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Locally nilpotent derivations of rings graded by an abelian group

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§1. Introduction

Among the most important tools in the study of locally nilpotent derivations (LNDs) of commutative rings are its $\mathbb{Z}$-gradings and the homogeneous derivations associated to them. Gradings which involve other totally ordered abelian groups have also been used to study LNDs, though to a lesser extent. Gradings which involve non-totally ordered abelian groups have been largely ignored in this context, since it is no longer possible to associate a highest-degree homogeneous derivation to a given derivation in this case. However, it turns out that one can still get valuable information about LNDs from such gradings. In [7], the second and third authors studied LNDs of certain rings graded by a finite cyclic group. Their results were applied to show that some families of Pham-Brieskorn threefolds are rigid, i.e., their coordinate rings have no nonzero LNDs.

In the present work, we generalize the theory developed in that paper to the case of rings graded by arbitrary abelian groups. Let B be a domain of characteristic zero graded by an abelian group G. For any subgroup H of G, let B_H be the subring of B generated by the nonzero homogeneous elements of B whose degrees belong to H. An element x of B is *G-critical* if it is homogeneous and nonzero, and if there exists a

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subgroup H of G such that $\deg x \notin H$ and $B = B_H[x]$. Our main result is *Theorem 6.2*, which states that $D^2x = 0$ whenever D is a homogeneous LND of B and $x \in B$ is a G-critical element. Moreover, if y is a second G-critical element and $xB \neq yB$, then either $Dx = 0$ or $Dy = 0$. As an application of this theorem, we settle some of the cases of Pham-Brieskorn threefolds which were out of the reach of the earlier paper. As a second application, we give a short proof of the fact that the Derksen invariant of Russell's cubic threefold is nontrivial.

We assume throughout that rings are commutative with identity. Given the ring B, the units of B are denoted by B^*. Given the integer $n \geq 0$, the polynomial ring in n variables over B is denoted $B^{[n]}$. If B is a domain, the field of fractions of B is denoted $\mathrm{frac}(B)$. The cyclic group of order n is indicated by $\mathbb{Z}_n$.

§2. Locally Nilpotent Derivations

We first recall a few basic definitions and facts concerning locally nilpotent derivations.

Let B be an integral domain containing $\mathbb{Q}$. The set of derivations of B is denoted by $\mathrm{Der}(B)$. If $D \in \mathrm{Der}(B)$, we write

$$\ker(D) = \{f \in B | D(f) = 0\} \text{ and } \mathrm{Nil}(D) = \{f \in B | D^n f = 0 \text{ for } n \gg 0\};$$

note that these are two subrings of B. We say that $D \in \mathrm{Der}(B)$ is *locally nilpotent* if $\mathrm{Nil}(D) = B$. The set of locally nilpotent derivations of B is denoted by $\mathrm{LND}(B)$. We say that B is *rigid* if $\mathrm{LND}(B) = \{0\}$.

Given $D \in \mathrm{LND}(B)$, $\ker D$ is factorially closed in B and $B^* \subset \ker D$. Each nonzero $D \in \mathrm{LND}(B)$ determines a degree function $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$, defined by

$$\deg_D f = \min\{n \in \mathbb{N} \mid D^{n+1} f = 0\} \text{ if } f \neq 0, \quad \text{and} \quad \deg_D 0 = -\infty.$$

(See *Def. 3.3*.) Any $t \in B$ with $\deg_D t = 1$ is called a *local slice* or *preslice* for D. If t is a local slice of D and $A = \ker D$, then:

$$B_{Dt} = A_{Dt}[t] = A_{Dt}^{[1]} \tag{1}$$

In this case, $\deg_D f$ for $f \in B$ is equal to the degree of f as a polynomial in t.

We make use of the following properties (see [6], Princ. 5 and Lemma 9.3).

Lemma 2.1. *Suppose that $D \in \mathrm{LND}(B)$.*

(a) *If $f \in B$ and the integer $n \geq 1$ are such that $D^n f \in fB$, then $D^n f = 0$.*

(b) *If $f, g \in B$ are such that $Df \in gB$ and $Dg \in fB$, then $Df = 0$ or $Dg = 0$.*

(c) *Suppose $u, v \in \ker D$ and $x, y \in B$ are nonzero, and a and b are integers with $a, b \geq 2$. Assume $ux^a + vy^b \neq 0$. If $D(ux^a + vy^b) = 0$, then $Dx = Dy = 0$.*

Definition 2.2. Given a derivation $D : B \to B$, let $\delta : R \to R$ be a derivation of a subring $R \subset B$. Then D is a *quasi-extension* of δ if there exists a nonzero $t \in B$ such that $Ds = t \cdot \delta s$ for all $s \in R$.

One of the main tools we use to study locally nilpotent derivations for rings graded by $\mathbb{Z}_n$ is the following.

Lemma 2.3. ([6], Lemma 5.38) *Let B be an integral domain containing $\mathbb{Z}$, and let $D : B \to B$ be a derivation which is a quasi-extension of a derivation $\delta : R \to R$ for some subring R. If $D \in \mathrm{LND}(B)$, then $\delta \in \mathrm{LND}(R)$.*

For a more extensive treatment of locally nilpotent derivations, the reader is referred to [6].

§3. *G*-Gradings of Rings

Definition 3.1. An abelian group G is *totally ordered* if G has a total order $\leq$ which is translation invariant: For all $x, y, z \in G$, $x + z \leq y + z$ whenever $x \leq y$.

Remark 3.2. One says that an abelian group is *orderable* if there exists a total order that makes it a totally ordered abelian group in the sense of *Def. 3.1.* It is well known that an abelian group is orderable if and only if it is torsion-free.

If G is a totally ordered abelian group, then $G \cup \{-\infty\}$ is the totally ordered set which extends the order of G by setting $-\infty < g$ for all $g \in G$.

Definition 3.3. Let B be an integral domain and let G be a totally ordered abelian group. A function $\deg : B \to G \cup \{-\infty\}$ is a *degree function* if:

(i) $\deg^{-1}(-\infty) = \{0\}$

(ii) $\deg(fg) = \deg f + \deg g$ for each nonzero $f, g \in B$

(iii) $\deg(f + g) \leq \max\{\deg f, \deg g\}$ for all $f, g \in B$

Definition 3.4. Let G be an abelian group, and let B be a ring. A *G-grading* of B is a family $\{B_g\}_{g\in G}$ of subgroups of $(B,+)$ such that:

(i) $B = \oplus_{g\in G} B_g$

(ii) $B_g B_h \subset B_{g+h}$ for all $g, h \in G$

If M is a submonoid of G then by an *M-grading* of B we mean a G-grading $\{B_g\}_{g\in G}$ of B satisfying $B_g = 0$ for all $g \in G \setminus M$.

3.5. Definitions and Observations. Assume that G is an abelian group, $B = \oplus_{g\in G} B_g$ is a G-graded ring, $H \subset G$ is a subgroup, and A is a subring of B.

