Explicit biregular/birational geometry of affine threefolds: completions of $\mathbb{A}^3$ into del Pezzo fibrations and Mori conic bundles

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Abstract.

We study certain pencils $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ of del Pezzo surfaces generated by a smooth del Pezzo surface S of degree less than or equal to 3 anti-canonically embedded into a weighted projective space $\mathbb{P}$ and an appropriate multiple of a hyperplane H. Our main observation is that every minimal model program relative to the morphism $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ lifting $\overline{f}$ on a suitable resolution $\sigma : \tilde{\mathbb{P}} \to \mathbb{P}$ of its indeterminacies preserves the open subset $\sigma^{-1}(\mathbb{P} \setminus H) \simeq \mathbb{A}^3$. As an application, we obtain projective completions of $\mathbb{A}^3$ into del Pezzo fibrations over $\mathbb{P}^1$ of every degree less than or equal to 4. We also obtain completions of $\mathbb{A}^3$ into Mori conic bundles, whose restrictions to $\mathbb{A}^3$ are twisted $\mathbb{A}^1_*$-fibrations over $\mathbb{A}^2$.

§ Introduction

A threefold Mori fiber space is a mildly singular projective threefold X equipped with an extremal contraction $\tau : X \to B$ over a lower dimensional normal projective variety B. More precisely, X has $\mathbb{Q}$-factorial terminal singularities, τ has connected fibers, the anti-canonical divisor $-K_X$ of X is ample on the fibers and the relative Picard number $\rho(X/B) = \mathrm{rk}(N_1(X)) - \mathrm{rk}(N_1(B))$ is equal to 1. These fiber spaces

are the possible outputs of Minimal Model Programs (MMP) ran from rational, or more generally uniruled, smooth projective threefolds and provide the natural higher dimensional analogues in this framework of the projective plane and the minimally ruled surfaces. Noting that rational minimally ruled surfaces $\mathbb{F}_n$, $n \geq 2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are smooth projective completions of the affine plane $\mathbb{A}^2$, it is natural to ask which total spaces of threefold Mori fiber spaces $\tau : X \to B$ are projective completions of $\mathbb{A}^3$. As the first step towards a potential geometric description of the structure of the automorphism group $\mathrm{Aut}(\mathbb{A}^3)$ of $\mathbb{A}^3$ from the point of view of the Sarkisov Program [3], it is also natural to try to classify these completions up to birational isomorphisms preserving the inner open subset $\mathbb{A}^3$.

In the case $\dim B = 0$, Fano threefolds of Picard number 1 containing $\mathbb{A}^3$ have received a lot of attention during the past decades: a complete classification is known in the smooth case (see e.g. [5] and the references therein) but the general picture in the singular case remains elusive. Much less seems to be known about completions of $\mathbb{A}^3$ into "strict" Mori fiber spaces, that is Mori fiber spaces $\tau : X \to B$, where $\dim B = 1, 2$. There are two cases: del Pezzo fibrations when $\dim B = 1$ and Mori conic bundles when $\dim B = 2$. Elementary examples of such completions are locally trivial projective bundles $\tau : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$ and $\tau : \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(m)) \to \mathbb{P}^2$ over $\mathbb{P}^1$ and $\mathbb{P}^2$, which come respectively as projective models of linear projections from $\mathbb{A}^3$ to $\mathbb{A}^1$ and $\mathbb{A}^2$. But in general, there is no reason that the restriction to $\mathbb{A}^3$ of the structure morphism $\tau : X \to B$ of a completion into a strict Mori fiber space has general fibers isomorphic to affine spaces. For instance, since by a result of Manin [7] a smooth del Pezzo surface of degree $d \leq 3$ over a non-closed fied with Picard number 1 is not rational, there cannot exist any completion of $\mathbb{A}^3$ into the total space of a del Pezzo fibration $\tau : X \to B = \mathbb{P}^1$ of degree $d \leq 3$ whose restriction to $\mathbb{A}^3$ is a fibration with generic fiber isomorphic to the affine plane $\mathbb{A}^2$ over the function field of B.

The main purpose of this article is to give examples of "twisted" completions of $\mathbb{A}^3$ into total spaces of strict Mori fiber spaces, that is completions $\tau : X \to B$ for which the general fibers of the restriction of τ to $\mathbb{A}^3$ are not isomorphic to affine spaces. One strategy to construct such examples is to start from a regular function $f : \mathbb{A}^3 \to \mathbb{A}^1$ with smooth rational general fibers which extends to a morphism $\tilde{f}' : X' \to \mathbb{P}^1$ with smooth general fibers on a smooth projective threefold X' and to run a relative MMP $\varphi : X' \dashrightarrow X$ over $\mathbb{P}^1$. The rationality of the fibers guarantees that the output $\tilde{f} : X \to \mathbb{P}^1$ is either a del Pezzo fibration or factors through a Mori conic bundle $\xi : X \to W$ over a normal projective

surface W. The main obstacle is that there is no reason in general that a relative MMP $\varphi : X' \dashrightarrow X$ preserves the open subset $\mathbb{A}^3 \subset X'$: such a process φ might contract divisors which are not supported on the boundary $X' \setminus \mathbb{A}^3$, inducing a nontrivial birational morphism between $\mathbb{A}^3$ and its image by φ which, in this case is in general again affine, and even worse, small contractions might occur outside the boundary with the effect that the image of $\mathbb{A}^3$ by φ is no longer affine. As a general fact, understanding the biregular geometry of an affine threefold via the birational geometry of its projective models requires to get some effective control on the birational maps appearing in MMP processes between these models. One solution in our situation is to consider functions $f : \mathbb{A}^3 \to \mathbb{A}^1$ extending to fibrations $\tilde{f}' : X' \to \mathbb{P}^1$ whose general fibers are already smooth del Pezzo surfaces. More precisely, the generic fiber of $\tilde{f}'$ is a smooth del-Pezzo surface S_η defined over the function field over $\mathbb{C}(\lambda)$ of $\mathbb{P}^1$. Since a relative MMP $\varphi : X' \dashrightarrow X$ restricts on S_η to a finite sequence of contractions of successive (-1)-curves defined over $\mathbb{C}(\lambda)$, we can expect to gain more control on the possible horizontal divisors contracted by φ, as well as on its flipping and flipped curves.

The functions we consider in this article are obtained as restrictions of pencils $\mathcal{L}$ generated by a smooth del Pezzo surface S of degree 1, 2 or 3 anti-canonically embedded into a weighted projective 3-space $\mathbb{P}$ and by an appropriate multiple eH of a hyperplane $H \in |\mathcal{O}_{\mathbb{P}}(1)|$. Namely, $\mathbb{P} \setminus H$ is isomorphic to $\mathbb{A}^3$ and $f : \mathbb{A}^3 \to \mathbb{A}^1$ is the restriction of the rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ defined by $\mathcal{L}$. For an appropriate class of resolutions $\sigma : \tilde{\mathbb{P}} \to \mathbb{P}$ of $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ restricting to an isomorphism over $\mathbb{P} \setminus H$ and for which $\sigma^{-1}(H)$ induces an anti-canonical divisor on the generic fiber of the induced morphism $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$, which we call *good resolutions*, we establish that every MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ relative to $\tilde{f}$ restricts to an isomorphism between $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \simeq \mathbb{A}^3$ and its image. The output $\tilde{\mathbb{P}}'$ is then a compactification of $\mathbb{A}^3$ either into a del Pezzo fibration $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ or into a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$ over a certain normal projective surface $q : W \to \mathbb{P}^1$, and we characterize each possible type of output in terms of the structure of the base locus of $\mathcal{L}$. Our main result can be summarized as follows:

Theorem. *Let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$ be the pencil generated by an anti-canonically embedded smooth del Pezzo surface $S \subset \mathbb{P}$ of degree $d \in \{1,2,3\}$ and a multiple of hyperplane $H \in |\mathcal{O}_{\mathbb{P}}(1)|$, let $\sigma : \tilde{\mathbb{P}} \to \mathbb{P}$ be a good resolution of the corresponding rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$, and let $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ be a MMP relative to the induced morphism $\tilde{f} = \overline{f} \circ \sigma : \tilde{\mathbb{P}} \to \mathbb{P}^1$, with output $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$. Then $\tilde{\mathbb{P}}'$ is a projective completion of*

$\mathbb{A}^3 = \mathbb{P} \setminus H$ *with $\mathbb{Q}$-factorial terminal singularities such that one of the following holds:*

a) If $H \cap S$ is irreducible, then the restriction of φ to the generic fiber S_η of $\tilde{f}$ is an isomorphism onto the generic fiber of $\tilde{f}'$, and $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is a del Pezzo fibration of degree d.

b) If $d = 2$ and $H \cap S$ is reducible, then the restriction of φ to the generic fiber S_η of $\tilde{f}$ consists of the contraction of the unique (-1)-curve on S_η supported on $\sigma^{-1}(H) \cap S_\eta$, and $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is del Pezzo fibration of degree $d + 1 = 3$.

c) If $H \cap S$ has three irreducible components, then the restriction of φ to the generic fiber S_η of $\tilde{f}$ consists of the contraction of precisely one of the (-1)-curves on S_η supported on $\sigma^{-1}(H) \cap S_\eta$, and $\tilde{\mathbb{P}}'$ has the structure of a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$ over a normal projective surface.

