Advanced Studies in Pure Mathematics 76, 2018
Representation Theory, Special Functions and Painlevé Equations — RIMS 2015
pp. 449–468

Embedding of the rank 1 DAHA into $Mat(2, \mathbb{T}_q)$ and its automorphisms

Marta Mazzocco

For the 60th birthday of Masatoshi Noumi

Abstract.

In this paper we show how the Cherednik algebra of type $\check{C}_1 C_1$ appears naturally as quantisation of the group algebra of the monodromy group associated to the sixth Painlevé equation. This fact naturally leads to an embedding of the Cherednik algebra of type $\check{C}_1 C_1$ into $Mat(2, \mathbb{T}_q)$, i.e. 2×2 matrices with entries in the quantum torus. For $q = 1$ this result is equivalent to say that the Cherednik algebra of type $\check{C}_1 C_1$ is Azumaya of degree 2 [31]. By quantising the action of the braid group and of the Okamoto transformations on the monodromy group associated to the sixth Painlevé equation we study the automorphisms of the Cherednik algebra of type $\check{C}_1 C_1$ and conjecture the existence of a new automorphism. Inspired by the confluences of the Painlevé equations, we produce similar embeddings for the confluent Cherednik algebras $\mathcal{H}_V, \mathcal{H}_{IV}, \mathcal{H}_{III}, \mathcal{H}_{II}$ and $\mathcal{H}_I$, defined in [27].

§1. Introduction

The Painlevé sixth equation [16, 33, 17] describes the monodromy preserving deformations of a rank 2 Fuchsian system with four simple poles a_1, a_2, a_3 and ∞. The solution of this Fuchsian system is in general a multi-valued analytic vector–function in the punctured Riemann sphere $\mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\}$ and its multivaluedness is described by the so-called monodromy group, i.e. a subgroup of $SL_2(\mathbb{C})$ generated by the images M_1, M_2, M_3 of the generators of the fundamental group under

Received December 12, 2015.
Revised March 10, 2016.
2010 *Mathematics Subject Classification.* 33D80, 33D52, 16T99.
Key words and phrases. Double Affine Hecke Algebra, Monodromy preserving deformations, Painlevé equations.

the anti-homomorphism:

$$\rho : \pi_1 \left(\mathbb{P}^1 \backslash \{a_1, a_2, a_3, \infty\}, \lambda_0\right) \to SL_2(\mathbb{C}).$$

In this paper, we introduce flat coordinates on a large open sub-set of the set of all possible monodromy groups obtained in this way (see Theorem 3). We then obtain a quantisation of the group algebra of the monodromy group by introducing a canonical quantisation for these flat coordinates. This quantum algebra is isomorphic to the Cherednik algebra of type $\check{C}_1 C_1$, i.e. the algebra $\mathcal{H}$ generated by four elements $V_0, V_1, \check{V}_0, \check{V}_1$ which satisfy the following relations [7, 32, 30, 34]:

$$(V_0 - k_0)(V_0 + k_0^{-1}) = 0 \tag{1}$$
$$(V_1 - k_1)(V_1 + k_1^{-1}) = 0 \tag{2}$$
$$(\check{V}_0 - u_0)(\check{V}_0 + u_0^{-1}) = 0 \tag{3}$$
$$(\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) = 0 \tag{4}$$
$$\check{V}_1 V_1 V_0 \check{V}_0 = q^{-1/2}, \tag{5}$$

where $k_0, k_1, u_0, u_1, q \in \mathbb{C}^\star$, such that $q^m \neq 1$, $m \in \mathbb{Z}_{>0}$.

As a consequence we obtain an embedding of the Cherednik algebra of type $\check{C}_1 C_1$ into $Mat(2, \mathbb{T}_q)$, i.e. 2×2 matrices with entries in the quantum torus:

Theorem 1. *The map*:

$$V_0 \to \begin{pmatrix} k_0 - k_0^{-1} - ie^{-S_3} & -i\,e^{-S_3} \\ k_0^{-1} - k_0 + i\,e^{-S_3} + i\,e^{S_3} & i\,e^{-S_3} \end{pmatrix} \tag{6}$$

$$V_1 \to \begin{pmatrix} k_1 - k_1^{-1} - i\,e^{S_2} & k_1 - k_1^{-1} - i\,e^{-S_2} - i\,e^{S_2} \\ i\,e^{S_2} & i\,e^{S_2} \end{pmatrix} \tag{7}$$

$$\check{V}_1 \to \begin{pmatrix} 0 & -ie^{S_1} \\ i\,e^{-S_1} & u_1 - u_1^{-1} \end{pmatrix} \tag{8}$$

$$\check{V}_0 \to \begin{pmatrix} u_0 & 0 \\ q^{\frac{1}{2}} s & -\frac{1}{u_0} \end{pmatrix}, \tag{9}$$

where S_1, S_2, S_3 *satisfy the following commutation relations*:

$$[S_1, S_2] = [S_2, S_3] = [S_3, S_1] = i\pi\hbar, \qquad u_0 = -i\,e^{-S_1 - S_2 - S_3}, \tag{10}$$

for $q = e^{-i\pi\hbar}$ *and*

$$s = \overline{k}_0 e^{-S_1-S_2} + \overline{k}_1 e^{-S_1+S_3} + \overline{u}_1 e^{S_2+S_3} + i\, e^{-S_1-S_2+S_3} + i\, e^{-S_1+S_2+S_3} - u_0,$$

gives and embedding of $\mathcal{H}$ *into* $Mat(2,\mathbb{T}_q)$*. In particular, the images of* $V_0, \check{V}_0, V_1, \check{V}_1$ *in* $GL(2,\mathbb{T}_q)$ *satisfy the relations* $(1,\dots,4)$ *and* (5)*, in the quantum ordering is dictated by the matrix product ordering*[1].

Note that this result was already proved in [27] (using a different presentation for $\mathcal{H}$). The purpose of the current paper is to explain this result in the Painlevé context and to draw parallels between the theory of the Painlevé equations and the theory of the Cherednik algebra of type $\check{C}_1C_1$.

In particular, we prove that all the known automorphisms of the Cherednik algebra of type $\check{C}_1C_1$ are a quantisation of the action of the braid group on monodromy matrices proposed in [9, 26] to describe the analytic continuation of the solutions to the sixth Painlevé equation.

Next we deal with the Okamoto transformations of the sixth Painlevé equation and their action on the monodromy group. By quantisation we conjecture the existence of an automorphism of the Cherednik algebra of type $\check{C}_1C_1$ which acts as follows on the parameters:

$$(u_1,u_0,k_1,k_0) \to \left(\frac{u_1}{\sqrt{u_1u_0k_1k_0}}, \frac{u_0}{\sqrt{u_1u_0k_1k_0}}, \frac{k_1}{\sqrt{u_1u_0k_1k_0}}, \frac{k_0}{\sqrt{u_1u_0k_1k_0}}\right)$$

We postpone the computation of the action this automorphism on $V_0, V_1, \check{V}_0, \check{V}_1$ to a subsequent publication.

