x,y)\,dx$ *holds for any differential operator $P \in D_{p+d}$ in the variables (a,y).*

(2) $\int_{\mathbb{R}^n} \partial_{x_i} u(a,x,y)\,dx = 0$ *holds for any $i = 1, \dots, n$.*

The proof is similar to the case without parameters.

Example 4.25. Set $x = (x_1, \dots, x_n)$ and let $a > 0$ and $b = (b_1, \dots, b_n) \in \mathbb{R}^n$. Let $f(x)$ be a real polynomial in x and set

$$u(x,t,a,b) = \left(\frac{a}{\pi}\right)^{\frac{n}{2}} \exp(-a(x-b)^2)Y(t - f(x))$$

with $(x-b)^2 = \sum_{j=1}^{n}(x_j - b_j)^2$. Then $u(x,t,a,b)$ belongs to $\mathcal{ESS}'(\Omega \times \mathbb{R}^n_x \times \mathbb{R}_t)$ with $\Omega = \{(a,b) \in \mathbb{R} \times \mathbb{R}^n \mid a > 0\}$. Moreover, $u(x,t,a,b)$

satisfies a holonomic system in the whole variables (x, t, a, b). Hence the integral

$$F(t, a, b) = \int_{\mathbb{R}^n} u(x, t, a, b)\, dx$$

is well-defined and holonomic as an element of $\mathcal{D}'(\mathbb{R}_t \times \Omega)$.

Example 4.26. Set $x = (x_1, \ldots, x_n)$ and let a_{ij} $(1 \le i, j \le n)$ and b_j $(1 \le j \le n)$ be real parameters such that $a_{ij} = a_{ji}$. Set $A = (a_{ij})$ and $b = (b_1, \ldots, b_n)$. Let $f(x)$ be a real polynomial in x. Then

$$u(x, t, A, b) = \exp\left(\sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i\right) Y(t - f(x))$$

belongs to $\mathcal{ESS}'(\Omega \times \mathbb{R}^n_x \times \mathbb{R}_t)$, where Ω is the set of (A, b) with $b \in \mathbb{R}^n$ and a negative definite n by n symmetric matrix $A = (a_{ij})$, i.e.,

$$(-1)^k \det \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} > 0 \qquad (1 \le k \le n).$$

The integral

$$F(t, A, b) = \int_{\mathbb{R}^n} u(x, t, A, b)\, dx$$

is well-defined and holonomic as an element of $\mathcal{D}'(\mathbb{R}_t \times \Omega)$ since the integrand $u(x, t, A, b)$ satisfies a holonomic system including the parameters.

§5. D-module theoretic integration algorithm

We first recall the notion of integration of D-modules, which is purely algebraic. The most crucial fact is that the integration preserves holonomicity. Then we recall an algorithm for precisely computing the D-module theoretic integration, which was first introduced in [26] and [27]. See also Chapter 5 of [29].

5.1. Integration as an operation on D-modules

Set $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_d)$. In this section we set $X = \mathbb{K}^{n+d}$ and $Y = \mathbb{K}^d$ to simplify the notation. Let $\varpi : X \ni (x, y) \longmapsto y \in Y$ be the projection. We denote by $D_X = D_{n+d}$ the ring of differential operators in the variables (x, y), and by $D_Y = D_d$ the ring of differential operators in the variables y.

The module

$$D_{Y \leftarrow X} := D_X / (\partial_{x_1} D_X + \cdots + \partial_{x_n} D_X)$$

has a structure of (D_Y, D_X)-bimodule. The *integral* of a left D_X-module M along the fibers of ϖ, or the *direct image* by ϖ is defined to be

$$\varpi_* M := D_{Y \leftarrow X} \otimes_{D_X} M = M/(\partial_{x_1} M + \cdots + \partial_{x_n} M).$$

This is a left D_Y-module since any element of D_Y commutes with ∂_{x_j}. For an element u of M, let $[u]$ be its residue class in $\varpi_* M$. If M is generated by $u_1, \ldots, u_r$ over D_X, then $\varpi_* M$ is generated by the set $\{x^\alpha [u_j] \mid 1 \leq j \leq r,\ \alpha \in \mathbb{N}^n\}$ over D_Y.

Now assume $\mathbb{K} = \mathbb{C}$ and let φ be a D_X-homomorphism from M to $\mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)$ with an open set $U \subset \mathbb{R}^d$, or else to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Let us define a $\mathbb{C}$-linear map φ' from M to $\mathcal{D}'(U)$ or to $\mathcal{S}'(\mathbb{R}^d)$ by

$$\varphi'(u) = \int_{\mathbb{R}^n} \varphi(u)\, dx \qquad (\forall u \in M),$$

which is D_Y-linear by Proposition 4.12. Moreover, Proposition 4.17 implies

$$\partial_{x_1} M + \cdots + \partial_{x_n} M \subset \operatorname{Ker} \varphi'.$$

Hence φ' induces a D_Y-homomorphism

$$\varpi_*(\varphi) : \varpi_* M \longrightarrow \mathcal{D}'(U) \quad \text{or} \quad \varpi_*(\varphi) : \varpi_* M \longrightarrow \mathcal{S}'(\mathbb{R}^d).$$

The generators $x^\alpha [u_j]$ of $\varpi_* M$ with $1 \leq j \leq r$ and $\alpha \in \mathbb{N}^n$ are sent by $\varpi_*(\varphi)$ to

$$\varpi_*(\varphi)(x^\alpha [u_j]) = \int_{\mathbb{R}^n} x^\alpha \varphi(u_j)\, dx.$$

In conclusion, we have defined $\mathbb{C}$-linear maps

$$\varpi_* \ :\ \operatorname{Hom}_{D_X}(M, \mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)) \longrightarrow \operatorname{Hom}_{D_Y}(\varpi_* M, \mathcal{D}'(U)),$$
$$\varpi_* \ :\ \operatorname{Hom}_{D_X}(M, \mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)) \longrightarrow \operatorname{Hom}_{D_Y}(\varpi_* M, \mathcal{S}'(\mathbb{R}^d)).$$

Theorem 5.1 (Bernstein, Kashiwara)**.** *If M is a holonomic D_X-module, then $\varpi_* M$ is a holonomic D_Y-module.*

Proof. We follow the argument in Chapter 1 of [4]. By induction on n, we have only to prove the holonomicity of $\varpi_* M$ in case $n = 1$. We may assume $M \neq 0$. First assume $\partial_{x_1} : M \to M$ is injective. Let $\{F_k(M)\}$ be a good $(\mathbf{1}; \mathbf{1})$-filtration on M. There exists a polynomial $H(k)$ of degree $d + 1$ such that $\dim_{\mathbb{K}} F_k(M) = H(k)$ for any $k \gg 0$. Set

$$F_k(\varpi_* M) := F_k(M)/(F_k(M) \cap \partial_{x_1} M) \qquad (k \in \mathbb{Z}).$$

Since $F_k(M) \cap \partial_{x_1} M$ contains $\partial_{x_1} F_{k-1}(M)$ and $\partial_{x_1} : M \to M$ is injective, we have

$$\begin{aligned}\dim_{\mathbb{K}} F_k(\varpi_* M) &= \dim_{\mathbb{K}} F_k(M) - \dim_{\mathbb{K}}(F_k(M) \cap \partial_{x_1} M) \\ &\leq \dim_{\mathbb{K}} F_k(M) - \dim_{\mathbb{K}} F_{k-1}(M) = H(k) - H(k-1)\end{aligned}$$

for sufficiently large k.

Let N be a finitely generated nonzero left D_Y-submodule of $\varpi_* M$. Then $F_k(N) := F_k(\varpi_* M) \cap N$ constitute a $(\mathbf{1}, \mathbf{1})$-filtration on N. There exists a good $(\mathbf{1}, \mathbf{1})$-filtration $\{F'_k(N)\}$ on N since it is finitely generated. Let $H'(k)$ be the associated Hilbert polynomial. Then by Lemma 2.10 there exists $k_0 \in \mathbb{N}$ such that

$$H'(k) = \dim_{\mathbb{K}} F'_k(N) \leq \dim_{\mathbb{K}} F_{k+k_0}(N) \leq H(k + k_0) - H(k + k_0 - 1)$$

holds if k is sufficiently large. Let $c_d k^d$ be the leading term of $H(k) - H(k-1)$. Then the inequality above implies that N is holonomic with $\operatorname{mult} N \leq d! c_d$. Hence we are done if $\varpi_* M$ is finitely generated over D_Y.

Otherwise, there exist finitely generated nonzero D_Y-submodules N_j ($j \in \mathbb{N}$) of $\varpi_* M$ such that $N_j \subsetneq N_{j+1}$. This implies that N_j are holonomic and $\operatorname{mult} N_j < \operatorname{mult} N_{j+1}$ holds in view of Proposition 2.19. This contradicts the inequality $\operatorname{mult} N_j \leq d! c_d$. Thus $\varpi_* M$ must be finitely generated over D_Y and hence holonomic.

In general case, set

$$N = \{u \in M \mid \partial_{x_1}^{\nu} u = 0 \text{ for some } \nu \in \mathbb{N}\}.$$

Then N is a left D_X-module since $\partial_{x_1}^{\nu} u = 0$ implies

$$\partial_{x_1}^{\nu+1}(x_1 u) = x_1 \partial_{x_1}^{\nu+1} u + (\nu + 1)\partial_{x_1}^{\nu} u = 0.$$

Let us show that $\partial_{x_1} : N \to N$ is surjective. Suppose $u \in N$ satisfies $\partial_{x_1} u = 0$. Then we have $\partial_{x_1} x_1 u = u$. Hence u belongs to $\partial_{x_1} M$. Now assume that for any $v \in N$, v belongs to $\partial_{x_1} M$ if $\partial_{x_1}^{\nu} v = 0$. Suppose $u \in N$ satisfies $\partial_{x_1}^{\nu+1} u = 0$. Then we have

$$\partial_{x_1}^{\nu}(\partial_{x_1} x_1 u - (\nu + 1)u) = x_1 \partial_{x_1}^{\nu+1} u = 0.$$

By the induction hypothesis, there exists $v \in N$ such that

$$\partial_{x_1} x_1 u - (\nu + 1)u = \partial_{x_1} v.$$

This implies that u belongs to $\partial_{x_1} N$.

From the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0,$$

we get an exact sequence

$$\varpi_* N \longrightarrow \varpi_* M \longrightarrow \varpi_*(M/N) \longrightarrow 0$$

of left D_Y-modules. Here $\varpi_* N = 0$ holds since $N = \partial_{x_1} N$. Hence $\varpi_* M$ is isomorphic to $\varpi_*(M/N)$, which is holonomic since $\partial_{x_1} : M/N \to M/N$ is injective. Q.E.D.

In particular, if a holonomic D_X-module M is generated by a single element u, then $\varpi_* M$ is generated by a finite number of residue classes $x^\alpha[u]$ with $\alpha \in \mathbb{N}^n$. In general, let M be a left D_X-module generated by u. Setting $I = \mathrm{Ann}_{D_X} u$, we have an isomorphism

$$\varpi_* M \cong D_X/(\partial_{x_1} D_X + \cdots + \partial_{x_n} D_X + I).$$

From a computational viewpoint, we are mainly interested in the submodule $D_Y[u]$ of $\varpi_* M$. The isomorphism above induces

$$D_Y[u] \cong D_Y/(D_Y \cap (\partial_{x_1} D_X + \cdots + \partial_{x_n} D_X + I)).$$

The map ϖ_* and the inclusion $D_Y[u] \to \pi_* M$ induces $\mathbb{C}$-linear maps

$$\begin{aligned} \varpi_* &: \mathrm{Hom}_{D_X}(M, \mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)) \longrightarrow \mathrm{Hom}_{D_Y}(D_Y[u], \mathcal{D}'(U)), \\ \varpi_* &: \mathrm{Hom}_{D_X}(M, \mathcal{S}\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)) \longrightarrow \mathrm{Hom}_{D_Y}(D_Y[u], \mathcal{S}'(\mathbb{R}^d)). \end{aligned}$$

This means that for a solution in $\mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)$ or in $\mathcal{S}\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ of a system M of differential equations, its integral with respect to x is a solution of $D_Y[u]$.

Example 5.2. The integral of the holonomic D_{n+d}-module $\mathbb{K}[x, y]$ along the fibers of the projection $\varpi : \mathbb{K}^{n+d} \ni (x, y) \mapsto y \in \mathbb{K}^d$ is $\{0\}$ as a D_d-module because $\partial_{x_j} : \mathbb{K}[x, y] \to \mathbb{K}[x, y]$ is surjective for $1 \le j \le n$.

Example 5.3. Set

$$M = D_{n+d}/(D_{n+d}x_1 + \cdots + D_{n+d}x_n + D_{n+d}y_1 + \cdots + D_{n+d}y_d)$$

and $\varpi : \mathbb{K}^{n+d} \ni (x, y) \mapsto y \in \mathbb{K}^d$ be the projection. Then there exists a natural isomorphism

$$\varpi_* M \cong D_d/(D_d y_1 + \cdots + D_d y_d).$$

In fact, by the definition we can write $\varpi_* M = D_{n+d}/N$ with

$$\begin{aligned} N := D_{n+d}x_1 + \cdots + D_{n+d}x_n + D_{n+d}y_1 + \cdots + D_{n+d}y_d \\ + \partial_{x_1} D_{n+d} + \cdots + \partial_{x_n} D_{n+d}. \end{aligned}$$

The ring extension $D_d \to D_{n+d}$ induces a homomorphism

$$\varphi : D_d/(D_d y_1 + \cdots + D_d y_d) \longrightarrow D_{n+d}/N$$

of left D_d-modules. Every element of $D_d/(D_d y_1 + \cdots + D_d y_d)$ is uniquely written as a linear combination of the residue classes $[\partial_y^\gamma]$ with $\gamma \in \mathbb{N}^d$. It follows that φ is injective. Every element of D_{n+d} is uniquely written as a linear combination of $\partial_x^\alpha x^\beta \partial_y^\gamma y^\delta$ with $\alpha, \beta \in \mathbb{N}^n$ and $\gamma, \delta \in \mathbb{N}^d$. It belongs to N unless $\alpha = \beta = 0$ and $\delta = 0$. This implies that φ is surjective.

Exercise 22. Set $n = d = 1$ and write $x = x_1$, $\partial_x = \partial_{x_1}$, $y = y_1$, $\partial_y = \partial_{y_1}$. Compute the integral of

$$M := D_2/(D_2\partial_y + D_2 x^2)$$

along the fibers of the projection $\varpi : \mathbb{K}^2 \ni (x, y) \mapsto y \in \mathbb{K}$. Note that $\varpi_* M$ is generated by [1] and $[x]$. Deduce a presentation of the submodule $D_1[u]$ of $\varpi_* M$.

Exercise 23. Let $(a, x, y) = (a_1, \dots, a_p, x_1, \dots, x_n, y_1, \dots, y_d) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^d$ with an open set Ω of $\mathbb{R}^p$. Let D_{p+n+d} and D_{p+d} be the ring of differential operators in the variables (a, x, y) and that in the variables (a, y) respectively. Let $\varpi : \mathbb{C}^{p+n+d} \ni (a, x, y) \mapsto (a, y) \in \mathbb{C}^{p+d}$ be the projection. Let M be a finitely generated left D_{p+n+d}-module. Construct a $\mathbb{C}$-linear map

$$\varpi_* : \mathrm{Hom}_{D_{p+n+d}}(M, \mathcal{ESS}'(\Omega\times\mathbb{R}^n\times\mathbb{R}^d)) \to \mathrm{Hom}_{D_{p+d}}(\varpi_* M, \mathcal{D}'(\Omega\times\mathbb{R}^d))$$

in terms of the integration.

5.2. An algorithm for integration

In what follows, we assume that a left module M over $D_X = D_{n+d}$ is generated by a single element u for the sake of simplicity; it is easy to extend the following arguments so as to work in the case where M is generated by several elements and as well to yield the torsion groups associated with the integration, which are nothing but the relative de Rham cohomology groups of M along the fibers of ϖ (see [26]).

Now let us fix the weight vector

$$w := (\underbrace{1, \dots, 1}_{n}, \underbrace{0, \dots, 0}_{d}; \underbrace{-1, \dots, -1}_{n}, \underbrace{0, \dots, 0}_{d}) \in \mathbb{Z}^{2(n+d)}$$

for D_{n+d} and set

$$\theta := -(\partial_{x_1} x_1 + \cdots + \partial_{x_n} x_n) = -x_1\partial_{x_1} - \cdots - x_n\partial_{x_n} - n.$$

That is, we define the weights of x_i and of ∂_{x_i} to be 1 and -1 respectively, and the weights of y_j and ∂_{y_j} to be 0. In fact, we could work with a more general weight vector

$$w = (w_1, \dots, w_n, 0, \dots, 0; -w_1, \dots, -w_n, 0, \dots, 0)$$

with positive integers $w_1, \dots, w_n$ as in Chapter 5 of [29] by modifying the following arguments accordingly. Set

$$F_k(M) := F_k^w(D_X)u, \qquad \mathrm{gr}_k(M) := F_k(M)/F_{k-1}(M) \qquad (k \in \mathbb{Z}).$$

Then $\{F_k(M)\}$ is a good w-filtration on M.