By the work of Pukhlikov [10], "most" three-dimensional fibrations in del Pezzo surfaces of degree $d \leq 3$ have non rational total spaces. In contrast, the del Pezzo fibrations $\tilde{\mathbb{P}}' \to \mathbb{P}^1$ we obtain all have rational total spaces since they contain $\mathbb{A}^3$ as an open subset by construction.

In the case where the output $\tilde{\mathbb{P}}'$ is a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$, we establish further that the restriction of ξ to the inner $\mathbb{A}^3$ is a *twisted $\mathbb{A}^1_*$-fibration* $\xi_0 : \mathbb{A}^3 \to \mathbb{A}^2$, that is, a flat fibration whose generic fiber is a nontrivial form of the punctured affine line $\mathbb{A}^1_*$ over the function field of $\mathbb{A}^2$. This contrasts with the situation for $\mathbb{A}^2$ for which no such type of $\mathbb{A}^1_*$-fibration can exist, essentially as a consequence of Tsen's theorem and the factoriality of $\mathbb{A}^2$ (see [8, Lemma 1.7.2]). We also provide a geometric interpretation of these fibrations in terms of the pair (S, H) initially chosen for the construction.

Acknowledgements

§1. Pencils of del Pezzo surfaces in weighted projective spaces

1.1. Basic facts on del Pezzo surfaces of degree ≤ 3

Recall that a smooth del Pezzo surface is a smooth projective surface S whose anti-canonical divisor $-K_S$ is ample. The integer $d = (-K_S^2) \in \{1, \ldots, 9\}$ is called the degree of S. Every such surface is either isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up of the projective plane $\mathbb{P}^2$ in $9 - d$ points in general position [7]. It is known from the structure of their anticanonical rings $\bigoplus_{m \geq 0} H^0(S, -mK_S)$ that smooth del Pezzo

surfaces of degree ≤ 3 are naturally embedded as hypersurfaces in certain weighted projective spaces. Their properties are summarized by the following proposition (see e.g. [7]):

Proposition 1.

1) Every smooth del Pezzo surface of degree 3 is isomorphic to a smooth cubic surface in $\mathbb{P}^3$, and conversely every smooth cubic surface S in $\mathbb{P}^3$ is a del Pezzo surface of degree 3. For every hyperplane $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$, $H|_S$ is a reduced anti-canonical divisor on S whose support is one of the following:

(i)   *An irreducible plane cubic,*

(ii)   *The union of a smooth conic C and a line ℓ intersecting each other twice, either transversally in two distinct points or tangentially in a unique point,*

(iii)   *A union of three lines, either in general position or intersecting each other in a unique point, which is then an Eckardt point of S.*

2) Every smooth del Pezzo surface of degree 2 is isomorphic to a smooth quartic hypersurface of the weighted projective space $\mathbb{P}(1,1,1,2)$. Conversely, every smooth quartic S in $\mathbb{P}(1,1,1,2)$ is a del Pezzo surface of degree 2. For every $H \in |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(1)|$, $H|_S$ is a reduced anti-canonical divisor on S whose support is one of the following:

(i)   *An irreducible plane cubic curve,*

(ii)   *A union of two (-1)-curves on S intersecting each other twice, either transversally in two distinct points or tangentially in a unique point.*

3) Every smooth del Pezzo surface of degree 1 is isomorphic to a smooth sextic hypersurface of the weighed projective space $\mathbb{P}(1,1,2,3)$, and conversely every smooth sextic in $\mathbb{P}(1,1,2,3)$ is a del Pezzo surface of degree 1. For every $H \in |\mathcal{O}_{\mathbb{P}(1,1,2,3)}(1)|$, $H|_S$ is an irreducible and reduced anti-canonical divisor on S whose support is isomorphic to a plane cubic curve.

In what follows, given an anti-canonically embedded smooth del Pezzo surface S of degree $d \leq 3$ as in Proposition 1 above, we use the same notation $\mathbb{P} = \mathrm{Proj}(\mathbb{C}[x,y,z,w])$ to denote the ambient spaces $\mathbb{P}^3$, $\mathbb{P}(1,1,1,2)$ and $\mathbb{P}(1,1,2,3)$ according to $d = 3$, 2 and 1, the variables x, y, z and w having degrees $(1,1,1,1)$, $(1,1,1,2)$ and $(1,1,2,3)$ respectively. The degree of S as a hypersurface of $\mathbb{P}$ is denoted by e. It is equal to 3, 4 and 6 according as $d = 3$, 2 and 1.

Lemma 2. *Let $S \subset \mathbb{P}$ be a smooth del Pezzo surface of degree $d \in \{1,2,3\}$, let $H \in |\mathcal{O}_{\mathbb{P}}(1)|$ be a hyperplane and let $D \subset \mathbb{P}$ be an*

irreducible and reduced hypersurface. Write $D|_S = D_0 + R$ where D_0 and R are effective Weil divisors such that $\mathrm{Supp}(D_0) \subset \mathrm{Supp}(H|_S)$ and such that no irreducible component of R is contained in $\mathrm{Supp}(H|_S)$. Then $\mathrm{Supp}(R)$ does not consist of a disjoint union of (-1)-curves.

Proof. We let $k \geq 1$ be the degree of D. In the case where $H \cap S$ is an irreducible curve C, we have $D_0 = aC$ for some $a \geq 0$, and then $R \sim (D - aH)|_S$ is either ample or trivial, hence in particular cannot be supported by a disjoint union of (-1)-curves.

In the case where $d = 2$ and $H \cap S$ consists of the union of two (-1)-curves ℓ_1 and ℓ_2, we have $D_0 = a_1\ell_1 + a_2\ell_2$ where, up to a permutation, $0 \leq a_1 \leq a_2$. If $a_1 \geq k$ then $R + (a_1 - k)\ell_1 + (a_2 - k)\ell_2 = (D - kH)|_S \sim 0$. Since $\mathrm{Pic}^0(S)$ is trivial, it follows that $a_1 = a_2 = k$ and that $R = 0$, and we are done. If $a_1 < k \leq a_2$ then using the identity

$$R + (a_1 - (k-1))\ell_1 + (a_2 - (k-1))\ell_2 = (D - (k-1)H)|_S \sim H|_S$$

and the fact that $H|_S$ is an anti-canonical divisor on S, we obtain

$$\begin{aligned} 1 = H|_S \cdot \ell_1 &= (a_1 - (k-1))\ell_1^2 + (a_2 - (d-1))\ell_2 \cdot \ell_1 + R \cdot \ell_1 \\ &= (k - 1 - a_1) + 2(a_2 - (k-1)) + R \cdot \ell_1. \end{aligned}$$

as $\ell_1 \cdot \ell_2 = 2$. This is absurd since $a_2 - (k-1) \geq 1$ and the other two terms are non-negative. Thus $a_1 \leq a_2 \leq k - 1$ and then

$$R \sim (k - a_1)\ell_1 + (k - a_2)\ell_2 \sim (k - a_2)H|_S + (a_2 - a_1)\ell_1$$

is big, hence cannot be supported by a disjoint union of (-1)-curves.

In the case where $d = 3$ and $H \cap S$ consists of two irreducible components C and ℓ with respective self-intersections 0 and -1, we have $D_0 = aC + b\ell$ and the following possibilities: $k \leq \min(a, b)$, $\max(a, b) < k$, $b < k \leq a$ or $a < k \leq b$. The first three cases follow from similar arguments as above. If $a < k \leq b$ then since $C^2 = 0$ and $\ell \cdot C = 2$, we have

$$\begin{aligned} 2 &= H|_S \cdot C = (D - (k-1)H)|_S \cdot C \\ &= (a - (k-1))C^2 + (b - (k-1))\ell \cdot C + R \cdot C \\ &= 2(b - k + 1) + R \cdot C. \end{aligned}$$

So $b = k$, $R \cdot C = 0$ and hence $R \sim (k - a)C$. It follows that $R^2 = 0$ and so, the support of R cannot consist of a disjoint union of (-1)-curves.

The case where $d = 3$ and $H \cap S$ consists of the union of three (-1)-curves ℓ_1, ℓ_2 and ℓ_3 follows in a similar way. Namely $D_0 = a_1\ell_1 + a_2\ell_2 + a_3\ell_3$ where, up to a permutation, $0 \leq a_1 \leq a_2 \leq a_3$. The sub-case

where $a_1 \geq k$ leads similarly as above to the conclusion that $R = 0$. If $a_1 < k \leq a_2$, then we reach the conclusion by considering the intersection product of ℓ_1 with $(D - (k-1)H)|_S \sim H|_S$. Finally if $a_2 < k \leq a_3$, then we have

$$2 = H|_S \cdot (\ell_1 + \ell_2) = 2(a_3 - (k-1)) + R \cdot (\ell_1 + \ell_2)$$

and so, $a_3 = k$ and $R \cdot (\ell_1 + \ell_2) = 0$. It follows that $R \sim (k - a_1)\ell_1 + (k - a_2)\ell_2$, and since $R \cdot \ell_1 = R \cdot \ell_2 = 0$, we have $a_1 = a_2$. Thus $R \sim (k - a_1)(\ell_1 + \ell_2)$, $R^2 = 0$ and so, $\mathrm{Supp}(R)$ does not consist of a disjoint union of (-1)-curves.   Q.E.D.

