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Remarks on the Capelli identities for reducible modules

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Dedicated to Professor Noumi on his 60th birthday

Abstract.

Inspired by the new type of Capelli identity associated with the regular representation of quaternions obtained by An Huang [6], we present a more basic type of Capelli identities for reducible modules of the matrix algebras. As a related subject in view of reducible modules, we also discuss the Capelli identities associated with group determinants, for which a precise formulation and an answer are given.

§ Introduction

The classical Capelli identity has been investigated from many points of view and extended to various directions. In this article, we would like to add one new possibility for some direct generalizations of the very classical Capelli identity, and beyond. The study started in 2011 January 23, when An Huang, who was a graduating PhD in physics at Berkeley, contacted the author asking if his Capelli type identities could be new. His identities [6] are related to the quaternions and the octonions, and he found them accidentally through his study of quantum field theory. To tell the truth, at first sight, they did not seem very striking, but actually revealed a new and potentially big view point. Aside from his two cases, the division algebras over the reals, similar identities hold for more fundamental one, the matrix algebra, and also some cases arising from the multiple of the irreducible representation of the matrix algebra. The main theme of article is about theses identities.

This direction also suggests and reminds us of other studies of an old but not published result, e.g. the Capelli identities associated with

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the group determinants. These related subjects are still open for new investigations.

§1. The classical Capelli identity

Let us first recall the very classical Capelli identity. To make things simple, we will always work over the complex number field $\mathbb{C}$, unless otherwise stated. On the matrix space $\mathrm{Mat}_{m\times n} = \mathrm{Mat}_{m\times n}(\mathbb{C})$, the two general linear groups GL_m and GL_n act respectively by the left and right multiplications. The infinitesimal actions of those Lie algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ on the polynomial functions on the $\mathrm{Mat}_{m\times n}$ are described as

$$\rho(E_{ij}) = \sum_{a=1}^{m} x_{ai}\partial_{aj}, \qquad \lambda(E^\circ_{kl}) = \sum_{b=1}^{n} x_{lb}\partial_{kb},$$

where x_{ab} are the coordinates of $\mathrm{Mat}_{n\times m}$ and $\partial_{ab} = \dfrac{\partial}{\partial x_{ab}}$ the corresponding partial differential operators, and E_{ij} and E°_{kl} are respectively the standard basis of $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$; the indices run over $1 \le a, k, l \le m; 1 \le b, i, j \le n$.

In the matrix notation with $\Pi_{ij} = \rho(E_{ij})$ and $\Pi^\circ_{kl} = \lambda(E^\circ_{kl})$, we can write these relations as

$$\Pi = {}^t X D, \qquad {}^t\Pi^\circ = X\,{}^t D,$$

where the entries of Π, Π°, X, D consist respectively of $\Pi_{ij}, \Pi^\circ_{kl}, x_{ab}, \partial_{ab}$ with the indices a, b, i, j, k, l running as above.

The Capelli identities are to describe the polynomial coefficient invariant differential operators $\mathscr{PD}(\mathrm{Mat}_{m\times n})^{GL_m\times GL_n}$ as the representation of the center of the universal enveloping algebras $\mathscr{U}(\mathfrak{gl}_n)$ and $\mathscr{U}(\mathfrak{gl}_m)$. The typical one is for $n = m$ (square case) is a non-commutative determinant multiplication formula as follows

$$\det(\Pi + \natural_n) = \det({}^t X)\,\det(D),$$

where $\natural_n = \mathrm{diag}(n-1, n-2, \cdots, 0)$ is the diagonal shift, and the determinant in the left-hand side with non-commutative entries means the 'column' determinant defined as

$$\det A = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}\,\sigma\, A_{\sigma(1)\,1}\cdots A_{\sigma(n)\,n},$$

for $A = (A_{ij})_{i,j=1}^n$. The central element of $\mathscr{U}(\mathfrak{gl}_n)$ corresponding to the left-hand side is called the Capelli element

$$C = \det(E + \natural_n),$$

as $\Pi = \rho(E)$. Let u be a formal parameter. We can introduce more generally

$$C(u) = \det(E + \natural_n + u \cdot 1_n),$$

where 1_n is the unit matrix of size n, and expand it in u as

$$C(u) = \sum_{r=0}^{n} (u)_r \, C_{n-r}$$

with $(u)_r = u(u+1)\cdots(u+r-1)$. These lower order Capelli elements C_k, also central in $\mathscr{U}(\mathfrak{gl}_n)$, are indeed to appear in the general Capelli identities. See [5], [10], [11] for more details. In this article, we will only be concerned with the 'highest' Capelli element C_n but need to use the element $C(u)$ with the parameter u.