Theorem 5.4. *If M is a holonomic D_X-module, then there exists a nonzero polynomial $b(s) \in \mathbb{K}[s]$ in s such that $b(\theta)\mathrm{gr}_0(M) = 0$. Such $b(s)$ of minimum degree is called the b-function of M with respect to the weight vector w and the filtration $\{F_k(M)\}$, or the b-function for integration along the fibers of ϖ. Moreover, $b(\theta + k)\mathrm{gr}_k(M) = 0$ holds for any $k \in \mathbb{Z}$.*

Proof. The graded module $\mathrm{gr}(M) = \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k(M)$ is also a holonomic module over $\mathrm{gr}^w(D_{n+d}) \cong D_{n+d}$ by Theorem 2.2.1 of [29]. From $(\theta + 1)x_j = x_j\theta$ and $(\theta - 1)\partial_{x_j} = \partial_{x_j}\theta$ it follows that $(\theta + k)P = P\theta$ holds if P is homogeneous of order k with respect to w. Hence the sum of $\theta + k : \mathrm{gr}_k(M) \to \mathrm{gr}_k(M)$ for each $k \in \mathbb{Z}$ defines an endomorphism of the left $\mathrm{gr}^w(D_{n+d})$-module $\mathrm{gr}(M)$. There exists the minimal polynomial $b(s) \in \mathbb{K}[s]$ of this endomorphism since the space of the endomorphisms of a holonomic D-module is finite dimensional as was first shown by Kashiwara. A direct statement of this fact is found e.g., as Theorem 4.45 of [15]; this also follows from Theorem 6.6 in Chapter 1 of [4] combined with Lemma 7.14 and Theorem 7.15 in Chapter 2 of [4]. Q.E.D.

Note that a non-holonomic D_{n+d}-module can have a b-function in the above sense. The following arguments only rely on the existence of the b-function hence applies also to such non-holonomic modules.

Let us begin with an algorithm for computing the intersection of a left ideal I of D_{n+d} with the subring $D_d[x_1\partial_{x_1}, \dots, x_n\partial_{x_n}]$ by using multi-homogenization (Proposition 4.3 of [27]). Let us set $\theta_j = x_j\partial_{x_j}$ for $1 \le j \le n$.

Algorithm 5.5. Input: A set G_0 of generators of a left ideal I of D_{n+d}.
Output: A set G of generators of the left ideal $I \cap D_d[\theta_1, \dots, \theta_n]$ of $D_d[\theta_1, \dots, \theta_n]$.

(1) Introducing new variables u_j, v_j for $j = 1, \dots, n$, let $h(P) \in D_{n+d}[u]$ be the multi-homogenization of $P \in D_{n+d}$; i.e., for each $j = 1, \dots, n$, $h(P)$ is homogeneous with respect to the weight in which x_j and u_j are of order -1, ∂_{x_j} is of order 1, while x_i, ∂_{x_i} for $i \neq j$ and y_i, ∂_{y_i} for all i are of order zero.

(2) Let J be the left ideal of $D_{n+d}[u, v]$ generated by the set

$$\{h(P) \mid P \in G_0\} \cup \{1 - u_j v_j \mid j = 1, \dots, n\}.$$

(3) Compute a set G_1 of generators of the ideal $J \cap D_{n+d}$ by eliminating u, v via an appropriate Gröbner basis.

(4) Since each element P of G_1 is multi-homogeneous and free of u, v, there exist unique $\nu_1, \dots, \nu_n \in \mathbb{Z}$ and $Q(\theta_1, \dots, \theta_n) \in D_d[\theta_1, \dots, \theta_n]$ such that

$$S_{1,\nu_1} \cdots S_{n,\nu_n} P = Q(\theta_1, \dots, \theta_n),$$

where we set $S_{j,\nu_j} = \partial_{x_j}^{\nu_j}$ if $\nu_j \geq 0$ and $S_{j,\nu_j} = x_j^{-\nu_j}$ otherwise. Let G be the set of such $Q(\theta_1, \dots, \theta_n)$ for each $P \in G_1$.

See the proof of Proposition 4.3 of [27] for the correctness of this algorithm. The following algorithm was also presented in [27] (Algorithm 4.6):

Algorithm 5.6 ($b(s)$ with respect to w)**.** Input: $I := \mathrm{Ann}_{D_{n+d}} u$. Output: The b-function $b(s)$ of $M = D_{n+d} u$ with respect to w if it exists. ‘None’ if it does not.

(1) Compute a Gröbner basis $G = \{P_1, \dots, P_r\}$ of I with respect to a monomial order which is adapted to the weight vector w defined above.

(2) Set $\sigma^w(G) = \{\sigma^w(P_1), \dots, \sigma^w(P_r)\}$ and let $\mathrm{gr}^w(I)$ be the left ideal of D_{n+d} generated by $\sigma^w(G)$.

(3) Compute a set of generators of $\mathrm{gr}^w(I) \cap D_d[\theta_1, \dots, \theta_n]$ by Algorithm 5.5.

(4) Compute the intersection

$$\mathrm{gr}^w(I) \cap \mathbb{K}[\theta_1, \dots, \theta_n] = (\mathrm{gr}^w(I) \cap D_d[\theta_1, \dots, \theta_n]) \cap \mathbb{K}[\theta_1, \dots, \theta_n]$$

by using a Gröbner basis.

(5) Setting $\theta = -\theta_1 - \cdots - \theta_n - n$, compute

$$B := \mathrm{gr}^w(I) \cap \mathbb{K}[\theta_1, \dots, \theta_n] \cap \mathbb{K}[\theta]$$

by using a Gröbner basis. If $B \neq \{0\}$, let $b(\theta)$ be a generator of B. If $B = \{0\}$, then there exists no b-function of M with respect to w.

If M is holonomic, or more generally if we know that there exists a (nonzero) b-function in advance, then we can employ more efficient algorithm by Noro [19] which calculates $b(s)$ directly as the minimal polynomial of θ with modular computation; this algorithm is available as a function named 'generic_bfct' in a computer algebra system Risa/Asir [20].

Proposition 5.7. *Suppose that a left D_X-module $M = D_X u = D_X/I$ has a b-function $b(s)$ with respect to the weight vector w as above and the good w-filtration $F_k(M) := F_k^w(D_X)u$. Let k_1 be the largest integer root, if any, of $b(s)$. Let k_1 be an arbitrary integer if $b(s)$ has no integral root. Then the exact sequence*

$$M^n \overset{(\partial_{x_1},\dots,\partial_{x_n})}{\longrightarrow} M \longrightarrow \varpi_* M \longrightarrow 0$$

induces an exact sequence

$$F_{k_1+1}(M)^n \overset{(\partial_{x_1},\dots,\partial_{x_n})}{\longrightarrow} F_{k_1}(M) \longrightarrow \varpi_* M \longrightarrow 0.$$

To prove this proposition we need a lemma on the Koszul complex. In general, let $L = \oplus_{k\in\mathbb{Z}} L_k$ be a graded module over $\mathrm{gr}^w(D_X) = D_{n+d}$ with L_k being the homogeneous part of order k. For any integer k, let us define the Koszul complex $\mathcal{K}^\bullet(L[k], \partial_{x_1}, \dots, \partial_{x_n})$ to be the complex

$$0 \longrightarrow L_{k+n} \otimes_{\mathbb{Z}} \wedge^0 \mathbb{Z}^n \overset{\delta_n}{\longrightarrow} L_{k+n-1} \otimes_{\mathbb{Z}} \wedge^1 \mathbb{Z}^n \overset{\delta_{n-1}}{\longrightarrow} \cdots \overset{\delta_1}{\longrightarrow} L_k \otimes_{\mathbb{Z}} \wedge^n \mathbb{Z}^n.$$

Here $\wedge^l \mathbb{Z}$ is the free $\mathbb{Z}$-module spanned by $e_{i_1} \wedge \cdots \wedge e_{i_l}$ with the unit vectors $e_1, \dots, e_n$ of $\mathbb{Z}^n$ satisfying $e_i \wedge e_j + e_j \wedge e_i = 0$. The homomorphism δ_l is defined by

$$\delta_l(v \otimes e_{i_1} \wedge \cdots \wedge e_{i_l}) = \sum_{j=1}^{n} (\partial_{x_j} v) e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_l}.$$

For example, we have $L_k \otimes_{\mathbb{Z}} \wedge^n \mathbb{Z}^n = L_k e_1 \wedge \cdots \wedge e_n \cong L_k$,

$$L_{k+1} \otimes_{\mathbb{Z}} \wedge^{n-1} \mathbb{Z}^n = \bigoplus_{j=1}^{n} L_{k+1} e_{\hat{j}} \cong (L_{k+1})^n,$$
$$L_{k+2} \otimes_{\mathbb{Z}} \wedge^{n-2} \mathbb{Z}^n = \bigoplus_{1\le i<j\le n} L_{k+2} e_{\hat{i}\hat{j}} \cong (L_{k+1})^{n(n-1)/2}$$

with

$$e_{\hat{j}} := e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n,$$
$$e_{\hat{i}\hat{j}} := e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n,$$

and

$$\delta_1\left(\sum_{j=1}^n v_j e_{\hat{j}}\right) = \sum_{j=1}^n (-1)^{j-1}(\partial_{x_j} v_j) e_1 \wedge \cdots \wedge e_n,$$

$$\delta_2\left(\sum_{1\le i<j\le n} v_{ij} e_{\hat{i}\hat{j}}\right) = \sum_{i=1}^n \left(\sum_{j=1}^{i-1}(-1)^{j-1}\partial_{x_j} v_{ji} + \sum_{j=i+1}^{n}(-1)^j \partial_{x_j} v_{ij}\right) e_{\hat{i}}.$$

Lemma 5.8. *Assume that there exists a nonzero polynomial* $b(s) \in \mathbb{K}[s]$ *such that* $b(\theta + j)L_j = 0$ *for any* $j \in \mathbb{Z}$. *If* $b(k) \neq 0$ *holds for an integer* k, *then the Koszul complex* $\mathcal{K}^\bullet(L[k], \partial_{x_1}, \ldots, \partial_{x_n})$ *is exact.*

Proof. We argue by induction on n. First let us prove the lemma for $n = 1$; i.e., that the homomorphism

$$\delta_1 : L_{k+1} \otimes_{\mathbb{Z}} \wedge^0 \mathbb{Z} \ni v \longmapsto (\partial_{x_1} v) e_1 \in L_k \otimes_{\mathbb{Z}} \wedge^1 \mathbb{Z}$$

is an isomorphism. Let v be an element of L_k. Then $b(\theta + k)v = 0$ holds. There exists a polynomial $c(\theta) \in \mathbb{K}[\theta]$ such that $b(\theta + k) - b(k) = \theta c(\theta)$. This implies

$$b(k)v = -\theta c(\theta)v = \partial_{x_1} x_1 c(-\partial_{x_1} x_1)v.$$

It follows that δ_1 is surjective. Next suppose $v \in L_{k+1}$ satisfies $\partial_{x_1} v = 0$ in L_k. Then we get

$$0 = b(\theta + k + 1)v = b(-x_1\partial_{x_1} + k)v = b(k)v,$$

and consequently $v = 0$ since $b(k) \neq 0$.

Now suppose $n \geq 2$ and that the lemma has been proved with n replaced by $n-1$. The Koszul complex $\mathcal{K}^\bullet(L[k], \partial_{x_1}, \ldots, \partial_{x_n})$ is isomorphic to the total complex associated with the double complex

$$\begin{array}{ccccccc} \cdots \longrightarrow & L_{k+2} \otimes_{\mathbb{Z}} \wedge^{n-2}\mathbb{Z}^{n-1} & \xrightarrow{\delta_1'} & L_{k+1} \otimes_{\mathbb{Z}} \wedge^{n-1}\mathbb{Z}^{n-1} & \longrightarrow 0 \\ & \downarrow \partial_{x_n} & & \downarrow \partial_{x_n} & \\ \cdots \longrightarrow & L_{k+1} \otimes_{\mathbb{Z}} \wedge^{n-2}\mathbb{Z}^{n-1} & \xrightarrow{\delta_1'} & L_k \otimes_{\mathbb{Z}} \wedge^{n-1}\mathbb{Z}^{n-1} & \longrightarrow 0 \end{array} \tag{12}$$

where the two horizontal sequences are $\mathcal{K}^\bullet(L[k+1], \partial_{x_1}, \ldots, \partial_{x_{n-1}})$ and $\mathcal{K}^\bullet(L[k], \partial_{x_1}, \ldots, \partial_{x_{n-1}})$ respectively. In fact, there is an isomorphism

$$\left(L_{k+l} \otimes_{\mathbb{Z}} \wedge^{n-l}\mathbb{Z}^{n-1}\right) \oplus \left(L_{k+l} \otimes_{\mathbb{Z}} \wedge^{n-l-1}\mathbb{Z}^{n-1}\right) \cong L_{k+l} \otimes_{\mathbb{Z}} \wedge^{n-l}\mathbb{Z}^n$$

for $0 \le l \le n$ (with the convention $\wedge^j \mathbb{Z}^n = 0$ if $j < 0$ or $j > n$) defined by the one-to-one correspondence

$$\Big(\sum_{1\le i_1<\cdots<i_{n-l}\le n-1} v_{i_1\dots i_{n-l}} e_{i_1} \wedge \cdots \wedge e_{i_{n-l}}, \\ \sum_{1\le i_1<\cdots<i_{n-l-1}\le n-1} v_{i_1\dots i_{n-l-1}} e_{i_1} \wedge \cdots \wedge e_{i_{n-l-1}} \Big) \\ \longleftrightarrow \sum_{1\le i_1<\cdots<i_{n-l}\le n-1} v_{i_1\dots i_{n-l}} e_{i_1} \wedge \cdots \wedge e_{i_{n-l}} \\ + \sum_{1\le i_1<\cdots<i_{n-l-1}\le n-1} v_{i_1\dots i_{n-l-1}} e_{i_1} \wedge \cdots \wedge e_{i_{n-l-1}} \wedge e_n,$$

in which the homomorphism ∂_{x_n} is given by

$$\partial_{x_n} \Big(\sum_{1\le i_1<\cdots<i_{n-l}\le n-1} v_{i_1\dots i_{n-l}} e_{i_1} \wedge \cdots \wedge e_{i_{n-l}} \Big) \\ = \sum_{1\le i_1<\cdots<i_{n-l}\le n-1} \partial_{x_n} v_{i_1\dots i_{n-l}} e_{i_1} \wedge \cdots \wedge e_{i_{n-l}} \wedge e_n.$$

Hence the homomorphism

$$\big(L_{k+l} \otimes_{\mathbb{Z}} \wedge^{n-l} \mathbb{Z}^{n-1}\big) \oplus \big(L_{k+l} \otimes_{\mathbb{Z}} \wedge^{n-l-1} \mathbb{Z}^{n-1}\big) \\ \longrightarrow \big(L_{k+l-1} \otimes_{\mathbb{Z}} \wedge^{n-l+1} \mathbb{Z}^{n-1}\big) \oplus \big(L_{k+l-1} \otimes_{\mathbb{Z}} \wedge^{n-l} \mathbb{Z}^{n-1}\big)$$

defined by $\big(\delta'_l \oplus (-1)^{n-l}\partial_{x_n}\big) \oplus \delta'_{l+1}$ corresponds to δ_l.

Let L'_j and L''_j be the kernel and the cokernel of $\partial_{x_n} : L_{j+1} \to L_j$ respectively and set $L' := \oplus_{j\in\mathbb{Z}} L'_j$ and $L'' := \oplus_{j\in\mathbb{Z}} L''_j$, which are graded D_{n+d-1}-modules. For any $v \in L'_j$ we have

$$b(-\partial_{x_1}x_1 - \cdots - \partial_{x_{n-1}}x_{n-1} + j)v \\ = b(-\partial_{x_1}x_1 - \cdots - \partial_{x_{n-1}}x_{n-1} - x_n\partial_{x_n} + j)v = b(\theta + j + 1)v = 0$$

by the assumption on L since L'_j is a subset of L_{j+1}.

Now let v be an element of L_j and $\overline{v}$ be its residue class in L''_j. Then we have $b(\theta + j)v = 0$ and

$$b(-\partial_{x_1}x_1 - \cdots - \partial_{x_{n-1}}x_{n-1} + j)v - b(\theta + j)v \in \partial_{x_n} L_{j+1}.$$

This implies

$$b(-\partial_{x_1}x_1 - \cdots - \partial_{x_{n-1}}x_{n-1} + j)\overline{v} = 0.$$

Hence $\mathcal{K}^\bullet(L'[k], \partial_{x_1}, \dots, \partial_{x_{n-1}})$ and $\mathcal{K}^\bullet(L''[k], \partial_{x_1}, \dots, \partial_{x_{n-1}})$ are both exact by the induction hypothesis if $b(k) \neq 0$.

