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Geometric inflexibility of hyperbolic cone-manifolds

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Abstract.

We prove 3-dimensional hyperbolic cone-manifolds are geometrically inflexible: a cone-deformation of a hyperbolic cone-manifold determines a bi-Lipschitz diffeomorphism between initial and terminal manifolds in the deformation in the complement of a standard tubular neighborhood of the cone-locus whose pointwise bi-Lipschitz constant decays exponentially in the distance from the cone-singularity. Estimates at points in the thin part are controlled by similar estimates on the complex lengths of short curves.

§1. Introduction

In our earlier paper [BB2], we developed an explicit realization of the qualitative idea that deformations at infinity of hyperbolic 3-manifolds have effect on the internal geometry that decays exponentially fast with the depth in the convex core. This notion of geometric inflexibility, suggested by McMullen and exhibited in the restrictive setting of injectivity bounds, proved sufficiently robust to give a new analytic proof of Thurston's *Double-Limit Theorem* for iteration of pseudo-Anosov mapping classes and a new "stand-alone" proof of the hyperbolization theorem for 3-manifolds that fiber over the circle with pseudo-Anosov monodromy.

This paper extends our inflexibility results to the setting where the change in the geometry is the result of a "cone-deformation," in which the cone-angle at a closed, geodesic singular locus is changed while the conformal structure at infinity is held fixed. Our results control the best pointwise bi-Lipschitz constant outside of a tubular neighborhood

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of the singular locus in domain and range. The optimal bi-Lipschitz constant decays to 1 exponentially fast with the distance from the tubular neighborhood of the singular locus.

Theorem 1.1. *Given $\alpha_0, L, K, \epsilon > 0$ and $B > 1$ there exists an $R > 0$ and a $d > 0$ such that the following holds. Let (M, g_α) be a geometrically finite hyperbolic cone-manifold with all cone-angles $\alpha < \alpha_0$ and the length of the singular locus is at most L. Then there exists a one-parameter family of geometrically finite hyperbolic cone-manifolds (M, g_t) defined for $t \in [0, \alpha]$ so that each component of the singular locus of (M, g_t) has cone-angle t and the conformal boundary is the same as the conformal boundary of (M, g_α) so that the following holds:*

(1) *If U_α is the R-tubular neighborhood of the singular locus in (M, g_α) and U_t is a tubular neighborhood of the singular locus in (M, g_t) such that* $\operatorname{area}(\partial U_t) = \operatorname{area}(\partial U_\alpha)$, *then there exists B-bi-Lipschitz diffeomorphisms*

$$\phi_t : M_\alpha \backslash U_\alpha \to M_t \backslash U_t$$

such that ϕ_t is the identity map on M in the ϵ-thick part of M_α.

(2) *If p is in the ϵ-thick part of (M, g_α) then the pointwise bi-Lipschitz constant of the maps*

$$\phi_t : M_\alpha \to M_t$$

satisfies

$$\log \operatorname{bilip}(\phi_t, p) \leq C_1 e^{-C_2 d_\alpha(p, M_\alpha \backslash U_\alpha)}$$

where the constants C_1 and C_2 depend on the α_0, L, K, ϵ and B.

Similar techniques control the behavior of the complex lengths of short geodesics in the manifold under the cone deformation, and once again the distortion decays exponentially in the distance from the tubular neighborhood of the cone-singularity.

Theorem 1.2. *Let $M_t = (M, g_t)$ be the one parameter family of geometrically finite cone-manifolds given by Theorem 1.1. Let γ be an essential simple closed curve in M and γ_t its geodesic representatives in M_t. Assume that $\ell_\alpha(\gamma) < \ell$ for some $\ell > 0$. Then there exists constants C_1 and C_2 depending on the constants α_0, L, K, ϵ and B from Theorem 1.1 and on ℓ such that the following holds:*

(1) *If* $\epsilon \le \ell_\alpha(\gamma) \le \ell$ *then*

$$\left|\log \frac{\ell_t(\gamma)}{\ell_\alpha(\gamma)}\right| \le C_1 e^{-C_2 d_\alpha(\gamma_\alpha, U_\alpha)}.$$