Finally, in [27], the author introduced confluent versions of the Cherednik algebra of type $\check{C}_1C_1$ by using a concatenation of Whittaker-type limits similar to those introduced in [8]. In this paper we explain the origin of these confluent Cherednik algebras from the point of view of the Painlevé theory. In [6] the confluence scheme of the Painlevé differential equations was explained in terms of certain geometric operations giving rise to specific asymptotic limits in the classical coordinates s_1, s_2, s_3 and parameters. Here, we quantise these asymptotic limits to obtain asymptotic limits for the quantum coordinates S_1, S_2, S_3 and of the parameters k_0, k_1, u_0, u_1 (see Fig. 1). By taking these limits in the matrices (6), ..., (9), we produce new matrices which turn out to provide embeddings for the confluent Cherednik algebras $\mathcal{H}_V, \mathcal{H}_{IV}, \mathcal{H}_{III}, \mathcal{H}_{II}$ and $\mathcal{H}_I$ in $Mat(2,\mathbb{T}_q)$.

[1] By this we mean that the product AB of two matrices A, B whose entries are in $\mathbb{T}_q$ is computed by keeping the entries of A on the left matrix of the entries of B.

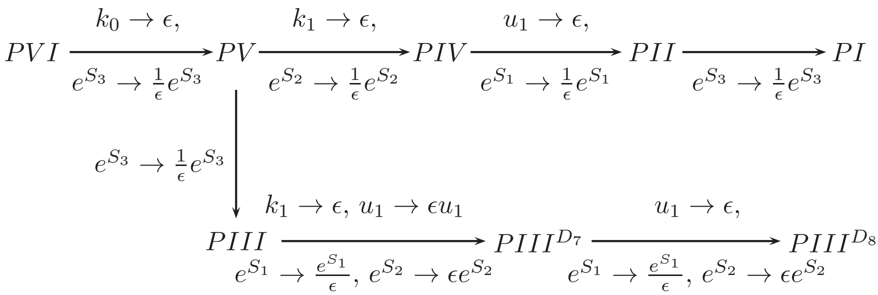

Fig. 1. The [6] confluence scheme for the Painlevé equations denoted here by PVI, PV, PIV, $PIII$, $PIII^{D_7}$, $PIII^{D_8}$, PII, PI and the corresponding rescaling of the quantum shifted shear coordinates S_1, S_2, S_3 such that $\lim_{\hbar\to 0} S_i = s_i + \frac{p_i}{2}$, $i = 1, 2, 3$.

§2. Flat coordinates for the monodromy group of the sixth Painlevé equation

2.1. Sixth Painlevé equation as isomonodromic deformation equation

We start by recalling without proof some very well known facts about the sixth Painlevé equation and its relation to the monodromy preserving deformations equations [22, 29].

The sixth Painlevé equation [16, 33, 17],

$$\begin{aligned} y_{tt} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)y_t^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)y_t+ \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right], \end{aligned} \tag{11}$$

describes the monodromy preserving deformations of a rank 2 meromorphic connection over $\mathbb{P}^1$ with four simple poles a_1, a_2, a_3 and ∞ (for example we may choose $a_1 = 0$, $a_2 = t$, $a_3 = 1$):

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \sum_{k=1}^{3}\frac{A_k(t)}{\lambda - a_k}\Phi, \tag{12}$$

where[2]

$$\text{eigen}(A_i) = \pm\frac{\theta_i}{2}, \quad \text{for } i=1,2,3, \quad A_\infty := -A_1 - A_2 - A_3 \tag{13}$$

$$A_\infty = \begin{pmatrix} \frac{\theta_\infty}{2} & 0 \\ 0 & -\frac{\theta_\infty}{2} \end{pmatrix}, \tag{14}$$

and the parameters θ_i, $i=1,2,3,\infty$ are related to the PVI parameters by

$$\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_1^2}{2}, \quad \gamma = \frac{\theta_3^2}{2}, \quad \delta = \frac{1-\theta_2^2}{2}.$$

The precise dependence of the matrices A_1, A_2, A_3 on the PVI solution $y(t)$ and its first derivative $y_t(t)$ can be found in [29].

The solution $\Phi(\lambda)$ of the system (12) is a multi-valued analytic function in the punctured Riemann sphere $\mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\}$ and its multivaluedness is described by the so-called monodromy matrices, i.e. the images of the generators of the fundamental group under the anti-homomorphism

$$\rho : \pi_1\left(\mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\}, \lambda_0\right) \to SL_2(\mathbb{C}).$$

In this paper we fix the base point λ_0 at infinity and the generators of the fundamental group to be l_1, l_2, l_3, where each l_i, $i=1,2,3$, encircles only the pole a_i once and l_1, l_2, l_3 are oriented in such a way that

$$M_1 M_2 M_3 M_\infty = \mathbb{1}, \tag{15}$$

where $M_\infty = \exp(2\pi i A_\infty)$.

2.2. Riemann-Hilbert correspondence and PVI monodromy manifold

Let us denote by $\mathcal{F}(\theta_1, \theta_2, \theta_3, \theta_\infty)$ the moduli space of rank 2 meromorphic connection over $\mathbb{P}^1$ with four simple poles a_1, a_2, a_3, ∞ of the form (12). Let $\mathcal{M}(G_1, G_2, G_3, G_\infty)$ denote the moduli of monodromy representations ρ up to Jordan equivalence, with the local monodromy data of G_i's prescribed by

$$G_i := \mathrm{Tr}(M_i) = 2\cos(\pi\theta_i), \quad i = 1,2,3,\infty.$$

[2]For simplicity sake, we are recalling here the main facts about the isomonodromic approach in the case when the parameters $\theta_1, \theta_2, \theta_3$ and θ_∞ are not integers. This is just a technical restriction, all the results proved in the paper are actually valid also when we lift such restriction.

Then the Riemann-Hilbert correspondence

$$\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_\infty)/\Gamma \to \mathcal{M}(G_1,G_2,G_3,G_\infty)/GL_2(\mathbb{C}),$$

where Γ is the gauge group [2], is defined by associating to each Fuchsian system its monodromy representation class. The representation space $\mathcal{M}(G_1,G_2,G_3,G_\infty)/GL_2(\mathbb{C})$ is realised as an affine cubic surface (see [21])

$$G_{12}^2+G_{23}^2+G_{31}^2+G_{12}G_{23}G_{31}-\omega_3G_{12}-\omega_1G_{23}-\omega_2G_{31}+\omega_\infty=0, \tag{16}$$

where G_{12}, G_{23}, G_{31} defined as:

$$G_{ij}=\operatorname{Tr}\left(M_iM_j\right), \qquad i,j=1,2,3,$$

and

$$\begin{aligned}\omega_{ij} &:= G_iG_j+G_kG_\infty, \quad k\neq i,j,\\ \omega_\infty &= G_0^2+G_t^2+G_1^2+G_\infty^2+G_0G_tG_1G_\infty-4.\end{aligned}$$

This cubic surface is called *monodromy manifold* of the sixth Painlevé equation and it is equipped with the following Poisson bracket:

$$\begin{aligned}\{G_{12},G_{23}\} &= G_{12}G_{23}+2G_{31}-\omega_2\\ \{G_{23},G_{31}\} &= G_{23}G_{31}+2G_{12}-\omega_3\\ \{G_{31},G_{12}\} &= G_{31}G_{12}+2G_{23}-\omega_1\end{aligned} \tag{17}$$

In [20], Iwasaki proved that the triple (G_{12},G_{23},G_{31}) satisfying the cubic relation (16) provides a set of coordinates on a large open subset $\mathcal{S}\subset\mathcal{M}(G_1,G_2,G_3,G_\infty)$. In the following sub-section, we restrict to such open set.