We have an exact sequence

$$\begin{aligned} 0 \to \mathcal{K}^\bullet(L'[k], \partial_{x_1}, \dots, \partial_{x_{n-1}}) &\to \mathcal{K}^\bullet(L[k+1], \partial_{x_1}, \dots, \partial_{x_{n-1}}) \\ &\xrightarrow{\partial_{x_n}} \mathcal{K}^\bullet(L[k], \partial_{x_1}, \dots, \partial_{x_{n-1}}) \to \mathcal{K}^\bullet(L''[k], \partial_{x_1}, \dots, \partial_{x_{n-1}}) \to 0 \end{aligned}$$

of chain maps. Here the central chain map defined by ∂_{x_n} is a quasi-isomorphism, i.e., induces isomorphisms of the homology groups since the leftmost and the rightmost complexes are exact. This implies that the total complex associated with the double complex (12) is exact, which can be verified by diagram chasing. Summing up, we have shown that $\mathcal{K}^\bullet(L[k], \partial_{x_1}, \dots, \partial_{x_n})$ is exact. Q.E.D.

Now let us prove Proposition 5.7. First let us show that $F_{k_1}(M) \to \varpi_* M$ is surjective. Let v be an element of $F_k(M)$ with $k > k_1$. Applying Lemma 5.8 to $\mathrm{gr}(M)$, we get $v_1, \dots, v_n \in F_{k+1}(M)$ such that

$$v - \partial_{x_1} v_1 - \cdots - \partial_{x_n} v_n \in F_{k-1}(M)$$

since $b(k) \neq 0$. By induction, we see that there exist $v'_1, \dots, v'_n \in F_{k+1}(M)$ such that

$$v - \partial_{x_1} v'_1 - \cdots - \partial_{x_n} v'_n \in F_{k_1}(M).$$

Thus $F_{k_1}(M) \to \varpi_* M$ is surjective.

Next, suppose the residue class $[v]$ in $\varpi_*(M)$ of $v \in F_{k_1}(M)$ vanishes. Then there exist $v_1, \dots, v_n \in F_{k+1}(M)$ with some k such that $v = \partial_{x_1} v_1 + \cdots + \partial_{x_n} v_n$. Assume $k > k_1$. Let $\overline{v}_j$ be the residue class of v_j in $\mathrm{gr}_{k+1}(M)$. Then we have $\partial_{x_1} \overline{v}_1 + \cdots + \partial_{x_n} \overline{v}_n = 0$ in $\mathrm{gr}_k(M)$. By Lemma 5.8, there exist $v_{ij} \in F_{k+2}(M)$ such that their residue classes $\overline{v}_{ij}$ in $\mathrm{gr}_{k+2}(M)$ satisfy

$$\overline{v}_i = \sum_{j=1}^{n} (-1)^{i+j-1} \partial_{x_j} \overline{v}_{ij}, \qquad \overline{v}_{ij} + \overline{v}_{ji} = 0.$$

Hence $v'_i := v_i - \sum_{j=1}^{n} (-1)^{i+j-1} \partial_{x_j} v_{ij}$ belongs to $F_k(M)$ and we get a new expression

$$v = \sum_{i=1}^{n} \partial_{x_i} \left(v'_i + \sum_{j=1}^{n} (-1)^{i+j-1} \partial_{x_j} v_{ij} \right) = \sum_{i=1}^{n} \partial_{x_i} v'_i.$$

Proceeding inductively, we can show that v belongs to $\partial_{x_1} F_{k_1+1}(M) + \cdots + \partial_{x_n} F_{k_1+1}(M)$. This completes the proof of Proposition 5.7.

Now let

$$(D_X)^r \xrightarrow{\psi} D_X \xrightarrow{\varphi} M \longrightarrow 0$$

be a presentation of M with

$$\varphi(P) = Pu \quad (\forall P \in D_X),$$
$$\psi((Q_1, \dots, Q_r)) = Q_1 P_1 + \cdots + Q_r P_r \qquad (\forall Q_1, \dots, Q_r \in D_X).$$

Here we assume that $P_1, \dots, P_r$ are a w-involutive basis of $I = \mathrm{Ann}_{D_X} u$ with $\mathrm{ord}_w(P_i) = m_i$. This implies that the sequence

$$\oplus_{i=1}^r F_{k-m_i}(D_X) \xrightarrow{\psi} F_k(D_X) \xrightarrow{\varphi} F_k(M) \longrightarrow 0$$

is exact for any $k \in \mathbb{Z}$. Set $F_k[\mathbf{m}]((D_{Y\leftarrow X})^r) := \oplus_{i=1}^r F_{k-m_i}(D_{Y\leftarrow X})$ with $\mathbf{m} = (m_1, \dots, m_r)$, and so on, where $\{F_k(D_{Y\leftarrow X})\}$ denotes the filtration induced by $\{F_k^w(D_X)\}$. Then ψ induces homomorphisms

$$\overline{\psi} : (D_{Y\leftarrow X})^r \longrightarrow D_{Y\leftarrow X}, \qquad \overline{\psi} : F_k[\mathbf{m}]((D_{Y\leftarrow X})^r) \longrightarrow F_k(D_{Y\leftarrow X}).$$

Let k_1 be an integer as in Proposition 5.7. Then we have a commutative diagram

$$\begin{array}{ccccccc}
F_{k_1+1}[\mathbf{m}]((D_X)^r)^n & \xrightarrow{(\psi,\dots,\psi)} & F_{k_1+1}(D_X)^n & \xrightarrow{(\varphi,\dots,\varphi)} & F_{k_1+1}(M)^n & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle (\partial_{x_1},\dots,\partial_{x_n})} & & \downarrow{\scriptstyle (\partial_{x_1},\dots,\partial_{x_n})} & & \\
F_{k_1}[\mathbf{m}]((D_X)^r) & \xrightarrow{\psi} & F_{k_1}(D_X) & \xrightarrow{\varphi} & F_{k_1}(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
F_{k_1}[\mathbf{m}]((D_{Y\leftarrow X})^r) & \xrightarrow{\overline{\psi}} & F_{k_1}(D_{Y\leftarrow X}) & \longrightarrow & \varpi_* M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

where the upper leftmost homomorphisms send

$$\begin{pmatrix} Q_{11} & \cdots & Q_{1r} \\ \vdots & & \vdots \\ Q_{n1} & \cdots & Q_{nr} \end{pmatrix} \in F_{k_1+1}[\mathbf{m}]((D_X)^r)^n$$

to

$$\begin{pmatrix} Q_{11} & \cdots & Q_{1r} \\ \vdots & & \vdots \\ Q_{n1} & \cdots & Q_{nr} \end{pmatrix} \begin{pmatrix} P_1 \\ \vdots \\ P_r \end{pmatrix} \in F_{k_1+1}(D_X)^n,$$

$$\begin{pmatrix} \partial_{x_1} & \cdots & \partial_{x_n} \end{pmatrix} \begin{pmatrix} Q_{11} & \cdots & Q_{1r} \\ \vdots & & \vdots \\ Q_{n1} & \cdots & Q_{nr} \end{pmatrix} \in F_{k_1}[\mathbf{m}]((D_X)^r)$$

respectively. In the commutative diagram, the three vertical sequences and the two horizontal sequences except the one at the bottom are exact in view of Proposition 5.7. This implies that the horizontal sequence at the bottom is also exact; i.e.,

$$\varpi_* M = \operatorname{coker}(\overline{\psi} : F_{k_1}[\mathbf{m}]((D_{Y \leftarrow X})^r) \longrightarrow F_{k_1}(D_{Y \leftarrow X})).$$

Note that

$$F_{k_1}(D_{Y \leftarrow X}) = \bigoplus_{|\alpha| \le k_1} x^\alpha D_Y, \;\; F_{k_1}[\mathbf{m}]((D_{Y \leftarrow X})^r) = \bigoplus_{i=1}^{r} \bigoplus_{|\alpha| \le k_1 - m_i} x^\alpha D_Y$$

as left D_Y-modules. Hence $\overline{\psi}$ is a homomorphism of free left D_Y-modules of finite rank so that coker $\overline{\psi}$ can be explicitly computed by linear algebra over D_Y. This gives the relations among the generators $\{x^\alpha[u] \mid |\alpha| \le k_1\}$ of $\varpi_* M$. In particular, we get

Proposition 5.9. *One has $\varpi_* M = 0$ if $b(k) \neq 0$ for any non-negative integer k.*

By elimination, we obtain $\mathrm{Ann}_{D_Y}[u]$ so that $D_Y[u] \cong D_Y/\mathrm{Ann}_{D_Y}[u]$ is a left D_Y-submodule of $\varpi_* M$. It is easy to see by the construction that

$$\mathrm{Ann}_{D_Y}[u] = D_Y \cap (\partial_{x_1} D_X + \cdots + \partial_{x_n} D_X + \mathrm{Ann}_{D_X} u)$$

holds; the right-hand side is called the integration ideal of $I = \mathrm{Ann}_{D_Y} u$. Summing up we have obtained

Algorithm 5.10 (integration ideal)**.** Input: A set G_0 of generators of $I := \mathrm{Ann}_{D_X} u$.
Output: A set G of generators of the integration ideal $\mathrm{Ann}_{D_Y}[u]$ of I.

(1) Compute a Gröbner basis $G_1 = \{P_1, \ldots, P_r\}$ of I with respect to a monomial order which is adapted to the weight vector w.

(2) Using G_1, compute the b-function $b(s)$ of $M = D_X/I$ with respect to w by steps (2)–(5) of Algorithm 5.6.

(3) If $b(s)$ has no non-negative integer root, then $[u] = 0$; quit. Otherwise let k_1 be the largest integral root of $b(s)$.

(4) Set $\mathbf{m} = (m_1, \dots, m_r)$ with $m_j = \mathrm{ord}_w(P_j)$.

(5) Express the homomorphism

$$(D_Y)^{l_1} \cong F_{k_1}[\mathbf{m}]((D_{Y\leftarrow X})^r) \xrightarrow{\overline{\psi}} F_{k_1}(D_{Y\leftarrow X}) \cong (D_Y)^{l_0}$$

of free D_Y-modules, which is induced by ${}^t(P_1, \dots, P_r)$, as an $l_1 \times l_0$ matrix $A = (A_{ij})$ with

$$l_1 = \sum_{i=1}^{r} \binom{n + k_1 - m_i}{n}, \quad l_0 = \binom{n+k_1}{n}, \quad A_{ij} \in D_Y.$$

(6) Compute a set G_2 of generators of the submodule

$$D_Y e_1 \cap \left\{ \sum_{i=1}^{l_1} Q_i(A_{i1}, \dots, A_{il_0}) \mid Q_i \in D_Y \right\}$$

of $D_Y^{l_0}$ with $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^{l_0}$, which corresponds to $\overline{1} \in F_{k_1}(D_{Y\leftarrow X})$, by a Gröbner basis with respect to what is called a 'position-over-term' order. Let G be the set of the first elements of G_2.

For practical computation of integration, computer algebra systems such as Risa/Asir [20], Macaulay2 [11], and Singular [7] are available. An implementation of the D-module theoretic integration algorithm was first supplied with Kan/sm1 [34] (see [26]). We make use of a Risa/Asir library 'nk_restriction.rr' by Hiromasa Nakayama for computing various examples in the next section. One can also use a Macaulay2 package 'Dmodules' by Anton Leykin and Harrison Tsai, or a Singular library 'dmodapp_lib' by Viktor Levandovskyy and Daniel Andres.

Example 5.11. Set $x = x_1$, $\partial_x = \partial_{x_1}$ and so on with $n = d = 1$ and $w = (1, 0, ; -1, 0)$. Let I be the left ideal of D_2 generated by $P_1 = y - x^2$, $P_2 = 2x\partial_y + \partial_x$ and set $M = D_2/I$. We denote by u the residue class of 1 in M. It is easy to see that P_1 and P_2 annihilate $\delta(y - x^2)$, which belongs to $\mathcal{E}'\mathcal{D}'(\mathbb{R}_x \times \mathbb{R}_y)$.

From Example 3.15, P_1, P_2, and $P_3 = 2y\partial_y + x\partial_x + 2$ are a w-involutive basis of I. The b-function $b(s)$ of $M = D/I$ with respect to w divides $s(s-1)$ since $\sigma^w(P_1) = -x^2$ and hence $\partial_x^2 x^2 = \partial_x x(\partial_x x + 1)$ annihilates $\mathrm{gr}_0(M) = F_0(M)/F_{-1}(M)$ with $F_k(M) := F_k^w(D_2)/(F_k^w(D_2) \cap$

I). Let $\varpi : X = \mathbb{C}^2 \ni (x, y) \mapsto x \in \mathbb{C} = Y$ be the projection. Since the w-orders of P_1, P_2, P_3 are $2, 1, 0$ respectively, we have an exact sequence

$$F_{-1}(D_{Y\leftarrow X}) \oplus F_0(D_{Y\leftarrow X}) \oplus F_1(D_{Y\leftarrow X}) \xrightarrow{\overline{\psi}} F_1(D_{Y\leftarrow X}) \to \varpi_* M \to 0,$$

where $\overline{\psi}$ is induced by ${}^t(P_1, P_2, P_3)$. With $F_{-1}(D_{Y\leftarrow X}) = \{0\}$ omitted, $\overline{\psi}$ is explicitly given by

$$\begin{aligned} \overline{\psi}((Q_1, Q_2 + Q_3 x)) &= [Q_1 P_2 + (Q_2 + Q_3 x) P_3] \\ &= [Q_1(2x\partial_y + \partial_x) + (Q_2 + Q_3 x)(2y\partial_y + \partial_x x + 1)] \\ &= [Q_2(2y\partial_y + 1) + 2(Q_1\partial_y + Q_3 y\partial_y)x] \end{aligned}$$

for $Q_1, Q_2, Q_3 \in D_Y = D_1$, where the bracket denotes the residue class in $D_{Y\leftarrow X}$. Hence the homomorphism $\overline{\psi}$ is represented by the matrix

$$A = \begin{pmatrix} 0 & 2\partial_y \\ 2y\partial_y + 1 & 0 \\ 0 & 2y\partial_y \end{pmatrix}.$$

Thus $\varpi_* M$ is isomorphic to the direct sum

$$\varpi_* M = D_1[u] \oplus D_1[xu] \cong D_1/D_1(2y\partial_y + 1) \,\oplus\, D_1/D_1\partial_y.$$

This implies that $f_0(y) := \int_{-\infty}^{\infty} \delta(y - x^2)\, dx$ and $f_1(y) := \int_{-\infty}^{\infty} x\delta(y - x^2)\, dx$ satisfies

$$(2y\partial_y + 1) f_0(y) = \partial_y f_1(y) = 0.$$

Noticing $f_0(y) = f_1(y) = 0$ for $y < 0$, we get $f_0(y) = C y_+^{-1/2}$ and $f_1(y) = 0$ (naturally!) with a constant C. We can use the formula

$$\delta(1 - x^2) = \delta((x-1)(x+1)) = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1)$$

to obtain $C = f_0(1) = 1$. We conclude that the distribution $f_0(y)$ coincides with the locally integrable function $y_+^{-1/2}$ on whole $\mathbb{R}$ because the differential equation $(2y\partial_y + 1) f(y) = 0$ has no distribution solution $f(y)$ whose support is $\{0\}$. See Proposition 6.1 in the next subsection.

Exercise 24. Find a differential equation for

$$g_0(y) := \int_{-\infty}^{\infty} Y(y - x^2)\, dx$$

by using the integration algorithm and determine $g_0(y)$ as a distribution on $\mathbb{R}$ explicitly.

Exercise 25. For a positive integer n, find a differential equation for $v(y) = \int_{-\infty}^{\infty} \delta(y - x^n)\,dx$ and determine $v(y)$ explicitly.

Exercise 26. Set $u(x,t) = e^{tx-x^3}Y(x)$, which belongs to the space $\mathcal{ESS}'(\mathbb{R}_t \times \mathbb{R}_x)$ introduced in 4.5.

(1) Find a holonomic system for $u(x,t)$; confirm that it is holonomic.

(2) Find a holonomic system, i.e., a linear ordinary differential equation for

$$v(t) := \int_0^{\infty} e^{tx-x^3}\,dx = \int_{-\infty}^{\infty} e^{tx-x^3}Y(x)\,dx.$$

§6. Integration of holonomic distributions

We first apply the integration algorithm in 5.2 and classes of distributions introduced in 4.3 to the integral of a holonomic distribution over the whole space. For integrals over domains defined by arbitrary polynomial inequalities, we need more sophisticated method in order to compute a holonomic system for the product of Heaviside functions and the given integrand, which will be introduced in 6.2 with correctness proofs. This method also provides us with an integration algorithm for functions satisfying difference-differential holonomic systems, which will be explained in 6.4. Algorithms in this section complement the ones introduced in [24] with more detailed arguments.

6.1. Integrals of holonomic distributions over the whole space

We assume $\mathbb{K} = \mathbb{C}$. Let $u(x,y) = u(x_1,\dots,x_n,y_1,\dots,y_d)$ be a distribution in $\mathcal{E}'\mathcal{D}'(\mathbb{R}^n \times U)$ with an open set U of $\mathbb{R}^d$, or in $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Suppose that $u(x,y)$ is holonomic and that we have a left ideal I of D_{n+d} which annihilates $u(x,y)$ such that D_{n+d}/I is holonomic. Then the integral $\varpi_* M = M/(\partial_{x_1}M + \cdots + \partial_{x_n}M)$ of M gives a holonomic system of linear differential equations for

$$v(y) := \int_{\mathbb{R}^d} u(x,y)\,dx,$$

which belongs to $\mathcal{D}'(U)$ or to $\mathcal{S}'(\mathbb{R}^d)$, as was explained so far.