1. B_0 is a subring of B, $1 \in B_0$, each B_g is a B_0-module, and $B = \oplus_{g\in G} B_g$ as B_0-modules. If R is a subring of B, then B is G-graded *over* R if $R \subset B_0$.
2. $f \in B$ is *G-homogeneous* if and only if $f \in B_g$ for some $g \in G$. If $f \neq 0$ then g is unique, we say that f is of *degree* g, and we write $\deg_G f = g$.
3. Given $f \in B$, there exists a unique family $(f_g)_{g\in G}$ such that (i) $f_g \in B_g$ for every $g \in G$, (ii) $\{\, g \in G \mid f_g \neq 0 \,\}$ is a finite set, and (iii) $f = \sum_{g\in G} f_g$.
4. If G is totally ordered, then $\deg_G$ (defined in 2.) extends to a function $\deg_G : B \to G \cup \{-\infty\}$ by defining:

$$\deg_G(f) = \max\{\, g \in G \mid f_g \neq 0 \,\} \text{ (if } f \neq 0) \text{ and } \deg_G(0) = -\infty.$$

   If B is a domain then this map is a degree function on B; we refer to it as *the degree function determined by the grading.*
5. Given $g \in G$, define $A_g = A \cap B_g$. We say that A is a *G-graded subring* of B if the family $\{A_g\}_{g\in G}$ defines a G-grading of A, or equivalently, if A is generated by G-homogeneous elements as an algebra over A_0.
6. If A is a G-graded subring of B then we write $G(A)$ for the subgroup of G generated by $\{\, g \in G \mid A_g \neq 0 \,\}$. Note that $G(A)$ is a subgroup of $G(B)$.
7. The subring $B_H \subset B$ defined by $B_H = \oplus_{g\in H} B_g$ is G-graded. Note that $G(B_H) \subset H$ and $A \subset B_{G(A)}$. If K is a subgroup of H, then $B_K \subset B_H$.
8. $D \in \mathrm{Der}(B)$ is *G-homogeneous* if and only if there exists $d \in G$ such that $DB_g \subset B_{g+d}$ for all $g \in G$. If $D \neq 0$, then d is unique, is called the *degree* of D, and is denoted by $\deg_G D$. The kernel of a G-homogeneous derivation of B is a G-graded subring of B.

9. Any group homomorphism $\alpha : G \to G'$ induces a G'-grading of B, namely:

$$B = \oplus_{\gamma \in G'} B'_\gamma \quad \text{where} \quad B'_\gamma = \oplus_{\alpha(g)=\gamma} B_g.$$

In this case, each G-homogeneous element of B is G'-homogeneous, and each G-homogeneous derivation of B is G'-homogeneous.

10. Suppose that B is an integral domain and let $a, b \in B \setminus \{0\}$. If ab and b are G-homogeneous, then a is also G-homogeneous. If G is torsion free, then ab G-homogeneous implies that a and b are G-homogeneous.
11. Suppose that G is totally ordered. Given nonzero $b \in B$, $\bar{b}$ denotes the highest-degree homogeneous summand of b. For any subring $R \subset B$, $\bar{R}$ denotes the subring of B generated by the set $\{\bar{r} \mid r \in R, r \neq 0\}$.

3.6. Associated Graded Rings. Suppose that R is an integral domain with a degree function $\deg : R \to G \cup \{-\infty\}$, where G is a totally ordered abelian group.

1. The pair $(R, \deg)$ determines the G-graded integral domain $\mathrm{Gr}(R) = \oplus_{g \in G} R_{\leq g}/R_{<g}$, called the *associated graded ring*, where for each $g \in G$ we set

$$R_{\leq g} = \left\{ f \in R \mid \deg(f) \leq g \right\} \text{ and } R_{<g} = \left\{ f \in R \mid \deg(f) < g \right\}.$$

One also defines the map $\mathrm{gr} : R \to \mathrm{Gr}(R)$ by stipulating that $\mathrm{gr}(0) = 0$ and that, given $x \in R \setminus \{0\}$, $\mathrm{gr}(x)$ is the nonzero element $x + R_{<g}$ of $R_{\leq g}/R_{<g}$, where $g = \deg(x)$. The map gr preserves multiplication but in general not addition.

2. Given a derivation $D : R \to R$, let $U(D) = \left\{ \deg(Dx) - \deg(x) \mid x \in R \setminus \{0\} \right\}$. Then $U(D)$ is a nonempty subset of the totally ordered set $G \cup \{-\infty\}$. If $U(D)$ has a greatest element, we define $\deg(D)$ to be that element; if $U(D)$ does not have a greatest element, we say that $\deg(D)$ is not defined. So the phrase "$\deg(D)$ is defined" is equivalent to $U(D)$ having a greatest element. Note that if $\deg(D)$ is defined then $\deg(D) \in G \cup \{-\infty\}$. Also note that if D is the zero derivation then $\deg(D)$ is defined and is equal to $-\infty$; conversely, $\deg(D) = -\infty$ implies $D = 0$.
3. It is well known that if $D : R \to R$ is a derivation such that $\deg(D)$ is defined then there is an associated G-homogeneous

derivation $\mathrm{gr}(D) : \mathrm{Gr}(R) \to \mathrm{Gr}(R)$ satisfying:
for each $x \in R \setminus \{0\}$,

$$(\mathrm{gr}\, D)(\mathrm{gr}\, x) = \begin{cases} \mathrm{gr}(Dx), & \text{if } \deg(Dx) - \deg x = \deg D, \\ 0, & \text{if } \deg(Dx) - \deg x < \deg D. \end{cases}$$

Moreover, $\mathrm{gr}(D)$ satisfies $\deg(\mathrm{gr}(D)) = \deg(D)$ and hence $\mathrm{gr}(D) = 0 \Leftrightarrow D = 0$. It is also easy to see that gr maps $\ker(D)$ into $\ker(\mathrm{gr}\, D)$ and that

if D is locally nilpotent then so is $\mathrm{gr}(D)$*.*

Remark. Parts 2. and 3. of paragraph 3.6 follow [5]. We stress that the condition "$\deg(D)$ is defined" (i.e., "$U(D)$ has a greatest element") is exactly what is needed for the existence of $\mathrm{gr}(D)$. In the special case $G = \mathbb{Z}$, the condition "$U(D)$ has a greatest element" is equivalent to "$U(D)$ has an upper bound in G", in which case one says that D respects the filtration (see [6, Sec. 1.1.5]).

In general, the ring $\mathrm{Gr}(R)$ can be difficult to work with. However, we are mainly interested in the case where the degree function comes from a G-grading, and the situation is easier to handle in this case.

Suppose that B is a G-graded integral domain, where G is totally ordered, and let $\deg_G$ be the degree function on B determined by the grading (cf. part 4. of paragraph 3.5). Then the associated graded ring $\mathrm{Gr}(B)$ determined by $(B, \deg_G)$ is naturally isomorphic to B. If $R \subset B$ is a (not necessarily graded) subring, then the restriction of $\deg_G$ to R gives rise to an associated graded ring $\mathrm{Gr}(R)$ which is easy to describe: $\mathrm{Gr}(R) = \bar{R}$ (cf. part 11. of paragraph 3.5).

