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A necessary condition for an edge ring to satisfy Serre's condition (S_2)

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Abstract.

Let G be a finite simple connected graph. We give a necessary condition for the edge ring of G to satisfy Serre's condition (S_2) in terms of a graph.

§1. Introduction

Let G be a finite simple connected graph. We denote by $V = V(G)$ the vertex set of G and $E(G)$ the edge set of G. We set $V = [d] := \{1, 2, \ldots, d\}$ and $E(G) = \{e_1, \ldots, e_r\}$, where $e_\ell = \{j_{\ell 1}, j_{\ell 2}\}$. Let K be a field. Consider the polynomial ring $K[\mathbf{t}]$ with d variables $t_1, t_2, \ldots, t_d$. The edge ring of G, denoted by $K[G]$ is the subring of $K[\mathbf{t}]$ generated by $\mathbf{t}^{e_1}, \ldots, \mathbf{t}^{e_r}$, where $\mathbf{t}^{e_\ell} := t_{j_{\ell 1}} t_{j_{\ell 2}}$. Let $\rho(e_\ell)$ be the $(0,1)$-vector of $\mathbb{R}^d$ which has 1 only $j_{\ell 1}$-, $j_{\ell 2}$-entries, and let S_G be the affine semigroup generated by $\rho(e_1), \ldots, \rho(e_r)$. Then $K[G]$ is the affine semigroup ring of S_G.

In [6], Ohsugi and Hibi characterized when $K[G]$ is normal in terms of the graph; see Section 3 for detail. Since $K[G]$ is an affine semigroup ring, by Hochster [5, Theorem 1], it is known that if $K[G]$ is normal then $K[G]$ is Cohen–Macaulay. Hence it is natural to ask when $K[G]$ is Cohen–Macaulay. For a general affine semigroup ring, the characterization of its Cohen–Macaulayness has been investigated by Goto and Watanabe [3] and Trung and Hoa [8, Theorem 4.1]. The purpose of our study is to illustrate the characterization in terms of the graph. Note that $K[G]$ is normal if and only if $K[G]$ satisfies Serre's conditions (R_1) and (S_2); see [2, Theorem 2.2.22]. Hibi and Katthän [4] give the characterization for $K[G]$ satisfying (R_1). On the other hand we try

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to characterize $K[G]$ which satisfies (S_2). Note that Serre's condition (S_2) is a necessary condition for $K[G]$ to be Cohen–Macaulay. The main result of the present report is a necessary condition for $K[G]$ to satisfy (S_2) in terms of a graph; see Theorem 4.1.

The organization of the present report is as follow. In Section 2, we recall the characterization of a Cohen–Macaulay affine semigroup ring. In Section 3, we recall the properties of an edge ring. Finally, in Section 4, we give a necessary condition for an edge ring to satisfy Serre's condition (S_2) in terms of a graph.

§2. Cohen–Macaulayness of affine semigroup rings

Recall that $K[G]$ is an affine semigroup ring, as stating at Introduction. In this section, we recall the characterization of a Cohen–Macaulay affine semigroup ring investigated by Goto and Watanabe [3] and Trung and Hoa [8].

Let $S \subset \mathbb{N}^d$ be an affine semigroup. Then the affine semigroup ring of S, denoted by $K[S]$, is the subring of the polynomial ring $K[\mathbf{t}]$ with d variables $t_1, t_2, \ldots, t_d$ generated by $\{\mathbf{t}^x : x \in S\}$, where $\mathbf{t}^x = t_1^{x_1} t_2^{x_2} \cdots t_d^{x_d}$ for $x = (x_1, x_2, \ldots, x_d)$. Let C_S be the convex rational polyhedral cone spanned by S in $\mathbb{Q}^d$. We may assume that C_S is d-dimensional. Let $F_1, \ldots, F_m$ be all the facets of C_S. Also let $\mathcal{G}$ be the group generated by S. We set