Let us first consider the standard normal distribution whose density function is given by $(2\pi)^{-n/2}\exp(-|x|^2/2)$. Let $f(x)$ be an arbitrary real polynomial in $x = (x_1,\dots,x_n)$. Then the cumulative function

$$F(t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\Bigl(-\frac{1}{2}|x|^2\Bigr) Y(t - f(x))\,dx$$

of the random variable $f(x)$ is well-defined as an element of $\mathcal{S}'(\mathbb{R})$ since the integrand belongs to $\mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}_t)$. It is also a continuous function if $f(x)$ is non-constant. The density function $F'(t)$ is given by the integral

$$F'(t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\Big(-\frac{1}{2}|x|^2\Big)\delta(t - f(x))\, dx$$

as an element of $\mathcal{S}'(\mathbb{R})$.

Since the integrands of $F(t)$ and of $F'(t)$ are holonomic by Proposition 4.20 and belong to $\mathcal{SS}'(R^n_x \times \mathbb{R}_t)$, we obtain linear ordinary differential equations which $F(t)$ and $F'(t)$ satisfy as elements of $\mathcal{S}'(\mathbb{R})$ by the integration algorithm.

The characteristic function of $v(t) := F'(t)$ is the tempered distribution $\hat{v}$ on $\mathbb{R}$ defined by

$$\langle \hat{v}, \varphi\rangle = \langle v, \hat{\varphi}\rangle, \qquad \hat{\varphi}(\tau) := \int_{-\infty}^{\infty} e^{i\tau t}\varphi(t)\, dt$$

for $\varphi \in \mathcal{S}(\mathbb{R})$ with $i = \sqrt{-1}$. Let us show that $\hat{v}(\tau)$ is given by

$$\hat{v}(\tau) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\Big(i\tau f(x) - \frac{1}{2}|x|^2\Big)\, dx.$$

Set $\psi(x) = (2\pi)^{-n/2} \exp\Big(-\frac{1}{2}|x|^2\Big)$. (Indeed ψ can be an arbitrary element of $\mathcal{S}(\mathbb{R}^n)$.) Then by the definition of the integral of an element of $\mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}_t)$ and Proposition 4.10, we get, for any $\chi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\langle \hat{v}, \chi\rangle &= \left\langle \int_{\mathbb{R}^n} \psi(x)\delta(t - f(x))\, dx,\ \hat{\chi} \right\rangle = \langle \delta(t - f(x)),\ \psi(x)\hat{\chi}(t)\rangle \\
&= \langle 1(x)\delta(t),\ \psi(x)\hat{\chi}(t + f(x))\rangle = \int_{\mathbb{R}^n} \psi(x)\hat{\chi}(f(x))\, dx \\
&= \int_{\mathbb{R}^n} \psi(x) \left(\int_{-\infty}^{\infty} e^{itf(x)}\chi(t)\, dt\right) dx \\
&= \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^n} \psi(x)e^{itf(x)}\, dx\right) \chi(t)\, dt.
\end{aligned}$$

This means the identity above. Moreover, $\hat{v}(\tau)$ belongs also to $C^\infty(\mathbb{R})$ since $\exp\Big(i\tau f(x) - \frac{1}{2}|x|^2\Big)$ belongs to $\mathcal{ES}(\mathbb{R}_\tau \times \mathbb{R}^n_x)$.

If $v(t)$ satisfies a differential equation $Pv = 0$ with $P = P(t, \partial_t) = \sum_{j=0}^m a_j(t)\partial_t^{m-j}$, then $\hat{v}$ satisfies $\hat{P}\hat{v} = 0$, where

$$\hat{P} = P(-i\partial_\tau, -i\tau) = \sum_{j=0}^m a_j(-i\partial_\tau)(-i\tau)^{m-j}.$$

Now let

$$P = a_0(t)\partial_t^m + a_1(t)\partial_t^{m-1} + \cdots + a_m(t) \tag{13}$$

be a linear ordinary differential operator with analytic functions $a_j(x)$ defined on a neighborhood of $t_0 \in \mathbb{R}$. In general, for an analytic function $f(t)$ near t_0, the order $\mathrm{ord}_{t_0} f(t)$ of $f(t)$ at t_0 is defined to be the smallest non-negative integer k such that $f^{(k)}(t_0) \neq 0$. The point t_0 is called a *regular singular point* of P if $\mathrm{ord}_{t_0} a_j(t) \geq \mathrm{ord}_{t_0} a_0(t) - j$ for $j = 1, \ldots, m$. Set $k = \mathrm{ord}_{t_0} a_0(t)$. With P being multiplied by a power of t or of ∂_t on the left, we may assume $k = m$. Then the *indicial polynomial* of P at a regular singular point t_0 is defined to be

$$b(s) := \sum_{j=0}^{m} c_j s(s-1)\cdots(s-m+j+1), \qquad c_j := \lim_{t\to t_0} \frac{a_j(t)}{(t-t_0)^{m-j}}.$$

This is nothing but the b-function with respect to the weight vector $(-1; 1)$. The roots of $b(s) = 0$ are called the *characteristic exponents* of P at t_0.

The following well-known facts often provide us with information on the behavior of the distribution solutions near a singular point:

Proposition 6.1. *Let $t_0 \in \mathbb{R}$ be a regular singular point of an ordinary differential operator* (13).

(1) *If P has no negative integer as a characteristic exponent, then P has no distribution* (*or even hyperfunction*) *solution whose support is $\{t_0\}$ on a neighborhood of t_0.*

(2) *If the real part of each characteristic exponent of P at t_0 is greater than -1, then any distribution* (*or even hyperfunction*) *solution of the differential equation $Pu = 0$ coincides with a Lebesgue integrable function on a neighborhood of t_0.*

The simplest proof of this proposition would be to consider a distribution as a hyperfunction, which is defined as the boundary value of a complex analytic function to the real line (see [30]), and employ the theory of ordinary differential equations with regular singularities in the complex domain.

As a first example, let us deduce the density function of the χ^2 distribution in statistics.

Example 6.2 (χ^2 distribution)**.** Set

$$u(x,t) = (2\pi)^{-\frac{n}{2}} \exp\Big(-\frac{1}{2}|x|^2\Big)\delta(t - |x|^2), \qquad v(t) = \int_{\mathbb{R}^n} u(x,t)\,dx$$

with $|x|^2 = x_1^2 + \cdots + x_n^2$. Then $u(x,t)$ belongs to $\mathcal{SS}'(\mathbb{R}_x^n \times \mathbb{R}_t)$ and thus $v(t)$ is well-defined as a tempered distribution on $\mathbb{R}$. Note that $v(t)$ is the density function of the χ^2 distribution. By Example 4.21, $u(x,t)$ satisfies a holonomic system

$$(t - |x|^2)u = (\partial_i + 2x_i\partial_t + x_i)u = 0 \qquad (i = 1, \ldots, n).$$

Since

$$\begin{aligned}
&\sum_{i=1}^n x_i(\partial_i + 2x_i\partial_t + x_i) + (1 + 2\partial_t)(t - |x|^2) \\
&= \sum_{i=1}^n x_i\partial_i + 2|x|^2\partial_t + |x|^2 + (1 + 2\partial_t)(t - |x|^2) \\
&= \sum_{i=1}^n x_i\partial_i + 2\partial_t t + t = \sum_{i=1}^n \partial_i x_i + 2t\partial_t + t - n + 2,
\end{aligned}$$

we know that $v(t)$ satisfies

$$(2t\partial_t + t - n + 2)v(t) = 0.$$

This differential equation has 0 as a regular singular point with the characteristic exponent $n/2 - 1$, which is greater than -1. Hence $v(t)$ is integrable on $\mathbb{R}$. Solving this equation by quadrature and noting that $v(t) = 0$ for $t < 0$, we conclude that

$$v(t) = Ce^{-t/2}t_+^{n/2-1}$$

with some constant C, which can be determined by

$$C = \left(\int_0^\infty e^{-t/2}t^{n/2-1}\,dt\right)^{-1} = \frac{1}{2^{n/2}\Gamma\left(\dfrac{n}{2}\right)}.$$

The characteristic function $\hat{v}(\tau) = \int_{-\infty}^\infty e^{i\tau t}v(t)\,dt$ satisfies

$$((2\tau + i)\partial_\tau + n)\hat{v}(\tau) = 0.$$

Together with $\hat{v}(0) = 1$, this implies

$$\hat{v}(\tau) = (1 - 2i\tau)^{-n/2}.$$

The following example was proposed by A. Takemura (see [18]):

Example 6.3 (sum of cubes of standard normal random variables)**.** Set

$$v(t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t - x_1^3 - \cdots - x_n^3)\, dx.$$

If $n = 2$, $v(t)$ satisfies the ordinary differential equation $Pv(t) = 0$ with

$$P = 729t^3{\partial_t}^6 + 6561t^2{\partial_t}^5 + 12555t{\partial_t}^4 + (81t^2 + 3240){\partial_t}^3 + 243t{\partial_t}^2 + 60\partial_t + 2t.$$

The origin is a regular singular point of P with the indicial polynomial $b(s) = s(s-1)^2(s-2)(3s+1)(3s-7)$ up to a constant multiple.

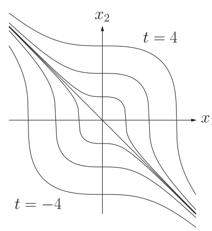

Fig. 3. Curves $x_1^3 + x_2^3 = t$ with $t = -4, -1, -1/8, 0, 1/8, 1, 4$

If $n = 3$, $v(t)$ satisfies the ordinary differential equation $Pv(t) = 0$ with

$$P = 6561t^4{\partial_t}^9 + 118098t^3{\partial_t}^8 + 607257t^2{\partial_t}^7 + (1458t^3 + 944055t){\partial_t}^6 + (13122t^2 + 280665){\partial_t}^5 + 25920t{\partial_t}^4 + (99t^2 + 8100){\partial_t}^3 + 297t{\partial_t}^2 + 90\partial_t + 2t.$$

Its indicial polynomial at 0 is

$$b(s) = s^2(s-1)(s-2)(s-3)(s-4)^2(3s-4)(3s-8)$$

up to a constant multiple. Hence in both cases, $v(t)$ is Lebesgue integrable on $\mathbb{R}$ and real analytic on $\mathbb{R} \setminus \{0\}$. In [18] it is proved that $v(t)$ satisfies a linear differential equation of order $3n$ with a regular singularity at the origin.

Example 6.4. Let us consider

$$v(t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t - x_1^4 - x_2^4 - \cdots - x_n^4)\, dx.$$

If $n = 2$, then $v(t)$ is annihilated by

$$128t^3{\partial_t}^4 + 768t^2{\partial_t}^3 + (-24t^2 + 864t){\partial_t}^2 + (-48t + 96)\partial_t + t - 6,$$

which has a regular singularity at 0 with the indicial polynomial $b(s) = s^2(2s-1)(2s+1)$ up to a constant multiple. If $n = 3$, $v(t)$ is annihilated by

$$\begin{aligned}&2048t^4{\partial_t}^6 + 24576t^3{\partial_t}^5 + (-768t^3 + 77568t^2){\partial_t}^4\\ &\quad + (-4608t^2 + 64512t){\partial_t}^3 + (88t^2 - 5328t + 7560){\partial_t}^2\\ &\quad + (176t - 720)\partial_t - 3t + 27,\end{aligned}$$

which has a regular singular point at 0 with the indicial polynomial

$$b(s) = s(s-1)(4s+1)(4s-1)(4s-3)(4s-5)$$

up to a constant multiple.

Example 6.5. Let us consider

$$v(t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t - x_1x_2\cdots x_n)\,dx.$$

If $n = 2$, then $v(t)$ is annihilated by $t{\partial_t}^2 + \partial_t - t$, which has 0 as a regular singular point with the indicial polynomial $b(s) = s^2$. If $n = 3$, then $v(t)$ is annihilated by $t^2{\partial_t}^3 + 3t{\partial_t}^2 + \partial_t + t$, which has 0 as a regular singular point with the indicial polynomial $b(s) = s^3$. If $n = 4$, then $v(t)$ is annihilated by $t^3{\partial_t}^4 + 6t^2{\partial_t}^3 + 7t{\partial_t}^2 + \partial_t - t$ with the indicial polynomial $b(s) = s^4$ at 0.

Example 6.6. Introducing parameters $a = (a_1, a_2)$, let us consider the density function

$$v(t, a) = (2\pi)^{-1} \int_{\mathbb{R}^2} \exp\Big(-\frac{1}{2}(x_1^2 + x_2^2)\Big)\delta(t - (x_1 + a_1)(x_2 + a_2))\,dx_1dx_2.$$

The integrand belongs to $\mathcal{SS}'(\mathbb{R}_x^2 \times (\mathbb{R}_t \times \mathbb{R}_a^2))$, hence $v(t, a)$ to $\mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_a^2)$. Integrating the holonomic module over D_5 for the integrand shows that $v(t, a)$ satisfies a holonomic system $P_jv(t, a) = 0$ $(1 \le j \le 6)$ with

$$\begin{aligned}
P_1 &= -4t\partial_{a_1} + \partial_{a_2}^3 + 4a_2\partial_{a_2}^2 + (4{a_2}^2 + 4)\partial_{a_2} + 4a_2,\\
P_2 &= (\partial_{a_2} + 2a_2)\partial_{a_1} + 2a_1\partial_{a_2} - 4t + 4a_2a_1,\\
P_3 &= 2t\partial_t + \partial_{a_2}^2 + 2a_2\partial_{a_2} + 2, \qquad P_4 = (\partial_{a_1} + 2a_1)\partial_t + 2\partial_{a_2},\\
P_5 &= (\partial_{a_2} + 2a_2)t\partial_t + 2\partial_{a_1}, \qquad P_6 = \partial_{a_1}^2 + 2a_1\partial_{a_1} - \partial_{a_2}^2 - 2a_2\partial_{a_2}.
\end{aligned}$$

The characteristic variety of $M := D_3/(D_3P_1 + \cdots + D_3P_6)$ is

$$\{(t, a_1, a_2; \tau, \xi_1, \xi_2) \in \mathbb{C}^6 \mid \xi_2^2 = \xi_1\xi_2 = \tau\xi_2 = \xi_1^2 = \tau\xi_1 = t\tau^2 = 0\}$$
$$= \{t = \xi_1 = \xi_2 = 0\} \cup \{\tau = \xi_1 = \xi_2 = 0\}.$$

The singular locus of M is $\{t = 0\}$. By elimination we obtain an operator

$$P = t\partial_t^4 + 3\partial_t^3 + (-2t - a_1a_2)\partial_t^2 + (a_1^2 + a_2^2 - 3)\partial_t + t - a_1a_2$$

in t with parameters a_1, a_2 which annihilates $v(t, a)$. The indicial polynomial of P at $t = 0$ is $s^2(s-1)(s-2)$. The Fourier transform gives us a differential equation

$$\Big((\tau^2 + 1)^2 \frac{d}{d\tau} + \tau^3 + ia_1a_2\tau^2 + (a_1^2 + a_2^2 + 1)\tau - ia_1a_2\Big)\hat{v}(\tau) = 0$$

for the characteristic function $\hat{v}(\tau)$. By quadrature we obtain

$$\hat{v}(\tau) = \frac{1}{\sqrt{\tau^2 + 1}} \exp\Big(\frac{2ia_1a_2\tau + a_1^2 + a_2^2}{2(\tau^2 + 1)} - \frac{1}{2}(a_1^2 + a_2^2)\Big).$$

Exercise 27. Compute a differential equation for

$$v(t) = \int_{\mathbb{R}^3} \exp\Big(-\frac{x_1^2 + x_2^2 + x_3^2}{2}\Big)\delta(t - x_1^2 + x_2^2 + x_3^2)\, dx_1dx_2dx_3$$

and one for its characteristic function $\hat{v}(\tau)$. Give an explicit formula for $\hat{v}(\tau)$.

Exercise 28. Set $u(x, t, a, b) = e^{tx}x_+^{a-1}(1-x)_+^{b-1}$ with positive real parameters a, b. We regard it as an element of $\mathcal{E}'\mathcal{D}'(\mathbb{R}_x \times \mathbb{R}_t)$ with a and b fixed. Deduce a linear differential equation (in t) for

$$v(t, a, b) := \int_0^1 e^{tx}x^{a-1}(1-x)^{b-1}\, dx = \int_{-\infty}^{\infty} u(x, t, a, b)\, dx$$

regarding a, b as parameters.

6.2. Powers of polynomials times a holonomic function

Let us begin with the simplest example: For a complex number λ with non-negative real part, the distribution x_+^λ on $\mathbb{R}$ satisfies a holonomic system $(x\partial_x - \lambda)x_+^\lambda = 0$. In particular, we have $x\partial_x Y(x) = 0$. This amounts to introducing the differential equation $(x\partial_x - s)x^s = 0$ for a formal function x^s, which corresponds to x_+^λ, and specializing the parameter s to λ. We cannot regard $x^0 = 1$ because x^0 does not correspond to the constant function 1 but to $Y(x) = x_+^0$.