2.3. Teichmüller theory of the 4-holed Riemann sphere

The real slice of moduli space $\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_\infty)$ of rank 2 meromorphic connections over $\mathbb{P}^1$ with four simple poles a_1,a_2,a_3,∞ can be obtained as a quotient of the Teichmüller space of the 4–holed Riemann sphere by the mapping class group. This fact allowed us to use the combinatorial description of the Teichmüller space of the 4–holed Riemann sphere in terms of fat-graphs to produce a good parameterisation of the monodromy manifold of PVI [4]. In this sub-section we recall the main ingredients of this construction.

We recall that according to Fock [13] [14], the fat graph associated to a Riemann surface $\Sigma_{g,n}$ of genus g and with n holes is a connected three–valent graph drawn without self-intersections on $\Sigma_{g,n}$ with a prescribed cyclic ordering of labelled edges entering each vertex; it must be

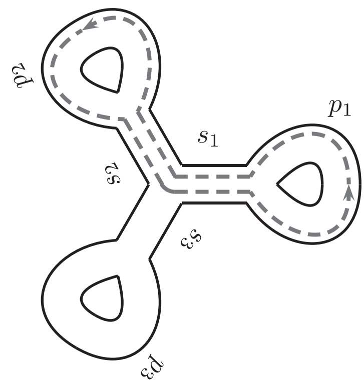

Fig. 2. The fat graph of the 4 holed Riemann sphere. The dashed geodesic corresponds to G_{12}.

a maximal graph in the sense that its complement on the Riemann surface is a set of disjoint polygons (faces), each polygon containing exactly one hole (and becoming simply connected after gluing this hole). In the case of a Riemann sphere $\Sigma_{0,4}$ with 4 holes, the fat–graph is represented in Fig.2 (the fourth hole is the outside of the graph).

The geodesic length functions, which are traces of hyperbolic elements in the Fuchsian group $\Delta_{g,n}$ such that in Poincaré uniformisation:

$$\Sigma_{g,n} \sim \mathbb{H}/\Delta_{g,n},$$

are obtained by decomposing each hyperbolic matrix $\gamma \in \Delta_{g,n}$ into a product of the so–called *right, left and edge matrices:* [13] [14]

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$
$$E_{s_i} := \begin{pmatrix} 0 & -\exp\left(\frac{s_i}{2}\right) \\ \exp\left(-\frac{s_i}{2}\right) & 0 \end{pmatrix}. \tag{18}$$

Let us consider the closed geodesics γ_{ij} encircling the i-th and j-th holes without self intersections (for example γ_{12} is drawn in Fig.1), then their geodesic length functions can be obtained as:

$$\begin{aligned} G_{23} &= -\mathrm{Tr}\left(RE_{s_2}RE_{p_2}RE_{s_2}RE_{s_3}RE_{p_3}RE_{s_3}R\right), \\ G_{31} &= -\mathrm{Tr}\left(LE_{s_3}RE_{p_3}RE_{s_3}RE_{s_1}RE_{p_1}RE_{s_1}\right), \\ G_{12} &= -\mathrm{Tr}\left(E_{s_1}RE_{p_1}RE_{s_1}RE_{s_2}RE_{p_2}RE_{s_2}L\right), \end{aligned} \tag{19}$$

which leads to:[3]

$$(20)\qquad \begin{aligned} G_{23} &= -e^{s_2+s_3} - e^{-s_2-s_3} - e^{-s_2+s_3} - G_2 e^{s_3} - G_3 e^{-s_2}\\ G_{31} &= -e^{s_3+s_1} - e^{-s_3-s_1} - e^{-s_3+s_1} - G_3 e^{s_1} - G_1 e^{-s_3},\\ G_{12} &= -e^{s_1+s_2} - e^{-s_1-s_2} - e^{-s_1+s_2} - G_1 e^{s_2} - G_2 e^{-s_1} \end{aligned}$$

where

$$G_i = e^{\frac{p_i}{2}} + e^{-\frac{p_i}{2}}, \qquad i = 1, 2, 3,$$

and

$$G_\infty = e^{s_1+s_2+s_3} + e^{-s_1-s_2-s_3}.$$

Due to the classical result that each conjugacy class in the fundamental group $\mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\}$ can be represented by a unique closed geodesic, we can make the following identification:

$$(21)\qquad G_{ij} := \mathrm{Tr}\left(M_i M_j\right),$$

and indeed it is a straightforward computation to show that G_{12}, G_{23}, G_{31} defined as in (20) indeed lie on the cubic (16).

Moreover, the Poisson algebra structure (17) is induced by the Poisson algebras of geodesic length functions constructed in [3] by postulating the Poisson relations on the level of the shear coordinates s_α of the Teichmüller space. In our case these are:

$$\{s_1, s_2\} = \{s_2, s_3\} = \{s_3, s_1\} = 1,$$

while the perimeters p_1, p_2, p_3 are assumed to be Casimirs.

Since the parameterisation (20) is analytic in s_1, s_2, s_3, we can complexify s_1, s_2, s_3 and G_1, G_2, G_3, G_∞ to claim that $s_1, s_2, s_3 \in \mathbb{C}$ provide a system of flat coordinates on the Friecke cubic (16).

2.4. Parameterisation of the monodromy group

In the case of *monodromy matrices*, Korotkin and Samtleben in [25] proposed an r-matrix structure of the Fock–Rosly type [15] which did not however satisfy Jacobi relations on monodromy matrices themselves but became consistent on the level of adjoint invariant elements. Therefore the problem of quantising the monodromy group remained open.

[3]Note that for simplicity we have actually shifted the shear coordinates $s_i \to s_i + \frac{p_i}{2}$, $i = 1, 2, 3$

In this section we show that thanks to the identification (21), we can impose

$$
\begin{array}{rcl}
M_1 &=& E_{s_1} R E_{p_1} R E_{s_1},\\
M_2 &=& -R E_{s_2} R E_{p_2} R E_{s_2} L,\\
M_3 &=& -L E_{s_3} R E_{p_3} R E_{s_3} R,
\end{array}
\tag{22}
$$

so that $s_1, s_2, s_3 \in \mathbb{C}$ provide a system of flat coordinates on the monodromy group, rather than only the monodromy manifold. This will allow as to quantise as we shall see in subsection 3.