$$\begin{aligned} S_i &:= S - S \cap F_i \\ &= \{x \in \mathcal{G} \ : \ \text{there exists } y \in S \cap F_i \text{ such that } x + y \in S\} \end{aligned}$$

and $S' = \bigcap_{i=1}^m S_i$. Let J be a subset of $[m] := \{1, 2, \ldots, m\}$. Set

$$G_J := \bigcap_{i \notin J} S_i \setminus (\bigcup_{j \in J} S_j).$$

Also set

$$\pi_J := \left\{ I \subset J \ : \ I \neq \emptyset, \bigcap_{i \in I} S \cap F_i \neq (0) \right\} \cup \{\emptyset\}.$$

Then π_J is a simplicial complex.

Theorem 2.1 (Trung and Hoa [8, Theorem 4.1])**.** *Let S be an affine semigroup ring. Assume that C_S has m facets. Then $K[S]$ is Cohen–Macaulay if and only if the following two conditions are satisfied*:

(i) $S' = S$;

(ii) $G_J = \emptyset$ *or* π_J *is acyclic for all proper subsets* J *of* $[m]$.

Remark 2.2. *The first condition of Theorem 2.1 corresponds to Serre's condition* (S_2); *see* [1, Exercise 4.17].

§3. Edge rings

As noted in Remark 2.2, the condition (i) of Theorem 2.1 corresponds to Serre's condition (S_2). Thus we investigate which graph G satisfies the condition. In order to do this, we need the facets of C_{S_G}. Fortunately, these have been studied by Ohsugi and Hibi [6]. In this section, we recall their result.

Let G be a finite simple connected graph. We denote $V(G)$ its vertex set and $E(G)$ its edge set. Let $W \subset V(G)$. The induced subgraph of G on W, denoted by G_W, is the graph whose vertex set is W and whose edge set is given by $\{e \in E(G) \ : \ e \subset W\}$. Also we set

$$N(G;W) := \{v \in V(G) \ : \ \{v,w\} \in E(G) \text{ for some } w \in W\}.$$

A subset T of $V(G)$ is said to be independent if $\{v_1, v_2\}$ is not an edge of G for any distinct vertices $v_1, v_2 \in T$. Let T be an independent set of G. Then we can consider the bipartite graph on $T \cup N(G;T)$ whose edge set is given by

$$\{\{v,w\} \in E(G) \ : \ v \in T, \ w \in N(T;G)\}.$$

We call such a graph the *bipartite graph induced by* T.

A pair (C_1, C_2) of odd cycles of G with no common vertex is said to be *exceptional* if it has no bridge, that is, for any vertices $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$, $\{v_1, v_2\}$ is not an edge of G. The graph G is said to satisfy the *odd cycle condition* if it has no exceptional pair. We have already known a graph G whose edge ring is normal:

Theorem 3.1 (Ohsugi and Hibi [6])**.** *Let* G *be a finite simple connected graph. Then* $K[G]$ *is normal if and only if* G *satisfies the odd cycle condition.*

Since we have already known a normal graph, we focus on a non-normal graph, namely, the graph whose edge ring is not normal, and in what follows, we always assume that the graph G is non-normal. Note that G is not bipartite.

In order to illustrate the facets of C_{S_G}, we need some notion.

Definition 3.2. *The vertex* $i \in V$ *is said to be regular in* G *if each connected component of* $G_{V\setminus\{i\}}$ *has at least one odd cycle.*

Definition 3.3. *A non-empty subset T of V is said to be fundamental in G if the following three conditions are satisfied*:

(i) *T is an independent set*;
(ii) *the bipartite graph induced by T is connected*;
(iii) *if $T \cup N(G;T) \neq V(G)$, then each connected component of $G_{V(G)\setminus(T\cup N(G;T))}$ has at least one odd cycle.*

The facets of C_{S_G} are given by the intersection of supporting hyperplanes of C_{S_G}, which are defined by the regular vertices in G and fundamental sets in G.