Now let $f_1(x), \dots, f_p(x)$ be non-constant real polynomials in the variables $x = (x_1, \dots, x_n)$. Let $v(x)$ be a holonomic locally integrable function on U. Then

$$\tilde{v}(x) = (f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} v(x)$$

is also locally integrable on U for complex numbers $\lambda_1, \dots, \lambda_p$ with non-negative real parts. Especially, we have $\tilde{v}(x) = Y(f_1) \cdots Y(f_p) v(x)$ if $\lambda_j = 0$ $(j = 1, \dots, p)$. Our purpose is to compute a holonomic system for $\tilde{v}(x)$.

Our strategy is as follows: First we work in a purely algebraic setting and consider the D-module generated by the tensor product $u \otimes f_1^{\lambda_1} \cdots f_p^{\lambda_p}$; we show that this D-module is holonomic and introduce an algorithm to compute its structure. Then we 'realize' these arguments and apply to the corresponding distribution $\tilde{v}(x)$, which lives in the 'real world'.

Let $\mathbb{K}$ be a field of characteristic zero and $f_1, \dots, f_p \in \mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_n]$ be non-constant polynomials. Let us consider a 'function' $f_1^{s_1} \cdots f_p^{s_p}$ with indeterminates (as parameters) $s = (s_1, \dots, s_p)$. More precisely, set

$$\mathcal{L} := \mathbb{K}[x, (f_1 \cdots f_p)^{-1}, s] f_1^{s_1} \cdots f_p^{s_p},$$

which is regarded as a free $\mathbb{K}[x, (f_1 \cdots f_p)^{-1}, s]$-module generated by the 'symbol' $f_1^{s_1} \cdots f_p^{s_p}$. Then $\mathcal{L}$ is a left $D_n[s]$-module with the natural derivations

$$\partial_{x_i}(f_1^{s_1} \cdots f_p^{s_p}) = \sum_{j=1}^{p} s_j \frac{\partial f_j}{\partial x_i} f_j^{-1} f_1^{s_1} \cdots f_p^{s_p} \qquad (i = 1, \dots, n).$$

In what follows we denote $f^s = f_1^{s_1} \cdots f_p^{s_p}$ for the sake of simplicity when there is no fear of confusion.

Let $M = D_n u = M/I$ be a holonomic left D_n-module generated by an element $u \in M$ with the left ideal $I = \mathrm{Ann}_{D_n} u$. Let us consider the tensor product $M \otimes_{\mathbb{K}[x]} \mathcal{L}$ as $\mathbb{K}[x]$-module, which has also a natural structure of left $D_n[s]$-module induced by the derivations

$$\partial_{x_i}(u' \otimes v) = (\partial_{x_i} u') \otimes v + u' \otimes (\partial_{x_i} v) \quad (u' \in M,\, v \in \mathcal{L},\, i = 1, \dots, n).$$

Our aim is to compute the annihilator (in $D_n[s]$) of $u \otimes f^s \in M \otimes_{\mathbb{K}[x]} \mathcal{L}$. For this purpose, define shift (difference) operators E_j by

$$E_j : \mathcal{L} \ni a(x, s_1, \dots, s_p) f^s \longmapsto a(x, s_1, \dots, s_j + 1, \dots, s_p) f_j f^s \in \mathcal{L}$$

for $j = 1, \dots, p$, which are bijective with the inverse shifts $E_j^{-1} : \mathcal{L} \to \mathcal{L}$.

Let $D_n\langle s, E, E^{-1}\rangle$ be the D_n-algebra generated by $s = (s_1, \dots, s_p)$, $E = (E_1, \dots, E_p)$, and $E^{-1} = (E_1^{-1}, \dots, E_p^{-1})$. We introduce new variables $t = (t_1, \dots, t_p)$ and the associated derivations $\partial_t = (\partial_{t_1}, \dots, \partial_{t_p})$. Let D_{n+p} be the ring of differential operators with respect to the variables $(x, t) = (x_1, \dots, x_n, t_1, \dots, t_p)$.

Let $\mu : D_{n+p} \to D_n\langle s, E, E^{-1}\rangle$ be the D_n-algebra homomorphism (Mellin transform) of D_n defined by

$$\mu(t_j) = E_j, \quad \mu(\partial_{t_j}) = -s_j E_j^{-1}.$$

This homomorphism is well-defined since

$$\begin{aligned}\mu(\partial_{t_j} t_j - t_j \partial_{t_j}) &= \mu(\partial_{t_j})\mu(t_j) - \mu(t_j)\mu(\partial_{t_j}) \\ &= -s_j E_j^{-1} E_j - E_{s_j}(-s_i) E_j^{-1} = 1.\end{aligned}$$

It can be extended to an isomorphism

$$\mu : D_{n+p}[(t_1 \cdots t_p)^{-1}] \xrightarrow{\sim} D_n\langle s, E, E^{-1}\rangle$$

such that $\mu(t_j^{-1}) = E_j^{-1}$. Hence we can regard D_{n+p} as a subring of $E\langle s, E, E^{-1}\rangle$ through μ. With this identification, we have

$$t_j = E_j, \qquad \partial_{t_j} = -s_j E_j^{-1}, \qquad s_j = -\partial_{t_j} t_j = -t_j \partial_{t_j} - 1.$$

Thus we have inclusions

$$D_n[s] \subset D_n\langle s, E\rangle \subset D_{n+p} \subset D_n\langle s, E, E^{-1}\rangle = D_{n+p}[(t_1 \cdots t_p)^{-1}]$$

of rings. Note that $M \otimes_{\mathbb{K}[x]} \mathcal{L}$ has a structure of left $D_n\langle s, E, E^{-1}\rangle$-module with s and E acting only on $\mathcal{L}$.

Lemma 6.7. *The submodule $D_{n+p} f^s$ of $\mathcal{L}$ is a free $\mathbb{K}[x]$-module generated by $\partial_t^\nu f^s$ with $\nu \in \mathbb{N}^p$.*

Proof. Let J be the left ideal of D_{n+p} generated by

$$P_i := \partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial_{t_j} \quad (i = 1, \dots, n), \quad t_j - f_j \quad (j = 1, \dots, p).$$

It is easy to see that J annihilates f^s. In order to show that J coincides with $\mathrm{Ann}_{D_{n+p}} f^s$, let $\prec$ be a lexicographic term order for D_{n+p} such that $\partial_{t_j} \prec \partial_{x_i}$, $x_j \prec \partial_{x_i}$, and $x_i \prec t_j$ for $1 \le i \le n$ and $1 \le j \le p$. Then by division with respect to $\prec$, every element P of D_{n+p} is written in the form

$$P = \sum_{i=1}^{n} U_i P_i + \sum_{j=1}^{p} V_j (t_j - f_j) + \sum_{\nu \in \mathbb{N}^p} r_\nu(x) \partial_t^\nu$$

with $U_i, V_j \in D_{n+p}$ and $r_\nu(x) \in \mathbb{K}[x]$. This implies

$$Pf^s = \sum_{\nu \in \mathbb{N}^p} r_\nu(x)\partial_t^\nu f^s = \sum_{\nu \in \mathbb{N}^p} (-1)^{|\nu|}[s]_\nu r_\nu(x) f^{s-\nu}$$

with

$$\begin{aligned}[s]_\nu &= [s_1]_{\nu_1} \cdots [s_p]_{\nu_p} \\ &= s_1(s_1-1)\cdots(s_1-\nu_1+1)\cdots s_p(s_p-1)\cdots(s_p-\nu_p+1).\end{aligned}$$

Hence $Pf^s = 0$ holds if and only if $r_\nu(x) = 0$ for each ν since $\mathcal{L}$ is a free $\mathbb{K}[s]$-module. This also implies that an element of $D_{n+p}f^s$ is uniquely written in the form $\sum_{\nu\in\mathbb{N}^p} r_\nu(x)\partial_t^\nu f^s$ with $r_\nu(x) \in \mathbb{K}[x]$. This completes the proof. Q.E.D.

Our primary purpose in the following algebraic arguments is to compute the $D_n[s]$-submodule $D_n[s](u \otimes f^s)$ of $M \otimes_{\mathbb{K}[x]} \mathcal{L}$. With this purpose in mind, we consider the following modules:

- The D_{n+p}-submodule $N := D_{n+p}(u \otimes f^s)$ of $M \otimes_{\mathbb{K}[x]} \mathcal{L}$,
- The $D_n[s]$-submodule $N_s := D_n[s](u \otimes f^s)$ of N,
- The D_{n+p}-submodule $N' := D_{n+p}(u \otimes f^s)$ of $M \otimes_{\mathbb{K}[x]} D_{n+p}f^s$,
- The $D_n[s]$-submodule $N'_s := D_n[s](u \otimes f^s)$ of N'.

We will see that N' coincides with $M \otimes_{\mathbb{K}[x]} D_{n+p}f^s$ in fact. The inclusion $D_{n+p}f^s \subset \mathcal{L}$ induces a natural homomorphism

$$\iota \,:\, M \otimes_{\mathbb{K}[x]} D_{n+p}f^s \longrightarrow M \otimes_{\mathbb{K}[x]} \mathcal{L}$$

such that $\iota(N') = N$ and $\iota(N'_s) = N_s$. Let us first determine the structure of N'.

Algorithm 6.8 ($N' = M \otimes_{\mathbb{K}[x]} D_{n+p}f^s$)**.** Input: A set G_0 of generators of I with $M = D_n/I$ and non-constant polynomials $f_1, \dots, f_p \in \mathbb{K}[x]$.

For $P = P(x, \partial_{x_1}, \dots, \partial_{x_n}) \in G_0$, set

$$\tau(P, f_1, \dots, f_p) := P\left(x,\, \partial_{x_1} + \sum_{j=1}^p \frac{\partial f_j}{\partial x_1}\partial_{t_j}, \dots, \partial_{x_n} + \sum_{j=1}^p \frac{\partial f_j}{\partial x_n}\partial_{t_j}\right)$$

following [35]. This substitution is well-defined in the ring D_{n+p} since the operators which are substituted for $\partial_{x_1}, \dots, \partial_{x_n}$ commute with one another.

Output: $G := \{\tau(P, f_1, \dots, f_p) \mid P \in G_0\} \cup \{t_j - f_j(x) \mid j = 1, \dots, p\}$ generates $J := \mathrm{Ann}_{D_{n+p}}(u \otimes f^s)$ and $M \otimes_{\mathbb{K}[x]} D_{n+p}f^s = N' = D_{n+p}/J$ is holonomic.

Proof. In view of the equality

$$\left(\partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i}\partial_{t_j}\right)(u \otimes f^s)$$
$$= (\partial_{x_i} u) \otimes f^s + u \otimes \left(\partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i}\partial_{t_j}\right) f^s$$
$$= (\partial_{x_i} u) \otimes f^s + u \otimes \left(\partial_{x_i} + \sum_{j=1}^{p} (-s_j) f_j^{-1} \frac{\partial f_j}{\partial x_i}\right) f^s = (\partial_{x_i} u) \otimes f^s$$

in $M \otimes_{\mathbb{K}[x]} \mathcal{L}$, we have, for any $P \in D_n$ and $j = 1, \dots, p$,

$$\tau(P, f_1, \dots, f_p)(u \otimes f^s) = (Pu) \otimes f^s, \quad (t_j - f_j)(u \otimes f^s) = u \otimes (t_j - f_j) f^s.$$

Hence J annihilates $u \otimes f^s$ in $M \otimes_{\mathbb{K}[x]} \mathcal{L}$. These relations also imply that $M \otimes_{\mathbb{K}[x]} D_{n+p} f^s$ is generated by $u \otimes f^s$ over D_{n+p} since $D_{n+p} f^s$ is generated by $\partial_t^\nu f^s$ ($\nu \in \mathbb{N}^p$) over $\mathbb{K}[x]$.

Conversely, suppose that $P \in D_{n+p}$ annihilates $u \otimes f^s$. We can rewrite P in the form

$$P = \sum_{\alpha \in \mathbb{N}^n, \nu \in \mathbb{N}^p} p_{\alpha,\nu}(x) \partial_t^\nu \Big(\partial_{x_1} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_1}\partial_{t_j}\Big)^{\alpha_1} \cdots \Big(\partial_{x_n} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_n}\partial_{t_j}\Big)^{\alpha_n} + \sum_{j=1}^{p} Q_j \cdot (t_j - f_j(x))$$

with $p_{\alpha,\nu}(x) \in \mathbb{K}[x]$ and $Q_j \in D_{n+p}$. Setting $P_\nu := \sum_{\alpha \in \mathbb{N}^n} p_{\alpha,\nu}(x)\partial_x^\alpha$, we get

$$0 = P(u \otimes f^s) = \sum_{\nu \in \mathbb{N}^p} \partial_t^\nu \tau(P_\nu, f_1, \dots, f_p)(u \otimes f^s) = \sum_{\nu \in \mathbb{N}^p} P_\nu u \otimes \partial_t^\nu f^s.$$

It follows in view of Lemma 6.7 that each P_ν belongs to I. Hence we have

$$P = \sum_{\nu \in \mathbb{N}^p} \partial_t^\nu \tau(P_\nu, f_1, \dots, f_p) + \sum_{j=1}^{p} Q_j \cdot (t_j - f_j(x)) \ \in J.$$

Finally, let us show that D_{n+p}/J is holonomic. Since D_n/I is holonomic, its characteristic variety $\mathrm{Char}(D_n/I)$ is an n-dimensional algebraic set of $\overline{\mathbb{K}}^{2n}$. By the definition, we have

$$\begin{aligned}
&\mathrm{Char}(D_{n+p}/J) \\
&\subset \Big\{(x,t,\xi,\tau) \in \overline{\mathbb{K}}^{2(n+p)} \mid \sigma(P)\Big(x, \xi_1 + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_1}\tau_j, \dots\Big) = 0 \ (\forall P \in I), \\
&\qquad t_j = f_j(x)\ (1 \le j \le p)\Big\} \\
&= \Big\{(x,t,\xi,\tau) \in \overline{\mathbb{K}}^{2(n+p)} \mid \Big(x, \xi_1 + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_1}\tau_j, \dots\Big) \in \mathrm{Char}(D_n/I), \\
&\qquad t_j = f_j(x)\ (1 \le j \le p)\Big\}.
\end{aligned}$$

Since the set on the last line is in one-to-one correspondence with the set $\mathrm{Char}(D_n/I) \times \overline{\mathbb{K}}^p$, the dimension of $\mathrm{Char}(D_{n+p}/J)$ is $n+p$, which implies that D_{n+p}/J is a holonomic module. This proves the correctness of the algorithm. Q.E.D.

Now that we have a set of generators of $J = \mathrm{Ann}_{D_{n+p}}(u \otimes f^s)$, we can compute $\mathrm{Ann}_{D_n[s]}(u \otimes f^s) = J \cap D_n[s]$ by using Algorithm 5.5. Thus we get an explicit presentation of the $D_n[s]$-submodule $N'_s = D_n[s](u \otimes f^s)$ of $M \otimes_{\mathbb{K}[x]} D_{n+p} f^s$. Finally let us specialize the parameters $s = (s_1, \dots, s_n)$ to $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{K}^p$. Set

$$\begin{aligned}
N'(\lambda) &= N'_s/((s_1 - \lambda_1)N'_s + \cdots + (s_p - \lambda_p)N'_s), \\
N(\lambda) &= N_s/((s_1 - \lambda_1)N_s + \cdots + (s_p - \lambda_p)N_s),
\end{aligned}$$

which are left D_n-modules.

Proposition 6.9. *Set $f = f_1 \cdots f_p$. The homomorphism*

$$\iota \,:\, N' = M \otimes_{\mathbb{K}[x]} D_{n+p} f_1^{s_1} \cdots f_p^{s_p} \longrightarrow M \otimes_{\mathbb{K}[x]} \mathcal{L}$$

is injective if and only if M is f-saturated, or f-torsion free, i.e., $f : M \to M$ is injective. In particular, ι induces isomorphisms $N' \cong N$ and $N'_s \cong N_s$ if M is f-saturated

Proof. By Lemma 6.7, an arbitrary element w of $M \otimes_{\mathbb{K}[x]} D_{n+p} f^s$ is uniquely written in a finite sum

$$w = \sum_{\nu \in \mathbb{N}^p} u_\nu \otimes \partial_t^\nu f_1^{s_1} \cdots f_p^{s_p}$$

with $u_\nu \in M$. Then we have

$$\iota(w) = \sum_{\nu\in\mathbb{N}^p} (-1)^{|\nu|}[s]_\nu u_\nu \otimes f_1^{s_1-\nu_1}\cdots f_p^{s_p-\nu_p}.$$

Since $\mathcal{L}$ is isomorphic to $\mathbb{K}[x,s,f^{-1}]$ with $f = f_1\cdots f_p$ as a $\mathbb{K}[x,s]$-module, $M\otimes_{\mathbb{K}[x]}\mathcal{L}$ is isomorphic to the localization $M[s,f^{-1}]$ of $M[s] := M\otimes_{\mathbb{K}}\mathbb{K}[s]$ with respect to f as a $\mathbb{K}[x,s]$-module. Hence $\iota(w)$ vanishes if and only if

$$\sum_{\nu\in\mathbb{N}^p} (-1)^{|\nu|}[s]_\nu f^{-\nu_1}\cdots f^{-\nu_p}u_\nu$$

vanishes in $M[s,f^{-1}]$. This amounts to the condition $u_\nu = 0$ in $M[f^{-1}]$ for all ν. Finally, u_ν vanishes in $M[f^{-1}]$ if and only if $f^{m_\nu}u_\nu = 0$ holds in M with a non-negative integer m_ν. This implies the assertion. Q.E.D.