The following two results are needed in *Section 8*.

Lemma 3.7. *Let G be a totally ordered abelian group and $B = \oplus_{g \in G} B_g$ a G-graded integral domain. Let $A = \oplus_{g \leq 0} B_g$, $x \in B$, and $R = A[x]$. Then $\bar{R} = A[\bar{x}]$.*

Proof. If $x \in A$ this is clear, since A is a G-graded subring of B. So assume $x \notin A$. Given nonzero $r \in R$, suppose that $r = \sum_{1 \leq i \leq n} a_i x^{e_i}$ for nonzero $a_1, ..., a_n \in A$ and distinct $e_1, ..., e_n \in \mathbb{N}$ ($n \geq 2$). If $\deg_G(a_i x^{e_i}) = \deg_G(a_j x^{e_j})$ for $e_i > e_j$, then $\deg_G(a_i x^{e_i - e_j}) = \deg_G(a_j) \leq 0$, which implies that $a_i x^{e_i - e_j} \in A$. We may therefore write r as a sum of $n-1$ terms of the form ax^e ($a \in A$, $e \in \mathbb{N}$). Since this kind of reduction can be carried out only a finite number of times, we can assume that:

$$r = \sum_{1 \leq i \leq n} a_i x^{e_i} \quad \text{and} \quad \deg_G(a_i x^{e_i}) \neq \deg_G(a_j x^{e_j}) \quad \text{when} \quad i \neq j$$

Let m $(1 \leq m \leq n)$ be such that $\deg_G(a_m x^{e_m}) = \max_{1\leq i\leq n} \deg_G(a_i x^{e_i})$. Then $\bar{r} = \bar{a}_m \bar{x}^{e_m}$. Since $\bar{a}_m \in A$, we see that $\bar{r} \in A[\bar{x}]$. Q.E.D.

Theorem 3.8. ([5], Thm 1.7(a) and L. 1.8) *Let k be a field of characteristic zero, G a totally ordered abelian group, B a G-graded k-affine integral domain, and $R \subset B$ a k-subalgebra such that B is a localization of R. Let $\deg : R \to G \cup \{-\infty\}$ be the restriction of the degree function on B determined by the grading. Then $\deg(D)$ is defined for every $D \in \mathrm{Der}_k(R)$.*

Remark. Let notations and assumptions be as in Thm 3.8. Then $(R, \deg)$ determines $\mathrm{Gr}(R)$, and Thm 3.8 (together with part 3. of paragraph 3.6) implies:

$$\mathrm{gr}(D) : \mathrm{Gr}(R) \to \mathrm{Gr}(R) \textit{ is defined for every } D \in \mathrm{Der}_k(R).$$

Definition 3.9. We shall say that a 4-tuple (B, R, t, n) is *pericyclic* if B is a ring, R is a subring of B, n is a positive integer, and $t \in B \setminus \{0\}$ is such that $t^n \in R$ and $B = \bigoplus_{0\leq i\leq n-1} Rt^i$.

Remarks 3.10. The notion of a pericyclic 4-tuple is convenient for avoiding repetitions in the text below. The reader should keep in mind that these 4-tuples have the following properties.

(1) If (R, B, t, n) is pericyclic, then the sum $B = \bigoplus_{0\leq i\leq n-1} Rt^i$ can be interpreted as a $\mathbb{Z}_n$-grading, $B = \bigoplus_{g\in\mathbb{Z}_n} B_g$, by choosing a generator ξ of the group $\mathbb{Z}_n$ and defining $B_{i\xi} = Rt^i$ for all i such that $0 \leq i < n$. Then $B_0 = R$, t is homogeneous of degree ξ, and $\mathbb{Z}_n(B) = \mathbb{Z}_n$.

Conversely, if $B = \bigoplus_{g\in\mathbb{Z}_n} B_g$ is a $\mathbb{Z}_n$-graded ring such that $\mathbb{Z}_n(B) = \mathbb{Z}_n$ and $B = B_0[t]$ for some homogeneous $t \in B \setminus \{0\}$, then (B, B_0, t, n) is a pericyclic 4-tuple.

(2) If (R, B, t, n) is a pericyclic 4-tuple and B is a domain, then $B \cong R[T]/(T^n - f)$ as an R-algebra, where T is an indeterminate over R and $f = t^n \in R \setminus \{0\}$.

Conversely, if R is a domain, $f \in R \setminus \{0\}$, and $n \geq 1$ are such that $B = R[T]/(T^n - f)$ is a domain, then $(B, \pi(R), \pi(T), n)$ is a pericyclic 4-tuple, where $\pi : R[T] \to B$ is the canonical homomorphism of the quotient ring.

Definition 3.11. Let (B, R, t, n) be a pericyclic 4-tuple, where B is a domain of characteristic zero. Then $\mathrm{Der}(B, R, t, n)$ denotes the set of all $D \in \mathrm{Der}(B)$ satisfying the following equivalent conditions (where we define $I = \{0, 1, \ldots, n-1\}$):

(i) for each $i \in I$, there exists $j \in I$ such that $D\big(Rt^i\big) \subseteq Rt^j$;

(ii) D is homogeneous with respect to the $\mathbb{Z}_n$-grading defined in *Remarks 3.10*(1).

In addition, $\mathrm{LND}(B,R,t,n) = \mathrm{LND}(B) \cap \mathrm{Der}(B,R,t,n)$.

Note that, in the definition above, (ii) clearly implies (i); the converse is an exercise left to the reader.

Theorem 3.12. ([7], Thm. 3.1) *Consider a pericyclic* 4*-tuple* (B,R,t,n) *where* B *is a domain of characteristic zero. Given* $D \in \mathrm{LND}(B,R,t,n)$*, there exist* $\delta \in \mathrm{LND}(R)$ *and* $m \in \mathbb{Z}$*,* $0 \le m \le n-1$*, such that the following conditions hold.*

(a) *D is the quasi-extension of δ given by $D|_R = t^m\delta$.*

(b) *$Dt, \delta f \in \ker\delta = R \cap \ker D$, where $f = t^n \in R$.*

(c) *If $Dt \neq 0$, then $m = n-1$ and $\ker D = \ker\delta$.*

§4. A Result on Homogeneous Prime Local Slices

An earlier version of the following result was given by the first author in [3, Prop. 2.1].