(2) *If* $\ell_\alpha(\gamma) \le \epsilon/B$ *then*

$$\left|\log \frac{\ell_t(\gamma)}{\ell_\alpha(\gamma)}\right| \le C_1 e^{-C_2 d_\alpha(U_\epsilon^\alpha(\gamma), U_\alpha)}.$$

The idea that complete hyperbolic 3-manifolds are increasingly inflexible as one takes basepoints deeper and deeper in the convex core is a natural outgrowth of Mostow and Sullivan rigidity. McMullen made this qualitative notion precise in the presence of injectivity bounds in [Mc], but his method made strong use of geometric limit arguments possible only in the complete setting. Our original argument for the complete case in [BB2] shows this pointwise exponential decay for points outside the thin part, which is an optimal result (each tubular thin part is controlled using the complex lengths of the core geodesics).

Here, the cone-deformation version generalizes the cone-rigidity theorems of Hodgson-Kerckhoff [HK1] and the second author, and enhances the bi-Lipschitz metric control away from the cone locus obtained in [BB1] to give explicit decay estimates in terms of the distance from a standard tubular neighborhood of the cone locus.

Inflexibility and ending laminations. Geometric inflexibility has provided a range of new tools to analyze the geometry and deformation theory of hyperbolic 3-manifolds. A key application of the work in the present paper will be an approach to the geometric classification of finitely generated Kleinian groups via their *ending laminations*, combinatorial invariants that are naturally associated to infinite volume geometric 'ends' of the convex core of a hyperbolic 3-manifold with finitely generated fundamental group, which we briefly describe. The ending lamination records the asymptotics of simple closed curves on a surface cutting of an end of a hyperbolic 3-manifold, whose geodesic representatives in the 3-manifold have an *a priori* length bound (and therefore must exit the end of the convex core).

A Theorem of Minsky [Min] guarantees that for any hyperbolic 3-manifold M in a Bers slice B_Y with the ending lamination λ there is an almost canonical (up to bounded choice at each stage) sequence of pants decompositions $P_n \to \lambda$ that arises with uniformly bounded total length $\ell_M(P_n) < L$ in M.

The notion of *grafting* [Brm3, BB1] may be employed with a covering argument similar to that of [BS], to allow us to drill the curves in P_n

in M with a cone-deformation that sends the cone angle to zero. This produces a *maximal cusp* $C_n \in B_Y$, and as the pants decompositions P_n move deeper and deeper into the convex core, the inflexibility theorem guarantees that the cone-deformations deform the geometry of a compact core $\mathcal{M}$ in a manner that decays with the distance of the geodesic representatives of the curves in P_n from $\mathcal{M}$. It follows that C_n limits to M, and as P_n depend only on λ, the lamination λ determines M. We take up this approach in [BBES].

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§2. Deformations

Let (M, g_t) be a one-parameter family of Riemannian manifolds. The time zero derivative η of this family of metrics is given by the formula

$$\frac{dg_t(v, w)}{dt}\,|_{t=0} = 2g(\eta(v), w).$$

This derivative is a symmetric tensor of type $(1, 1)$. We can define a pointwise norm of η by fixing an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and setting

$$\|\eta\|^2 = \sum_i g(\eta(e_i), \eta(e_i)).$$

As the L^2-norm bounds the sup norm we have the inequality

$$\|\eta(v)\| \leq \|\eta\|\|v\|$$

which will be useful in controlling the change in geometry throughout the flow.

In this paper we will be interested in the case when (M, g_t) is a family of hyperbolic 3-manifolds and the derivative η is a *harmonic strain field*. Loosely speaking, η is harmonic if it locally minimizes the L^2-norm. Here is a precise definition. Every point p in M has a chart U and a smooth family of maps $\phi_t : U \to \mathbb{H}^3$ such that on U the hyperbolic metric g_t is the ϕ_t-pullback of the hyperbolic metric on $\mathbb{H}^3$. For each $q \in U$, $\phi_t(q)$ is a smooth path in $\mathbb{H}^3$ and the time zero tangent vector of this path defines a vector field on $\phi_0(U)$. Let v be the ϕ_0-pullback of this vector to U. If D is the covariant derivative for g then $\eta = \operatorname{sym} Dv$. The infinitesimal change in volume is measured by the trace of $\operatorname{sym} DV$, the *divergence* of the vector field. The traceless, symmetric part, $\operatorname{sym}_0 Dv$