It is easy to see that

$$\det({}^tX)^{-u} \, \rho(C) \, \det({}^tX)^u = \det \rho(C(u)),$$

because $\det({}^tX)^{-u} \, \Pi_{ij} \, \det({}^tX)^u = \Pi_{ij} + u \cdot \delta_{ij}$ holds; δ_{ij} being the Kronecker's delta. From this and the Capelli identity, we have

$$\det({}^tX)^s \, \det(D)^s = \det \rho(C(-s+1)) \cdots \det \rho(C(-1)) \det \rho(C(0)).$$

Instead of giving the full proof of this equality, let us demonstrate the case for $s = 2$, from which one can easily deduce the inductive process for more general cases:

$$\begin{aligned}
\det({}^tX)^2 \, \det(D)^2 &= \det({}^tX) \det({}^tX) \, \det(D) \, \det(D) \\
&= \det({}^tX) \, \det \rho(C(0)) \, \det(D) \\
&= \det({}^tX) \, \det \rho(C(0)) \, \det({}^tX)^{-1} \, \det({}^tX) \, \det(D) \\
&= \det \rho(C(-1)) \, \det({}^tX) \, \det(D) \\
&= \det \rho(C(-1)) \, \det \rho(C(0)).
\end{aligned}$$

If we introduce

$$\Pi(u) = \Pi + \natural_n + u \cdot 1_n,$$

then the above formula is

$$\det(\Pi(0)) \det(\Pi(-1)) \cdots \det(\Pi(-s+1)) = \det({}^tX)^s \, \det(D)^s.$$

§2. The problem

In the setting of the classical Capelli identity (square case), one may explain the role of coordinates x_{ij} as extracted from the matrix elements of the representation of the algebra $A = \mathrm{Mat}_n(\mathbb{C})$ on $V = \mathbb{C}^n$. Here V is the (unique) irreducible module of the simple algebra A, and the determinant is the "reduced norm" of the algebra A. If we replace the role of the representation V with the regular representation on A itself, we get the nth power of the determinant as its "norm", where n is the rank of A over the ground field $\mathbb{C}$. This is just seen from the fact that the regular representation of A is the n times multiple of the irreducible V.

The attempt, as mentioned in the Introduction above, by An Huang was to get the Capelli identity for the case of the regular representation of the quaternions (simple algebra over the reals), and of the octonions (non-associative division ring over the reals). He actually got the Capelli-type identity as a non-commutative multiplication formula (with a suitable correction by diagonal shift for the left-hand side) of the determinant. One would imagine that this kind of Capelli identities might be deduced from the usual (irreducible) one. But this is not quite the case: a new point of view appeared in this result.

Let us explain the problem for more basic situation with $A = \mathrm{Mat}_n$. When we write σ_V to denote the matrix expression for the irreducible representation σ on the vector space V, the r times multiple of σ on $\mathbb{C}^r \otimes V$ will be written as the Kronecker product $1_r \otimes \sigma_V$, which looks like

$$1_r \otimes \sigma_V = \begin{bmatrix} \sigma_V & 0 & 0 & \cdots & 0 \\ 0 & \sigma_V & 0 & \cdots & 0 \\ 0 & 0 & \sigma_V & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_V \end{bmatrix}.$$

We remark that $\mathbb{C}^r$ is the multiplicity space, and if we exchange the place of multiplicity space in the tensor multiplication as $V \otimes \sigma$, then the matrix expression accordingly comes to $\sigma_V \otimes 1_r$. Let us illustrate the difference between those two more explicitly for $\dim V = 2$ and $r = 2$. Writing

$$\sigma_V = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have

$$1_2 \otimes \sigma_V = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}, \qquad \sigma_V \otimes 1_2 = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

In both cases, when a, b, c, d are commutative, the determinants of $1_2 \otimes \sigma_V$ and $\sigma_V \otimes 1_2$ are just $(\det \sigma_V)^2$. This is sort of trivial because two Kronecker products $A \otimes B$ and $B \otimes A$ are similar. But once we transfer to non-commutative variables from commutative ones, nothing could be obvious.