Theorem 2. *Given any quadruple of diagonalisable matrices $M_1, M_2, M_3, M_\infty \in SL_2(\mathbb{C})$, such that $M_1M_2M_3M_\infty = \mathbb{I}$, the group $\langle M_1, M_2, M_3\rangle$ is irreducible and none of the matrices M_1, M_2, M_3, M_∞ is a multiple of the identity, we can find $s_1, s_2, s_3, p_1, p_2, p_3 \in \mathbb{C}$ such that the following relations hold true* (*up to global conjugation and cyclic permutation*):

$$
\begin{array}{rcl}
M_1 &=& \begin{pmatrix} 0 & -e^{s_1} \\ e^{-s_1} & -e^{\frac{p_1}{2}} - e^{-\frac{p_1}{2}} \end{pmatrix},\\
M_2 &=& \begin{pmatrix} -e^{\frac{p_2}{2}} - e^{-\frac{p_2}{2}} - e^{s_2} & -e^{\frac{p_2}{2}} - e^{-\frac{p_2}{2}} - e^{s_2} - e^{-s_2} \\ e^{s_2} & e^{s_2} \end{pmatrix},\\
M_3 &=& \begin{pmatrix} -e^{\frac{p_3}{2}} - e^{-\frac{p_3}{2}} - e^{-s_3} & -e^{-s_3} \\ e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}} + e^{-s_3} + e^{-s_3} & e^{-s_3} \end{pmatrix},\\
M_\infty &=& \begin{pmatrix} -e^{-s_1-s_2-s_3} & 0 \\ s_\infty & -e^{s_1+s_2+s_3} \end{pmatrix},
\end{array}
$$

where

$$
s_\infty = \left(e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}}\right) e^{-s_1-s_2} + \left(e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}}\right) e^{-s_1+s_3} + \left(e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}}\right) e^{s_2+s_3}
+ e^{-s_1-s_2-s_3} + e^{-s_1-s_2+s_3} + e^{-s_1+s_2+s_3}.
$$

Note that in this parameterisation

$\mathrm{eigen}(M_j) = -e^{\pm\frac{p_j}{2}}$, $j=1,2,3$, so that $\mathrm{Tr}(M_i) = G_i = e^{\frac{p_i}{2}} + e^{-\frac{p_i}{2}}$, $i=1,2,3$,

and $M_\infty = (M_1M_2M_3)^{-1}$ is not diagonal but has eigenvalues $e^{\pm(s_1+s_2+s_3)}$.

§3. Quantisation

In [5] the proper quantum ordering for a special class of geodesic functions corresponding to geodesics going around exactly two holes was constructed and it was proved that for each such geodesic, the matrix entries of the corresponding element in the Fuchsian group satisfy a deformed version of the quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ relations.

In this section we use the same quantum ordering for quantising matrix elements of the monodromy group.

In [4], the quantum Painlevé cubic can be obtained by introducing the Hermitian operators S_1, S_2, S_3 subject to the commutation inherited from the Poisson bracket of s_i:

$$[S_i, S_{i+1}] = i\pi\hbar\{s_i, s_{i+1}\} = i\pi\hbar, \quad i = 1, 2, 3, \ i + 3 \equiv i,$$

while the central elements, i.e. perimeters p_1, p_2, p_3 and $S_1 + S_2 + S_3$ remain non–deformed, so that the constants $\omega_i^{(d)}$ remain non-deformed [4].

The Hermitian operators $G_{23}^\hbar$, $G_{31}^\hbar$, $G_{12}^\hbar$ corresponding to G_{23}, G_{31}, G_{12} are introduced as follows: consider the classical expressions for G_{23}, G_{31}, G_{12} is terms of s_1, s_2, s_3 and p_1, p_2, p_3. Write each product of exponential terms as the exponential of the sum of the exponents and replace those exponents by their quantum version, for example the classical G_{23} is

$$G_{23} = -e^{s_2+s_3} - e^{-s_2-s_3} - e^{-s_2+s_3} - G_2 e^{s_3} - G_3 e^{-s_2},$$

and its quantum version is defined as

$$G_{23}^\hbar = -e^{S_2+S_3} - e^{-S_2-S_3} - e^{-S_2+S_3} - G_2 e^{S_3} - G_3 e^{-S_2}.$$

Then $G_{23}^\hbar, G_{31}^\hbar, G_{12}^\hbar$ satisfy the following quantum algebra [4]:

$$\begin{aligned} q^{-1/2} G_{12}^\hbar G_{23}^\hbar - q^{1/2} G_{23}^\hbar G_{12}^\hbar &= (q^{-1} - q) G_{13}^\hbar + (q^{-1/2} - q^{1/2})\omega_2 \\ q^{-1/2} G_{23}^\hbar G_{13}^\hbar - q^{1/2} G_{13}^\hbar G_{23}^\hbar &= (q^{-1} - q) G_{12}^\hbar + (q^{-1/2} - q^{1/2})\omega_3 \qquad (23) \\ q^{-1/2} G_{13}^\hbar G_{12}^\hbar - q^{1/2} G_{12}^\hbar G_{13}^\hbar &= (q^{-1} - q) G_{23}^\hbar + (q^{-1/2} - q^{1/2})\omega_1 \end{aligned}$$

and satisfy the following quantum cubic relations:

$$\begin{aligned} \mathcal{C}^\hbar &= q^{-1/2} G_{12}^\hbar G_{23}^\hbar G_{13}^\hbar - q^{-1}\big(G_{12}^\hbar\big)^2 - q\big(G_{23}^\hbar\big)^2 - q^{-1}\big(G_{13}^\hbar\big)^2 \\ &\quad - q^{-1/2}\omega_3 G_{12}^\hbar - q^{1/2}\omega_1 G_{23}^\hbar - q^{-1/2}\omega_2 G_{13}^\hbar, \qquad (24) \end{aligned}$$

where $\mathcal{C}^\hbar$ is a central element is the quantum algebra (23).

We now quantise the monodromy matrices in the same way:

Theorem 3. *The following matrices*

$$\begin{aligned}
M_1^\hbar &= \begin{pmatrix} 0 & -e^{S_1} \\ e^{-S_1} & -e^{\frac{p_1}{2}} - e^{-\frac{p_1}{2}} \end{pmatrix}, \\
M_2^\hbar &= \begin{pmatrix} -e^{\frac{p_2}{2}} - e^{-\frac{p_2}{2}} - e^{S_2} & -e^{\frac{p_2}{2}} - e^{-\frac{p_2}{2}} - e^{S_2} - e^{-S_2} \\ e^{S_2} & e^{S_2} \end{pmatrix}, \\
M_3^\hbar &= \begin{pmatrix} -e^{\frac{p_3}{2}} - e^{-\frac{p_3}{2}} - e^{-S_3} & -e^{-S_3} \\ e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}} + e^{-S_3} + e^{-S_3} & e^{-S_3} \end{pmatrix}, \\
M_\infty^\hbar &= \begin{pmatrix} -e^{-S_1-S_2-S_3} & 0 \\ s_\infty^\hbar & -e^{S_1+S_2+S_3} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
s_\infty^\hbar = &\left(e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}}\right) e^{-S_1-S_2} + \left(e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}}\right) e^{-S_1+S_3} + \left(e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}}\right) e^{S_2+S_3} \\
&+ e^{-S_1-S_2-S_3} + e^{-S_1-S_2+S_3} + e^{-S_1+S_2+S_3},
\end{aligned}$$