Theorem 3.4 (Ohsugi and Hibi [6, Theorem 1.7])**.** *Let G be a non-normal finite connected simple graph. Then all the supporting hyperplanes of C_{S_G} are the following ones*:

(i) *$\mathcal{H}_i = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0\}$, where i is a regular vertex in G.*
(ii) *$\mathcal{H}_T = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \sum_{i\in T} x_i = \sum_{j\in N(G;T)} x_j\}$, where T is a fundamental set in G.*

We close the present section by recalling the characterization of the graph G whose edge ring satisfies Serre's condition (R_1) given by Hibi and Katthän [4].

Theorem 3.5 (Hibi and Katthän [4, Theorem 2.1])**.** *Let G be a finite simple connected graph. Assume that G is not bipartite. Then $K[G]$ satisfies Serre's condition (R_1) if and only if G satisfies the following two conditions*:

(i) *$G \setminus \{i\}$ is connected for all regular vertex i in G*;
(ii) *for each fundamental set T in G with $T \cup N(G;T) \neq V(G)$, the induced subgraph $G_{V(G)\setminus(T\cup N(G;T))}$ is connected.*

We note that if a non-normal graph G satisfies Serre's condition (R_1), then G does not satisfy Serre's condition (S_2).

§4. Main result

Our purpose of the study is to characterize a Cohen–Macaulay edge ring $K[G]$ in terms of the graph G. As the first step of our study, we try to characterize the Serre's condition (S_2); see Theorem 2.1 and Remark 2.2. In this section, we give a necessary condition for G satisfying the Serre's condition (S_2), namely, satisfying $S_G = S'_G$.

Before stating our result, we define one notation. Let C be an odd cycle. Set

$$\mathbf{e}_C := \sum_{i\in V(C)} \mathbf{e}_i \in \mathbb{N}^d,$$

where $\mathbf{e}_i$ is the i-th elementary vector of $\mathbb{R}^d$.

The following theorem is the main result of the present report.

Theorem 4.1. *Let G be a finite simple connected graph. Suppose that there exists an exceptional pair (C_1, C_2) satisfying following two conditions*:

(i) *for each regular vertex $i \in V(G) \setminus (V(C_1) \cup V(C_2))$ in G, both C_1 and C_2 belong to the same connected components of $G \setminus \{i\}$*;

(ii) *for each fundamental set T in G with $(V(C_1) \cup V(C_2)) \cap (T \cup N(G;T)) = \emptyset$, both C_1 and C_2 belong to the same connected components of $G_{V(G)\setminus(T \cup N(G;T))}$.*

Then $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} \in S'_G$. In particular, $S_G \neq S'_G$.

Theorem 4.1 is restated as follows:

Theorem 4.2. *Let G be a finite simple connected graph. If $S_G = S'_G$, then for each exceptional pair (C_1, C_2), one of the following is satisfied*:

(i) *there exists a regular vertex i in G such that both C_1 and C_2 belong to different connected components of $G \setminus \{i\}$.*

(ii) *there exists a fundamental set T in G such that both C_1 and C_2 belong to different connected components of $G_{V(G)\setminus(T \cup N(G;T))}$.*

Before proving Theorem 4.1, we consider which elements can be the elements of S_G or S'_G. Note that $S_G \subset S'_G$ holds in general. Hence we would like to know which element belongs to $S'_G \setminus S_G$.