To make things more specific, we start from the basic relation

$$\Pi = {}^t X\, D$$

and form its r times multiple representation:

$$1_r \otimes \Pi = (1_r \otimes {}^t X)(1_r \otimes D).$$

In this case, the determinants of the factors in the right-hand side are respectively $(\det {}^t X)^r$ and $(\det D)^r$, so that the product of these two is seen to be

$$\det(\Pi(0)) \det(\Pi(-1)) \cdots \det(\Pi(-r+1))$$

from the formula at the end of the section 1. This is just an equality of the invariant differential operators. However, we see that the Π side is also written as a non-commutative (column) determinant: its just the determinant of the matrix

$$\begin{bmatrix} \Pi(0) & 0 & 0 & \cdots & 0 \\ 0 & \Pi(-1) & 0 & \cdots & 0 \\ 0 & 0 & \Pi(-2) & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Pi(-r+1) \end{bmatrix}.$$

The proof is quite obvious, because, this matrix is a block diagonal one. We may remark that this is

$$1_r \otimes \Pi + \operatorname{diag}(\natural_n, \natural_n - 1_n, \cdots, \natural_n - (r-1)1_n),$$

a simple correction of $1_r \otimes \Pi$ by diagonal shift. This may be called a *trivial* Capelli identity for the r multiple of the irreducible module $\mathbb{C}^n$.

So far, nothing is interesting nor striking. But when we transfer from the equality $1_r \otimes \Pi = (1_r \otimes {}^t X)(1_r \otimes D)$ to its 'flip' $\Pi \otimes 1_r = ({}^t X \otimes 1_r)(D \otimes 1_r)$, the situation can drastically change.

Problem: For the multiplication of the matrix

$$\Pi \otimes 1_r = ({}^t X \otimes 1_r)(D \otimes 1_r),$$

does the multiplication formula of the column determinant with suitable correction for the left-hand side, preferably by a diagonal shift?

The answer is actually 'yes'. We may call this result by the name of a *non-trivial* Capelli identity for the r multiple of the irreducible module $\mathbb{C}^n$. To be more precise, the diagonal shift will be the re-ordering of

$$\mathrm{diag}(\natural_n, \natural_n - 1_n, \cdots, \natural_n - (r-1)1_n)$$

appearing in the formula into

$$\mathrm{diag}(\natural_r + (n-r)1_r, \natural_r + (n-r-1)1_r, \cdots, \natural_r + (n-r-(n-1))1_r).$$

In other words, the numbers are just the 'contents'

$$\begin{array}{cccc} 0 & -1 & \cdots & -r+1 \\ 1 & 0 & \cdots & -r+2 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ n-1 & n-2 & \cdots & n-r \end{array}$$

filled in the rectangular Young diagram (r^n) to read off lexicographically, on one hand vertical first, and on the other hand horizontal first.

For example, in case $n = 2, r = 2$, the $\mathrm{diag}(1, 0, 0-1)$ is both for $1_2 \otimes \Pi$ and $\Pi \otimes 1_2$. But the equality

$$\det(1_2 \otimes \Pi + \mathrm{diag}(1, 0, 0, -1)) = \det(\Pi \otimes 1_2 + \mathrm{diag}(1, 0, 0, -1))$$

is not obvious. We will show this later. Also for $n = 2$, $r = 3$, the diagonal shift $\mathrm{diag}(1, 0, 0, -1, -1, -2)$ is for $1_3 \otimes \Pi$ and the $\mathrm{diag}(1, 0, -1, 0, -1, -2)$ is for $\Pi \otimes 1_3$. For $n = 3$, $r = 2$, the $\mathrm{diag}(2, 1, 0, 1, 0, -1)$ is for $1_2 \otimes \Pi$ and the $\mathrm{diag}(2, 1, 1, 0, 0, -1)$ is for $\Pi \otimes 1_2$. In general, for the case $n = r$, i.e, the regular representation case, the correction for the determinant with the diagonal shifts in both $1_n \otimes \Pi$ and $\Pi \otimes 1_n$ are the same.