are elements of $SL(2,\mathbb{T}_q)$ and satisfy the following relations:

$$\begin{aligned}
(M_1^\hbar + e^{\frac{p_1}{2}}\mathbb{I})(M_1^\hbar + e^{\frac{-p_1}{2}}\mathbb{I}) &= 0, \\
(M_2^\hbar + e^{\frac{p_2}{2}}\mathbb{I})(M_2^\hbar + e^{\frac{-p_2}{2}}\mathbb{I}) &= 0, \\
(M_3^\hbar + e^{\frac{p_2}{2}}\mathbb{I})(M_3^\hbar + e^{\frac{-p_3}{2}}\mathbb{I}) &= 0, \\
(M_\infty^\hbar + e^{S_1+S_2+S_3}\mathbb{I})(M_\infty^\hbar + e^{-S_1-S_2-S_3}\mathbb{I}) &= 0, \\
M_\infty^\hbar M_1^\hbar M_2^\hbar M_3^\hbar &= q^{-\frac{1}{2}}\mathbb{I}, \qquad (25)
\end{aligned}$$

where $\mathbb{I}$ is the 2×2 identity matrix.

This theorem shows that we can interpret the Cherednik algebra as quantisation of the group algebra of the monodromy group of the sixth Painlevé equation, in fact the matrices defined by (6), (7), (8), (9) are simply obtained as $iM_3^\hbar$, $iM_2^\hbar$, $iM_1^\hbar$ and $iM_\infty^\hbar$ respectively so that Theorem 1 can be stated as follows:

Theorem 4. *The map*:

$$V_0 \to iM_3^\hbar, \quad V_1 \to iM_2^\hbar, \quad \check{V}_1 \to iM_1^\hbar, \quad \check{V}_0 \to iM_\infty^\hbar, \qquad (26)$$

where $M_1^\hbar, M_2^\hbar, M_3^\hbar, M_\infty^\hbar$ are defined as in (25), *gives and embedding of $\mathcal{H}$ into $Mat(2,\mathbb{T}_q)$. In other words, the matrices $iM_3^\hbar$, $iM_2^\hbar$, $iM_1^\hbar$ and $iM_\infty^\hbar$ in $GL(2,\mathbb{T}_q)$ satisfy the relations* (1,2,3) *and* (4), *in which the quantum ordering is dictated by the matrix product ordering and*

$$u_1 = -ie^{-\frac{p_1}{2}}, \quad k_0 = -ie^{-\frac{p_3}{2}}, \quad k_1 = -ie^{-\frac{p_2}{2}}, \quad u_0 = -ie^{-S_1-S_2-S_3}.$$

Proof of Theorem 1. To prove this theorem, we use the fact that the algebras $\mathcal{H}$ is the algebra generated by five elements $T^{\pm 1}, X^{\pm 1}, Y^{\pm 1}$ with the following relations:

$$XW = WX = 1, \tag{27}$$
$$YZ = ZY = 1, \tag{28}$$
$$XT + abT^{-1}W + a + b = 0, \tag{29}$$
$$ZT + \frac{q}{cd}T^{-1}Y + 1 + \frac{q}{cd} = 0, \tag{30}$$
$$(T + ab)(T + 1) = 0, \tag{31}$$
$$YX = -\frac{q}{ab}T^2XY - q\left(\frac{1}{a} + \frac{1}{b}\right)TY - \left(1 + \frac{cd}{q}\right)TX + (c + d)T, \tag{32}$$

where

$$a = -\frac{u_1}{k_1}, \qquad b = u_1 k_1, \qquad c = -\sqrt{q}\frac{k_0}{u_0}, \qquad d = \sqrt{q}u_0 k_0,$$

and

$$X = \sqrt{q}V_0\check{V}_0, \qquad Y = k_0 u_1 \check{V}_1 V_0, \qquad T = u_1\check{V}_1,$$

and viceversa

$$\check{V}_1 = \frac{1}{u_1}T, \qquad V_0 = \frac{1}{k_0}T^{-1}Y, \qquad \check{V}_0 = \frac{k_0}{\sqrt{q}}Y^{-1}TX, \qquad V_1 = \frac{1}{u_1}TX^{-1}.$$

We then use Theorem 5.2 from [27] to prove formulae (6,7,8,9). □

Q.E.D.

§4. Classical limit of the automorphisms of the Cherednik algebra

In [9] the following action of the braid group on monodromy matrices was proposed to describe the analytic continuation of solutions to the sixth Painlevé equation:

$$\beta_1(M_1, M_2, M_3, M_\infty) = (M_1 M_2 M_1^{-1}, M_1, M_3, M_\infty),$$
$$\beta_2(M_1, M_2, M_3, M_\infty) = (M_1, M_2 M_3 M_2^{-1}, M_2, M_\infty). \tag{33}$$

In [26], this action was expended by adding the following involution:

$$r(M_1, M_2, M_3, M_\infty) = (M_3^{-1}, M_2^{-1}, M_1^{-1}, M_\infty^{-1}). \tag{34}$$

In this section we prove that this extended braid group action gives rise to the automorphisms of the Cherednik algebra of type $\check{C}_1 C_1$ which were studied in [30, 34]. Here we list them as they appear in [31]:

$$\sigma(\check{V}_1, V_1, V_0, \check{V}_0) = (V_0, V_1, V_1^{-1}\check{V}_1 V_1, V_0\check{V}_0 V_0^{-1}), \tag{35}$$

$$\tau(\check{V}_1, V_1, V_0, \check{V}_0) = (\check{V}_1, V_1, V_0\check{V}_0 V_0^{-1}, V_0) \tag{36}$$

$$\eta(\check{V}_1, V_1, V_0, \check{V}_0) = (V_1^{-1}\check{V}_1^{-1} V_1, V_1^{-1}, V_0^{-1}, V_0\check{V}_0^{-1} V_0^{-1}), \tag{37}$$

$$\pi(\check{V}_1, V_1, V_0, \check{V}_0) = (\check{V}_0, \check{V}_1, V_1, V_0). \tag{38}$$

Indeed by quantising (33) and (34) we obtain

$$\beta_1^\hbar(\check{V}_1, V_1, V_0, \check{V}_0) = (\check{V}_1 V_1 \check{V}_1^{-1}, \check{V}_1, V_0, \check{V}_0),$$
$$\beta_2^\hbar(\check{V}_1, V_1, V_0, \check{V}_0) = (\check{V}_1, V_1 V_0 V_1^{-1}, V_1, \check{V}_0),$$
$$r^\hbar(\check{V}_1, V_1, V_0, \check{V}_0) = (V_0^{-1}, V_1^{-1}, \check{V}_1^{-1}, \check{V}_0^{-1}).$$

It is not hard to check that $\sigma = \beta_2^\hbar\beta_1^\hbar\beta_2^\hbar$, $\tau = \pi^2\beta_1^\hbar\pi^{-2}$ and $\eta = r^\hbar\beta_2^\hbar\beta_1^\hbar\beta_2^\hbar$, so that we can claim that the automorphisms of the Cherednik algebra of type $\check{C}_1 C_1$ are indeed the quantisation of the extended modular group action described in [9, 26].