1.2. Pencils of del Pezzo surfaces of degree ≤ 3

Definition 3. Let $S \subset \mathbb{P}$ be a smooth del Pezzo surface of degree $d \in \{1, 2, 3\}$ and let $H \in |\mathcal{O}_{\mathbb{P}}(1)|$ be a hyperplane. We denote by $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$ the pencil generated by S and eH. We let $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ be the corresponding rational map.

1.2.1. A member $S_{[\alpha:\beta]}$, $[\alpha : \beta] \in \mathbb{P}^1$, of $\mathcal{L}$ is defined up to a linear transformation of $\mathbb{P}$ by the vanishing of a weighted-homogeneous polynomial $F \in \mathbb{C}[x, y, z, w]$ of degree e of the form

$$F = \beta s(x, y, z, w) - \alpha x^e,$$

where S and H are defined respectively by the vanishing of $s(x, y, z, w)$ and x. The scheme-theoretic base locus of $\mathcal{L}$ is equal to the closed sub-scheme of $\mathbb{P}$ defined by the weighted-homogeneous ideal $(s(x, y, z, w), x^e)$ of $\mathbb{C}[x, y, z, w]$. Its support is equal to $H \cap S$. With this description, the rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ coincides with that defined by $[x : y : z : w] \mapsto [s(x, y, z, w) : x^e]$. The complement of H is isomorphic to $\mathbb{A}^3$ with inhomogeneous coordinates $Y = x^{-1}y$, $Z = x^{-a}z$ and $W = x^{-b}w$, where $(a, b) = (1, 1)$, $(1, 2)$ and $(2, 3)$ according to $d = 3, 2$ and 1 respectively, and letting $\infty = [1 : 0] = \overline{f}_*(H) \in \mathbb{P}^1$, the restriction of $\overline{f}$ to $\mathbb{P} \setminus H$ coincides with the regular function

$$f : \mathbb{A}^3 = \mathbb{P} \setminus H \quad \rightarrow \quad \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\} \simeq \mathrm{Spec}(\mathbb{C}[\lambda]),$$
$$(X, Y, Z) \quad \mapsto \quad s(1, Y, Z, W).$$

The *generic member* S_η of $\mathcal{L}$, that is, the closure in $\mathbb{P}_{\mathbb{C}(\lambda)} = \mathrm{Proj}(\mathbb{C}(\lambda)[x, y, z, w])$ of the fiber of f over the generic point η of $\mathbb{P}^1$, is isomorphic to the projective surface over $\mathbb{C}(\lambda)$ defined by the vanishing of the weighted-homogeneous polynomial $s(x, y, z, w) + \lambda x^e \in \mathbb{C}(\lambda)[x, y, z, w]$. Since S is smooth, it follows from the Jacobian criterion that S_η is smooth, hence is a smooth del Pezzo surface of degree

d defined over the function field $\mathbb{C}(\lambda)$ of $\mathbb{P}^1$. This implies in particular that the general member of $\mathcal{L}$ is a smooth del Pezzo surface of degree d. Some members of $\mathcal{L}$ can be singular (see Example 5 below) but all members of $\mathcal{L}$ except eH are integral schemes:

Lemma 4. *All members of $\mathcal{L}$ except eH are irreducible and reduced.*

Proof. We consider each degree $d = 3, 2, 1$ separately. If $d = 3$ and $S' \in \mathcal{L} \setminus \{S, 3H\}$ is either reducible or non reduced, then one of its irreducible components is necessarily a hyperplane, say H', which is different from H as $\mathcal{L}$ does not have any fixed component. Since the restriction map

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(S, \mathcal{O}_{\mathbb{P}^3}(1)|_S) \simeq H^0(S, \mathcal{O}_S(-K_S))$$

is an isomorphism, $H' \cap S$ is distinct from $H \cap S$, hence is strictly contained in it as $H \cap S$ coincides with the support of the base locus of $\mathcal{L}$. This is absurd in view of 1) in Proposition 1.

In the case $d = 2$, a member $S' \in \mathcal{L} \setminus \{S, 4H\}$ which is either reducible or non reduced contains an irreducible component of degree one or two. Note that the restriction maps

$$\begin{aligned} H^0(\mathbb{P}(1,1,1,2), \mathcal{O}_{\mathbb{P}(1,1,1,2)}(j)) \\ \to H^0(S, \mathcal{O}_{\mathbb{P}(1,1,1,2)}(j)|_S) \simeq H^0(S, \mathcal{O}_S(-jK_S)), \quad j = 1, 2 \end{aligned}$$

are both isomorphisms. So in the first case, we would have again a hyperplane $H' \in |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(1)|$ distinct from H for which $H' \cap S$ is contained in $H \cap S$, which is absurd by virtue of 2) in Proposition 1. In the second case, S' would be the union of two irreducible quadric hypersurfaces Q_1 and Q_2 of $\mathbb{P}(1,1,1,2)$, necessarily distinct from each other since otherwise every member of $\mathcal{L}$ would be reducible. Since the restriction map for $j = 2$ is an isomorphism, both intersections $Q_i \cap H$, $i = 1, 2$ are then strictly contained in $H \cap S$. Indeed, if $Q_i \cap H = H \cap S$ then $Q_i|_S = 2H|_S$ and then $Q_i = 2H$ contradicting the irreducibility of Q_i. This implies in turn by virtue of 2) in Proposition 1 that $Q_i|_S$ is supported on a (-1)-curve, which is absurd as $Q_i|_S$ has non negative self-intersection.

Finally, if $d = 1$ and $S' \in \mathcal{L} \setminus \{S, 6H\}$ is not integral, then it contains an irreducible component P of degree 1, 2 or 3. Because of the isomorphisms

$$\begin{aligned} H^0(\mathbb{P}(1,1,2,3), \mathcal{O}_{\mathbb{P}(1,1,2,3)}(j)) \\ \xrightarrow{\sim} H^0(S, \mathcal{O}_{\mathbb{P}(1,1,2,3)}(j)|_S) \simeq H^0(S, \mathcal{O}_S(-jK_S)), \quad j = 1, 2, 3, \end{aligned}$$

the same argument as in the previous case implies that $P \cap S$ is strictly contained in $H \cap S$, which is absurd since the latter is irreducible by virtue of 3) in Proposition 1. Q.E.D.

Example 5. (See also [1])

a) The sextics S_1 and S_2 in $\mathbb{P}(1, 1, 2, 3) = \text{Proj}_{\mathbb{C}}(\mathbb{C}[x, y, z, w])$ defined respectively by the equations $z^3 + w^2 + xy^5 = 0$ and $z^3 + w^2 + x^2(x^3y + z^2) = 0$ are normal del Pezzo surfaces with a unique singular point of type E_8 at $p_1 = [1 : 0 : 0 : 0]$ and $p_2 = [0 : 1 : 0 : 0]$ respectively. The general members of the pencil $\overline{f}_1 : \mathbb{P}(1, 1, 2, 3) \dashrightarrow \mathbb{P}^1$, generated by S_1 and $6H_1$ where $H_1 = \{x + by = 0\} \in |\mathcal{O}_{\mathbb{P}(1,1,2,3)}(1)|$, $b \in \mathbb{C}$, are smooth del Pezzo surfaces of degree 1. The intersection $S_1 \cap H_1$ is either a rational cuspidal cubic if $b = 0$ or a smooth elliptic curve otherwise. The general members of the pencil $\overline{f}_2 : \mathbb{P}(1, 1, 2, 3) \dashrightarrow \mathbb{P}^1$, generated by S_2 and $6H_2$ where $H_2 = \{ax + y = 0\} \in |\mathcal{O}_{\mathbb{P}(1,1,2,3)}(1)|$, $a \in \mathbb{C}$, are also smooth del Pezzo surfaces of degree 1. The intersection $S_2 \cap H_2$ is either a nodal cubic if $a = 0$ or a smooth elliptic curve otherwise.

b) The quartic surface $S = \{w^2 + yz^3 + xy^3 = 0\}$ in $\mathbb{P}(1, 1, 1, 2) = \text{Proj}_{\mathbb{C}}(\mathbb{C}[x, y, z, w])$ is a normal del Pezzo surface with a unique singular point of type E_7 at $[1 : 0 : 0 : 0]$. The general members of the pencil $\overline{f} : \mathbb{P}(1, 1, 1, 2) \dashrightarrow \mathbb{P}^1$ generated by S and $4H$ where $H = \{x + ay + bz = 0\} \in |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(1)|$, $a, b \in \mathbb{C}$ are smooth del Pezzo surfaces of degree 2. The intersection of H with S is either a cuspidal cubic if $b = 0$ or a smooth elliptic curve otherwise.

c) The cubic surfaces $S_1(\lambda) = \{x^3 + w(\lambda x^2 + y^2 + wz) = 0\}$, $\lambda \in \mathbb{C}$, and $S_2 = \{xyz + y^3 + w^2z = 0\}$ in $\mathbb{P}^3$ are normal del Pezzo surfaces respectively with a unique singularity of type E_6 at $[0 : 0 : 1 : 0]$ and a pair of singular points $[1 : 0 : 0 : 0]$ and $[0 : 0 : 1 : 0]$ of types A_1 and A_5. The general members of the pencils $\overline{f}_i : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$, $i = 1, 2$, generated respectively by $S_1(\lambda)$ and $3H_1$, where $H_1 = \{z = 0\}$, and by S_2 and $3H_2$, where $H_2 = \{x + z = 0\}$, are smooth cubic surfaces.