is the *strain* of v and it measures the infinitesimal change in conformal structure. A vector field is *harmonic* if it satisfies the equation

$$D^* Dv + 2v = 0$$

where D^* is the formal adjoint of D. While it might be more natural to define v to be harmonic when $D^* Dv = 0$ we include the 0-th order term as we want infinitesimal isometries to be harmonic. This extra term comes from the fact that the Ricci curvature of hyperbolic space is -2. We then say that η is a *harmonic strain field* if $\eta = \operatorname{sym} Dv$ where v is a divergence free, harmonic vector field.

On a hyperbolic 3-manifold with boundary, a global bound on the norm of a harmonic strain field leads to exponential decay, in distance from the boundary, of the pointwise norm in the thick part of the manifold. Before we state the main results from [BB2] we make some more definitions. Let $M_t = (M, g_t)$ be a one-parameter family of hyperbolic 3-manifolds. Then $M_t^{\geq \epsilon}$ is the ϵ-thick part of M_t, those points where the injectivity radius is $\geq \epsilon$. Here is a key structural theorem from [BB2].

Theorem 2.1. *Let g_t be a one-parameter family of hyperbolic metrics on a 3-manifold M with $t \in [a, b]$. Let η_t be the time t derivative of the metrics g_t and let N_t be a family of submanifolds of M such that η_t is a harmonic strain field on N_t. Also assume that*

$$\int_{N_t} \|\eta_t\|^2 + \|D_t \eta_t\|^2 \leq K^2$$

for some $K > 0$. Let p be a point in M such that for all $t \in [a, b]$, p is in $M_t^{\geq \epsilon}$ and

$$d_{M_t}(p, M \backslash N_t) \geq d$$

where $d > \epsilon$. Then

$$\log \operatorname{bilip}(\Phi_t, p) \leq (t - a) K A(\epsilon) e^{-d}$$

where Φ_t is the identity map from M_a to M_t,

$$A(\epsilon) = \frac{3e^{\epsilon}\sqrt{2\operatorname{vol}(B)}}{4\pi f(\epsilon)}$$

and

$$f(\epsilon) = \cosh(\epsilon)\sin(\sqrt{2}\epsilon) - \sqrt{2}\sinh(R)\cos(\sqrt{2}R).$$

In the thin part of the manifold, close to a short geodesic, we lack this level of control. Instead, we control the length of the short geodesic where the change will decay exponentially in the depth of certain tubular neighborhoods of the short curves. More specifically, given a short geodesic γ we will measure the depth of a tubular neighborhood U of γ where the area of ∂U is bounded below.

Theorem 2.2. *Let g_t be a one-parameter family of hyperbolic metrics on a 3-manifold M with $t \in [a,b]$. Let η_t be the time t derivative of the metrics g_t and let N_t be a family of submanifolds of M such that η_t is a harmonic strain field on N_t. Also assume that*

$$\int_{N_t} \|\eta_t\|^2 + \|D_t\eta_t\|^2 \leq K^2$$

for some $K > 0$. Let γ_t be the geodesics representative on (M, g_t) of a closed curve γ and let $\ell_\gamma(t)$ be the length of γ.

(1) *Assume that γ_t is in $M_t^{\geq\epsilon}$ for all $t \in [a,b]$, and that*

$$d_{M_t}(\gamma_t, M \backslash N_t) \geq d.$$

Then

$$\left|\log \frac{\ell_\gamma(b)}{\ell_\gamma(a)}\right| \leq \sqrt{2/3}A(\epsilon)(b-a)Ke^{-d}.$$