The following theorem was proved in case $p = 1$ by Kashiwara [13] in the analytic setting.

Theorem 6.10. *For any $\lambda\in\mathbb{K}^p$, $N(\lambda)$ is a holonomic D_n-module.*

Proof. First, let us show that $N'(\lambda)$ is holonomic if either no component of λ belongs to $\mathbb{N}$ or M is f-saturated with $f = f_1\cdots f_p$.

We may assume $M \neq \{0\}$. Set $F_k(N') = F_k^{(\mathbf{1};\mathbf{1})}(D_{n+p})(u\otimes f^s)$ for $k\in\mathbb{Z}$. Since N' is holonomic, there exists a polynomial $H_0(k)$ of degree $n+p$ such that

$$\dim_{\mathbb{K}} F_k(N') = H_0(k) \qquad (\forall k \gg 0).$$

We define a filtration $F_k(D_n[s])$ on the ring $D_n[s]$ by

$$F_k(D_n[s]) = \Big\{\sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma}x^\alpha\partial_x^\beta s^\gamma \mid |\alpha|+|\beta|+2|\gamma|\le k\Big\}.$$

Then $F_k(N'_s) := F_k(D_n[s])(u\otimes f^s)$ defines a good filtration on N'_s as a filtered module over the filtered ring $D_n[s]$. The associated graded module $\mathrm{gr}(N'_s)$ is a graded module over the graded ring $\mathbb{K}[x,\xi,s]$ in which x_i, ξ_i are of order one, and s_j are of order two. Hence (see e.g., [5], [8], [12]) there exists $Q(T)\in\mathbb{Z}[T,T^{-1}]$ such that

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\dim_{\mathbb{K}}\mathrm{gr}_k(N'_s)T^k &= \sum_{k=-\infty}^{\infty}(\dim_{\mathbb{K}} F_k(N'_s) - \dim_{\mathbb{K}} F_{k-1}(N'_s))T^k \\ &= \frac{Q(T)}{(1-T)^{2n}(1-T^2)^p}\end{aligned}$$

holds as a formal Laurent series in an indeterminate T. It follows that there exists a polynomial $H_1(k) \in \mathbb{Q}[k]$ such that

$$\dim_{\mathbb{K}} F_{2k}(N'_s) = H_1(k) \qquad (\forall k \gg 0).$$

The degree of $H_1(k)$ is at most $n+p$ since $F_k(N'_s) \subset F_k(N')$.

Now, let us consider the $D_n[s]$-endomorphism of N'_s defined by $s_p - \lambda_p = -\partial_{t_p} t_p - \lambda_p$. Let v be an element of N'_s such that $(s_p - \lambda_p)v = 0$. As an element of N', v is written uniquely in the form

$$v = \sum_{\nu \in \mathbb{N}^p} v_\nu \otimes \partial_t^\nu f^s$$

with $v_\nu \in M$. Then $(s_p - \lambda_p)v = 0$ is equivalent to

$$(14) \qquad (\nu_p - \lambda_p)v_\nu - f_p v_{\nu - e_p} = 0 \qquad (\forall \nu = (\nu_1, \dots, \nu_p) \in \mathbb{N}^p)$$

with $e_p = (0, \dots, 0, 1) \in \mathbb{N}^p$ since

$$s_p \partial_{t_p}^j = -\partial_{t_p} t_p \partial_{t_p}^j = -\partial_{t_p}^{j+1} t_p + j\partial_{t_p}^j, \qquad t_p f^s = f_p f^s.$$

First let us assume that $\lambda_p \neq 0, 1, 2, \dots$. Then from (14) we deduce $v_\nu = 0$ for any $\nu \in \mathbb{N}^p$ by induction on ν_p. Hence $s_p - \lambda_p : N'_s \to N'_s$ is injective.

Next, assume that $\lambda_p \in \mathbb{N}$ and that M is f-saturated. There exists d such that $v_\nu = 0$ if $\nu_p \geq d$. Then from (14), it follows that $f^j v_{(\nu', d-j)} = 0$ for all $0 \leq j \leq d$ and $\nu' \in \mathbb{N}^{p-1}$ by induction on j. Hence we have $v_\nu = 0$ for all $\nu \in \mathbb{N}^p$ if M is f-saturated.

Hereafter we suppose that $s_p - \lambda_p$ defines an injective endomorphism of N'_s. Then we have an exact sequence

$$0 \longrightarrow N'_s \stackrel{s_p - \lambda_p}{\longrightarrow} N'_s \longrightarrow N'_s/(s_p - \lambda_p)N'_s \longrightarrow 0$$

of left $D_n[s']$-modules with $s' = (s_1, \dots, s_{p-1})$. Let us regard $N'' := N'/(s_p - \lambda_p)N'$ as a filtered module over the filtered ring $D_n[s_1, \dots, s_{p-1}]$ and define a good filtration on N'' by

$$F_k(N'') = F_k(N'_s)/(F_k(N'_s) \cap (s_p - \lambda_p)N'_s).$$

Since $(s_p - \lambda_p)F_{2k-2}(N'_s)$ is contained in $F_{2k}(N'_s) \cap (s_p - \lambda_p)N'_s$, we have

$$\begin{aligned}\dim_{\mathbb{K}} F_{2k}(N'') &= \dim_{\mathbb{K}} F_{2k}(N'_s) - \dim_{\mathbb{K}}(F_{2k}(N'_s) \cap (s_p - \lambda_p)N'_s) \\ &\leq \dim_{\mathbb{K}} F_{2k}(N'_s) - \dim_{\mathbb{K}} F_{2k-2}(N'_s) = H_1(k) - H_1(k-1)\end{aligned}$$

for sufficiently large k. Note that $H_1(k) - H_1(k-1)$ is a polynomial of degree at most $n+p-1$. Proceeding inductively, we obtain a polynomial $H_p(k)$ of degree at most n such that

$$\dim_{\mathbb{K}} F_{2^p k}(N'(\lambda)) = H_p(k) \qquad (\forall k \gg 0)$$

with a good $(\mathbf{1};\mathbf{1})$-filtration

$$F_k(N'(\lambda)) := F_k(N'_s)/(F_k(N'_s) \cap ((s_1 - \lambda_1)N'_s + \cdots + (s_p - \lambda_p)N'_s))$$

on $N'(\lambda)$. On the other hand, there exists a polynomial $G(k)$ such that

$$\dim_{\mathbb{K}} F_k(N'(\lambda)) = G(k) \qquad (\forall k \gg 0).$$

Hence $H_p(k) = G(2^p k)$ holds for sufficiently large k. This implies

$$\dim N'(\lambda) = \deg G(k) = \deg H_p(k) \leq n.$$

Thus $N'(\lambda)$ is a holonomic D_n-module under the assumption above.

Finally, let us show $N(\lambda)$ is holonomic. The localization $M[f^{-1}]$ has a natural structure of left D_n-module and is holonomic as such ([13], [4]; see also Theorem 2.14 in [25] for a constructive proof). Moreover, $M[f^{-1}]$ is f-saturated by the definition. Let $\rho : M \ni Pu \mapsto Pu \otimes 1 \in M[f^{-1}]$ be the canonical homomorphism. Then by Proposition 6.9, the canonical homomorphism

$$\iota \,:\, \rho(M) \otimes_{\mathbb{K}[x]} D_{n+p} f^s \longrightarrow \rho(M) \otimes_{\mathbb{K}[x]} \mathcal{L}$$

is injective. Hence the submodule $\tilde{N}_s := D_n[s](\rho(u) \otimes f^s)$ of $\rho(M) \otimes_{\mathbb{K}[x]} \mathcal{L}$ is isomorphic to the submodule $\tilde{N}'_s := D_n[s](\rho(u) \otimes f^s)$ of $\rho(M) \otimes_{\mathbb{K}[x]} D_{n+p} f^s$. On the other hand, ρ induces an isomorphism $M \otimes_{\mathbb{K}[x]} \mathcal{L} \to \rho(M) \otimes_{\mathbb{K}[x]} \mathcal{L}$ by the definition. This induces an isomorphism $N_s \cong \tilde{N}_s$. Summing up, we obtain an isomorphism

$$N(\lambda) \cong \tilde{N}'(\lambda) := \tilde{N}'_s/((s_1 - \lambda_1)\tilde{N}'_s + \cdots + (s_p - \lambda_p)\tilde{N}'_s).$$

The module of the right-hand side is holonomic by the argument above. This completes the proof. Q.E.D.

Note that if $\mathbb{K}$ is algebraically closed then Theorem 6.10 holds under a weaker assumption that M be holonomic on $\{x \in \mathbb{K}^n \mid f_1(x) \cdots f_p(x) \neq 0\}$ since it implies that $M[f^{-1}]$ is a holonomic D_n-module ([13]; see also Theorem 3.14 of [25] for an elementary proof). We do not know if $N'(\lambda)$, which can be computed directly, is always holonomic; the proof of Theorem 4 in [24] is insufficient. In general, $N(\lambda)$ is stronger and more suited to our application below than $N'(\lambda)$. Summing up we have obtained

Algorithm 6.11 ($N(\lambda)$)**.** Input: A set G_0 of generators of I with $M = D_n/I = D_n u$, non-constant polynomials $f_1, \dots, f_p \in \mathbb{K}[x]$, and $\lambda \in \mathbb{K}^p$.
Output: $N(\lambda) = D_n/J$.

(1) Compute the D_n-submodule $\rho(M) = D_n\rho(u)$ of the localization $M[(f_1 \cdots f_p)^{-1}]$ with $\rho(u) = u \otimes 1$ by using the localization algorithm of [25], which is a modification of the one in [28].
(2) Compute a set G_1 of generators of the annihilator of $\rho(u) \otimes f^s$ in $\rho(M) \otimes_{\mathbb{K}[x]} D_{n+p} f^s$ by Algorithm 6.8.
(3) Compute generators $P_1(s), \dots, P_r(s)$ of the annihilator

$$\mathrm{Ann}_{D_n[s]}(\rho(u) \otimes f^s) = D_n[s] \cap \mathrm{Ann}_{D_{n+p}}(\rho(u) \otimes f^s)$$

by using G_1 and Algorithm 5.5.
(4) Let J be the left ideal of D_n generated by $P_1(\lambda), \dots, P_r(\lambda)$.

In practice, we can skip step (1) and proceed to steps (2),(3),(4) with $\rho(M)$ replaced by M. This gives us $N'(\lambda)$ which might be weaker than $N(\lambda)$.

Now let us return to the 'real world'. Assume that $f_1, \dots, f_p \in \mathbb{R}[x]$ and let $v(x)$ be a locally integrable holonomic function on an open set U of $\mathbb{R}^n$. Then

$$\tilde{v}(x, \lambda) := v(x)(f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p}$$

is well-defined as a locally integrable function on U if the real parts of $\lambda_1, \dots, \lambda_p$ are non-negative. More precisely, $\tilde{v}(x, \lambda)$ belongs to the space $\mathcal{O}(\mathbb{C}_+^p, \mathcal{D}'(U))$ of $\mathcal{D}'(U)$-valued holomorphic functions on

$$\mathbb{C}_+^p = \{\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{C}^p \mid \mathrm{Re}\ \lambda_j > 0\ (1 \le j \le p)\},$$

that is, $\langle \tilde{v}(x, \lambda), \varphi(x) \rangle$ is a holomorphic function of $\lambda = (\lambda_1, \dots, \lambda_p)$ on $\mathbb{C}_+^p$ for any $\varphi \in C_0^\infty(U)$. Moreover, $\langle \tilde{v}(x, \lambda), \varphi(x) \rangle$ is continuous on the closure $\overline{\mathbb{C}_+^p}$.

Let I be a left ideal of D_n which annihilates $v(x)$ such that $M := D_n/I$ is holonomic. Set $M = D_n/I = D_n u$ with $u = \overline{1}$ and $\mathcal{L} = \mathbb{K}[x, (f_1 \cdots f_p)^{-1}, s] f^s$ with $\mathbb{K} = \mathbb{C}$. In order to apply the algebraic arguments so far to $\tilde{v}(x, \lambda)$, we need the following fact:

Theorem 6.12. *Suppose that $P(s) \in D_n[s]$ annihilates $u \otimes f^s$ in $M \otimes_{\mathbb{C}[x]} \mathcal{L}$ and that the real parts of the components of $\lambda \in \mathbb{C}^p$ are non-negative. Then $P(\lambda)\tilde{v}(x, \lambda)$ vanishes as a distribution on U. Hence $\tilde{v}(x, \lambda)$ is a solution of the holonomic system $N(\lambda)$.*

Before proving this theorem, let us begin with

Lemma 6.13. *Suppose that $P(s) \in D_n[s]$ annihilates $u \otimes f^s$ in $M \otimes_{\mathbb{C}[x]} \mathcal{L}$. Then $P(\lambda)\tilde{v}(x,\lambda)$ vanishes as a distribution on $U_f := \{x \in U \mid f(x) \neq 0\}$ with $f = f_1 \cdots f_p$ for any $\lambda \in \mathbb{C}^p_+$.*

Proof. We denote $f_+^\lambda = (f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p}$. The $\mathbb{C}$-bilinear homomorphism

$$\Phi \,:\, M \times \mathcal{L} \ni (Pu,\, a(x,s,f^{-1})f^s) \longmapsto a(x,\lambda,f^{-1})f_+^\lambda Pv(x) \in \mathcal{O}(\mathbb{C}^p_+, \mathcal{D}'(U_f))$$

with $P \in D_n$, $a \in \mathbb{C}[x,s,t]$ is well-defined since $f_1,\ldots,f_p$ do not vanish on U_f. Moreover, Φ is $\mathbb{C}[x]$-balanced in the sense that

$$\Phi(c(x)Pu,\, a(x,s,f^{-1})) = \Phi(Pu,\, c(x)a(x,s,f^{-1})) \qquad (\forall c(x) \in \mathbb{C}[x]).$$

Hence Φ induces a $\mathbb{C}[x]$-homomorphism

$$\Psi \,:\, M \otimes_{\mathbb{C}[x]} \mathcal{L} \longrightarrow \mathcal{O}(\mathbb{C}^p_+, \mathcal{D}'(U_f))$$

such that

$$\Psi((Pu) \otimes a(x,s,f^{-1})f^s) = \Phi(Pu,\, a(x,s,f^{-1})f^s) = a(x,\lambda,f^{-1})f_+^\lambda Pv$$

because of the universality of the tensor product. It is easy to see that

$$\Psi((Pu) \otimes s_j a(x,s,f^{-1})f^s) = \lambda_j \Psi((Pu) \otimes a(x,s,f^{-1})f^s) \quad (1 \leq j \leq p),$$

and

$$\begin{aligned}
&\Psi(\partial_i((Pu) \otimes a(x,s,f^{-1})f^s)) = \Psi((\partial_i Pu) \otimes a(x,s,f^{-1})f^s) \\
&\quad + \Psi((Pu) \otimes (\partial_i a(x,s,f^{-1}))f^s)) \\
&\quad + \Psi\Big((Pu) \otimes \sum_{j=1}^p s_j f_j^{-1} \frac{\partial f_j}{\partial x_i} a(x,s,f^{-1})f^s\Big) \\
&= a(x,\lambda,f^{-1})f_+^\lambda \partial_i Pv(x) + \partial_i a(x,\lambda,f^{-1})f_+^\lambda Pv(x) \\
&\quad + \sum_{j=1}^p \lambda_j f_j^{-1} \frac{\partial f_j}{\partial x_i} a(x,\lambda,f^{-1})f_+^\lambda Pv(x) \\
&= \partial_i \left(a(x,\lambda,f^{-1})f_+^\lambda Pv(x)\right) \qquad (1 \leq i \leq n)
\end{aligned}$$

hold as distributions on U_f since f_+^λ is real analytic in x there. This implies that

$$\Psi(P(s)(u \otimes f^s)) = P(\lambda)\Psi(u \otimes f^s) = P(\lambda)(f_+^\lambda v(x))$$

holds for any $P(s) \in D_n[s]$. Hence the right-hand side vanishes as an element of $\mathcal{O}(\mathbb{C}^p_+, \mathcal{D}'(U_f))$ if $P(s)(u \otimes f^s) = 0$ in $M \otimes_{\mathbb{C}[x]} \mathcal{L}$. Q.E.D.

