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Dipoles and pixie dust

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Abstract.

Every closed subset of the Riemann sphere can be approximated in the Hausdorff topology by the Julia set of a rational map.

§1. Dipoles

In [6], Kathryn Lindsey gives an elegant construction to show that any Jordan curve in the complex plane can be approximated in the Hausdorff topology by Julia sets of polynomials (another such construction was given subsequently by Oleg Ivrii [4]), and further, that any finite collection of disjoint Jordan domains can be approximated by the basins of attraction of a rational map. The proof depends on an interpolation result due to Curtiss [1]. Some other relevant references are [2], [3] and [5].

In this note, we give a direct geometric (and computation-free) proof of the following result:

Theorem 1.1. *Any closed nonempty subset of the Riemann sphere can be approximated in the Hausdorff topology by the Julia set of a rational map.*

It is worth pointing out that Lindsey's construction allows one to control the topology of the approximating Julia set, so our method does not supersede her work.

In the theory of electomagnetism, a *dipole* refers to a pair of oppositely charged particles with charges of equal magnitude. The electric field of a dipole falls off at a rate of $1/r^3$ because of the approximate cancellation of the fields at distances large compared to the separation of the particles (unlike the usual inverse square law for a system with nonzero net charge). The point is that the dipole is effectively electrically neutral on large scales.

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By analogy, we define a *dipole* to be a degree 1 rational function $(z-a)/(z-b)$ with a zero and pole at distinct non-zero a, b, for which $|a-b|$ is "small". The dipole is uniformly close to 1 outside a small disk containing a and b. We now explain how to use dipoles to build designer Julia sets.

Let's suppose we want to build a rational function whose Julia set approximates X, a closed subset of the Riemann sphere. If X is equal to the entire sphere, then it is known that X is already the Julia set of the Lattès map $z \to (z^2+1)^2/4z(z^2-1)$; otherwise we can suppose (by applying a Möbius transformation) that X is disjoint from ∞.

We will show for any fixed point p, and any positive ϵ that we can find a rational map whose Julia set is ϵ close to $X \cup p$ in the (spherical) Hausdorff metric. Taking p very close to X and taking $\epsilon \to 0$ will prove the theorem.

The first step is to pick some finite collection Y of discrete points ("pixels") which approximate X in the Hausdorff metric to within $\epsilon/3$.

Next, consider the map $f : z \to C(z-p)^N + p$ for some big fixed integer N and a suitable constant C. For any fixed N, if we pick a suitable C the Julia set $J(f)$ of f will be an arbitrarily small circle centered at p. So for any N we can choose C with $J(f)$ contained in the $\epsilon/3$-neighborhood of p. Note that f has p and ∞ as superattractive fixed points, so we can suppose X is contained in the basin of ∞.

Now for any positive δ we can build a new rational function g_δ by multiplying f by a dipole centered at each point in Y, whose zero and pole are δ-close. As $\delta \to 0$ the Julia sets of g_δ converge uniformly in the Hausdorff topology to $\hat{Y}$, which is equal to the union of Y together with its preimages under iterates of f, together with the Julia set of f. To see this, observe that functions g_δ converge uniformly to f on compact subsets of the complement of Y, while the presence of the dipole near each point $y \in Y$ guarantees a point in the Julia set of each g_δ near y and (thus) near its preimages under f. If N is sufficiently big, the preimages of Y under iterates of f are all as close as we desire to the Julia set of f which itself is close to the point p. Thus for such an N, the Julia sets of g_δ converge to a set $\hat{Y}$ which is ϵ-close to $X \cup p$.

§2. Acknowledgments

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