Proposition 4.1. *Let $B = \bigoplus_{g\in G} B_g$ be a G-graded $\mathbb{Q}$-domain, where G is an abelian group. Suppose that $D \in \mathrm{LND}(B)$ is G-homogeneous and that p is a G-homogeneous prime element of B satisfying $\deg(p) \notin G(\ker D)$. Then the following hold.*

(a) *p is a local slice of D that divides every G-homogeneous local slice of D.*

(b) *The ideal $\ker(D) \cap D(B)$ of $\ker(D)$ is the principal ideal generated by $D(p)$.*

(c) *If B is a UFD and G is torsion-free then $B = B_H[p]$, where $H = G(\ker D)$.*

Proof. Let $A = \ker D$ and $H = G(A)$, and note that $\deg(p) \notin H$ and $p \notin A$ (in particular $D \neq 0$). Since D is homogeneous, locally nilpotent and nonzero, there exists a homogeneous local slice of D, i.e., a homogeneous element $v \in B$ such that $Dv \neq 0$ and $D^2v = 0$. Let $\alpha = Dv \in A \setminus \{0\}$, then $B[\alpha^{-1}] = A[\alpha^{-1}][v] = A[\alpha^{-1}]^{[1]}$. In particular there exists an integer $m \ge 0$ such that $\alpha^m p \in A[v]$. So we may write $\alpha^m p = \sum_{i\in I} a_i v^i$, where I is a nonempty finite subset of $\mathbb{N}$ and, for each $i \in I$, $a_i \in A \setminus \{0\}$ is homogeneous and $\deg(a_i v^i) = \deg(\alpha^m p)$. Since $\deg(a_i), \deg(\alpha) \in H$ and $\deg(p) \notin H$, we get $i\deg(v) \notin H$ and hence $i > 0$ for each $i \in I$. It follows that

$$\alpha^m p = bv, \tag{2}$$

for some $b \in B \setminus \{0\}$. We claim that $p \mid v$ (in B). Indeed, if $p \nmid v$ then $p \mid b$ (because p is a prime element of B), so equation (2) gives $v \mid \alpha^m$, so $v \in A$ since A is factorially closed in B, contradicting $Dv \neq 0$. Hence $p \mid v$ (showing that p divides every homogeneous local slice of D). Now $\deg_D(p) \leq \deg_D(v) = 1$ and $p \notin A$ imply that p is a local slice of D. Assertion (a) is proved, and (b) immediately follows.

To prove (c), we assume that G is torsion-free; then G is orderable and consequently the grading of B has the following property: *every factor of a nonzero homogeneous element of B is itself homogeneous* (observe in particular that all units of B are homogeneous). We also assume that B is a UFD, so every nonzero homogeneous element of B is a finite product of homogeneous elements that are either units or prime. So it will suffice to show that all units and all homogeneous prime elements belong to $B_H[p]$. Each unit belongs to A and hence to B_H. If q is a homogeneous prime such that $q \notin B_H$ then $\deg q \notin H$, so, by applying (a) to q, we obtain that q is a homogeneous local slice that divides every homogeneous local slice. This implies that p, q are associates, and since we already know that all units are in B_H we obtain $q \in B_H[p]$. Q.E.D.

Corollary 4.2. *Let B be a G-graded $\mathbb{Q}$-domain, where G is an abelian group. Let $n \geq 2$ and suppose that $x_1, \dots, x_n$ are homogeneous prime elements of B satisfying:*

(i) *for each subset $I \subset \{1, \dots, n\}$ of cardinality $n-1$, $\big\{ \deg(x_i) \mid i \in I \big\}$ generates $G(B)$;*

(ii) *for any choice of distinct $i, j \in \{1, \dots, n\}$, x_i, x_j are not associates.*

Then $G(\ker D) = G(B)$ for all G-homogeneous $D \in \mathrm{LND}(B)$.

Proof. If $G(\ker D) \neq G(B)$ then there exist distinct $i, j \in \{1, \dots, n\}$ such that $\deg(x_i), \deg(x_j) \notin G(\ker D)$; then *Prop. 4.1(a)* implies that $x_i \mid x_j$ and $x_j \mid x_i$, a contradiction. Q.E.D.

The next two results follow immediately from *Cor. 4.2*.

Corollary 4.3. *Let $B = k[x_1, \dots, x_n] = k^{[n]}$ where k is a field of characteristic zero and $n \geq 2$, and let B be graded by an abelian group G in such a way that each x_i is G-homogeneous and*

for each subset $I \subset \{1, \dots, n\}$ of cardinality $n-1$,

$$\big\{ \deg(x_i) \mid i \in I \big\} \text{ generates } G(B).$$

Then $G(\ker D) = G(B)$ for all G-homogeneous $D \in \mathrm{LND}(B)$.

Corollary 4.4. *Suppose that k is a field of characteristic zero. Given the integer $n \geq 4$, let $k[X_1, ..., X_n] = k^{[n]}$ and define*

$$B = k[X_1, \dots, X_n]/(X_1^{a_1} + \cdots + X_n^{a_n})$$

where $a_1, \dots, a_n \in \mathbb{Z}$ are positive. Let $\pi : k[X_1, ..., X_n] \to B$ be the standard surjection and set $x_i = \pi(X_i)$, $1 \leq i \leq n$. Define a $\mathbb{Z}$-grading on B by declaring that x_i is homogeneous of degree L/a_i, $1 \leq i \leq n$, where $L = \mathrm{LCM}(a_1, \dots, a_n)$. Assume that the following property holds.

> *For every prime number p, every positive integer e and every $i \in \{1, \dots, n\}$, if $p^e \mid a_i$ then $p^e \mid a_j$ for some $j \in \{1, \dots, n\} \setminus \{i\}$.*

Then $\mathbb{Z}(\ker D) = \mathbb{Z}(B) = \mathbb{Z}$ for all $\mathbb{Z}$-homogeneous $D \in \mathrm{LND}(B)$.

Proof. To prove *Cor. 4.4*, first note that the hypothesis that $n \geq 4$ implies that each x_i is a prime element in B. Also, if i and j are distinct, then x_i and x_j are not associates. Finally, for any prime number p, let e be maximal such that p^e divides the least common multiple of $a_1, \dots, a_n$. Then, by the hypothesis, it divides at least two distinct integers a_i and a_j. Thus p divides neither the degree of x_i nor the degree of x_j. Thus the result follows from *Cor. 4.2*. Q.E.D.

§5. G-Critical Elements

Definition 5.1. Let G be an abelian group and let B be a G-graded ring. A nonzero homogeneous element x of B is *G-critical* if it satisfies the following equivalent conditions:

(i) there exists a graded subring A of B such that $G(A) \neq G(B)$ and $B = A[x]$,

(ii) there exists a subgroup H of $G(B)$ such that $\deg x \notin H$ and $B = B_H[x]$,

(iii) there exists a subgroup H of G such that $\deg x \notin H$ and $B = B_H[x]$.

Definition 5.2. Let B be a G-graded ring, where G is an abelian group, and let x be a G-critical element of B. Any subgroup $H \subset G(B)$ satisfying

$$B = B_H[x], \quad \deg_G(x) \notin H, \quad \text{and} \quad p \deg_G(x) \in H \text{ for some prime } p$$

is called a *G-critical subgroup associated to x.*

Remark 5.3. The prime number p that appears in the above definition satisfies $G(B)/H \cong \mathbb{Z}_p$, and so is uniquely determined by H.