The Painlevé sixth equation admits also an affine group of bi-rational transformations as described in table 1:

	θ_1	θ_2	θ_3	θ_∞	y	t
w_1	$-\theta_1$	θ_2	θ_3	θ_∞	y	t
w_2	θ_1	$-\theta_2$	θ_3	θ_∞	y	t
w_3	θ_1	θ_2	$-\theta_3$	θ_∞	y	t
w_∞	θ_1	θ_2	θ_3	$2-\theta_\infty$	y	t
w_ρ	$\theta_1+\rho$	$\theta_2+\rho$	$\theta_3+\rho$	$\theta_\infty+\rho$	$y+\frac{\rho}{p}$	t
r_1	$\theta_\infty-1$	θ_3	θ_2	θ_1+1	t/y	t
r_2	θ_3	$\theta_\infty-1$	θ_1	θ_2+1	$\frac{t(y-1)}{y-t}$	t
r_3	θ_2	θ_1	$\theta_\infty-1$	θ_3+1	$\frac{y-t}{y-1}$	t
π_{13}	θ_3	θ_2	θ_1	θ_∞	$1-y$	$1-t$
$\pi_{1\infty}$	$\theta_\infty-1$	θ_2	θ_3	θ_1+1	$1/y$	$1/t$
π_{12}	θ_2	θ_1	θ_3	θ_∞	$\frac{t-y}{t-1}$	$\frac{t}{t-1}$

Table 1: Bi-rational transformations for Painlevé VI,
$\rho = \frac{2-\theta_1-\theta_2-\theta_3-\theta_\infty}{2}$.

We note that w_1, w_2, w_3, w_∞ have no effect on the monodromy matrices and therefore on the generators of $\mathcal{H}$, while r_1, r_2, r_3 act as combinations of cyclic permutations and the transformation r, while the permutations $\pi_{13}, \pi_{1\infty}, \pi_{12}$ correspond to a combination of braids. The only transformation which does not have a simple explanation in terms of monodromy matrices M_1, M_2, M_3, M_∞ is w_ρ, which was explained in terms of isomonodromic deformations of a 3×3 linear system with one irregular singularity and one simple pole in [28]. On the parameters (u_1, u_0, k_1, k_0) this transformation acts as follows:

$$(u_1, u_0, k_1, k_0) \to \left(\frac{u_1}{\sqrt{u_1 u_0 k_1 k_0}}, \frac{u_0}{\sqrt{u_1 u_0 k_1 k_0}}, \frac{k_1}{\sqrt{u_1 u_0 k_1 k_0}}, \frac{k_0}{\sqrt{u_1 u_0 k_1 k_0}}\right)$$

We postpone the computation of the action this automorphism on V_0, V_1, $\check{V}_0$, $\check{V}_1$ to a subsequent publication, this will involve a quantum version of the middle convolution operation discussed in [12].

§5. Embedding of the confluent Cherednik algebras into $Mat(2, \mathbb{T}_q)$

The confluent limits of the Cherednik algebra of type $\check{C}_1 C_1$ were introduced in [27], in terms of a different presentation which is equivalent to the following (see Theorem 3.2 in [27]):

Definition 5.1. *Let $k_1, u_0, u_1, q \in \mathbb{C}^\star$, such that $q^m \neq 1$, $m \in \mathbb{Z}_{>0}$. The confluent Cherednik algebras $\mathcal{H}_V, \mathcal{H}_{IV}, \mathcal{H}_{III}, \mathcal{H}_{II}, \mathcal{H}_I$ are the algebras generated by four elements $V_0, V_1, \check{V}_0, \check{V}_1$ satisfying the following relations respectively:*

- $\mathcal{H}_V$:

$$V_0^2 + V_0 = 0, \tag{39}$$
$$(V_1 - k_1)(V_1 + k_1^{-1}) = 0, \tag{40}$$
$$\check{V}_0^{\,2} + u_0^{-1}\check{V}_0 = 0, \tag{41}$$
$$(\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) = 0, \tag{42}$$
$$q^{1/2}\check{V}_1 V_1 V_0 = \check{V}_0 + u_0^{-1}, \tag{43}$$
$$q^{1/2}\check{V}_0\check{V}_1 V_1 = V_0 + 1. \tag{44}$$

- $\mathcal{H}_{IV}$:

$$V_0^2 + V_0 = 0, \tag{45}$$
$$V_1^2 + V_1 = 0, \tag{46}$$
$$\check{V}_0^{\,2} + \frac{1}{u_0}\check{V}_0 = 0, \tag{47}$$
$$(\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) = 0, \tag{48}$$
$$q^{1/2}\check{V}_1 V_1 V_0 = \check{V}_0 + u_0^{-1}, \tag{49}$$
$$\check{V}_0\check{V}_1 V_1 = 0, \tag{50}$$
$$V_0\check{V}_0 = 0. \tag{51}$$

- $\mathcal{H}_{III}$:

$$V_0^2 = 0, \tag{52}$$
$$(V_1 - k_1)(V_1 + k_1^{-1}) = 0, \tag{53}$$
$$\check{V_0}^2 + \frac{1}{\sqrt{q}}\check{V_0} = 0, \tag{54}$$
$$(\check{V_1} - u_1)(\check{V_1} + u_1^{-1}) = 0, \tag{55}$$
$$q^{1/2}\check{V_1}V_1V_0 = \check{V_0} + \frac{1}{\sqrt{q}} \tag{56}$$
$$q^{1/2}\check{V_0}\check{V_1}V_1 = V_0. \tag{57}$$

- $\mathcal{H}_{II}$:

$$V_0^2 + V_0 = 0, \tag{58}$$
$$V_1^2 = 0, \tag{59}$$
$$\check{V_0}^2 + \check{V_0} = 0, \tag{60}$$
$$(\check{V_1} - u_1)(\check{V_1} + u_1^{-1}) = 0, \tag{61}$$
$$q^{1/2}\check{V_1}V_1V_0 = \check{V_0} + 1, \tag{62}$$
$$\check{V_0}\check{V_1}V_1 = 0, \tag{63}$$
$$V_0\check{V_0} = 0. \tag{64}$$

- $\mathcal{H}_{I}$:

$$V_0^2 + V_0 = 0, \tag{65}$$
$$V_1^2 = 0, \tag{66}$$
$$\check{V_0}^2 + \check{V_0} = 0, \tag{67}$$
$$\check{V_1}^2 + \check{V_1} = 0, \tag{68}$$
$$q^{1/2}\check{V_1}V_1V_0 = \check{V_0} + 1, \tag{69}$$
$$\check{V_0}\check{V_1} = 0, \tag{70}$$
$$V_0\check{V_0} = 0. \tag{71}$$

All these algebras $\mathcal{H}_V, \mathcal{H}_{IV}, \mathcal{H}_{III}, \mathcal{H}_{II}, \mathcal{H}_I$ admit embeddings in $Mat(2, \mathbb{T}_q)$ (see Theorems 5.2, 5.3, 5.4, 5.5 and 5.5 in [27]). Here we report these embeddings for the generators $\check{V}_1, V_1, V_0, \check{V}_0$ in order to clarify the confluence scheme in accordance with Figure 1. Note that in Figure 1, we also have the algebras $\mathcal{H}_{III}^{D_7}$ and $\mathcal{H}_{III}^{D_8}$ for which we don't have a Noumi Stokman [30] representation and for which we can't prove the embedding into $Mat(2, \mathbb{T}_q)$, so that the geometric explanation behind these algebras remains conjectural.