Next we generalize a lemma by Kashiwara and Kawai [16], which corresponds to the case $p = 1$:

Proposition 6.14. *Let U be an open set of $\mathbb{R}^n$ and let $f_1, \dots, f_p$ be real-valued real analytic functions on U such that $\{x \in U \mid f_1(x) > 0, \dots, f_p(x) > 0\}$ is not empty. Let $v(x)$ be a locally integrable function on U. Set $f = f_1 \cdots f_p$ and $U_f = \{x \in U \mid f(x) \neq 0\}$. Let $s_1, \dots, s_p$ be indeterminates and $\lambda_1, \dots, \lambda_p$ be complex variables. Assume that $P(s_1, \dots, s_p) \in D_n[s_1, \dots, s_p]$ satisfies*

$$P(\lambda_1, \dots, \lambda_p)((f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} v) = 0 \quad \textit{in } \mathcal{D}'(U_f)$$

if $\mathrm{Re}\, \lambda_j$ *$(j = 1, \dots, p)$ are sufficiently large. Then one has*

$$P(\lambda_1, \dots, \lambda_p)((f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} v) = 0 \quad \textit{in } \mathcal{D}'(U)$$

for any $\lambda_j \in \mathbb{C}$ with $\mathrm{Re}\, \lambda_j \geq 0$ *$(j = 1, \dots, p)$.*

Proof. We argue by induction on p. First let us set $p = 1$ and recall the proof of Lemma 2.9 in [16]. We denote $s = s_1$, $f = f_1$, and $\lambda = \lambda_1$. Let φ be an element of $C_0^\infty(U)$. Then its support $K = \mathrm{supp}\, \varphi$ is a compact subset of U. Let $\chi(t)$ be a C^∞ function of a variable t such that $\chi(t) = 1$ for $|t| \leq 1/2$ and $\chi(t) = 0$ for $|t| \geq 1$. Let τ be a real number with $0 < \tau < 1$. Since the support of $\big(1 - \chi(\frac{f}{\tau})\big)\varphi$ is contained in U_f, we have by the assumption that

$$\begin{aligned} \big\langle P(\lambda)(f_+^\lambda v),\ \varphi \big\rangle &= \Big\langle P(\lambda)(f_+^\lambda v),\ \chi\big(\frac{f}{\tau}\big)\varphi \Big\rangle \\ &= \int_K {}^t P(\lambda)\Big(\chi\big(\frac{f}{\tau}\big)\varphi\Big)(f_+^\lambda v)\, dx, \end{aligned}$$

where ${}^t P(\lambda)$ denotes the adjoint operator of $P(\lambda)$. Let m be the order of $P(s)$ and d be the degree of $P(s)$ in s. Then there exists a constant $C > 0$ such that

$$\sup_{x \in K} \left| {}^t P(\lambda)\Big(\chi\big(\frac{f(x)}{\tau}\big)\varphi(x)\Big) \right| \leq C(1 + |\lambda|)^d \tau^{-m} \qquad (0 < \forall \tau < 1).$$

Assume $\mathrm{Re}\,\lambda > m$ and $0 < \tau < 1$. Then we have

$$\begin{aligned}&\left|\int_K {}^tP(\lambda)\Big(\chi(\frac{f}{\tau})\varphi\Big)(f_+^\lambda v)\,dx\right|\\&\le C(1+|\lambda|)^d\tau^{-m}\int_{\{x\in K\,|\,0\le f(x)\le\tau\}}|f_+^\lambda v(x)|\,dx\\&\le C(1+|\lambda|)^d\tau^{-m+\mathrm{Re}\,\lambda}\int_K|v(x)|\,dx.\end{aligned}$$

This implies

$$\langle P(\lambda)(f_+^\lambda v),\ \varphi\rangle = \lim_{\tau\to+0}\int_K {}^tP(\lambda)\Big(\chi(\frac{f}{\tau})v\Big)(f_+^\lambda v)\,dx = 0.$$

By the uniqueness of analytic continuation with respect to λ, we know that $P(\lambda)(f_+^\lambda v) = 0$ in $\mathcal{D}'(U)$ if $\mathrm{Re}\,\lambda \ge 0$.

Now suppose that the assertion of the proposition is proved with p replaced by $p-1$. We use the notation $s = (s_1,\dots,s_p)$ and $\lambda = (\lambda_1,\dots,\lambda_p)$. Set

$$V = \{x\in U \mid f_1(x) = \cdots = f_p(x) = 0\}.$$

With the assumption of the proposition, we have

$$P(\lambda)((f_1)_+^{\lambda_1}\cdots(f_p)_+^{\lambda_p}v) = 0$$

on $U\setminus V$ if $\mathrm{Re}\,\lambda_j \ge 0$ for $j = 1,\dots,p$. In fact, for a point x_0 of $U\setminus V$, we may assume $f_p(x_0) > 0$. Then replacing v by $f_p^{\lambda_p}v$ and U by a neighborhood U_{x_0} of x_0, we conclude that $P(\lambda)((f_1)_+^{\lambda_1}\cdots(f_p)_+^{\lambda_p}v)$ vanishes as an element of $\mathcal{D}'(U_{x_0})$ if $\mathrm{Re}\,\lambda_j \ge 0$ $(1\le j\le p)$ by the induction hypothesis.

Since the support of $\big(1-\chi(\frac{f_1}{\tau})\cdots\chi(\frac{f_p}{\tau})\big)\varphi$ is contained in $U\setminus V$, we have

$$\begin{aligned}&\Big\langle P(\lambda)((f_1)_+^{\lambda_1}\cdots(f_p)_+^{\lambda_p}v),\ \varphi\Big\rangle\\&=\Big\langle P(\lambda)(f_1)_+^{\lambda_1}\cdots(f_p)_+^{\lambda_p}v),\ \chi(\frac{f_1}{\tau})\cdots\chi(\frac{f_p}{\tau})\varphi\Big\rangle\\&=\int_U {}^tP(\lambda)\Big(\chi(\frac{f_1}{\tau})\cdots\chi(\frac{f_p}{\tau})\varphi\Big)(f_1)_+^{\lambda_1}\cdots(f_p)_+^{\lambda_p}v\,dx.\end{aligned}$$

Let m be the order of $P(s)$ and d be the total degree of $P(s)$ in s. Then there exists a constant $C > 0$ such that

$$\sup_{x\in K}\left|{}^tP(\lambda)\Big(\chi(\frac{f_1(x)}{\tau})\cdots\chi(\frac{f_p(x)}{\tau})\varphi(x)\Big)\right| \le C(1+|\lambda_1|+\cdots+|\lambda_p|)^d\tau^{-m}$$

holds if $0 < \forall\tau < 1$. Set $K(\tau) := \{x \in K \mid 0 \le f_j(x) \le \tau \ (j = 1, \dots, p)\}$ with $K = \operatorname{supp}\varphi$. Then we have

$$\begin{aligned}
&\left|\left\langle P(\lambda)((f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} v),\ \varphi\right\rangle\right| \\
&\le C(1 + |\lambda_1| + \cdots + |\lambda_p|)^d \tau^{-m} \int_{K(\tau)} |(f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} v(x)|\, dx \\
&\le C(1 + |\lambda_1| + \cdots + |\lambda_p|)^d \tau^{\mathrm{Re}\ \lambda_1 + \cdots + \mathrm{Re}\ \lambda_p - m} \int_K |v(x)|\, dx
\end{aligned}$$

if $0 < \forall\tau < 1$. This implies that $\left\langle P(\lambda)((f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p} u),\ \varphi\right\rangle$ vanishes if $\mathrm{Re}\ \lambda_j \ge 0$ $(1 \le j \le p)$ and $\mathrm{Re}\ \lambda_1 + \cdots + \mathrm{Re}\ \lambda_p > m$, and hence only if $\mathrm{Re}\ \lambda_j \ge 0$ $(1 \le j \le p)$ by the uniqueness of analytic continuation. Q.E.D.

Proof of Theorem 6.12: The first statement follows from Lemma 6.13 and Proposition 6.14. Let us fix $\lambda^0 = (\lambda_1^0, \dots, \lambda_p^0) \in \overline{\mathbb{C}_+^p}$. Suppose $P \in D_n$ annihilates the residue class $(u \otimes f^s)|_{s=\lambda^0}$ in $N(\lambda^0)$ of $u \otimes f^s \in M \otimes_{\mathbb{K}[x]} \mathcal{L}$. Then there exist $P(s), Q_j(s) \in D_n[s]$ such that

$$P = P(s) + (s_1 - \lambda_1^0)Q_1(s) + \cdots + (s_p - \lambda_p^0)Q_p(s), \qquad P(s)(u \otimes f^s) = 0.$$

It follows from the first statement that

$$P\tilde{v}(x, \lambda) = (\lambda_1 - \lambda_1^0)Q_1(\lambda)\tilde{v}(x, \lambda) + \cdots + (\lambda_p - \lambda_p^0)Q_p(\lambda)\tilde{v}(x, \lambda)$$

for $\lambda \in \overline{\mathbb{C}_+^p}$, and hence $P\tilde{v}(x, \lambda^0) = 0$. This implies that there exists a D_n-linear map from $N(\lambda^0)$ to $\mathcal{D}'(U)$ which sends $(u \otimes f^s)|_{s=\lambda_0}$ to $\tilde{v}(x, \lambda^0)$. This completes the proof of Theorem 6.12.

6.3. Integrals over the domain defined by polynomial inequalities

We assume $\mathbb{K} = \mathbb{C}$. As in 4.3, set $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_d)$. Assume that $f_1, \dots, f_p$ are real polynomials in (x, y) and let $v(x, y)$ be a holonomic locally integrable function on an open set of $\mathbb{R}^{n+d}$ and let $\lambda_1, \dots, \lambda_p$ be complex numbers with non-negative real parts. We assume that

$$\tilde{v}(x, y) := v(x, y)(f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p}$$

belongs to $\mathcal{E}'\mathcal{D}'(\mathbb{R}_x^n \times U)$ with an open set U of $\mathbb{R}_y^d$, or to $\mathcal{SS}'(\mathbb{R}_x^n \times \mathbb{R}_y^d)$. Let I be a left ideal of D_{n+d} which annihilates $v(x, y)$ such that $M := D_{n+d}/I$ is holonomic. Set $M = M/I = D_{n+d}u$ with $u = \overline{1}$ and $\mathcal{L} = \mathbb{C}[x, y, (f_1 \cdots f_p)^{-1}, s]f^s$. Then Algorithm 6.11 yields a holonomic system for $\tilde{v}(x,y)$ by virtue of Theorem 6.12. Thus by the integration algorithm, we get a holonomic system for

$$w(y) := \int_{\mathbb{R}^n} v(x,y)(f_1)_+^{\lambda_1} \cdots (f_p)_+^{\lambda_p}\, dx.$$

In particular, setting $\lambda_1 = \cdots = \lambda_p = 0$, we obtain a holonomic system for

$$w(x) = \int_{D(y)} v(x,y)\, dx = \int_{\mathbb{R}^n} v(x,y) Y(f_1(x,y)) \cdots Y(f_p(x,y))\, dx$$

with

$$D(y) = \{x \in \mathbb{R}^n \mid f_j(x,y) \geq 0 \ (1 \leq j \leq p)\}.$$

As examples, let us consider truncated multi-dimensional normal distributions: Let $f_1, \dots, f_p$ be real polynomials in $x = (x_1, \dots, x_n)$ and set

$$D = \{x \in \mathbb{R}^n \mid f_j(x) \geq 0 \ (1 \leq j \leq p)\}.$$

Then $\exp\Big(-\dfrac{|x|^2}{2}\Big) Y(f_1) \dots Y(f_p)$ is, up to a constant multiple, the probability density function of the standard normal distribution truncated by D. Let $f(x)$ be a real polynomial, which we regard as a random variable. Then the cumulative and the density functions of $f(x)$ are given by

$$\begin{aligned} F(t) &= \int_D \exp\Big(-\frac{|x|^2}{2}\Big) Y(t - f(x))\, dx \\ &= \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big) Y(t - f(x)) Y(f_1(x)) \cdots Y(f_p(x))\, dx \end{aligned}$$

and

$$F'(t) = \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big) \delta(t - f(x)) Y(f_1(x)) \cdots Y(f_p(x))\, dx$$

respectively up to constant multiples. The integrands belong to the space $\mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}_t)$.

Example 6.15. Setting $f(x) = |x|^2$ and

$$D = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \ (1 \leq i \leq n),\ x_1 + \cdots + x_n \leq 1\},$$

let us consider the density function

$$v(t) = \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big) \delta(t - |x|^2) Y(x_1) \cdots Y(x_n) Y(1 - x_1 - \cdots - x_n)\, dx$$

(up to a constant multiple) of the random variable $|x|^2$. If $n = 2$, then $v(t)$ satisfies a differential equation

$$\{4t(t-1)(2t-1){\partial_t}^2+4(-2t^3+6t^2-5t+1)\partial_t+2t^3-9t^2+9t-2\}v(t) = 0.$$

Its indicial polynomials at 0, 1, and 1/2 are s^2, $s(s-1)$, and $s(2s-1)$ respectively. It can be verified that 1 is an apparent singular point e.g., by an algorithm described in Chapter 1 of [23]. Hence $v(t)$ belongs to $L^1(\mathbb{R})$ and real analytic on $\mathbb{R} \setminus \{0, 1/2\}$.

If $n = 3$, $v(t)$ is annihilated by

$$\begin{aligned} &8t(t-1)(2t-1)(3t-1){\partial_t}^3 + (-72t^4 + 276t^3 - 308t^2 + 116t - 12){\partial_t}^2 \\ &+ (36t^4 - 210t^3 + 308t^2 - 162t + 28)\partial_t - 6t^4 + 47t^3 - 83t^2 + 53t - 11. \end{aligned}$$

Its indicial polynomials at $0, 1, 1/2, 1/3$ are $s(s-1)(2s-1)$, $s(s-1)(s-2)$, $s(s-1)(2s-3)$, $s(s-1)^2$ respectively up to constant multiples. The point 1 is an apparent singular point.

Example 6.16. Set $n = 2$ and

$$v(t) = \int_{\mathbb{R}^2} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t - x_1 - x_2)Y(1 - x_1^2 - x_2^2)\,dx.$$

Then $v(t)$ is annihilated by two operators

$$\begin{aligned} P_1 &= 2t(t^2 - 2){\partial_t}^2 + (-t^4 + 2t^2 + 4)\partial_t - t^3, \\ P_2 &= 4(t^2 - 2){\partial_t}^3 + 12t{\partial_t}^2 + (-t^4 - 8t^2 + 12)\partial_t - t^3 - 6t, \end{aligned}$$

neither of which is a multiple of the other. The singular locus of the D_1-module $D_1/(D_1P_1 + D_1P_2)$ is $\{t \mid t^2 - 2 = 0\} = \{\sqrt{2}, -\sqrt{2}\}$.

Example 6.17. Set $n = 2$, $D = \{x = (x_1, x_2) \mid x_1^3 - x_2^2 \geq 0\}$ and consider

$$v(t) = \int_{\mathbb{R}^2} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t - f(x))Y(x_1^3 - x_2^2)\,dx_1dx_2$$

for a real polynomial $f(x)$. If $f(x) = x_1$, then $v(t)$ is annihilated by

$$2t{\partial_t}^2 + (-3t^3 - 4t^2 - 1)\partial_t + 3t^4 + 2t^3 - t;$$

Its indicial polynomial at 0 is $s(2s-3)$.

If $f(x) = x_1^2 + x_2^2$, then $v(t)$ is annihilated by

$$16t^3(27t-4){\partial_t}^4 + (-864t^4 + 3368t^3 - 320t^2){\partial_t}^3 \\ + (648t^4 - 4956t^3 + 5724t^2 - 268t){\partial_t}^2 \\ + (-216t^4 + 2462t^3 - 5484t^2 + 1654t - 12)\partial_t \\ + 27t^4 - 409t^3 + 1351t^2 - 760t + 6.$$

The indicial polynomials at 0 and 27/4 are $s^2(4s-1)(4s-3)$ and $s(s-1)(s-2)(s-3)$ respectively up to constant multiples. The point 27/4 is an apparent singular point.

Example 6.18. Set

$$v(t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp\Bigl(-\frac{x_1^2 + x_2^2}{2}\Bigr)\delta(t - x_1x_2)\,dx_1dx_2.$$

This is the density function of the random variable x_1x_2 with the standard normal distribution (x_1, x_2) and satisfies $(t\partial_t^2 + \partial_t - t)v(t) = 0$. Consider the density function

$$w(t) = \int_{\mathbb{R}^n} \delta(t - x_1^2 - x_2^2)v(x_1)v(x_2)\,dx_1dx_2$$

of $x_1^2 + x_2^2$, where (x_1, x_2) is the random vector with the probability density function $v(x_1)v(x_2)$. The integrand belongs to $\mathcal{E}'\mathcal{D}'(\mathbb{R}_x^2 \times \mathbb{R}_t)$, a holonomic system for which can be computed by using Algorithm 6.11. The integration algorithm gives

$$(8t^3{\partial_t}^4 + 48t^2{\partial_t}^3 - (6t^2 - 56t){\partial_t}^2 - (12t-8)\partial_t + t - 2)w(t) = 0.$$

The indicial equation at 0 is s^4.