(2) *Assume γ_t has a tubular neighborhood U_t of radius $\geq R$ and the area of ∂U_t is $\geq B$. Also assume that*

$$d_{M_t}(U_t, M \backslash N_t) \geq d$$

for all $t \in [a,b]$. Then

$$\left|\log \frac{\ell_\gamma(b)}{\ell_\gamma(a)}\right| \leq \frac{C(R)(b-a)Ke^{-d}}{\sqrt{B}}$$

where

$$1/C(R) = 2\tanh R\left(2 + \frac{1}{\cosh^2 R}\right).$$

The Margulis lemma provides an embedded tubular neighborhood about a sufficiently short geodesic in a hyperbolic 3-manifold: there is a ε_0 such that if γ is a primitive closed geodesic and length$(\gamma) < \epsilon < \varepsilon_0$ then the component of the ϵ-thin part that contains γ will be a tubular neighborhood which we denote $U_\epsilon(\gamma)$. This is the *ϵ-Margulis tube* about γ and the area of $\partial U_\epsilon(\gamma)$ is bounded below by $\pi\epsilon^2$. In particular we can apply (2) of the above theorem to such tubes. In this paper, we will be studying singular hyperbolic manifolds so we will need to adapt this slightly to find our tubes.

§3. Cone-manifolds

We now turn our attention to deformations of hyperbolic cone-manifolds. We begin with a definition. We let $\tilde{\mathbb{H}}^3$ be the set

$$\{(r,\theta,z)|r>0,\theta,z\in\mathbb{R}\}$$

with the incomplete Riemannian metric

$$dr^2+\sinh^2 rd\theta^2+\cosh^2 rdz^2.$$

Then $\tilde{\mathbb{H}}^3$ is isometric to the lift to the universal cover of the hyperbolic metric on $\mathbb{H}^3\setminus\ell$ where ℓ is a complete geodesic. For each $\alpha>0$, let $\mathbb{H}^3_\alpha$ be the metric completion of the quotient of $\tilde{\mathbb{H}}^3$ under the isometry $(r,\theta,z)\mapsto(r,\theta+\alpha,z)$. Note that $\mathbb{H}^3_\alpha$ is a topological ball. Let N be a compact 3-manifold with boundary and g a complete metric on the interior of N. The metric g is a *hyperbolic cone-metric* if every point in the interior of N has a neighborhood isometric to a neighborhood of a point in $\mathbb{H}^3_\alpha$ for some $\alpha>0$. The pair (N,h) is a *hyperbolic cone-manifold.* Let $\mathcal{C}$ be the subset of N where the metric h is singular. Then $\mathcal{C}$ will be a collection of isolated simple curves in N. In this paper we will assume that $\mathcal{C}$ is compact which implies that it is a finite collection of disjoint simple closed curves.

Let c be a component of $\mathcal{C}$. Then there is a unique $\alpha>0$ such that each point p in c has a neighborhood isometric to the neighborhood of a singular point in $\mathbb{H}^3_\alpha$. This α is the cone-angle of the component c.

Recall that $\mathbb{H}^3$ is naturally compactified by $\widehat{\mathbb{C}}$. The union is a closed 3-ball and isometries of $\mathbb{H}^3$ extend continuously to conformal automorphisms of $\widehat{\mathbb{C}}$. Let $\partial_0 N$ be the components of ∂N that are not tori. Then (N,g) is a *geometrically finite cone-manifold* if each point p in $\partial_0 N$ has a neighborhood V in N and a chart $\phi: V\to\bar{\mathbb{H}}^3$ such that ϕ restricted to $V\cap\mathrm{int}(N)$ is an isometry and ϕ restricted to $V\cap\partial N$ is a map into $\partial\bar{\mathbb{H}}^3=\widehat{\mathbb{C}}$. Note that the restriction of the charts to $\partial_0 N$ defines an atlas for a conformal structure on $\partial_0 N$. In fact, as we will be important in the next section, this conformal atlas determines a *complex projective structure* on $\partial_0 N$.

Theorem 3.1. *Given $\alpha_0, L, K, \epsilon>0$ and $B>1$ there exists an $R>0$ and a $d>0$ such that the following holds. Let $M_\alpha=(M,g_\alpha)$ be a geometrically finite hyperbolic cone-manifold with all cone-angles $\alpha<\alpha_0$, each component of the singular locus has an embedded tubular neighborhood of radius R and the length of the singular locus is at most L. Then there exists a one-parameter family of geometrically finite* *hyperbolic cone-manifolds* $M_t = (M, g_t)$ *defined for* $t \in [0, \alpha]$ *with the following properties:*

(1) *Each component of the singular locus of* M_t *has cone-angle* t *and the conformal boundary is the same as the conformal boundary of* M_α.