We need some more notation. Let $\Gamma = (e_{j_1}, \ldots, e_{j_q})$ be a walk. We set

$$\mathbf{e}_{\Gamma,o} = \sum_{\ell} \mathbf{e}_{j_{2\ell+1}}, \qquad \mathbf{e}_{\Gamma,e} = \sum_{\ell} \mathbf{e}_{j_{2\ell}}.$$

Note that $\mathbf{e}_\ell$ is the ℓ-th elementary vector of $\mathbb{R}^r$ and $\mathbf{e}_{\Gamma,o}, \mathbf{e}_{\Gamma,e} \in \mathbb{N}^r$. Also for $f = (f_1, \ldots, f_r) \in \mathbb{N}^r$, we set

$$\rho(f) := \sum_{j=1}^{r} f_j \rho(\mathbf{e}_j).$$

Let i be a regular vertex in G. Then we denote by Γ_i, the subsequence of Γ whose edges contain i. Let T be a fundamental set in G. Then we denote by Γ_T, the subsequence of Γ whose edges contain an element of $N(G;T)$ and do not contain any element of T.

For $f = (f_1, \ldots, f_r), g = (g_1, \ldots, g_r) \in \mathbb{N}^r$, we denote $g \leq f$ if $g_i \leq f_i$ for all i. Katthän proved the following proposition.

Proposition 4.3 (Katthän)**.** *Let G be a finite simple connected non-normal graph. Let (C_1, C_2) be an exceptional pair of odd cycles of G and $f \in \mathbb{N}^r$. Then $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) \in S_G$ if and only if there exists a walk Γ which combine C_1 and C_2 with either $\mathbf{e}_{\Gamma,o} \leq f$ or $\mathbf{e}_{\Gamma,e} \leq f$.*

We give a proof of this proposition for the convenience of the reader.

Proof. "If" part is obvious. We consider "Only if" part.

Take $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$. Set $e' := e_{r+1} := \{v_1, v_2\}$. (Recall that $E(G) = \{e_1, \ldots, e_r\}$.) Then $e' \notin E(G)$ since (C_1, C_2) is exceptional. Let G' be the graph obtained by adding an edge e' to the graph G. Then there exists a $(0,1)$-vector $g = (g_1, \ldots, g_{r+1}) \in \mathbb{N}^{r+1}$ with $g_{r+1} = 1$ such that $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} = \rho(g)$ for the graph G'.

On the other hand, since $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) \in S_G$, there exists $h \in \mathbb{N}^r$ with $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) = \rho(h)$ for the graph G.

Identifying an element $\mathbb{N}^r$ with the element $\mathbb{N}^{r+1}$ whose $(r+1)$-th entry is 0, we have $\rho(g+f) = \rho(h)$ for the graph G'. Then the binomial $\mathbf{t}^{\rho(g+f)} - \mathbf{t}^{\rho(h)}$ belongs to the toric ideal of G'. By [7, Lemma 1.1], there exists an even closed walk $\Gamma' = (e', e_{j_2}, \ldots, e_{j_q})$ with $\mathbf{e}_{\Gamma',o} \leq g + f$. Let e_{j_α} be the last edge of Γ' which contains the vertex of C_2 and let e_{j_β} be the first edge of Γ' which contains the vertex of C_1 after e_{j_α}. Then the subwalk Γ of Γ' starting at e_{j_α} and ending at e_{j_β} has the desired property. Q.E.D.

On the other hand, we have the following proposition.

Proposition 4.4. *Let G be a graph with the same conditions as in Proposition 4.3. Let (C_1, C_2) be an exceptional pair and $f \in \mathbb{N}^r$. Then $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) \in S'_G$ if and only if the following two conditions are satisfied*:

(i) *for any regular vertex i in G such that C_1 and C_2 belong to different connected components of $G \setminus \{i\}$, there exists a walk $\Gamma^{(i)}$ which combine C_1 and C_2 with either $\mathbf{e}_{\Gamma_i^{(i)},o} \leq f$ or $\mathbf{e}_{\Gamma_i^{(i)},e} \leq f$;*

(ii) *for any fundamental set T in G such that C_1 and C_2 belong to different connected components of $G_{V(G)\setminus(T \cup N(G;T))}$, there exists a walk $\Gamma^{(T)}$ which combine C_1 and C_2 with either $\mathbf{e}_{\Gamma_T^{(T)},o} \leq f$ or $\mathbf{e}_{\Gamma_T^{(T)},e} \leq f$.*

Proof. "If" part is obvious. We consider "Only if" part.