§2. Good resolutions and relative MMPs

In this section, we introduce particular resolutions $\sigma : \tilde{\mathbb{P}} \to \mathbb{P}$ of the indeterminacy of the rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ associated to a pencil as in Definition 3 above. These have the property to restrict to isomorphisms over the open subset $\mathbb{A}^3 = \mathbb{P} \setminus H$, and we show that every MMP process $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ relative to the induced morphism $\tilde{f} = \overline{f} \circ \sigma : \tilde{\mathbb{P}} \to \mathbb{P}^1$ again preserves $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \simeq \mathbb{P} \setminus H$, inducing an isomorphism between $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \simeq \mathbb{P} \setminus H$ and $\tilde{\mathbb{P}}' \setminus \varphi_*(\sigma^{-1}(H))$.

2.1. Good resolutions of del Pezzo pencils

Let $S \subset \mathbb{P}$ be a smooth del Pezzo surface of degree $d \leq 3$, let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$ be the pencil generated by S and eH for some $H \in |\mathcal{O}_{\mathbb{P}}(1)|$ and let $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ be the corresponding rational map as in Definition 3. Similarly as in § 1.2.1, we let $\infty = \overline{f}_*(H) \in \mathbb{P}^1$.

Definition 6. A *good resolution* of $\overline{f}$ is a triple $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ consisting of a normal projective threefold $\tilde{\mathbb{P}}$, a birational morphism $\sigma : \tilde{\mathbb{P}} \to \mathbb{P}$ and a morphism $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ satisfying the following properties:

(a)   The diagram

$$\begin{array}{ccc} \tilde{\mathbb{P}} & \xrightarrow{\;\sigma\;} & \mathbb{P} \\ {\scriptstyle \tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f}} \\ \mathbb{P}^1 & = \!\!=\!\!= & \mathbb{P}^1 \end{array}$$

commutes.

(b) $\tilde{\mathbb{P}}$ has at most $\mathbb{Q}$-factorial terminal singularities and is smooth outside $\tilde{f}^{-1}(\infty)$.

(c) $\sigma : \tilde{\mathbb{P}} \to \mathbb{P}$ is a sequence of blow-ups whose successive centers lie above the base locus of $\mathcal{L}$, inducing an isomorphism $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \xrightarrow{\sim} \mathbb{P} \setminus H$, and whose restriction to every closed fiber of $\tilde{f}$ except $\tilde{f}^{-1}(\infty)$ is an isomorphism onto its image.

It follows from the definition that all irreducible divisors in the exceptional locus $\text{Exc}(\sigma)$ of a good resolution σ that are vertical for $\tilde{f}$ are contained in $\tilde{f}^{-1}(\infty)$. Furthermore, since the restriction of σ to the generic fiber of $\tilde{f}$ is an isomorphism onto the generic member of $\mathcal{L}$, $\text{Exc}(\sigma)$ contains exactly as many irreducible horizontal divisors as there are irreducible components in $H \cap S$. Indeed, there is a one to one correspondence between irreducible horizontal divisors in $\text{Exc}(\sigma)$ and irreducible components of the intersection of $\sigma^{-1}(H)$ with the generic fiber S_η of $\tilde{f}$. By assumption, S_η is isomorphic to the smooth del Pezzo surface of degree d in $\mathbb{P}_{\mathbb{C}(\lambda)}$ with equation $s(x, y, z, w) - \lambda x^e = 0$ (see § 1.2.1), and the definition of $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ implies that it intersects $\sigma^{-1}(H)$ along the curve $D_\eta \simeq (H \cap S) \times_{\text{Spec}(\mathbb{C})} \text{Spec}(\mathbb{C}(\lambda))$ with equation $s(0, y, z, w) = 0$ in $\text{Proj}(\mathbb{C}(\lambda)[y, z, w])$. In particular, D_η is an anti-canonical divisor on S_η with the same number of irreducible components as $H \cap S$, all of them being defined over $\mathbb{C}(\lambda)$.

Since by definition of a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$, σ restricts to an isomorphism between $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H)$ and $\mathbb{P} \setminus H \simeq \mathbb{A}^3$, it follows that the irreducible components of $\sigma^{-1}(H)$ form a basis of the Néron-Severi group of $\tilde{\mathbb{P}}$. The fiber $\tilde{f}^{-1}(\infty)$ of $\tilde{f}$ consists precisely of the union of the proper

transform of H and the vertical exceptional divisors of σ and since the numerical classes of prime exceptional divisors of σ are linearly independent, it follows that these components together with the r horizontal exceptional divisors of σ form a basis of the Néron-Severi group of $\tilde{\mathbb{P}}$. The Picard number $\rho(\tilde{\mathbb{P}})$ of $\tilde{\mathbb{P}}$ is thus equal to $r+e_v+1$, where e_v denotes the number of vertical exceptional divisors of σ.

In view of Proposition 1, the possibilities for reducible D_η are the following:

a) $d = 3$ and D_η consists of:

(i) The union of a (-1)-curve C_1 and of a 0-curve C_2 both defined over $\mathbb{C}(\lambda)$, intersecting each other twice, either with multiplicity 2 at a unique $\mathbb{C}(\lambda)$-rational point, or transversally at a pair of distinct $\mathbb{C}(\lambda)$-rational points, or at unique point whose residue field is a quadratic extension of $\mathbb{C}(\lambda)$.

(ii) A union of three (-1)-curves C_1, C_2 and C_3 defined over $\mathbb{C}(\lambda)$ and intersecting each others transversally at $\mathbb{C}(\lambda)$-rational points.

b) $d = 2$ and D_η consists of the union of two (-1)-curves C_1 and C_2 both defined over $\mathbb{C}(\lambda)$, intersecting each other twice, either with multiplicity 2 at a unique $\mathbb{C}(\lambda)$-rational point, or transversally at a pair of distinct $\mathbb{C}(\lambda)$-rational points, or at unique point whose residue field is a quadratic extension of $\mathbb{C}(\lambda)$.

Note also that the intersection of $\sigma^{-1}(H)$ with a closed fiber $\tilde{f}^{-1}(c)$ distinct from $\tilde{f}^{-1}(\infty)$ is isomorphic to the intersection of H with the corresponding member $\sigma(\tilde{f}^{-1}(c))$ of $\mathcal{L}$.

A good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ of $f : \mathbb{P} \dashrightarrow \mathbb{P}^1$ always exists. For instance, let $\tau : X \to \mathbb{P}$ be the blow-up of scheme-theoretic base locus of $\mathcal{L}$. Then X is isomorphic to the hypersurface in $\mathbb{P} \times \mathrm{Proj}(\mathbb{C}[\alpha, \beta])$ defined by the weighted bi-homogeneous equation $\beta s(x, y, z, w) - \alpha x^e = 0$, and we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \tau\ } & \mathbb{P} \\
& {\scriptstyle \pi = \mathrm{pr}_2|_X} \searrow & \big\downarrow {\scriptstyle \bar{f}} \\
& & \mathbb{P}^1
\end{array}
$$

The morphism τ restricts on each fiber of π to an isomorphism onto the corresponding member of $\mathcal{L}$ and $X \setminus \tau^{-1}(H) \simeq \mathbb{P} \setminus H$. Furthermore, since S is smooth, it follows from the Jacobian criterion that X is smooth outside $\pi^{-1}(\infty)$. Letting $\tau_1 : \tilde{\mathbb{P}} \to X$ be any resolution of the singularities

of X, the triple $(\tilde{\mathbb{P}}, \tau \circ \tau_1, \pi \circ \tau_1)$ is a good resolution of $\overline{f}$ for which $\tilde{\mathbb{P}}$ is even smooth.

2.2. Basic properties of relative MMPs ran from good resolutions

Let $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ be a good resolution of the rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ associated to a pencil $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$ as above. Recall [6, 3.31] that a MMP $\varphi : \tilde{\mathbb{P}}_0 = \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}' = \tilde{\mathbb{P}}_n$ relative to $\tilde{f}_0 = \tilde{f} : \tilde{\mathbb{P}}_0 \to \mathbb{P}^1$ consists of a finite sequence $\varphi = \varphi_n \circ \cdots \circ \varphi_1$ of birational maps

$$
\begin{array}{ccc}
\tilde{\mathbb{P}}_{k-1} & \overset{\varphi_k}{\dashrightarrow} & \tilde{\mathbb{P}}_k \\
\tilde{f}_{k-1} \downarrow & & \downarrow \tilde{f}_k \qquad k = 1, \ldots, n, \\
\mathbb{P}^1 & = & \mathbb{P}^1
\end{array}
$$

where each φ_k is associated to an extremal ray R_{k-1} of the closure $\overline{NE}(\tilde{\mathbb{P}}_{k-1}/\mathbb{P}_1)$ of the relative cone of curves of $\tilde{\mathbb{P}}_{k-1}$ over $\mathbb{P}^1$. Each of these birational maps φ_k is either a divisorial contraction or a flip whose flipping and flipped curves are contained in the fibers of $\tilde{f}_{k-1}$ and $\tilde{f}_k$ respectively. Letting $\Delta_0 = \sigma^{-1}(H)$ and $\Delta_k = (\varphi_k)_*(\Delta_{k-1})$ for every $k = 1, \ldots, n$, the next result asserts in particular that every relative MMP ran from a good resolution of $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ preserves the open subset $\sigma^{-1}(\mathbb{P} \setminus H) \simeq \mathbb{P} \setminus H \simeq \mathbb{A}^3$.