(2) *The derivative* η_t *of* g_t *is a family of harmonic strain fields outside of a radius* $\sinh^{-1}(1/\sqrt{2})$ *neighborhood of the singular locus.*

(3) *Let* U_α *be the* R*-tubular neighborhood of the singular locus in* M_α *and let* U_t *be a tubular neighborhood of the singular locus in* M_t *such that* $\mathrm{area}(\partial U_t) = \mathrm{area}(\partial U_\alpha)$. *Then*

$$\int_{M_t \backslash U_t} \|\eta_t\|^2 + \|D_t \eta_t\|^2 \leq K^2.$$

(4) *There exists* B*-bi-Lipschitz diffeomorphisms* $\phi_t : M_\alpha \backslash U_\alpha \to M_t \backslash U_t$ *such that* ϕ_t *is the identity map on* M *in the* ϵ*-thick part of* M_α.

(5) *If* $p \in (M_\alpha \backslash U_\alpha)^{\geq \epsilon}$ *then* $p \in (M_t \backslash U_t)^{\geq \epsilon / B}$ *and*

$$d_t(p, U_t) \geq \frac{d_\alpha(p, U_\alpha)}{B}.$$

(6) *If* γ *is a closed curve in* M *then*

$$d_t(\gamma_t, U_t) \geq \frac{d_\alpha(\gamma_\alpha, U_\alpha)}{B} - d.$$

(7) *If* γ *is a closed curve in* M *with* $\ell_\alpha(\gamma) < \epsilon/B$ *then*

$$d_t(U^t_\epsilon(\gamma), U_t) \geq \frac{d_\alpha(U^\alpha_\epsilon(\gamma), U_\alpha)}{B} - d.$$

Proof. Statements (1)-(4) are proven in [Brm2] (see Theorem 5.3 and its proof). When the singular locus is sufficiently short this was proven in [Brm1, BB1] building on Hodgson and Kerckhoff's foundational work on deformations of hyperbolic cone-manifolds in [HK1, HK2, HK3].

Statement (5) follows directly from (4). Statements (6) and (7) are more difficult. To prove them we need to modify the metrics g_α and g_t in U_α and U_t so that they are complete metrics of pinched negative curvature and by then extending the map ϕ_t to a bi-Lipschitz map for these new metrics.

The construction of such metrics is straightforward: they are doubly warped products using cylindrical coordinates. Given an $r_0 > 0$ define a metric on $\mathbb{R}^3$ by

$$dr^2 + f_{r_0}(r)^2 d\theta^2 + g_{r_0}(r)^2 dz^2$$

where $f_{r_0}(r)$ and $g_{r_0}(r)$ are convex functions with $f_{r_0}(r) = \sinh r$ and $g_{r_0}(r) = \cosh r$ for $r \in [r_0/2, r_0]$ and $f_{r_0}(r) = g_{r_0}(r) = \frac{1}{2}e^r$ for $r \leq r_0/4$. We can also assume that $\sinh r \leq f_{r_0}(t) \leq \frac{1}{2}e^r$ and $\frac{1}{2}e^r \leq g_{r_0}(r) \leq \cosh r$. When $r \geq r_0/2$ or $r \leq r_0/4$ then this metric is hyperbolic. For $r \in (r_0/4, r_0/2)$ the sectional curvature will be pinched within δ of -1 where δ only depends on r_0 and $\delta \to 0$ as $r_0 \to \infty$. Details of this calculation can be found in Section 1.2 of [Koj] where the construction is attributed to Kerckhoff.