(Proof: Let $\tau = \deg_G(x)$; then $B = B_H[x]$ implies that $G(B) = H + \langle\tau\rangle$, and we also have $\tau \notin H$ and $p\tau \in H$.)

Theorem 5.4. *Let $B = \bigoplus_{g\in G} B_g$ be a G-graded ring, where G is an abelian group, and let x be a G-critical element of B.*

(a) *If H_0 is any subgroup of $G(B)$ satisfying $\deg(x) \notin H_0$ and $B = B_{H_0}[x]$, then there exists a G-critical subgroup H associated to x and satisfying $H_0 \subset H$.*

In particular, there exists a G-critical subgroup associated to x.

(b) *Let H be a G-critical subgroup associated to x and consider the prime number $p = |G(B)/H|$. Then (B, B_H, x, p) is a pericyclic 4-tuple.*

Remark 5.5. Assertion (b) entails in particular that $B=\bigoplus_{i=0}^{p-1} B_H\cdot x^i$. It follows from *Remarks 3.10* that (i) this direct sum can be interpreted as a $\mathbb{Z}_p$-grading of B over B_H in which x is homogeneous of nonzero degree, and (ii) if B is a domain then $B \cong B_H[T]/(T^p - x^p)$ as B_H-algebras, where T is an indeterminate over B_H.

Proof of Thm 5.4. Let $\tau = \deg_G x \in G$. There exists a subgroup H_0 of $G(B)$ satisfying $\deg(x) \notin H_0$ and $B = B_{H_0}[x]$. Consider any such H_0 and note that $G(B)$ is generated by $H_0 \cup \{\tau\}$ (because $B = B_{H_0}[x]$). As $\Gamma = \{n \in \mathbb{Z} \mid n\tau \in H_0\}$ is a proper subgroup of $\mathbb{Z}$, we may choose a prime number $p > 0$ such that $\Gamma \subset p\mathbb{Z}$. Define the subgroup $H \subset G(B)$ by

$$H = H_0 + \langle p\tau\rangle$$

and note that $p\tau \in H$ and that $\tau \notin H$ (since $\Gamma \subset p\mathbb{Z}$). Since $H_0 \subset H$, we have $B_{H_0} \subset B_H$ and hence $B = B_H[x]$. So H is a G-critical subgroup associated to x and satisfying $H_0 \subset H$, proving part (a).

To prove (b), consider a G-critical subgroup H associated to x, let $p = |G(B)/H|$, and let $\bar\tau = \tau + H \in G(B)/H$. Then $G(B)/H$ is generated by $\bar\tau$ and $\bar\tau$ has order p. It is clear that a $G(B)/H$-grading $B = \oplus_{k\in G(B)/H} S_k$ of B is defined by setting $S_k = \oplus_{g\in\pi^{-1}(k)} B_g$ for all $k \in G(B)/H$, where $\pi : G(B) \to G(B)/H$ is the canonical homomorphism of the quotient group. We have $B_H x^i \subseteq S_{i\bar\tau}$ for $i = 0, \dots, p-1$, and $B = \sum_{i=0}^{p-1} B_H \cdot x^i$ because $B = B_H[x]$ and $x^p \in B_H$; so $B_H x^i = S_{i\bar\tau}$ for all $i = 0, \dots, p-1$ and consequently $B = \bigoplus_{i=0}^{p-1} B_H \cdot x^i$. Assertion (b) follows. Q.E.D.

Lemma 5.6. *Let G be an abelian group, and let B be a G-graded integral domain. Assume that there exists a pair $x, y \in B$ of G-critical elements such that x and y are not associates and $x, y \notin B^*$. Let $H \subset$*

$G(B)$ be a G-critical subgroup associated to x, and let $K \subset G(B)$ be a G-critical subgroup associated to y. Then $x \in B_K$ and $y \in B_H$.

Proof. By Thm 5.4(b), there exist prime integers p and q such that the decompositions

$$B = \bigoplus_{0 \le i < p} B_H \cdot x^i = \bigoplus_{0 \le j < q} B_K \cdot y^j$$

define gradings of B by $\mathbb{Z}_p$ and $\mathbb{Z}_q$, respectively. Let $a \in B_H$, $b \in B_K$ and non-negative integers i, j be such that $y = ax^i$ and $x = by^j$. By observation 3.5(11), a and b are G-homogeneous. We have:

$$y = a(by^j)^i = ab^i y^{ij}$$

Since $y \notin B^*$, it follows that either $ij = 0$, or $i = j = 1$ and $ab = 1$. However, the latter case cannot occur, since x and y are not associates. Therefore, $ij = 0$.

Suppose that $i = 0$. Then $y = a \notin B^*$. Since b is G-homogeneous, there exist $c \in B_H$ and $l \ge 0$ with $b = cx^l$. So $x = a^j c x^l$, which implies that either $l = 0$ or $l = 1$ (since $x \notin B^*$). If $l = 0$, then $x \in B_H$, a contradiction. Therefore, $l = 1$ and $a^j c = 1$, which implies that $j = 0$ (since $y = a \notin B^*$).

On the other hand, if $j = 0$, then the symmetric argument shows $i = 0$. Therefore, $i = j = 0$ in all cases. Q.E.D.

If the hypotheses of *Lemma 5.6* are weakened to allow x to be a unit, then one can find examples where x and y are not associates (so y is not a unit) and $x \in B_K$, but $y \notin B_H$. For example, let $B = \mathbb{C}[x, x^{-1}, y]$ be graded by $G = \mathbb{Z}^2$, where x and y are homogeneous of degrees $(1, 0)$ and $(0, 1)$, respectively. Let H be the subgroup generated by $(2, 0)$ and $(1, 1)$ and K be the subgroup generated by $(1, 0)$ and $(0, 2)$. Then:

$$G/H \cong \mathbb{Z}/2\mathbb{Z}, G/K \cong \mathbb{Z}/2\mathbb{Z}, B_H = \mathbb{C}[x^2, x^{-2}, xy], B_K = \mathbb{C}[x, x^{-1}, y^2]$$

We have that $B_H[x] = B_K[y] = B$, and $y \notin B_H$. Note that, in this example, xy, which is an associate of y, belongs to B_H.

§6. LNDs of G-Graded Domains

The following is a consequence of *Thm. 5.4* and *Thm. 3.12*.

Corollary 6.1. *Let G be an abelian group and B a G-graded integral domain of characteristic zero. Suppose that $D \in \mathrm{LND}(B)$ is G-homogeneous and $x \in B$ is a G-critical element with $Dx \neq 0$. For every*

G-critical subgroup $H \subset G(B)$ associated to x, we have:

$$D(x) \in \ker D \subset B_H \quad \textit{and} \quad D\big(B_H\big) \subset B_H x^{p-1}$$

where $p = |G(B)/H|$.

Proof. By *Thm 5.4*, (B, B_H, x, p) is a pericyclic 4-tuple. The result follows by applying *Thm. 3.12* to (B, B_H, x, p): part (b) implies that $D(x) \in \ker(D)$ and part (c) implies that $D\big(B_H\big) \subset B_H x^{p-1}$ and $\ker(D) \subset B_H$. Q.E.D.