§3. About the proof

For the proof, we utilize the exterior algebra as for the classical Capelli identities. We need r copies of the anti-commuting variables.

First consider the exterior algebra

$$\Lambda_n^r = \Lambda(e_{pq};\, 1 \le p \le r, 1 \le q \le n)$$

of rn generators with the commutation relations

$$e_{pq}e_{kl} + e_{kl}e_{pq} = 0.$$

We work on the algebra

$$A = \Lambda_n^r \otimes \mathscr{P}\mathscr{D}(\mathrm{Mat}_n),$$

the ring of polynomial coefficient differential operators on the matrix space extended by the anti-commuting formal variables. Let us put

$$\zeta_{pi} = \sum_{a=1}^{n} e_{pa}\Pi_{ai}$$

and also, with parameter u,

$$\zeta_{pi}(u) = \sum_{a=1}^{n} e_{pa}(\Pi_{ai} + u\delta_{ai}) = \zeta_{pi} + u\, e_{pi}.$$

We have the following commutation relations among these elements.

Commutation relations: For any parameter u and v, we have

$$\zeta_{pi}(u)\,\zeta_{qj}(v) + \zeta_{qj}(v)\,\zeta_{pi}(u) = \zeta_{pj}(w)e_{qi} + \zeta_{qi}(w)e_{pj}$$

with arbitrary w independent of u and v. Notice that the left-hand side is the anti-commutator $[\,\zeta_{pi}(u), \zeta_{qj}(v)\,]_+$ of the two elements.

From this we see the two typical commutation relations, which are essentially the same as in the proof for the classical Capelli identities as follows:

(1) For the case $p = q$, we see

$$\zeta_{pi}(u)\,\zeta_{pj}(u-1) + \zeta_{pj}(u)\,\zeta_{pi}(u-1) = 0.$$

(2) For the case $i = j$, we see

$$\zeta_{pi}(u)\,\zeta_{qi}(u-1) + \zeta_{qi}(u)\,\zeta_{pi}(u-1) = 0.$$

(3) As the corollaries of these two, we have

$$\zeta_{pi}(u)\,\zeta_{pi}(u-1) = 0.$$

Instead of giving the proof for the general case, we show only for the simplest case $r = 2, n = 2$. The product

$$\zeta_{11}(1)\,\zeta_{12}(0)\,\zeta_{21}(0)\,\zeta_{22}(-1)$$

gives us $e_{11}e_{12}e_{21}e_{22}\ \det(\Pi(0))\ \det(\Pi(-1))$, because

$$\zeta_{11}(1)\,\zeta_{12}(0) = e_{11}e_{12}\det(\Pi(0))$$

and

$$\zeta_{21}(0)\,\zeta_{22}(-1) = e_{21}e_{22}\det(\Pi(-1)).$$

This order of product is thus for

$$\det(1_2 \otimes \Pi + \mathrm{diag}(1, 0, 0, -1)).$$

To get

$$\det(\Pi \otimes 1_2 + \mathrm{diag}(1, 0, 0, -1),$$

we need to make the product (up to the signature)

$$\zeta_{11}(1)\,\zeta_{21}(0)\,\zeta_{12}(0)\,\zeta_{22}(-1).$$

Our goal for this minimal non-trivial case, we have only to change the order in the middle. By the commutation relation above, we see

$$\zeta_{12}(0)\,\zeta_{21}(0) = -\zeta_{21}(0)\,\zeta_{12}(0) + \zeta_{11}(w)e_{22} + \zeta_{22}(w)e_{11}$$

for any w, so that

$$\begin{aligned}\zeta_{11}(1)\,\zeta_{12}(0)\,\zeta_{21}(0)\,\zeta_{22}(-1) = &- \zeta_{11}(1)\,\zeta_{21}(0)\,\zeta_{12}(0)\,\zeta_{22}(-1)\\ &+ \zeta_{11}(1)\,\zeta_{11}(w)\,e_{22}\,\zeta_{22}(-1)\\ &+ \zeta_{11}(1)\,\zeta_{22}(w)\,e_{11}\,\zeta_{22}(-1).\end{aligned}$$

Then putting $w = 0$, we get the vanishing factors $\zeta_{11}(1)\,\zeta_{11}(0)$ and $\zeta_{22}(0)\,\zeta_{22}(-1)$. in the second and the the third terms respectively. Thus we obtain

$$\zeta_{11}(1)\,\zeta_{12}(0)\,\zeta_{21}(0)\,\zeta_{22}(-1) = -\zeta_{11}(1)\,\zeta_{21}(0)\,\zeta_{12}(0)\,\zeta_{22}(-1)$$

as desired.