The map $(r, \theta, z) \mapsto (r, \theta + x, z + y)$ is an isometry in this metric. If we take the quotient of the set of points with $r \in (-\infty, r_0]$ by isometries $(r, \theta, z) \mapsto (r, \theta + t, z)$ and $(r, \theta + x, z + \ell)$ we get a complete metric on $T^2 \times (-\infty, r_0]$. If $r_0 = R_t$ is the tube radius of U_t and $\ell + \imath x$ is the complex length of the singular locus of (M, g_t) then the $R_t/2$-neighborhood of the boundary is isometric to the $R_t/2$-neighborhood of ∂U_t. We then define g'_t on U_t by replacing the original metric with the above metric. Since the two metrics agree in a collar neighborhood of ∂U_t the metric g'_t is smooth and g'_t is a complete metric on M with sectional curvature within δ of -1.

We now construct a bi-Lipschitz diffeomorphism $\phi'_t : (M, g'_\alpha) \to (M, g'_t)$ by extending the map ϕ_t from (4). The original map ϕ_t restricted to ∂U_α is a B-bi-Lipschitz diffeomorphism from ∂U_α to ∂U_t. This map can then be extended to a map on (U_α, g'_α) in the obvious way. Namely there are nearest point projections of (U_α, g'_α) and (U_t, g'_t) onto ∂U_α and ∂U_t respectively. Then on U_α, ϕ'_t is the unique map that commutes with these projections and that takes a point distance r from ∂U_α to a point distance r from ∂U_t. We need to calculate the bi-Lipschitz constant of this map.

To do so we make a few observations. First the functions $f_R(r)$ and $g_R(r)$ converge uniformly to $\frac{1}{2}e^r$ as $R \to \infty$. Second we note that by construction the derivative of the map is an isometry in the r-direction. For a vector v tangent to the tori of fixed r-coordinate a direction calculation shows that

$$\frac{1}{B}\frac{f_{R_t}(r')}{f_{R_t}(R_t)}\frac{g_{R_\alpha}(R_\alpha)}{g_{R_\alpha}(r)}\|v\| \leq \|\left(\phi'_t\right)_* v\| \leq B\frac{g_{R_t}(r')}{g_{R_t}(R_t)}\frac{f_{R_\alpha}(R_\alpha)}{f_{R_\alpha}(r)}\|v\|$$

where $R_\alpha - r = R_t - r'$. Therefore the map is B'-bi-Lipschitz where B' is the maximum of the factor on the right side of the inequality and the inverse of the factor on left side of the inequality. Since the functions $f_R(r)$ and $g_R(r)$ converge uniformly to $\frac{1}{2}e^r$, the quotients $f_R(r_1)/f_R(r_0)$ and $g_R(r_1)/g_R(r_0)$ converge uniformly to $e^{r_1-r_0}$. By Theorem 2.7 of [HK2] the length of the singular locus is an increasing function of t. This implies that R_t is a decreasing function in t and therefore the bi-Lipschitz constant, B', depends only on B and R.

By the Morse Lemma (see e.g. [BH]) the ϕ_t-image of a geodesic is contained in the d-neighborhood of a geodesic where d only depends on B' and the curvature bounds of the modified metric (which we have uniformly controlled). In particular, $\phi_t(\gamma_\alpha)$ is contained in the d-neighborhood of γ_t. Since ϕ_t is B-bi-Lipschitz on $M_\alpha \backslash U_\alpha$ and $\phi_t(U_\alpha) = U_t$ we have $d_t(\phi_t(\gamma_\alpha), U_t) \geq d_\alpha(\gamma_\alpha, U_\alpha)/B$ and therefore $d_t(\gamma_t, U_t) \geq d_\alpha(\gamma_\alpha, U_\alpha)/B - d$ which is (6).

For (7) we choose ϵ such that the $B\epsilon$ is less than than Margulis constant for manifolds with curvature pinched between $-1-\delta$ and $-1+\delta$. Then if $\ell_\alpha(\gamma) < \epsilon/B$ we have that $\ell_t(\gamma) < \epsilon < B\epsilon$ and both $U^t_{B\epsilon}(\gamma)$ and $U^t_{\epsilon/B}(\gamma)$ will be embedded tubular neighborhoods. Furthermore we have $U^t_{\epsilon/B}(\gamma) \subseteq \phi_t(U^\alpha_\epsilon(\gamma)) \subseteq U^t_{B\epsilon}(\gamma)$. By [BM] the width of the collar $U^t_{B\epsilon}(\gamma) - U^t_{\epsilon/B}(\gamma)$ is bounded by a constant that is independent of $\ell_t(\gamma)$. This gives uniform control of the distance between $\phi_t(U^\alpha_\epsilon(\gamma))$ and $U^t_\epsilon(\gamma)$ and then (7) follows in a similar manner as (6). Q.E.D.