Proposition 7. *Let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$ be as above and let $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ be any good resolution of the corresponding rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$. Then every MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ relative to $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ restricts to an isomorphism $\mathbb{A}^3 \simeq \tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \overset{\sim}{\to} \tilde{\mathbb{P}}' \setminus \varphi_*(\sigma^{-1}(H))$. More precisely, the following hold at each intermediate step:*

a) The threefold $\tilde{\mathbb{P}}_k$ is smooth outside $\tilde{f}_k^{-1}(\infty)$,

b) The birational map $\varphi_k : \tilde{\mathbb{P}}_{k-1} \dashrightarrow \tilde{\mathbb{P}}_k$ restricts to an isomorphism $\tilde{\mathbb{P}}_{k-1} \setminus \Delta_{k-1} \to \tilde{\mathbb{P}}_k \setminus \Delta_k$,

c) The restriction of φ_k to a general closed fiber of $\tilde{f}_{k-1}$ is either an isomorphism onto its image, or the contraction of finitely many disjoint (-1)-curves.

Proof. Since by virtue of Lemma 4, all members of $\mathcal{L}$ except eH are irreducible and reduced, the fact that $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ is a good resolution guarantees that all fibers of $\tilde{f}_0$ except maybe $\tilde{f}_0^{-1}(\infty)$ are irreducible and reduced. This implies in turn that the divisors contracted by $\varphi : \tilde{\mathbb{P}}_0 \dashrightarrow \tilde{\mathbb{P}}_n$ are contained in $\tilde{f}_0^{-1}(\infty)$ or horizontal for $\tilde{f}_0$. Let $\varphi_0 = \mathrm{id}_{\tilde{\mathbb{P}}_0}$.

1) If φ_k, $k \geq 1$, is a flip, then since its flipping curves must pass through a singular point of $\tilde{\mathbb{P}}_{k-1}$ [2, 14.6.4], they are contained in

$\tilde{f}_{k-1}^{-1}(\infty)$. The flipped curves of φ_k are thus contained in $\tilde{f}_k^{-1}(\infty)$ and φ_k restricts to an isomorphism between $\tilde{\mathbb{P}}_{k-1} \setminus \tilde{f}_{k-1}^{-1}(\infty)$ and $\tilde{\mathbb{P}}_k \setminus \tilde{f}_k^{-1}(\infty)$, which is thus again smooth.

2) If φ_k, $k \geq 1$, is the contraction of a divisor $E_{k-1} \subset \tilde{\mathbb{P}}_{k-1}$ onto a curve $B_k \subset \tilde{\mathbb{P}}_k$, then by the previous observation, E_{k-1} is either contained in $\tilde{f}_{k-1}^{-1}(\infty)$ or horizontal for $\tilde{f}_{k-1}$. In the second case, E_{k-1} is the proper transform in $\tilde{\mathbb{P}}_{k-1}$ of an irreducible divisor $E \subset \tilde{\mathbb{P}}_0$, which is necessarily contained in the support of Δ_0. Indeed, by induction hypothesis, the restriction $\varphi_{k-1} \circ \cdots \varphi_1 \circ \varphi_0 : S_{c,0} = \tilde{f}_0^{-1}(c) \to S_{c,k-1} = \tilde{f}_{k-1}^{-1}(c)$ to a general closed fiber of $\tilde{f}_0$ is either an isomorphism or a sequence of contractions of (-1)-curves. Since $E_{k-1} \cap S_{c,k-1}$ consists of a disjoint union of (-1)-curves, it follows that $E \cap S_{c,0}$ is a curve C on $S_{c,0}$ that can be contracted to a finite number of smooth points, hence consists of a disjoint union of (-1)-curves because $S_{c,0}$ is a smooth del Pezzo surface. But on the other hand, if E were not σ-exceptional, the hypothesis that σ maps $S_{c,0}$ isomorphically onto its image in $\mathbb{P}$ would imply that the proper transform $\sigma_* E$ of E in $\mathbb{P}$ is an ample divisor whose intersection with $\sigma(S_{c,0})$ is equal to the union of the curve $\sigma(C)$ with an effective divisor D_0, possibly zero, supported on $H \cap S_{c,0}$. Indeed, the support of the intersection $\sigma_*(E) \cap \sigma_*(S_{c,0})$ is equal to the union of the image of $C = E \cap S_{c,0}$ and that of

$$(\sigma^*(\sigma_*(E)) - E) \mid_{S_{c,0}} \subset \mathrm{Exc}(\sigma) \mid_{S_{c,0}},$$

and by construction, $\sigma(\mathrm{Exc}(\sigma))$ is contained in the support of the base locus of $\mathcal{L}$, which is equal to $H \cap S_{c,0}$. Since $\sigma(C)$ consists again of a disjoint union of (-1)-curves, this would contradict Lemma 2. Thus E is contained in Δ_0 and hence E_{k-1} is contained in Δ_{k-1}. Furthermore, since $\tilde{\mathbb{P}}_{k-1} \setminus \tilde{f}_{k-1}^{-1}(\infty)$ is smooth by hypothesis, it follows that $\tilde{\mathbb{P}}_k \setminus \tilde{f}_k^{-1}(\infty)$ is still smooth along $B_k \setminus (B_k \cap \tilde{f}_k^{-1}(\infty))$. More precisely, it follows from [9, Lemmas 3.20 and 3.21] that $B_k \setminus (B_k \cap \tilde{f}_k^{-1}(\infty))$ is smooth and that

$$\varphi_k \mid_{\tilde{\mathbb{P}}_{k-1} \setminus \tilde{f}_{k-1}^{-1}(\infty)} : \tilde{\mathbb{P}}_{k-1} \setminus \tilde{f}_{k-1}^{-1}(\infty) \to \tilde{\mathbb{P}}_k \setminus \tilde{f}_k^{-1}(\infty)$$

coincides with the blow-up of $\tilde{\mathbb{P}}_k \setminus \tilde{f}_k^{-1}(\infty)$ along $B_k \setminus (B_k \cap \tilde{f}_k^{-1}(\infty))$. Since the restriction of φ_k to a general closed fiber of $\tilde{f}_{k-1}$ is either an isomorphism onto its image, or the contraction of finitely many disjoint (-1)-curves, its image by φ_k is again a smooth del Pezzo surface. Q.E.D.

Corollary 8. *With the notation of Proposition 7, the following hold:*

a) *If $H \cap S$ is irreducible, then φ does not contract the horizontal irreducible component of $\sigma^{-1}(H)$.*

b) *If $H \cap S$ is reducible, then φ contracts at most one horizontal irreducible component of $\sigma^{-1}(H)$.*

Proof.

a) If $H \cap S$ is irreducible then $\sigma^{-1}(H)$ has a unique horizontal irreducible component, whose intersection with the generic fiber S_η of $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ is an irreducible anti-canonical divisor with self-intersection d. It follows that at each intermediate step $\varphi_k : \tilde{\mathbb{P}}_{k-1} \dashrightarrow \tilde{\mathbb{P}}_k$ of φ, the intersection of Δ_{k-1} with the generic fiber of $\tilde{f}_{k-1} : \tilde{\mathbb{P}}_{k-1} \to \mathbb{P}^1$ is an irreducible curve with non negative self-intersection, which is therefore not contracted by φ_k. So φ does not contract the unique horizontal irreducible component of $\sigma^{-1}(H)$.

b) Otherwise, if $H \cap S$ is reducible then $d = 2$ or $d = 3$ and we have the following possibilities:

(i) If $d = 2$ then $\sigma^{-1}(H)$ consists of two horizontal irreducible components and its intersection with the generic fiber S_η of $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ is the union of two (-1)-curves C_1 and C_2 (see 2.1). If φ contracts one of the components, say the one intersecting S_η along C_1, then, letting $\varphi_k : \tilde{\mathbb{P}}_{k-1} \dashrightarrow \tilde{\mathbb{P}}_k$ be the intermediate step of φ at which this contraction occurs, the induced morphism $\varphi_{k,\eta} : S_{k-1,\eta} \to S_{k,\eta}$ between the generic fibers of $\tilde{f}_{k-1} : \tilde{\mathbb{P}}_{k-1} \to \mathbb{P}^1$ and $\tilde{f}_k : \tilde{\mathbb{P}}_k \to \mathbb{P}^1$ coincides with the contraction of C_1. So $S_{k,\eta}$ is a smooth del Pezzo surface of degree 3 defined over $\mathbb{C}(\lambda)$, which intersects the proper transform Δ_k of $\sigma^{-1}(H)$ along the image of C_2. The image of C_2 being an irreducible $\mathbb{C}(\lambda)$-rational curve with self-intersection 3, the same argument as in the previous case implies that the corresponding horizontal irreducible component of Δ_k cannot be contracted at any further step $\varphi_{k'}$, $k' \geq k + 1$, of φ.

(ii) If $d = 3$ and $\sigma^{-1}(H)$ consists of two horizontal irreducible components whose intersection with S_η are respectively a (-1)-curve C_1 and of a 0-curve C_2 both defined over $\mathbb{C}(\lambda)$, then the same argument as in the previous case implies that φ can at most contract the horizontal component of $\sigma^{-1}(H)$ intersecting S_η along C_1.