The general case needs much more intricate manipulations of commutation relations. Cancellation of the terms yielded by the exchanges of elements could be tricky. We will omit the details about them here.

We remark here that the commutation relations above have some similarity with those used in the paper [13] on the Koszul complex, in which both commuting and anti-commuting formal variables are utilized. There some difference operator plays some roles, which we may expect to work well also in our cases.

§4. Some related studies — Capelli type identity for group determinant

The starting fact was brought by Dr. An Huang for the case of the regular representations of quaternions (and of octonions) [6]. Inspired by his results, we can make more general problems in this direction such as "Capelli type identities" for (semi-)simple algebras (over "any" ground fields). As we treat non-commutative variables, the choice, even their order of arrangement in the matrix (for the determinant) are by no means obvious for any good formulations.

The matrix case and quaternion case, we could make use of some distinguished basis of the algebra, and the column determinant suffices for the good identities. For this point, we should say we are lucky enough.

This formulation reminds us of other old but not published study of my own on Capelli type identities for group determinant. Let us explain about it.

Let G be a finite group of order n. For a function φ on G, we associate an $n \times n$ matrix $R(\varphi)$ as follows:

$$R(\varphi) = (\varphi(gh^{-1}))_{g,h \in G}.$$

This is a matrix expression of the regular representation of G. For the functions φ, ψ, we define their convolution product by

$$(\varphi * \psi)(g) = \sum_{k \in G} \varphi(gk^{-1})\psi(k).$$

Then it is easy to see

$$R(\varphi * \psi) = R(\varphi)\, R(\psi).$$

The values of those functions are not necessarily supposed to be commutative, but are allowed to be in a non-commutative (associative) algebra. The operation on function

$$\check{\varphi}(g) = \varphi(g^{-1})$$

is corresponding to the transposition for R, i.e.,

$$R(\check{\varphi}) = {}^{t}R(\varphi).$$

Now, for each element $g \in G$, consider the indeterminate x_g, and also the corresponding partial differential operator ∂_g. Let $\mathscr{A} = \mathscr{P}\mathscr{D}[\mathbb{C}^n]$ be the ring of polynomial coefficient differential operators on $\mathbb{C}^n$ for which the x_g's are the standard coordinates. In what follows, the functions on G are to take values in $\mathscr{A}$. Furthermore, we define

$$\mathit{\Pi}_g = \sum_{a \in G} x_{ag^{-1}}\, \partial_a,$$

a kind of polarization operators, which are similar but different from Π_{ij}'s introduced in section 1, so that we use a slightly modified notation as $\mathit{\Pi}_g$'s. We now have three $\mathscr{A}$-valued functions

$$\check{x}\,;\, g \mapsto x_{g^{-1}}, \qquad \partial \,:\, g \mapsto \partial_g, \qquad \mathit{\Pi} \,:\, g \mapsto \mathit{\Pi}_g,$$

and, by definition,

$$\mathit{\Pi} = \check{x} * \partial,$$

so that

$$R(\mathit{\Pi}) = {}^t R(x)\, R(\partial).$$

We see this looks like the basic relation for the infinitesimal action of $\mathfrak{gl}_n$ on the matrix space. It is natural to raise the following questions:

Problems:

(1) Describe the relation among the determinants of those three matrices.

(2) Express the differential operator $\det {}^t R(x) \det R(\partial)$ in terms of $\mathit{\Pi}_g$'s.

In the problem (1), the meaning of the determinant for $R(\mathit{\Pi})$ is not clear, so that question is more or less vague. However, in case G is abelian, since entries in each matrices are commutative, there are nothing to be worried about. The result in this case is

Theorem 1: Suppose G to be abelian, then we have

$$\det R(\mathit{\Pi}) = \det {}^t R(x) \det R(\partial).$$

Note that contrary to the classical Capelli identity, no correction (for example by diagonal shift) in the left-hand side is needed. Although the result looks like the multiplication formula for the commutative entries, it is not quite obvious, because x_g's and ∂_g's do not commute.