Theorem 5. *The map*:

$$V_0 \to \begin{pmatrix} -1 & 0 \\ 1 + i\, e^{S_3} & 0 \end{pmatrix} \tag{72}$$

$$V_1 \to \begin{pmatrix} k_1 - k_1^{-1} - i\, e^{S_2} & k_1 - k_1^{-1} - i\, e^{-S_2} - i\, e^{S_2} \\ i\, e^{S_2} & i\, e^{S_2} \end{pmatrix} \tag{73}$$

$$\check{V}_1 \to \begin{pmatrix} 0 & -\, i e^{S_1} \\ i\, e^{-S_1} & u_1 - u_1^{-1} \end{pmatrix} \tag{74}$$

$$\check{V}_0 \to \begin{pmatrix} 0 & 0 \\ q^{\frac{1}{2}} s & -\frac{1}{u_0} \end{pmatrix}, \tag{75}$$

where

$$s = e^{-S_1 - S_2} + \left(\frac{1}{k_1} - k_1\right) e^{-S_1 + S_3} + \left(\frac{1}{u_1} - u_1\right) e^{S_2 + S_3} + i\, e^{-S_1 - S_2 + S_3} + i\, e^{-S_1 + S_2 + S_3}.$$

gives and embedding of $\mathcal{H}_V$ *into* $Mat(2, \mathbb{T}_q)$. *The images of* $V_0, \check{V}_0, V_1, \check{V}_1$ *in* $Mat(2, \mathbb{T}_q)$ *satisfy the relations* (39), (40), (41), (42), (43), (44) *in which the quantum ordering is dictated by the matrix product ordering.*

Theorem 6. *The map*:

$$V_0 \to \begin{pmatrix} -1 & 0 \\ 1 + i\, e^{S_3} & 0 \end{pmatrix} \tag{76}$$

$$V_1 \to \begin{pmatrix} -1 - i\, e^{S_2} & -1 - i\, e^{S_2} \\ i\, e^{S_2} & i\, e^{S_2} \end{pmatrix} \tag{77}$$

$$\check{V}_1 \to \begin{pmatrix} 0 & -\, i e^{S_1} \\ i\, e^{-S_1} & u_1 - u_1^{-1} \end{pmatrix} \tag{78}$$

$$\check{V}_0 \to \begin{pmatrix} 0 & 0 \\ q^{\frac{1}{2}} s & -\frac{1}{u_0} \end{pmatrix}, \tag{79}$$

where

$$s = e^{-S_1 + S_3} + \left(\frac{1}{u_1} - u_1\right) e^{S_2 + S_3} + i\, e^{-S_1 + S_2 + S_3}.$$

gives and embedding of $\mathcal{H}_{IV}$ into $Mat(2,\mathbb{T}_q)$. The images of $V_0, \check{V}_0, V_1, \check{V}_1$ in $Mat(2,\mathbb{T}_q)$ satisfy the relations (45), (46), (47), (48), (49), (51) *in which the quantum ordering is dictated by the matrix product ordering.*

Theorem 7. *The map*:

$$V_0 \to \begin{pmatrix} 0 & 0 \\ i\,e^{S_3} & 0 \end{pmatrix} \tag{80}$$

$$V_1 \to \begin{pmatrix} k_1 - k_1^{-1} - i\,e^{S_2} & k_1 - k_1^{-1} - i\,e^{-S_2} - i\,e^{S_2} \\ i\,e^{S_2} & i\,e^{S_2} \end{pmatrix} \tag{81}$$

$$\check{V}_1 \to \begin{pmatrix} 0 & -\,ie^{S_1} \\ i\,e^{-S_1} & u_1 - u_1^{-1} \end{pmatrix} \tag{82}$$

$$\check{V}_0 \to \begin{pmatrix} 0 & 0 \\ q^{\frac{1}{2}} s & -1 \end{pmatrix}, \tag{83}$$

where

$$s = e^{-S_1-S_2} + \left(\frac{1}{k_1} - k_1\right) e^{-S_1+S_3} + \left(\frac{1}{u_1} - u_1\right) e^{S_2+S_3} + i\,e^{-S_1-S_2+S_3} + i\,e^{-S_1+S_2+S_3}.$$

gives and embedding of $\mathcal{H}_{III}$ into $Mat(2,\mathbb{T}_q)$. The images of $V_0, \check{V}_0, V_1, \check{V}_1$ in $Mat(2,\mathbb{T}_q)$ satisfy the relations (52), (53), (54), (55), (56), (57), *in which the quantum ordering is dictated by the matrix product ordering.*

Theorem 8. *The map*:

$$V_0 \to \begin{pmatrix} -1 & 0 \\ 1 + i\,e^{S_3} & 0 \end{pmatrix} \tag{84}$$

$$V_1 \to \begin{pmatrix} -i\,e^{S_2} & -i\,e^{S_2} \\ i\,e^{S_2} & i\,e^{S_2} \end{pmatrix} \tag{85}$$

$$\check{V}_1 \to \begin{pmatrix} 0 & -\,ie^{S_1} \\ 0 & -1 \end{pmatrix} \tag{86}$$

$$\check{V}_0 \to \begin{pmatrix} 0 & 0 \\ i\sqrt{q}e^{-S_1}e^{S_2}e^{S_3} - \sqrt{q}\left(u_1 - \frac{1}{u_1}\right) e^{S_2}e^{S_3} & 1 \end{pmatrix}, \tag{87}$$

gives and embedding of $\mathcal{H}_{II}$ *into* $Mat(2,\mathbb{T}_q)$*. The images of* $V_0, \check{V}_0, V_1, \check{V}_1$ *in* $Mat(2,\mathbb{T}_q)$ *satisfy the relations* (58), (59), (60), (62), (63), (64), *in which the quantum ordering is dictated by the matrix product ordering.*

Theorem 9. *The map*:

$$V_0 \to \begin{pmatrix} -1 & 0 \\ 1 + i\, e^{S_3} & 0 \end{pmatrix} \tag{88}$$

$$V_1 \to \begin{pmatrix} -i\, e^{S_2} & -i\, e^{S_2} \\ i\, e^{S_2} & i\, e^{S_2} \end{pmatrix} \tag{89}$$

$$\check{V}_1 \to \begin{pmatrix} 0 & -\, ie^{S_1} \\ 0 & -1 \end{pmatrix} \tag{90}$$

$$\check{V}_0 \to \begin{pmatrix} 0 & 0 \\ q^{\frac{1}{2}} e^{S_2} e^{S_3} & 1 \end{pmatrix}, \tag{91}$$

gives and embedding of $\mathcal{H}_I$ *into* $Mat(2,\mathbb{T}_q)$*. The images of* $V_0, \check{V}_0, V_1, \check{V}_1$ *in* $Mat(2,\mathbb{T}_q)$ *satisfy the relations* (65), (66), (67), (68), (69), (71), *in which the quantum ordering is dictated by the matrix product ordering.*

Proof. The proof of this Theorem is very similar to the proof if Theorem 8, and is therefore omitted. Q.E.D.