(iii) If $d = 3$ and $\sigma^{-1}(H)$ consists of three horizontal irreducible components whose intersection with S_η are (-1)-curves C_1, C_2 and C_3 defined over $\mathbb{C}(\lambda)$ intersecting each others transversally at $\mathbb{C}(\lambda)$-rational points, the same argument implies that φ contracts at most one horizontal component of $\sigma^{-1}(H)$. Namely, if φ contracts the irreducible component intersecting S_η along C_1 then at some intermediate step, the proper transforms of C_2 and C_3 in the image of S_η by the induced

contraction are 0-curves intersecting each other twice at $\mathbb{C}(\lambda)$-rational points, and so the corresponding horizontal components of the image of $\sigma^{-1}(H)$ cannot be contracted at any further step. Q.E.D.

§3. Outputs of relative MMPs

Since a general member of a pencil $\mathcal{L} \subset |\mathcal{O}_\mathbb{P}(e)|$ as in Definition 3 above is a rational surface, the output $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ of a relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ ran from a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ of the corresponding rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ is a relative Mori fiber space: more precisely, $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is either a del Pezzo fibration with relative Picard number 1, or it factors through a Mori conic bundle over a certain normal projective surface W over $\mathbb{P}^1$, say $\tilde{f}' = q \circ \xi : \tilde{\mathbb{P}}' \to W \to \mathbb{P}^1$ where $\xi : \tilde{\mathbb{P}}' \to W$ is a morphism of relative Picard number 1, with connected fibers and such that $-K_{\tilde{\mathbb{P}}'}$ is relatively ample. In each case, it follows from Proposition 7 that $\tilde{\mathbb{P}}'$ is a projective completion of $\mathbb{A}^3$ with at most $\mathbb{Q}$-factorial terminal singularities. The following theorem shows in particular that except maybe in the case where $d = 3$ and $H \cap S$ consists of two irreducible components, the structure of $\tilde{\mathbb{P}}'$ as a Mori fiber space depends only on the base locus of $\mathcal{L}$. In particular, it depends neither on the chosen good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ nor on the relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$.

Theorem 9. *Let $\mathcal{L} \subset |\mathcal{O}_\mathbb{P}(e)|$ be the pencil generated by a smooth del Pezzo surface $S \subset \mathbb{P}$ of degree $d \in \{1, 2, 3\}$ and $H \in |\mathcal{O}_\mathbb{P}(1)|$, let $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ be a good resolution of the corresponding rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$, and let $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ be a relative MMP. Then the following hold:*

a) If $H \cap S$ is irreducible, then $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is a del Pezzo fibration of degree d.

b) If $d = 2$ and $H \cap S$ is reducible, then $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is a del Pezzo fibration of degree $d + 1 = 3$.

c) If $H \cap S$ has three irreducible components, then $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ factors through a Mori conic bundle over a normal projective surface.

Proof.

a) Suppose that $H \cap S$ is irreducible. Then since by virtue of Corollary 8 a), φ does not contract any horizontal irreducible component of $\sigma^{-1}(H)$, it follows that φ restricts to an isomorphism between the generic fibers of $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ and $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$. The former is a smooth del Pezzo surface of degree d over the function field $\mathbb{C}(\lambda)$ of $\mathbb{P}^1$ by virtue of § 2.1. On the other hand, Lemma 10 below implies that $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ cannot be factored through a Mori conic bundle, and so $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is a del Pezzo fibration of degree d.

b) Now assume that $d = 2$ and that $H \cap S$ is reducible. Then the intersection of $\sigma^{-1}(H)$ with the generic fiber S_η of $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ consists of the union of two (-1)-curves C_1 and C_2 defined over $\mathbb{C}(\lambda)$ (see §2.1). These two curves being independent in the Néron-Severi group of S_η, the Picard number $\rho(S_\eta)$ is bigger than or equal to 2. By virtue of Corollary 8 b), at most one of the horizontal irreducible components of $\sigma^{-1}(H)$ is contracted by φ. It suffices to show that φ does indeed contract one of these components, say the one intersecting S_η along C_1. Indeed, if so, then the generic fiber S'_η of $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is isomorphic to the image of S_η by the contraction of C_1 hence is a smooth del Pezzo surface of degree 3 defined over $\mathbb{C}(\lambda)$. We then deduce again from Lemma 10 that $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ cannot factor through Mori conic bundle, and so $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is a del Pezzo fibration of degree 3.

Now suppose for contradiction that φ does not contract any of the horizontal components of $\sigma^{-1}(H)$. Then it restricts to an isomorphism between S_η and S'_η and since $\rho(S'_\eta) = \rho(S_\eta) \geq 2$, this implies that $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ factors through a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$ over a normal projective surface $q : W \to \mathbb{P}^1$. Furthermore, the general fibers of $\tilde{f}'$ being rational, so are the general fibers of q, implying that $q : W \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration. Restricting ξ over the generic point η of $\mathbb{P}^1$, we obtain a Mori conic bundle $\xi_\eta : S'_\eta \to W_\eta \simeq \mathbb{P}^1_{\mathbb{C}(\lambda)}$ defined over $\mathbb{C}(\lambda)$. Letting C'_1 and C'_2 be the images of C_1 and C_2 respectively in S'_η, we have $-K_{S'_\eta} \sim C'_1 + C'_2$ and since $(-K_{S'_\eta} \cdot \ell) = 2$ for every general $\mathbb{C}(\lambda)$-rational fiber ℓ of ξ_η, it follows either that C'_1 and C'_2 are both sections of ξ_η or that, up to a permutation, C'_1 is a 2-section of ξ_η while C'_2 is contained in a fiber. The second possibility is excluded because a Mori conic bundle over $\mathbb{P}^1_{\mathbb{C}(\lambda)}$ does not contain any (-1)-curve defined over $\mathbb{C}(\lambda)$ in its closed fibers. In the first case, since the relative Picard number $\rho(S'_\eta/\mathbb{P}^1_{\mathbb{C}(\lambda)})$ is equal to 1, we would have $C'_2 \sim C'_1 + a\ell$ for some $a \in \mathbb{Q}$ such that $2 = C'_1 \cdot C'_2 = (C'_1)^2 + a = -1 + a$ and $-1 = (C'_2)^2 = (C'_1)^2 + 2a = -1 + 2a$, which is absurd.

c) Finally, assume that $d = 3$ and that $H \cap S$ has three irreducible components. Then the intersection of $\sigma^{-1}(H)$ with S_η is a reduced anti-canonical divisor on S_η whose support consists of the union of three (-1)-curves C_1, C_2 and C_3 defined over $\mathbb{C}(\lambda)$ and intersecting each other transversally at $\mathbb{C}(\lambda)$-rational points. If φ does not contract any horizontal irreducible component of $\sigma^{-1}(H)$, then it induces an isomorphism between S_η and the generic fiber S'_η of $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$. The latter is thus a smooth del Pezzo surface of degree 3 defined over $\mathbb{C}(\lambda)$ and having the sum $C'_1 + C'_2 + C'_3$ of the images of the C_i's as an anti-canonical divisor. The Picard number of S'_η is thus strictly bigger than one, and so

$\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ factors again through a Mori conic bundle, restricting over the generic point η of $\mathbb{P}^1$ to a Mori conic bundle $\xi_\eta : S'_\eta \to \mathbb{P}^1_{\mathbb{C}(\lambda)}$ defined over $\mathbb{C}(\lambda)$. Since $(-K_{S'_\eta} \cdot \ell) = 2$ for every general $\mathbb{C}(\lambda)$-rational fiber ℓ of ξ_η, either two of the C'_i are sections of ξ_η and the third one is contained in a fiber or one of the C'_i is a 2-section of ξ_η and the two other ones are contained in a fiber. In each case, there would exist a closed fiber of $\xi_\eta : S'_\eta \to \mathbb{P}^1_{\mathbb{C}(\lambda)}$ containing a (-1)-curve defined over $\mathbb{C}(\lambda)$, which is impossible. Together with Corollary 8 b), this implies that φ contracts exactly one horizontal irreducible component of $\sigma^{-1}(H)$, say the one intersecting S_η along C_1. Then S'_η is isomorphic to the image of S_η by the contraction of C_1, hence is a smooth del Pezzo surface of degree 4 defined over $\mathbb{C}(\lambda)$, having the sum $C'_2 + C'_3$ of the images of C_2 and C_3 as an anti-canonical divisor. The Picard number $\rho(S'_\eta)$ is thus bigger or equal to 2 and so, $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ necessarily factors through a Mori conic bundle. Q.E.D.