Theorem 6.2. *Let G be an abelian group, let B be a G-graded integral domain containing $\mathbb{Z}$, and let $D \in \mathrm{LND}(B)$ be G-homogeneous.*

(a) *For every G-critical element $x \in B$, $D^2 x = 0$.*

(b) *For every pair $x, y \in B$ of non-associated G-critical elements, either $Dx = 0$ or $Dy = 0$.*

Proof. Assertion (a) follows from *Cor. 6.1*.

Note that assertion (b) is trivial if x or y is a unit. Assume that x, y are not units and let H (resp. K) be a G-critical subgroup associated to x (resp. to y). Let $p = |G(B)/H|$ and $q = |G(B)/K|$. *Lemma 5.6* implies that $x \in B_K$ and $y \in B_H$. Now if $Dx \neq 0$, then *Cor. 6.1* gives that $D(B_H) \subset B_H x^{p-1}$. Thus x^{p-1} divides Dy. Also, by part (a), $D^2 y = 0$, and thus Dy is in the kernel of D. Since x^{p-1} is not in the kernel of D, the only possibility is that $Dy = 0$. This proves part (b). Q.E.D.

The next result generalizes Thm. 5.1 of [7] and Thm. 2.6.3 of [8].

Corollary 6.3. *Let G be an abelian group, $B = \oplus_{g \in G} B_g$ a G-graded integral domain containing $\mathbb{Z}$, where B is finitely generated as a B_0-algebra. Then we can write*

$$B = B_0[x_1, ..., x_n]$$

where $x_i \neq 0$ is homogeneous of degree $d_i \neq 0$ for each i. Let $H_i = \langle d_1, \dots, \hat{d_i}, \dots, d_n \rangle$ for each $i \in \{1, \dots, n\}$. Then for every G-homogeneous $D \in \mathrm{LND}(B)$ the following conditions hold.

(a) *For each $i \in \{1, \dots, n\}$ such that $H_i \neq G(B)$, x_i is a G-critical element of B and $D^2 x_i = 0$.*

(b) *For every choice of distinct $i, j \in \{1, \dots, n\}$ such that $H_i \neq G(B)$ and $H_j \neq G(B)$, we have $Dx_i = 0$ or $Dx_j = 0$.*

Observe that, in the statement of this corollary, the subgroup $\langle d_1, \dots, d_n \rangle$ of G generated by $d_1, \dots, d_n$ is equal to $G(B)$.

Proof. If $i \in \{1, \dots, n\}$ is such that $H_i \neq G(B)$ then $\deg x_i \notin H_i$ and $B = B_{H_i}[x_i]$, so x_i is a G-critical element of B and assertion (a) follows from *Thm 6.2(a)*.

Consider distinct $i, j \in \{1, \dots, n\}$ such that $H_i \neq G(B)$ and $H_j \neq G(B)$. Then x_i and x_j are G-critical elements of B. By *Thm 5.4*(a), we may choose a G-critical subgroup H associated to x_i and satisfying $H_i \subset H$, and a G-critical subgroup K associated to x_j and satisfying $H_j \subset K$. Then $d_i \in H_j \subset K$ and $d_j \in H_i \subset H$, so

$$d_i \in K \setminus H \quad \text{and} \quad d_j \in H \setminus K. \tag{3}$$

By contradiction, assume that $Dx_i \neq 0$ and $Dx_j \neq 0$. Then *Cor 6.1* implies that $\ker D \subset B_H \cap B_K$. If $x_j = ux_i$ for some $u \in B^*$, then $u \in \ker D$, so $d_j - d_i = \deg_G u \in H \cap K$, which contradicts (3); so x_i and x_j are not associates. Now *Thm 6.2(b)* implies that $Dx_i = 0$ or $Dx_j = 0$, a contradiction. Q.E.D.

Example 6.4. (See [7], Example 4.1) Let k be a field of characteristic zero. Given the integer $n \geq 1$, suppose that $e_i \in \mathbb{Z}$, $0 \leq i \leq n$, is a sequence with $e_i \geq 2$ for each i and $\gcd(e_i, e_j) = 1$ for each pair $i \neq j$. Set $e = e_0 \cdots e_n$. Define a $\mathbb{Z}$-grading over k of the ring

$$B = k[x_0, ..., x_n]/(x_0^{e_0} + \cdots + x_n^{e_n})$$

such that, for each i, x_i is homogeneous and $\deg x_i = e/e_i$. Suppose that $D \in \mathrm{LND}(B)$ is homogeneous. By *Cor. 6.3(b)*, it follows that $Dx_i = 0$ or $Dx_j = 0$ for every pair $i \neq j$. But then $Dx_i \neq 0$ for at most one i, $0 \leq i \leq n$. Since x_i is algebraically dependent on $x_0, ..., \hat{x_i}, ..., x_n$, it must also be the case that $Dx_i = 0$. Therefore, $D = 0$, and this implies that B is rigid (by *Thm 3.8*, if there exists a nonzero lnd of B then there exists a nonzero *homogeneous* lnd of B). The ring B is the coordinate ring of a Pham-Brieskorn variety; see *Section 7*.

Rings without Homogeneous LNDs

As noted earlier, when B is an affine k-domain over a field k of characteristic zero, and B is graded by a totally ordered abelian group, then any nonzero $D \in \mathrm{LND}(B)$ induces a nonzero homogeneous element $\bar{D} \in \mathrm{LND}(B)$. However, for a grading which uses an abelian group G that is not orderable (see 3.2), it can happen that there are no nonzero G-homogeneous elements of $\mathrm{LND}(B)$, even though $\mathrm{LND}(B) \neq \{0\}$. We give the following two examples, with the same non-rigid ring B in the two cases.

Proposition 6.5. *If* $B = \mathbb{C}[x,y,z]/(x^2+y^2+z^3)$*, then* $\mathrm{LND}(B) \neq \{0\}$*. Define a* $\mathbb{Z}_2$*-grading of* B *by declaring that* x, y, z*, are homogeneous with* $\deg x = \deg z = 0$ *and* $\deg y = 1$*. If* $D \in \mathrm{LND}(B)$ *is* $\mathbb{Z}_2$*-homogeneous, then* $D = 0$*.*

Proof. B is the coordinate ring of a Danielewski surface over $\mathbb{C}$, and the locally nilpotent derivations of B are well-known; see for example [4, 9]. In particular, $\mathrm{LND}(B) \neq \{0\}$, and the kernel of any nonzero locally nilpotent derivation of B is isomorphic to a univariate polynomial ring $\mathbb{C}^{[1]}$.

Let $G = \mathbb{Z}_2$. By contradiction, suppose that $D \in \mathrm{LND}(B)$ is G-homogeneous and nonzero. Then $\ker D = \mathbb{C}[t]$ for G-homogeneous $t \in B$. If $H = \{0\} \subset G$, then:

$$B_H = \mathbb{C}[x,z] = \mathbb{C}^{[2]} \quad \text{and} \quad B = B_H[y] = B_H \oplus B_H \cdot y$$

So y is a G-critical element and H is a G-critical subgroup associated to y.