6.4. Integrals with auxiliary difference parameters

Let us take as an example the integral

$$v(t;a,b) = \frac{1}{B(a,b)}\int_{-\infty}^{\infty} \delta(t - x + x^2)x_+^{a-1}(1-x)_+^{b-1}\,dx$$

for positive real numbers a, b. The integrand is holonomic in x and t, but not in (x, t, a, b) if we regard a, b as variables. Let us apply the algorithm of the preceding sections: first,

$$u(x,t;a,b) := \frac{1}{B(a,b)}\delta(t - x + x^2)x_+^{a-1}(1-x)_+^{b-1}$$

is annihilated by two operators

(15) $t-x+x^2, \quad (-x^2+x)\partial_x+(2x^3-3x^2+x)\partial_t+(a+b-2)x-a+1.$

Since the initial part of $t-x+x^2$ with respect to the weight vector $(1,0;-1,0)$ for $(x,t,\partial_x,\partial_t)$ is x^2 and $\partial_x^2 x^2 = \partial_x x(\partial_x x+1)$, we know that the b-function $b(s)$ with respect to this weight vector divides, in fact equals, $s(s-1)$, which does not depend on a, b. Hence the integration algorithm can be safely applied although the integrand is not holonomic in the variables (x,t,a,b) and produces a differential equation

$$(16) \quad (t^2(1-4t){\partial_t}^2 + ((4a+4b-18)t^2+(-a-b+3)t)\partial_t \\ + (-a^2+(-2b+7)a-b^2+7b-12)t+(b-1)a-b+1)v(t;a,b) = 0.$$

It has regular singularities at $t=0$ and $t=1/4$. The characteristic exponents at 0 are $a-1$ and $b-1$; those at $1/4$ are 0 and $-1/2$. Note that $v(t;a,b)$ vanishes if $t<0$ or $t>1/4$ and its explicit formula for $0<t<1/4$ can be obtained directly. The computation above assures us that $v(t;a,b)$ satisfies (16) on the whole $\mathbb{R}$ as a distribution in t.

In what follows we treat the case where the integrand has some auxiliary parameters with respect to which the integrand satisfies difference equations. In general, let D_n be the ring of differential operators defined over $\mathbb{K}=\mathbb{C}$. As in 6.2 define the D_n-algebra homomorphism $\mu : D_{n+p} \to D_n\langle a, E_a, E_a^{-1}\rangle$ by

$$\mu(t_j) = E_{a_j}, \quad \mu(\partial_{t_j}) = -a_j E_{a_j}^{-1} \qquad (1 \le j \le p),$$

where E_{a_j} denotes the shift operator $a_j \mapsto a_j+1$. Conversely, we define a D_n-algebra homomorhphism $\hat{\mu} : D_n\langle a, E_a\rangle \to D_{n+p}$ by

$$\hat{\mu}(a_j) = -\partial_{t_j} t_j, \quad \hat{\mu}(E_{a_j}) = t_j \qquad (1 \le j \le p).$$

Then $\mu\circ\hat{\mu}$ coincides with the inclusion map $D_n\langle a, E_a\rangle \subset D_n\langle a, E_a, E_a^{-1}\rangle$.

Definition 6.19. A left ideal I of $D_n\langle a, E_a\rangle$ is called a *holonomic* $D_n\langle a, E_a\rangle$-ideal if

$$J := \mu^{-1}(D_n\langle a, E_a, E_a^{-1}\rangle I) = \{P \in D_{n+p} \mid \mu(P) \in D_n\langle a, E_a, E_a^{-1}\rangle I\}$$

is a holonomic ideal of D_{n+p}, i.e., D_{n+p}/J is a holonomic D_{n+p}-module.

Definition 6.20. A subset Ω of $\mathbb{C}^p$ is said to be *shift-invariant* if $a \in \Omega$ implies that $a+(1,0,\dots,0), \dots, a+(0,\dots,0,1)$ also belong to Ω.

Definition 6.21. Let Ω be a shift-invariant subset of $\mathbb{C}^p$. We define a pair of classes $(\mathcal{F}_{n,d}(\Omega), \mathcal{F}_{0,d}(\Omega))$ as one which satisfies the following properties:

(1) Any element $u = u(x, y, a)$ of $\mathcal{F}_{n,d}(\Omega)$ is a map from Ω to $\mathcal{D}'(\mathbb{R}^n_x \times \mathbb{R}^d_y)$; any element $v = v(y, a)$ of $\mathcal{F}_{0,d}(\Omega)$ is a map from Ω to $\mathcal{D}'(\mathbb{R}^d_y)$.

(2) $\mathcal{F}_{n,d}(\Omega)$ is a left $D_{n+d}\langle a, E_a\rangle$-module and $\mathcal{F}_{0,d}(\Omega)$ is a left $D_d\langle a, E_a\rangle$-module.

(3) Let $u = u(x, y, a)$ be an arbitrary element of $\mathcal{F}_{n,d}(\Omega)$. Then the integral $\int_{\mathbb{R}^n} u(x, y, a)\, dx$ is well-defined and belongs to $\mathcal{F}_{0,d}(\Omega)$. Moreover,

$$P \int_{\mathbb{R}^n} u(x, y, a)\, dx = \int_{\mathbb{R}^n} Pu(x, y, a)\, dx, \quad \int_{\mathbb{R}^n} \partial_{x_i} u(x, y, a)\, dx = 0$$

hold for any $u \in \mathcal{F}_{n,d}(\Omega)$, $P \in D_d\langle a, E_a\rangle$, and $i = 1, \ldots, n$.

(4) $E_{a_j} : \mathcal{F}_{0,d}(\Omega) \to \mathcal{F}_{0,d}(\Omega)$ defines an injective $\mathbb{C}$-linear map for each $j = 1, \ldots, p$.

As the first examle, let Ω be a shift-invariant open subset of $\mathbb{C}^p$ and define the space $\mathcal{O}(\Omega, \mathcal{S}'(\mathbb{R}^d))$ to be the set of $\mathcal{S}'(\mathbb{R}^d)$-valued holomorphic functions:

$$\begin{aligned} &\mathcal{O}(\Omega, \mathcal{S}'(\mathbb{R}^d)) \\ &\qquad = \{u : \Omega \to \mathcal{S}'(\mathbb{R}^d) \mid \langle u(y, a), \varphi(y)\rangle_y \in \mathcal{O}(\Omega)\ (\forall \varphi \in \mathcal{S}(\mathbb{R}^d))\}. \end{aligned}$$

Let $\mathcal{O}(\Omega, \mathcal{S}\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d))$ be the set of functions from Ω to $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ of the form

$$u(x, y, a) = \sum_{j=1}^m u_j(x) v_j(x, y, a)$$

with $m \in \mathbb{N}$, $u_j \in \mathcal{S}(\mathbb{R}^n)$, $v_j \in \mathcal{O}(\Omega, \mathcal{S}'(\mathbb{R}^{n+d}))$. Then the integral of $u(x, y, a)$ with respect to x is defined by

$$\left\langle \int_{\mathbb{R}^n} u(x, y, a)\, dx,\ \varphi(y) \right\rangle = \sum_{j=1}^m \langle v_j(x, y, a),\ u_j(x)\varphi(y)\rangle_{(x,y)}$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, which is well-defined as an element of $\mathcal{O}(\Omega, \mathcal{S}'(\mathbb{R}^d))$ independent of the expression of $u(x, y, a)$ above. Thus the pair

$$\mathcal{F}_{n,d}(\Omega) = \mathcal{O}(\Omega, \mathcal{S}\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)), \quad \mathcal{F}_{0,d}(\Omega) = \mathcal{O}(\Omega, \mathcal{S}'(\mathbb{R}^d))$$

satisfies the conditions of Definition 6.21.

The second example is the pair

$$\mathcal{F}_{n,d}(\Omega) = \{u : \Omega \to \mathcal{S}\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)\}, \quad \mathcal{F}_{0,d}(\Omega) = \{u : \Omega \to \mathcal{S}'(\mathbb{R}^d)\}$$

with a subset Ω of $\mathbb{C}^p$ such that $a \in \Omega$ implies $(a_1, \dots, a_j \pm 1, \dots, a_p) \in \Omega$ for $j = 1, \dots, p$.

Algorithm 6.22 (difference-differential equations for an integral)**.**
Input: A set G_0 of generators of a holonomic ideal I of $D_{n+d}\langle a, E_a\rangle$ which annihilates $u(x, y, a) \in \mathcal{F}_{n,d}(\Omega)$ with a shift-invariant subset Ω of $\mathbb{C}^p$.
Output: A set G of generators of a holonomic ideal of $D_d\langle a, E_a\rangle$ which annihilates $v(y, a) = \int_{\mathbb{R}^n} u(x, y, a)\, dx$.

(1) Let J be the left ideal of D_{n+d+p} generated by $\hat{\mu}(G_0)$.

(2) Compute $\tilde{J} := \{P \in D_{n+d+p} \mid t^\nu P \in J\ (\exists \nu \in \mathbb{N}^p)\}$ as the annihilator of $\overline{1} \otimes 1$ in the localization

$$(D_{n+d+p}/J)[(t_1 \cdots t_p)^{-1}] = (D_{n+d+p}/J) \otimes_{\mathbb{K}[x,y,t]} \mathbb{K}[x, y, t, (t_1 \cdots t_p)^{-1}],$$

where $\overline{1}$ is the residue class of 1 in D_{n+d+p}/J, by using the localization algorithm of [25].

(3) Compute a set G_1 of generators of the integration ideal

$$N := D_{d+p} \cap (\partial_{x_1} D_{n+d+p} + \cdots + \partial_{x_n} D_{n+d+p} + \tilde{J})$$

of $\tilde{J}$ by Algorithm 5.10.

(4) Let P be an element of G_1. Then there exists a (componentwise) minimal $\nu = (\nu_1, \dots, \nu_p) \in \mathbb{Z}^p$ such that $Q := E_a^\nu \mu(P)$ belongs to $D_d\langle a, E_a\rangle$. (Set $\nu_j = 0$ if $\mu(P)$ does not contain E_{a_j}.) Let us denote this Q by $\mathrm{nm}(\mu(P))$. Set $G := \{\mathrm{nm}(\mu(P)) \mid P \in G_1\}$.

Proof. First, let us prove that $D_{n+d+p}/\tilde{J}$ is holonomic. It suffices to show that $\tilde{J}$ contains $\mu^{-1}(D_{n+d}\langle a, E_a, E_a^{-1}\rangle I)$, which is a holonomic ideal by the assumption. Let P be an element of this set. Then there exist $Q \in I$ and $\nu \in \mathbb{N}^p$ such that

$$\mu(t^\nu P) = E_a^\nu \mu(P) = Q = \mu(\hat{\mu}(Q)).$$

Hence $t^\nu P = \hat{\mu}(Q)$ belongs to J. This implies $P \in \tilde{J}$.

Next let us prove that each element P of G annihilates $v(y, a)$. By the definition of G, there exist $Q_i \in D_{n+d+p}$, $R \in \tilde{J}$ and $\nu \in \mathbb{N}^p$ such that

$$E_a^\nu P = \sum_{i=1}^{n} \partial_{x_i} \mu(Q_i) + \mu(R). \tag{17}$$

Taking the components of ν large enough, we may also assume that $\mu(Q_i)$ belong to $D_d\langle a, E_a\rangle$ and $\mu(R)$ belongs to I so that $\mu(Q_i)u \in \mathcal{F}_{n,d}(\Omega)$ and $\mu(R)u = 0$. Thus (17) implies

$$E_a^\nu Pv(y,a) = \int_{\mathbb{R}^n} E_a^\nu Pu(x,y,a)\,dx = \sum_{i=1}^n \int_{\mathbb{R}^n} \partial_{x_i}\mu(Q_i)u(x,y,a)\,dx = 0$$

by (2) and (3) of Definition 6.21, and hence $Pv = 0$ by (4).

Finally, let us show the ideal $\tilde{I}$ of $D_d\langle a, E_a\rangle$ generated by G is holonomic. Theorem 5.1 assures that D_{d+p}/N is holonomic. Hence it suffices to show that $\mu^{-1}(D_d\langle a, E_a, E_a^{-1}\rangle\tilde{I})$ contains N, which is easy to see by the definition of $\tilde{I}$. This completes the correctness proof of the algorithm. Q.E.D.

In practice we can skip step (2) if J of step (1) is already holonomic.

Example 6.23. Let us come back to the example given at the beginning of this subsection. First note that the integrand

$$u(x,t;a,b) = \frac{1}{B(a,b)}\delta(t - x + x^2)x_+^{a-1}(1-x)_+^{b-1},$$

which belongs to $\mathcal{O}(\Omega, \mathcal{E}'\mathcal{D}'(\mathbb{R}_x \times \mathbb{R}_t))$ with $\Omega = \{(a,b) \in \mathbb{C}^2 \mid \mathrm{Re}\ a > 0,\ \mathrm{Re}\ b > 0\}$, satisfies difference equations

$$(aE_a - (a+b)x)u(x,t;a,b) = 0, \quad (bE_b - (a+b)(1-x))u(x,t;a,b) = 0$$

with the shift operators $E_a : a \mapsto a+1$ and $E_b : b \mapsto b+1$, in addition to (15). With these inputs, Algorithm 6.22 returns a set of generators of a holonomic ideal of $D_1\langle a, b, E_a, E_b\rangle$ which annihilates $v(t;a,b) = \int_{-\infty}^{\infty} u(x,t;a,b)\,dx$. For example, $v(t;a,b)$ is annihilated by

$$\begin{aligned}(4E_bE_at^2 - E_bE_at)\partial_t + \{&(b(-2E_b+1) + 2E_b + 1)E_a \\ &+ a(-E_b+1) + b(2E_b^2 - 2E_b + 1) + 2E_b^2 - E_b\}t.\end{aligned}$$

Computing the intersection with the subring $D_1[a,b]$, we get (16) again. If only differential equations in t is needed, we could have ignored the factor, i.e, the reciprocal of $B(a,b)$ at first.

Finally, let us consider the multivariate gamma distribution with the density function

$$\begin{aligned}u_n(x;a) &= u(x_1,\dots,x_n;a_1,\dots,a_n) \\ &:= \frac{1}{\Gamma(a_1)\cdots\Gamma(a_n)}(x_1)_+^{a_1-1}\cdots(x_n)_+^{a_n-1}e^{-x_1-\cdots-x_n}\end{aligned}$$

with $x = (x_1, \dots, x_n)$ and $a = (a_1, \dots, a_n)$. It is annihilated by

$$x_i\partial_{x_i} + x_i - a_i + 1, \quad a_iE_{a_i} - x_i \qquad (1 \le i \le n)$$

which generate a holonomic $D_n\langle a, E_a\rangle$-ideal. Hence Algorithm 6.22 produces a holonomic ideal of $D_1\langle a, E_a\rangle$ which annihilates the density function

$$v(t; a) := \int_{\mathbb{R}^n} \delta(t - f(x))u_n(x; a)\, dx$$

for an arbitrary real polynomial $f(x)$ as a random variable. Here we can regard the integrand as an element of $\mathcal{O}(\Omega, \mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}_t))$ with

$$\Omega = \{a \in \mathbb{C}^n \mid \mathrm{Re}\ a_j > 0\ (1 \le j \le n)\}.$$

Example 6.24. Set

$$v(t; a_1, a_2) := \int_{\mathbb{R}^2} \delta(t - x_1x_2)u_2(x; a_1, a_2)\, dx_1dx_2.$$

By Algorithm 6.22 we know that it is annihilated by the differential operator

$$t^2{\partial_t}^2 + (-a_1 - a_2 + 3)t\partial_t - t + a_1a_2 - a_1 - a_2 + 1$$

whose indicial polynomial at 0 is $(s - a_1 + 1)(s - a_2 + 1)$, as well as by difference-differential operators

$$t\partial_t + a_1(E_{a_1} - 1) + 1, \qquad t\partial_t + a_2(E_{a_2} - 1) + 1.$$

These three operators generate a holonomic ideal of $D_1\langle a_1, a_2, E_{a_1}, E_{a_2}\rangle$.

Example 6.25. Set

$$v(t; a_1, a_2) := \int_{\mathbb{R}^2} \delta(t - x_1^2 - x_2^2)u_1(x; a_1, a_2)\, dx_1dx_2.$$

It is annihilated by the differential operator

$$\begin{aligned}
P &= 32t^4{\partial_t}^6 + (-64a_1 - 64a_2 + 480)t^3{\partial_t}^5 \\
&+ \{-32t^3 + (48a_1^2 + (96a_2 - 624)a_1 + 48a_2^2 - 624a_2 + 2040)t^2\}{\partial_t}^4 \\
&+ \{(40a_1 + 40a_2 - 256)t^2 + (-16a_1^3 + (-48a_2 + 264)a_1^2 \\
&\quad + (-48a_2^2 + 528a_2 - 1448)a_1 - 16a_2^3 + 264a_2^2 - 1448a_2 + 2640)t\}{\partial_t}^3 \\
&+ \{10t^2 + (-16a_1^2 + (-32a_2 + 172)a_1 - 16a_2^2 + 172a_2 - 456)t \\
&\quad + 2a_1^4 + (8a_2 - 36)a_1^3 + (12a_2^2 - 108a_2 + 238)a_1^2 \\
&\quad + (8a_2^3 - 108a_2^2 + 476a_2 - 684)a_1 \\
&\quad + 2a_2^4 - 36a_2^3 + 238a_2^2 - 684a_2 + 720\}{\partial_t}^2 \\
&+ \{(-6a_1 - 6a_2 + 30)t + 2a_1^3 + (6a_2 - 26)a_1^2 \\
&\quad + (6a_2^2 - 52a_2 + 108)a_1 + 2a_2^3 - 26a_2^2 + 108a_2 - 144\}\partial_t \\
&- t + a_1^2 - 5a_1 + a_2^2 - 5a_2 + 10.
\end{aligned}$$

The indicial polynomial of P at 0 is

$$b(s) = s(s-1)(2s-a_1-a_2)(2s-a_1-a_2-1)(2s-a_1-a_2+1)(2s-a_1-a_2+2)$$

up to a constant multiple.

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