We should remark that the Capelli identity for the group determinant treats highly reducible case. If the regular representation on $\mathbb{C}[G]$ is decomposed into irreducibles in the form

$$\mathbb{C}^n = \mathbb{C}[G] \simeq \bigoplus_{\rho \in \hat{G}} d_\rho\, V_\rho,$$

where $\hat{G}$ is the set of equivalence classes of irreducible G-modules, and V_ρ is the representation space of ρ and d_ρ its degree ($= \dim_{\mathbb{C}} V_\rho$), the Fourier transform of the functions $x, \partial, \mathit{\Pi}$ give us quite the same situation for the classical Capelli identity, and also with the multiplicity d_ρ, the case we treated above in this paper. The (operator valued) Fourier transform of a function φ is defined as

$$\rho(\varphi) = \sum_{g \in G} \varphi(g)\rho(g) \in \mathscr{A} \otimes \mathrm{End}(V_\rho),$$

for an irreducible representation ρ. Then just like the regular representation, we have

$$\rho(\mathit{\Pi}) = \rho(\check{x})\, \rho(\partial),$$

and this is quite the same as the classical situation. The Capelli element associated to ρ is defined as

$$C_\rho = \det(\rho(\mathit{\Pi}) + \varepsilon_\rho \natural_{d_\rho})$$

with $\varepsilon_\rho = n/d_\rho$. More generally, we introduce the Capelli element with parameter shift as

$$C_\rho(u) = \det(\rho(\mathit{\Pi}) + \varepsilon_\rho(u + \natural_{d_\rho})).$$

Furthermore, a sort of factorial power for C_ρ is defined as

$$C_\rho^{(s)}(u) = C_\rho(u)C_\rho(u-1)\cdots C_\rho(u-s+1).$$

Under these notations, we have an answer for the Problem (2) as

Theorem 2: For a general G, we have

$$\det {}^t R(x)\, \det R(\partial) = \prod_{\rho \in \hat{G}} C_\rho^{(d_\rho)}(0).$$

In the formula here, the right-hand side is not expressed as a single determinant using the original matrix $R(\mathit{\Pi})$. It should be too good to be true if the right-hand side could be expressed in a form $\det(R(\mathit{\Pi}) + \natural)$, because we see no natural order in arranging the elements $g \in G$. Also, we cannot tell what kind of non-commutative determinant would fit for this right-hand side.

These problems are already in the author's mind around 2000, but any essential progress has not been made since then. With the motivation for the reducible Capelli identities dealt in the first part of this article, we may formulate more general problem like "the Capelli identity associated with (semi-)simple algebras (over any field)", where the choice for the natural base could be more specific. For example, the quaternion algebra (over the reals) has a privileged basis $1, i, j, k$, which are "group-like" and something similar to the generators of group algebra, but might have more natural meaning for the Capelli identities.

There seems more problems to solve still for reducible modules.

§ Appendix: Sketch of the proof for Theorem 2

We recall some fundamental facts as preliminaries. Let us start with the irreducible decomposition of the regular representation of the finite group G in a slightly different form:

$$\mathbb{C}G \simeq \bigoplus_{\rho \in \hat{G}} \mathrm{End}(V_\rho),$$

where the left-hand side is the group algebra of G. This isomorphism is not only between the vector spaces but (1) as $G \times G$ modules, and also (2) as $\mathbb{C}$ algebras. The transition from the left to the right is just the Fourier transform explained above. There is one more view point to look at this isomorphism. In both spaces, there are sort of canonical linear forms. On the group algebra $\mathbb{C}G$, we have the 'co-unit' ϵ (the evaluation at the unit 1_G) defined as

$$\epsilon(g) = \begin{cases} 1 & (g = 1_G), \\ 0 & \text{otherwise}. \end{cases}$$