If $Dy = 0$ then $D(x^2+z^3) = 0$, so *Lemma 2.1(c)* implies $Dx = Dz = 0$ and hence $D = 0$, contradicting our assumption. So $Dy \neq 0$.

Let $S \subset \mathbb{C}[t]$ be the set of nonzero elements of $\mathbb{C}[t]$ and $F = \mathbb{C}(t)$. Since $Dy \neq 0$, *Cor. 6.1* gives $S \subset \mathbb{C}[t] \subset B_H$, and we have:

$$S^{-1}B = S^{-1}B_H \oplus S^{-1}B_H \cdot y$$

On the other hand, we see from equation (1) (just before 2.1) that:

$$S^{-1}B = F[y] = F[y^2] \oplus F[y^2] \cdot y$$

Since $S^{-1}B_H \supset F[y^2]$ it follows that $S^{-1}B_H = F[y^2]$, and consequently:

$$\begin{aligned} \mathbb{C}(x,z) = \mathrm{frac}(B_H) &= \mathrm{frac}(S^{-1}B_H) = \mathrm{frac}(F[y^2]) \\ &= F(y^2) = F(x^2+z^3) = \mathbb{C}(t, x^2+z^3) \end{aligned}$$

But this is a contradiction, since x^2+z^3 is not a field generator of $\mathbb{C}(x,z) = \mathbb{C}^{(2)}$. Q.E.D.

Proposition 6.6. *Let* $B = \mathbb{C}[x,y,z]/(x^2+y^2+z^3)$ *and* $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$*. Define a* G*-grading of* B *by declaring that* x, y, z*, are homogeneous with* $\deg x = (1,0,0)$*,* $\deg y = (0,1,0)$ *and* $\deg z = (0,0,1)$*. If* $D \in \mathrm{LND}(B)$ *is* G*-homogeneous, then* $D = 0$*.*

Proof. Since $\langle \deg(y), \deg(z) \rangle$, $\langle \deg(x), \deg(z) \rangle$ and $\langle \deg(x), \deg(y) \rangle$ are proper subgroups of $G(B)$, *Cor. 6.3* implies that if $D \in \mathrm{LND}(B)$ is G-homogeneous then at least two of Dx, Dy, Dz equal 0. Since each of x, y, z is algebraic over the subring generated by the other two, it follows that $Dx = Dy = Dz = 0$. Q.E.D.

Note that we could also have proved *Prop. 6.6* by using *Prop. 6.5*: Since the $\mathbb{Z}_2$-grading of B used in *Prop. 6.5* is induced by the projection $G \to \mathbb{Z}_2$ mapping (a, b, c) to b, any D which is G-homogeneous is also $\mathbb{Z}_2$-homogeneous, hence $D = 0$.

§7. Pham-Brieskorn Threefolds

In this section, we assume that the field k is algebraically closed and of characteristic zero.

Given $n \geq 1$ and positive integers a_i, $0 \leq i \leq n$, the corresponding *Pham-Brieskorn variety* over k is the hypersurface $V \subset \mathbb{A}_k^{n+1}$ defined by:

$$x_0^{a_0} + x_1^{a_1} + \cdots + x_n^{a_n} = 0$$

These hypersurfaces have been of interest in topology and algebraic geometry for decades. See [11] for a survey of these varieties.

It is known that V is non-rigid in the following two cases: $a_i = 1$ for some i, or $a_i, a_j \leq 2$ for some pair i, j with $i \neq j$. For the Pham-Brieskorn surface S defined by $x_0^{a_0} + x_1^{a_1} + x_2^{a_2} = 0$, these are the only cases in which S is non-rigid. See [7], Thm. 7.1. The expectation is that this condition extends to all Pham-Brieskorn varieties, i.e., that V is non-rigid if and only if $a_i = 1$ for some i or $a_i, a_j \leq 2$ for some pair i, j with $i \neq j$. The following result confirms this expectation for certain families of Pham-Brieskorn threefolds. [1,2]

Theorem 7.1. ([7], Thm 8.1; [2], Cor. 1.9) *Given integers $a, b, c, d \geq 2$ such that at most one of a, b, c, d is equal to 2, the ring*

$$B = k[x_0, x_1, x_2, x_3]/(x_0^a + x_1^b + x_2^c + x_3^d)$$

is rigid in each of the following cases.

(a) $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} + \dfrac{1}{d} \leq \dfrac{1}{2}$

(b) $\gcd(abc, d) = 1$

(c) $(a, b, c, d) = (a, 3, 3, 3)$

[1]Part (a) is slightly more general than what is presented in [7], where the case $\min\{a, b, c, d\} \geq 8$ is given. This more general case is implied by Theorem 8 of [1], arguing as we did in the proof of [7], Thm 8.1(b).

[2]The case $a = 3$ in part (c) is especially important. It is due to Cheltsov, Park and Won, whose proof uses geometric techniques quite different from the algebraic approach taken herein.

(d) $(a,b,c,d) = (2,b,c,d)$, *where* $b,c,d \geq 3$*;* b *is even;* $\gcd(b,c) \geq 3$*; and* $\gcd(d, \mathrm{lcm}(b,c)) = 2$*.*

Our next result uses *Cor. 6.3* to capture many cases not included in this theorem, for example, $(a,b,c,d) = (4,9,6,6)$.

Let B be as in *Thm. 7.1*, with $a,b,c,d \geq 2$ and at most one of a,b,c,d equal to 2. Set $L = \mathrm{lcm}(a,b,c,d)$, and define a $\mathbb{Z}$-grading of B over k by declaring that each x_i is homogeneous and:

$$\deg_{\mathbb{Z}} x_0 = L/a \ , \ \deg_{\mathbb{Z}} x_1 = L/b \ , \ \deg_{\mathbb{Z}} x_2 = L/c \ , \ \deg_{\mathbb{Z}} x_3 = L/d$$

If $d_i = \deg x_i$ for each i, then $\gcd(d_0, d_1, d_2, d_3) = 1$.

Theorem 7.2. *Let B be the $\mathbb{Z}$-graded Pham-Brieskorn threefold defined as above.*

(a) *Suppose that there exists a prime integer p and a positive integer r such that:*

$$a \equiv 0 \,(\mathrm{mod}\, p^r) \quad \text{and} \quad b,c,d \not\equiv 0 \,(\mathrm{mod}\, p^r)$$

Then $D^2 x_0 = 0$ for every homogeneous $D \in \mathrm{LND}(B)$.

(b) *Suppose that there exist distinct primes p and q, and positive integers r and s, such that:*

$$a \equiv 0 \,(\mathrm{mod}\, p^r) \quad \text{and} \quad b,c,d \not\equiv 0 \,(\mathrm{mod}\, p^r)$$

and

$$b \equiv 0 \,(\mathrm{mod}\, q^s) \quad \text{and} \quad a,c,d \not\equiv 0 \,(\mathrm{mod}\, q^s)$$

Then B is rigid.