Acknowledgements. The author is specially grateful to P. Etingof, T. Koornwinder, M. Noumi and J. Stokman for interesting discussions on this subject.

References

[1] Askey R., Wilson J. A., Some basic hypergeometric orthogonal polynomials that generalise Jacobi polynomials, *Memoirs of the AMS*, **319** (1985).

[2] Bolibruch A.A., The 21-st Hilbert problem for linear Fuchsian systems, *Developments in mathematics*: *the Moscow school* Chapman and Hall, London, (1993).

[3] Chekhov L., Fock V., *A quantum Techmüller space*, Theor. and Math. Phys. **120** (1999), 1245–1259, http://arxiv.org/abs/math.QA/9908165math.QA/9908165.

[4] Chekhov L., Mazzocco M., Shear coordinate description of the quantised versal unfolding of a D_4 singularity, J. Phys. A: Math. Theor. **43**, (2010) 442002, 13 pages.

[5] Chekhov L., Mazzocco M., Quantum ordering for quantum geodesic functions of orbifold Riemann surfaces, in *Topology, Geometry, Integrable Systems, and Mathematical Physics: Novikov's Seminar 2012-2014*, American Mathematical Society Translations–Series 2, **234** (2014).

[6] Chekhov L., Mazzocco M., Rubtsov V., Painlevé monodromy manifolds, decorated character varieties and cluster algebras, *arXiv:1511.03851*, (2015).

[7] Cherednik I., Double affine Hecke algebras, Knizhnik-Zamolodchikov equations and Macdonald's operators, *Int. Math. Res. Not.* (1992), no.9:171–180.

[8] Cherednik I., Whittaker limits of difference spherical functions, *Int. Math. Res. Not.* (2009), no.20:3793–3842.

[9] Dubrovin B.A., Mazzocco M., Monodromy of certain Painlevé-VI transcendents and reflection group, *Invent. Math.* **141** (2000), 55–147.

[10] Eshmatov A. and Eshmatov F., Notes on isomorphism between skein algebra and DAHA, *private communication*, (2012).

[11] Etingof P., Oblomkov A. and Rains E., Generalised double affine Hecke algebras of rank 1 and quantised del Pezzo surfaces, *Adv. Math.*, **212** (2007), 749–796.

[12] Filipuk G., On the middle convolution and birational symmetries of the sixth Painlev? equation, *Kumamoto J. Math.* **19** (2006), 15–23.

[13] Fock V.V., Combinatorial description of the moduli space of projective structures, http://arxiv.org/abs/hep-th/9312193hep-th/9312193.

[14] Fock V.V., Dual Teichmüller spaces, http://arxiv.org/abs/dg-ga/9702018dg-ga/9702018.

[15] V. V. Fock and A. A. Rosly, *Moduli space of flat connections as a Poisson manifold*, *Internat. J. Modern Phys. B* **11** (1997), no. 26-27, 3195–3206.

[16] Fuchs R., Ueber lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären, *Stellen. Math. Ann.*, **63** (1907) 301–321.

[17] Garnier R., Solution du probleme de Riemann pour les systemes différentielles linéaires du second ordre, *Ann. Sci. Ecole Norm.. Sup.*, **43** (1926) 239–352.

[18] Inaba M., Iwasaki K., Saito M., Dynamics of the sixth Painlevé equation, in *Théories asymptotiques et équations de Painlevé*, *Sémin. Congr.*, **14** (2006) 103–167.

[19] Ito T. and Terwillinger P., Double affine Hecke algebras of rank 1 and the $\mathbb{Z}_3$-symmetric Askey–Wilson relations, *SIGMA* **6** (2010) 065, 9 pages.

[20] Iwasaki K., An Area-Preserving Action of the Modular Group on Cubic Surfaces and the Painlevé VI Equation, *Comm. Math. Phys.* **242** (2003) 185–219.

[21] Jimbo M., Monodromy Problem and the Boundary Condition for Some Painlevé Equations, *Publ. RIMS, Kyoto Univ.*, **18** (1982) 1137–1161.

[22] Jimbo M., Miwa T. and Ueno K., Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I, *Physica 2D*, **2**, (1981), no. 2, 306–352
[23] Koekoek R., Lesky P., Swarttouw R. F., Hypergeometric Orthogonal Polynomials and Their q-Analogues, *Spinger Monographs in Mathematics*, Springer-Verlag, Berlin, (2010).
[24] Koornwinder T. H., The relationship between Zhedanov's algebra $AW(3)$ and the double affine Hecke algebra in the rank one case, *SIGMA* **3** (2007), 063, 15 pp.
[25] Korotkin D. and Samtleben H., Quantization of coset space σ-models coupled to two-dimensional gravity, *Comm. Math. Phys.* **190** (1997), no. 2, 411–457.
[26] Lisovyy O. Tykhyy Yu., Algebraic solutions of the sixth Painlevé equation, *J. Geom. Phys.* **85** (2014), 124–163.
[27] Mazzocco M., Confluences of the Painlevé equations, Cherednik algebras and q-Askey scheme, *Nonlinearity* **29** (2016), 2565–2608.
[28] Mazzocco M., Irregular isomonodromic deformations for Garnier systems and Okamoto's canonical transformations, *J. London Math. Soc.* (2), **70** (2004), no. 2, 405–419.
[29] Jimbo M. and Miwa T., Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II, *Physica 2D*, **2** (1981), no. 3, 407–448.
[30] Noumi M., and Stokman J. V., Askey–Wilson polynomials: an affine Hecke algebraic approach, *Laredo Lectures on Orthogonal Polynomials and Special Functions*, *Adv. Theory Spec. Funct. Orthogonal Polynomials, Nova Sci. Publ., Hauppauge, NY* (2004) 111–144.
[31] Oblomkov A., Double affine Hecke algebras of rank 1 and affine cubic surfaces, *IMRN* **2004**, no.18:877–912.
[32] Sahi S. Nonsymmetric Koornwinder polynomials and duality, *Ann. of Math.* (2) **150**, (1999) no1:267–282.
[33] Schlesinger L., Ueber eine Klasse von Differentsial System Beliebliger Ordnung mit Festen Kritischer Punkten, *J. fur Math.*, **141**, (1912), 96–145.
[34] Stokman J. V., Difference Fourier transforms for nonreduced root systems, *Selecta Math.* (*N.S.*) **9** (2003) no3:409–494.
[35] Terwilliger P., The universal Askey-Wilson algebra *SIGMA* **7** (2011) 069, 24 pages, arXiv:1104.2813.

Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, United Kingdom.
E-mail address: m.mazzocco@lboro.ac.uk.