Let i be a regular vertex such that C_1 and C_2 belong to different connected components of $G \setminus \{i\}$. Since $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) \in S'_G \subset$

$S_G - S_G \cap \mathcal{H}_i$, we can write

$$\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) = \sum_{e \in E(G)} a_e \rho(e) - \sum_{e' \in E(G),\, i \notin e'} b_{e'} \rho(e').$$

Then $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) + \sum_{e'} b_{e'} \rho(e') = \sum_{e \in E(G)} a_e \rho(e) \in S_G$. Let $f' = (f'_1, \ldots, f'_r)$ be the element of $\mathbb{N}^r$ such that $\rho(f') = \sum_{e'} b_{e'} \rho(e')$. By Proposition 4.3, there exists a walk $\Gamma^{(i)}$ combining C_1 and C_2 with either $\mathbf{e}_{\Gamma^{(i)},o} \le f + f'$ or $\mathbf{e}_{\Gamma^{(i)},e} \le f + f'$. Note that $f'_\ell = 0$ if $i \in e_\ell$. Then focusing on the entry ℓ with $i \in e_\ell$, we see that $\mathbf{e}_{\Gamma_i^{(i)},o} \le f$ or $\mathbf{e}_{\Gamma_i^{(i)},e} \le f$.

Let T be a fundamental set such that C_1 and C_2 belong to different connected components of $G_{V(G) \setminus (T \cup N(G;T))}$. Since $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) \in S'_G \subset S_G - S_G \cap \mathcal{H}_T$, we can write

$$\begin{aligned} &\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) \\ &= \sum_{e \in E(G)} a_e \rho(e) - \sum_{e' \in E(G),\, e' \cap (T \cup N(G;T)) = \emptyset \text{ or } e' \cap T \neq \emptyset} b_{e'} \rho(e'). \end{aligned}$$

Then $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(f) + \sum_{e'} b_{e'} \rho(e') = \sum_{e \in E(G)} a_e \rho(e) \in S_G$. Let $f' = (f'_1, \ldots, f'_r)$ be the element of $\mathbb{N}^r$ such that $\rho(f') = \sum_{e'} b_{e'} \rho(e')$. By Proposition 4.3, there exists a walk $\Gamma^{(T)}$ combining C_1 and C_2 with either $\mathbf{e}_{\Gamma^{(T)},o} \le f + f'$ or $\mathbf{e}_{\Gamma^{(T)},e} \le f + f'$. Focus on entries ℓ with $e_\ell \cap T = \emptyset$ and $e' \cap N(G;T) \neq \emptyset$. Since $f'_\ell = 0$, we have $\mathbf{e}_{\Gamma_T^{(T)},o} \le f$ or $\mathbf{e}_{\Gamma_T^{(T)},e} \le f$. Q.E.D.

Propositions 4.3 and 4.4 show the gap between the sets S_G and S'_G. In these propositions, we consider the elements which are of the form $e_{C_1} + e_{C_2} + \rho(f)$. Although these look special, in the proof of the characterization of a normal edge ring given by Ohsugi and Hibi [6] (Theorem 3.1), they reduced the discussion to the case of only one exceptional pair. Hence we believe that we can also reduce the discussion to such a case and the above proposition would be the first step for the characterization of a non-normal edge ring which satisfies Serre's condition (S_2).

Theorem 4.1 is an easy consequence of these propositions.

Proof of Theorem 4.1. By Proposition 4.3, we have $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} = \mathbf{e}_{C_1} + \mathbf{e}_{C_2} + \rho(\mathbf{0}) \notin S_G$. On the other hand, we have $\mathbf{e}_{C_1} + \mathbf{e}_{C_2} \in S'_G$ by Proposition 4.4. Q.E.D.

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