Proof. Part (a) Under the given hypotheses, we see that p divides $\gcd(d_1, d_2, d_3)$. Therefore, the conclusion follows from *Cor. 6.3(a)*.

Part (b): Note that, because B is affine and has a $\mathbb{Z}$-grading, it will suffice to show that $D = 0$ for every homogeneous $D \in \mathrm{LND}(B)$. Let homogeneous $D \in \mathrm{LND}(B)$ be given.

Under the given hypotheses, we see that p divides $\gcd(d_1, d_2, d_3)$ and q divides $\gcd(d_0, d_2, d_3)$. Therefore, *Cor. 6.3(b)* implies that $Dx_0 = 0$ or $Dx_1 = 0$.

Assume that $Dx_0 = 0$. If $D \neq 0$, then we may assume that D is irreducible. In this case, the quotient derivation $\bar{D} \in \mathrm{LND}(B/x_0B)$ is nonzero, where:

$$B/x_0B = k[x_1, x_2, x_3]/(x_1^b + x_2^c + x_3^d)$$

But this is a contradiction, since B/x_0B is a rigid ring. By symmetry, we also arrive at a contradiction if we assume that $Dx_1 = 0$ and $D \neq 0$.

Therefore, $D = 0$ in either case. Q.E.D.

Although these results cover multiple cases, there are still many open cases for small exponents. For example, are the rings

$k[x_0, x_1, x_2, x_3]/(x_0^2+x_1^3+x_2^3+x_3^4)$ and $k[x_0, x_1, x_2, x_3]/(x_0^2+x_1^3+x_2^3+x_3^6)$

rigid? Note that, for the first ring, *Thm. 7.2(a)* implies that $D^2x_3 = 0$ for any $\mathbb{Z}$-homogeneous LND D.

In attempting to settle some of these cases, the following type of question comes up: The ring

$$B = k[x_0, x_1, x_2, x_3]/(x_0^2 + x_1^2 + x_2^3 + x_3^3)$$

is non-rigid. Does there exist $D \in \mathrm{LND}(B)$ with $\deg_D x_0 = 1$?

§8. The Russell Cubic Threefold

The coordinate ring R_0 of the Russell cubic threefold over the field k is defined by:

$$R_0 = k[x, y, z, t]/(x + x^2y + z^2 + t^3)$$

In this section, we use *Cor. 6.3* to show that $\mathcal{D}(R_0) \neq R_0$ under the assumption that the characteristic of k is zero (recall that the Derksen invariant $\mathcal{D}(R)$ of a k-algebra R is the subalgebra of R generated by the set $\{ f \in R \mid D(f) = 0 \text{ for some } D \in \mathrm{LND}(R) \setminus \{0\} \}$). Our proof of this fact can be viewed as a more concise version of Makar-Limanov's original proof given in [10]. The main implication is that $R_0 \neq k^{[3]}$. Since the proof is identical for all Koras-Russell threefolds of the first kind, we treat this more general case:

$$R = k[x, y, z, t]/(x + x^dy + z^k + t^l)$$

where $d, k, l \geq 2$ and k and l are relatively prime.

Lemma 8.1. *Let k, l and d be integers greater than or equal to two, and suppose that k and l are relatively prime. Let Q be the ring:*

$$Q = k[x, y, z, t]/(x^dy + z^k + t^l).$$

Let $G = \mathbb{Z}^2$ and put a G-grading on Q by declaring that x, y, z, t are G-homogeneous and:

$$\deg_G(x, y, z, t) = ((-1, 0), (d, -kl), (0, -l), (0, -k)).$$

Then for every nonzero G*-homogeneous* $D \in \mathrm{LND}(Q)$*, either* $\ker D = k[x,z]$ *or* $\ker D = k[x,t]$*.*

Proof. Since $\langle (d,-kl),(0,-l),(0,-k)\rangle$, $\langle (-1,0),(d,-kl),(0,-k)\rangle$ and $\langle (-1,0),(d,-kl),(0,-l)\rangle$ are proper subgroups of $G(Q) = \mathbb{Z}^2$, *Cor. 6.3(b)* implies that at least two of Dx, Dz, Dt must be 0.

Suppose that $Dz = Dt = 0$. Then $D(x^d y) = 0$ implies $D = 0$. Therefore, if $D \neq 0$, then either $Dx = Dz = 0$ or $Dx = Dt = 0$. Since there exist $D_1, D_2 \in \mathrm{LND}(Q)$ with $\ker D_1 = k[x,z]$ and $\ker D_2 = k[x,t]$, it follows that either $\ker D = k[x,z]$ or $\ker D = k[x,t]$. Q.E.D.

Theorem 8.2. *Let* R *be the ring*

$$R = k[x,y,z,t]/(x + x^d y + z^k + t^l)$$

where $d,k,l \geq 2$ *and* k *and* l *are relatively prime. Then* $\mathcal{D}(R) = k[x,z,t]$*.*

Proof. Let $G = \mathbb{Z}^2$ and define a total order $\preceq$ on G by lexicographical ordering.

Let $A = k[x,z,t] \cong k^{[3]}$ and define a G-grading on the ring $B = k[x,x^{-1},z,t]$ such that x,z,t are homogeneous and:

$$\deg_G(x,z,t) = ((-1,0),(0,-l),(0,-k))$$

Then $A = \left\{ f \in B \mid \deg(f) \preceq (0,0) \right\}$. The degree function $\deg_G$ on B restricts to R, where $\deg_G y = (d,-lk)$. According to *Lemma 3.7*, we have:

$$\mathrm{Gr}(R) = \bar{R} = A[\bar{y}] = k[x,z,t,\bar{y}]$$

Since $y = -x^{-d}(x + z^k + t^l)$ in B, we see that $\bar{y} = -x^{-d}(z^k + t^l)$, i.e., $x^d\bar{y} + z^k + t^l = 0$.

Let $D \in \mathrm{LND}(R)$ be a nonzero locally nilpotent derivation of R, and let $f \in \ker D$ be given. By *Thm. 3.8*, the induced homogeneous derivation $\bar{D}$ of $\bar{R}$ is nonzero and locally nilpotent. Therefore, by *Lemma 8.1*, $\ker \bar{D} \subset A$. Since $\bar{f} \in \ker \bar{D}$, we see that:

$$\deg_G f = \deg_G \bar{f} \preceq (0,0)$$

Therefore, $f \in A$, and $\mathcal{D}(R) \subset A$.

The reverse inclusion follows from the observation that the derivations

$$kz^{k-1}\frac{\partial}{\partial y} - x^d\frac{\partial}{\partial z} \quad \text{and} \quad lt^{l-1}\frac{\partial}{\partial y} - x^d\frac{\partial}{\partial t}$$

are elements of $\mathrm{LND}(R)$.

Q.E.D.

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