In the proof of Theorem 9 above, we used the following criterion for the output of a relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ to be a Mori conic bundle:

Lemma 10. *With the notation above, let $r \in \{1, 2, 3\}$ and $h_\varphi \in \{0, 1\}$, respectively, be the number of irreducible components of $H \cap S$ and the number of horizontal irreducible components of $\sigma^{-1}(H)$ contracted by $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$. If $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ factors through a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$ over a normal projective surface $q : W \to \mathbb{P}^1$ then $r = h_\varphi + 2$.*

Proof. We first observe that the inverse image by ξ of every irreducible curve $C \subset W$ is again irreducible. Indeed, assuming on the contrary that $\xi^{-1}(C)$ has at least two irreducible components F_1 and F_2 such that $F_1 \cap F_2 \neq \emptyset$, we can choose an irreducible curve $\ell_1 \subset F_1$ whose class $[\ell_1]$ in $\overline{NE}(\tilde{\mathbb{P}}')$ belongs to the extremal ray giving rise to ξ and such that $\ell_1 \cap F_2 \neq \emptyset$. Then for a general fiber ℓ of ξ, we have by definition $[\ell] = a[\ell_1]$ for some $a > 0$, but since ℓ is disjoint from F_2, this would lead to the contradiction $0 = F_2 \cdot \ell = aF_2 \cdot \ell_1 > 0$. Since all fibers of $\tilde{f}'$ except maybe $(\tilde{f}')^{-1}(\infty)$ are irreducible and rational, it follows that $q : W \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration with $q^{-1}(\infty)$ as a unique possibly reducible fiber. This fibration lifts to a $\mathbb{P}^1$-fibration $\tilde{q} = \tau \circ q : \tilde{W} \to W$ on the minimal desingularization $\tau : \tilde{W} \to W$ of W, having $\tilde{q}^{-1}(\infty)$ as a unique possibily reduced fiber. The Néron-Severi group of $\tilde{W}$ is thus freely generated by a section of $\tilde{q}$ and the irreducible components of $\tilde{q}^{-1}(\infty)$. Since W has rational singularities, it follows that the Néron-Severi group of W is generated by a section of q and the irreducible components of $q^{-1}(\infty)$.

So $\rho(W) = \nu_\infty + 1$, where ν_∞ denotes the number of irreducible components of $q^{-1}(\infty)$, which by the previous observation, is equal to the number of irreducible components of $(\tilde{f}')^{-1}(\infty)$.

Since $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ is a good resolution, the Picard number $\rho(\tilde{\mathbb{P}})$ of $\tilde{\mathbb{P}}$ is equal to $r + e_v + 1$, where e_v denote the number of vertical exceptional divisors of σ, all of them being contained in $\tilde{f}^{-1}(\infty)$ (see § 2.1). Since divisorial contractions decrease the Picard number by one, and flips leave it unchanged, we obtain

$$
\begin{aligned}
\nu_\infty + 1 = \rho(W) &= \rho(\tilde{\mathbb{P}}') - 1 = 1 + r + e_v - h_\varphi - v_\varphi - 1 \\
&= (1 + e_v - v_\varphi) + (r - h_\varphi) - 1 = \nu_\infty + (r - h_\varphi) - 1
\end{aligned}
$$

where v_φ denotes the number of vertical component of $\sigma^{-1}(H)$ contracted by φ. So $r = h_\varphi + 2$.   Q.E.D.

The remaining case where $d = 3$ and $H \cap S$ has two irreducible components is more intricate. Here given a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ of the rational map $\overline{f} : \mathbb{P} = \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$, the intersection of $\sigma^{-1}(H)$ with the generic fiber S_η of $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ is a reduced anti-canonical divisor whose support consists of the union of a (-1)-curve C_1 and of a 0-curve C_2 both defined over $\mathbb{C}(\lambda)$, and Corollary 8 b) implies that a relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ can at most contract the horizontal component of $\sigma^{-1}(H)$ intersecting S_η along C_1.

Proposition 11. *In the situation above, the following alternative hold:*

a) If φ contracts a horizontal irreducible component of $\sigma^{-1}(H)$ then $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is a del Pezzo fibration of degree 4.

b) Otherwise, $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ factors through a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$ over a normal projective surface $q : W \to \mathbb{P}^1$. Furthermore, up to a permutation, the images of C_1 and C_2 are respectively a 2-section of the restriction ξ_η of ξ over the generic point of $\mathbb{P}^1$ and a full fiber of it.

Proof. Indeed, if φ contracts a horizontal component then the image of S_η by the induced birational morphism is a smooth del Pezzo surface of degree 4 defined over $\mathbb{C}(\lambda)$ and the output $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is a del Pezzo fibration of degree 4 by virtue of Lemma 10.

Otherwise, if φ does not contract any horizontal irreducible component of $\sigma^{-1}(H)$ then φ restricts to an isomorphism between S_η and the generic fiber S'_η of $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$. Since $C_1 + C_2$ is an anti-canonical divisor on S_η, $\rho(S_\eta) \geq 2$ and so $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ necessarily factors through a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$ over a normal projective surface $q : W \to \mathbb{P}^1$. The restriction of $q \circ \xi$ over the generic point η of $\mathbb{P}^1$ is a Mori conic

bundle $\xi_\eta : S'_\eta \to W_\eta \simeq \mathbb{P}^1_{\mathbb{C}(\lambda)}$ defined over $\mathbb{C}(\lambda)$. Since $(-K_{S'_\eta} \cdot \ell) = 2$ for every general $\mathbb{C}(\lambda)$-rational fiber ℓ of ξ_η and C_1 is a (-1)-curve defined over $\mathbb{C}(\lambda)$, hence cannot be contained in a fiber of ξ_η, the only possibilities are that C_1 and C_2 are both sections of ξ_η and that C_1 is a 2-section of ξ_η while C_2 is a full fiber of it. Similarly as in the case $d = 2$ in the proof of Theorem 9 above, the first possibility is excluded by the fact that $\rho(S'_\eta / \mathbb{P}^1_{\mathbb{C}(\lambda)}) = 1$: indeed, we would have $C_2 \sim C_1 + a\ell$ for some $a \in \mathbb{Q}$ satisfying simultaneously the identities $0 = C_2^2 = C_1^2 + 2a = -1 + 2a$ and $2 = C_2 \cdot C_1 = C_1^2 + a = -1 + a$, which is impossible.   Q.E.D.

In contrast with the case $d = 2$, the possibility that the images of C_1 and C_2 are respectively a 2-section of ξ_η and a full fiber of it cannot be excluded. Actually a smooth cubic surface $S'_\eta \subset \mathbb{P}^3_{\mathbb{C}(\lambda)}$ containing a (-1)-curve C_1 defined over $\mathbb{C}(\lambda)$ always admit a conic bundle structure $\pi : S'_\eta \to \mathbb{P}^1_{\mathbb{C}(\lambda)}$ with five degenerate fibers, defined by the mobile part of the restriction to S'_η of the pencil of hyperplanes in $\mathbb{P}^3_{\mathbb{C}(\lambda)}$ containing C_1.

This suggests that in this case the structure of the output $\tilde{\mathbb{P}}'$ might depend on the chosen good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ and on the relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$. Partial results on the structure of $\tilde{\mathbb{P}}'$ can be obtained by a more careful study of relative MMPs ran from particular explicit good resolutions $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$, but a complete discussion would lead us far beyond the intended aim of this article. The following result, which we mention without proof referring the reader to the forthcoming paper [4] for the detail, asserts the existence of relative MMPs whose outputs are del Pezzo fibrations of degree 4. In contrast, we do not know examples for which the output is a Mori conic bundle (see also Remark 16 below).

Proposition 12. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface, let $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$ be a hyperplane intersecting S along the union of a line and smooth conic, let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(3)|$ be the pencil generated by S and $3H$ and let $\overline{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ be the corresponding rational map. Then there exists a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ and a MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ relative to $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ whose output is a del Pezzo fibration $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ of degree 4.*

§4.   Mori conic bundles and twisted $\mathbb{A}^1_*$-fibrations

In this section, we investigate more closely the case where a relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ ran from a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ terminates with a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$ over a normal projective surface W. According to Theorem 9 and Proposition 11, this occurs for all pencils $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(3)|$ generated by a smooth cubic surface $S \subset \mathbb{P}^3$ and three

times a hyperplane $H \subset \mathbb{P}^3$ such that $H \cap S$ consists of three lines, and possibly for pencils for which $H \cap S$ consists of a line and smooth conic when $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ does not contract any horizontal irreducible component of $\sigma^{-1}(H)$.

Theorem 13. *Let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(3)|$ be a pencil as above and let $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ be a relative MMP ran from a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ of the corresponding rational map $\overline{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ whose output is a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$ over a normal projective surface $q : W \to \mathbb{P}^1$. Then there exists an open subset $U \subset W$ isomorphic to $\mathbb{A}^2$ such that the induced morphism $\xi_0 = \xi \circ \varphi \circ \sigma^{-1} : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \to W$ factors through a twisted $\mathbb{A}_*^1$-fibration over U.*