On the space $\mathrm{End}(V_\rho)$, we have the trace τ_ρ. The relation between those linear forms via the above isomorphism is given by the Plancherel formula for G:

$$\epsilon = \sum_{\rho \in \hat{G}} \frac{d_\rho}{n} \, \tau_\rho.$$

The dual of this takes form of the Schur orthogonality for the matrix elements. For the vectors $u \in V_\rho$, $\lambda \in V_\rho^*$, we put

$$\rho_{u,\lambda}(g) = \langle \lambda, \rho(g) u \rangle.$$

Here $\langle \, , \rangle$ means the canonical pairing of V_ρ and its dual V_ρ^*. The Schur orthogonality relations are

$$\sum_{g \in G} \rho_{u,\lambda}(g) \, \sigma_{v,\mu}(g^{-1}) = \begin{cases} 0 & (\rho \not\sim \sigma), \\ \dfrac{n}{d_\rho} \langle \lambda, v \rangle \langle \mu, u \rangle & (\rho = \sigma). \end{cases}$$

Now, let us return to our main concern, the commutation relations for the non-commutative matrix elements of $\rho(\check{x}), \rho(\partial)$, and $\rho(\mathit{\Pi})$. By the Schur orthogonality, we see first

$$\begin{aligned}
[\,\rho_{u,\lambda}(\partial), \rho_{v,\mu}(\check{x})\,] &= \sum_{g,h\in G} \left[\,\partial_g\, \rho_{u,\lambda}(g),\, x_h\, \rho_{v,\mu}(h^{-1})\,\right] \\
&= \sum_{g,h\in G} [\,\partial_g, x_h\,]\; \rho_{u,\lambda}(g)\, \rho_{v,\mu}(h^{-1}) \\
&= \sum_{g,h\in G} \delta_{g,h}\;\rho_{u,\lambda}(g)\, \rho_{v,\mu}(h^{-1}) \\
&= \sum_{g\in G} \rho_{u,\lambda}(g)\, \rho_{v,\mu}(g^{-1}) = \frac{n}{d_\rho}\, \langle \lambda, v\rangle\, \langle \mu, u\rangle
\end{aligned}$$

Quite the same computations, based also on the Schur orthogonality, give us the matrix elements $\rho_{u,\lambda}(\partial)$ and $\sigma_{v,\mu}(\check{x})$ commute when ρ and σ are not equivalent.

For the representation space V_ρ, take a linear basis (also its dual basis) indexed by the letters i, j etc. Then the above commutation relations read

$$[\,\rho_{ij}(\partial), \rho_{kl}(\check{x})\,] = \frac{n}{d_\rho}\; \delta_{il}\, \delta_{jk}.$$

This means that the matrix elements for the irreducible ρ, the matrix elements of $\rho(\partial)$ and the transposed of $\rho(\check{x})$ are the "dual" in the sense that they have the canonical commutation relations with the coupling constant $\varepsilon_\rho = n/d_\rho$. The matrix composition $\rho(\mathit{\Pi}) = \rho(\check{x})\,\rho(\partial)$ tells us that the elements $\rho_{ij}(\mathit{\Pi})$'s behave like the usual polarization operators Π_{ij} in the section 1, only with the correction of coupling constant ε_ρ instead of 1.

The rest of the details are quite parallel to the known process, as, for example found in [14] (see also the section 3 in this article). To be more precise, using the exterior algebra Λ_ρ with d_ρ anti-commuting generators, we put

$$\begin{aligned}
\xi_i &= \sum_{a=1}^{d_\rho} e_a\, \rho_{ai}(\check{x}), \\
\zeta_j &= \sum_{i=1}^{d_\rho} \xi_i\, \rho_{ij}(\partial) = \sum_{a=1}^{d_\rho} e_a\, \rho_{aj}(\mathit{\Pi}).
\end{aligned}$$

Also with the parameter u, we define

$$\zeta_j(u) = \zeta_j + \varepsilon_\rho \, u \, e_j = \sum_{a=1}^{d_\rho} e_a \, (\rho_{aj}(\Pi) + \varepsilon_\rho \, u \, \delta_{aj}).$$

Then the following commutation relations are proved:

$$\zeta_q(u+1)\xi_i + \xi_i\zeta_q(u) = 0.$$

With these relations, the computation of the product

$$\zeta_1(d_\rho - 1) \, \zeta_2(d_\rho - 2) \cdots \zeta_{d_\rho}(0)$$

gives us the desired Capelli identity on the space V_ρ.

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