Proof. Recall that by virtue of Proposition 7, the composition $\varphi \circ \sigma^{-1} : \mathbb{P}^3 \setminus H \to \tilde{\mathbb{P}}' \setminus \varphi_*(\sigma^{-1}(H))$ is an isomorphism. As observed in the proof of Lemma 10, $q : W \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration with $\eta^{-1}(\infty)$ as a unique possibly reducible fiber, where $\infty = \overline{f}_*(H)$. So the restriction of q over $\mathbb{P}^1 \setminus \{\infty\}$ is isomorphic to the trivial bundle $\mathbb{P}^1 \setminus \{\infty\} \times \mathbb{P}^1$. The union of all vertical components of $\varphi_*(\sigma^{-1}(H))$ is equal to $(\tilde{f}')^{-1}(\infty)$ (see § 2.1) and on the other hand, it follows from the proof of Theorem 9 and § 11 that the restrictions of the two horizontal irreducible components E_1 and E_2 of $\varphi_*(\sigma^{-1}(H))$ to the generic fiber S'_η of $\tilde{f}'$ are either a pair of 0-curves C_1 and C_2 defined over $\mathbb{C}(\lambda)$ and intersecting each other twice at $\mathbb{C}(\lambda)$-rational points if $H \cap S$ consist of three irreducible components, or the union of a (-1)-curve C_1 and a 0-curve C_2 defined over $\mathbb{C}(\lambda)$ with $(C_1 \cdot C_2) = 2$ in the case where $H \cap S$ consists of two irreducible components. In the first case, one of the curves C_i is a 2-section of the induced conic bundle $\xi_\eta : S'_\eta \to W_\eta \simeq \mathbb{P}_{\mathbb{C}(\lambda)}$ while the other one is a full fiber of it, and in the second case, C_1 is a 2-section of ξ_η while C_2 is a full fiber. So up to a permutation, we may assume that in both cases, E_1 is a rational 2-section of $\xi : \tilde{\mathbb{P}}' \to W$ while E_2 is mapped by ξ onto a section D of $q : W \to \mathbb{P}^1$. The open subset $U = W \setminus \xi(E_2 \cup (\tilde{f}')^{-1}(\infty)) = W \setminus (D \cup \eta^{-1}(\infty))$ of W is thus isomorphic to $\mathbb{A}^2$, and by construction, the composition $\xi_0 = \xi \circ \varphi \circ \sigma^{-1} : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \to W$ factors through U. Since E_1 is an irreducible birational 2-section of the conic bundle $\xi : \tilde{\mathbb{P}}' \to W$, the generic fiber of ξ_0 is a nontrivial form of the punctured affine line over the function field of W, so $\xi_0 : \mathbb{A}^3 \to U$ is a twisted $\mathbb{A}_*^1$-fibration.   Q.E.D.

The twisted $\mathbb{A}_*^1$-fibrations $\xi_0 : \mathbb{A}^3 \to \mathbb{A}^2$ obtained in Theorem 13 above can be described in terms of the initial data consisting of the smooth cubic surface $S \subset \mathbb{P}^3$ and the hyperplane $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$. In both cases, it follows from the description given in the proof of Theorem 9 that

a relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ ran a from a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ of $\overline{f} :$ $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ contracts exactly one of the horizontal irreducible component of $\sigma^{-1}(H)$ corresponding to a line in the support of $H \cap S$. More precisely, in the case where $d = 3$ and $H \cap S$ consists of the union of three lines ℓ, ℓ_2 and ℓ_3, we may assume up to permutation that φ contracts the horizontal irreducible component E_3 of $\sigma^{-1}(H)$ corresponding to ℓ_3 and that the image in $\tilde{\mathbb{P}}'$ of the component E corresponding to ℓ is a rational 2-section of the Mori conic bundle $\xi_\eta : S'_\eta \to W_\eta \simeq \mathbb{P}^1_{\mathbb{C}(\lambda)}$ factoring $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$. In the case where $d = 2$ then φ contracts the unique horizontal irreducible component of $\sigma^{-1}(H)$ corresponding to the line ℓ in the support of $H \cap S$. Letting $\Theta_\ell : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ be the projection from the line ℓ, we have the following description.

Proposition 14. *With the notation above, the twisted $\mathbb{A}^1_*$-fibration $\xi_0 : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \to \mathbb{A}^2$ coincides with the restriction to $\mathbb{P}^3 \setminus H$ of the rational map $\overline{f} \times \Theta_{\ell_1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. If $d = 3$ and $H \cap S = \ell \cup \ell_2 \cup \ell_3$ then the intersection with the generic fiber S'_η of $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ of the proper transforms of E and of the irreducible component of $\sigma^{-1}(H)$ corresponding to ℓ_2 are respectively a 2-section and a fiber of the induced Mori conic bundle structure $\xi_\eta :$ $S'_\eta \to W_\eta \simeq \mathbb{P}^1_{\mathbb{C}(\lambda)}$. Therefore ξ_η coincides with the proper transform by the restriction φ_η of φ of the conic bundle $\theta : S_\eta \to \mathbb{P}^1_{\mathbb{C}(\lambda)}$ defined by the mobile part of the restriction to S_η of the pencil of hyperplanes in $\mathbb{P}^3_{\mathbb{C}(\lambda)}$ containing $E|_{S_\eta}$. So $\xi_0 : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \to \mathbb{A}^2$ coincides with the restriction to $\mathbb{P}^3 \setminus H$ of $\overline{f} \times \Theta_\ell$.

The case where $H \cap S$ consists of the union of a line ℓ and a smooth conic follows from similar argument using the description given in Proposition 11. Q.E.D.

Example 15. Let $S \subset \mathbb{P}^3 = \mathrm{Proj}_{\mathbb{C}}(\mathbb{C}[x, y, z, w])$ be the smooth cubic surface defined by the vanishing of the polynomial $F = w^2 z + y^2 x + w x^2 + z^3$, let $\overline{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ be the pencil generated by S and $3H$, where $H = \{x = 0\}$ and let

$$f : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \simeq \mathrm{Spec}(\mathbb{C}[y, z, w]) \to \mathbb{A}^1, \ (y, z, w) \mapsto w^2 z + y^2 + w + z^3$$

be the induced morphism. The intersection $H \cap S$ consists of three lines $\ell_1 = \{x = z = 0\}$, $\ell_2 = \{x = w + iz = 0\}$ and $\ell_3 = \{x = w - iz = 0\}$ meeting in the Eckardt point $[0 : 1 : 0 : 0]$ of S, and the morphism $\xi_0 = (f, \mathrm{pr}_z) : \mathbb{A}^3 \to \mathbb{A}^2$ is a surjective twisted $\mathbb{A}^1_*$-fibration induced by the restriction of $\overline{f} \times \Theta_{\ell_1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. The fact that ξ_0 is twisted can be seen directly as follows: its generic fiber is isomorphic

to the curve $C \subset \mathbb{A}^2_{\mathbb{C}(\lambda, z)} = \mathrm{Spec}(\mathbb{C}(\lambda, z)[y, w])$ defined by the equation $w^2 z + y^2 + w + z^3 - \lambda = 0$. Extending the scalars to the quadratic extension $K = \mathbb{C}(\lambda, z)[v]/(v^2 - z)$, we have

$$
\begin{aligned}
C_K & \simeq \mathrm{Spec}(K[y, w]/(w^2 v^2 + y^2 + w + v^6 - \lambda)) \\
& \simeq \mathrm{Spec}(K[y, w]/((wv + \frac{1}{2v})^2 + y^2 - (\frac{1}{4v^2} - v^6 + \lambda))) \\
& \simeq \mathrm{Spec}(K[U, V]/(UV - (\frac{1}{4v^2} - v^6 + \lambda))) \\
& \simeq \mathrm{Spec}(K[U^{\pm 1}])
\end{aligned}
$$

where

$$
U = wv + \frac{1}{2v} + iy \text{ and } V = wv + \frac{1}{2v} - iy,
$$

on which the Galois group $\mathrm{Gal}(K/\mathbb{C}(\lambda, z)) \simeq \mathbb{Z}_2$ acts by $U \mapsto -U^{-1}$. So C is a nontrivial $\mathbb{C}(\lambda, z)$-form of the punctured affine line over $\mathbb{C}(\lambda, z)$.

Remark 16. In the case where $d = 3$ and $H \cap S$ consists of a line ℓ and smooth conic, the fact that the projection $\Theta_\ell : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ gives rise to a twisted $\mathbb{A}^1_*$-fibration $\xi_0 = (\overline{f}, \Theta_\ell)|_{\mathbb{P}^3 \setminus H} : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \to \mathbb{A}^2$ does not necessarily imply that a relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ ran from a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ of $\overline{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ terminates with a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to W$ inducing ξ_0 (see Proposition 12). Note that since the base locus of Θ_ℓ is contained in the indeterminacy locus of $\overline{f}$, we can choose a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ of $\overline{f}$ which simultaneously resolves the indeterminacies of Θ_ℓ. Every MMP $\psi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}_1$ relative to the morphism $(\tilde{f}, \Theta_\ell \circ \sigma) : \tilde{\mathbb{P}} \to \mathbb{P}^1 \times \mathbb{P}^1$ being also a part of a MMP relative to $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$, it preserves the open subset $\mathbb{A}^3 = \tilde{\mathbb{P}} \setminus \sigma^{-1}(H)$ by virtue of Proposition 7. Such a MMP process ψ does not contract any horizontal irreducible component of $\sigma^{-1}(H)$ and terminates with a Mori conic bundle $\xi_1 : \tilde{\mathbb{P}}'' \to \mathbb{P}^1 \times \mathbb{P}^1$, whose restriction to $\mathbb{A}^3$ coincides with ξ_0 by construction. But there is no guarantee in general that $\tilde{f}_1 = \mathrm{pr}_1 \circ \xi_1 : \tilde{\mathbb{P}}'' \to \mathbb{P}^1$ coincides with the final output of a MMP relative to $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$: there could exist a relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ which factorizes through ψ and for which the induced rational map $\psi' = \varphi \circ \psi^{-1} : \tilde{\mathbb{P}}'' \dashrightarrow \tilde{\mathbb{P}}'$ contracts an irreducible component of $\psi_*(\sigma^{-1}(H))$ that is horizontal for $\tilde{f}_1$.

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