We can now prove the bi-Lipschitz inflexibility theorem for cone-manifolds.

Theorem 3.2. *Let $M_t = (M, g_t)$ be the one-parameter family of geometrically finite cone-manifolds given by Theorem 3.1. If p is in the ϵ-thick part of (M, g_α) then the pointwise bi-Lipschitz constant of the maps*

$$\phi_t : M_\alpha \to M_t$$

satisfies

$$\log \operatorname{bilip}(\phi_t, p) \leq C_1 e^{-C_2 d_\alpha(p, U_\alpha)}$$

where the constants C_1 and C_2 depend on the α_0, L, K, ϵ and B as in Theorem 3.1.

Proof. We apply Theorem 2.1 to M_t with $N_t = M_t \backslash U_t$. By (2) of Theorem 3.1 the derivative η_t of M_t is a harmonic strain field on N_t and by (3) we have that

$$\int_{N_t} \|\eta_t\|^2 + \|D_t \eta_t\|^2 \leq K^2.$$

Let $B > 1$ be the bi-Lipschitz constant given by (4) and then, by (5), a point $p \in M_\alpha^{\geq \epsilon}$ will be in $M_t^{\geq \epsilon/B}$ and

$$d_t(p, U_t) \geq d_\alpha(p, U_\alpha)/B.$$

The result then follows from Theorem 2.1 with $C_1 = \alpha K A(\epsilon/B)$ and $C_2 = 1/B$. Q.E.D.

Next we state and prove the length inflexibility statement.

Theorem 3.3. *Let $M_t = (M, g_t)$ be the one parameter family of geometrically finite cone-manifolds given by Theorem 3.1. Let γ be an essential simple closed curve in M and γ_t its geodesic representative in M_t. Assume that $\ell_\alpha(\gamma) < \ell$ for some $\ell > 0$. Then there exists constants C_1 and C_2 depending on the constants α_0, L, K, ϵ and B from Theorem 3.1 and on ℓ such that the following holds.*

(1) *If $\epsilon \leq \ell_\alpha(\gamma) \leq \ell$ then*

$$\left|\log \frac{\ell_t(\gamma)}{\ell_\alpha(\gamma)}\right| \leq C_1 e^{-C_2 d_\alpha(\gamma_\alpha, U_\alpha)}.$$

(2) *If $\ell_\alpha(\gamma) \leq \epsilon/B$ then*

$$\left|\log \frac{\ell_t(\gamma)}{\ell_\alpha(\gamma)}\right| \leq C_1 e^{-C_2 d_\alpha(U_\epsilon^\alpha(\gamma), U_\alpha)}.$$

Proof. As in the proof of Theorem 3.2 we let $N_t = M_t/U_t$ and then by (2) and (3) of Theorem 3.1 the derivative of M_t on N_t is a harmonic strain field η_t with

$$\int_{N_t} \|\eta_t\|^2 + \|D_t \eta_t\|^2 \leq K^2.$$

If $B > 1$ is the bi-Lipschitz constant from (4) then by (6) there is a constant $d > 0$ such that

$$d_t(\gamma_t, U_t) \geq d_\alpha(\gamma_\alpha, U_\alpha)/B - d.$$

The first inequality the follows from (1) of Theorem 2.2 with

$$C_1 = \sqrt{\frac{2}{3}} A\left(\frac{\epsilon}{B}\right) \alpha K e^{-d}$$

and $C_2 = 1/B$.

The second inequality is proved similarly but we use (7) of Theorem 3.1 instead of (6). Q.E.D.

§4. Schwarzian derivatives

As was noted when defining geometrically finite hyperbolic cone-manifolds, the conformal boundary of a hyperbolic cone-manifold also has a projective structure. While the conformal boundary will be fixed throughout the deformations given by Theorem 3.1, the projective structure will vary. The variation in a projective structure is measured by the Schwarzian derivative and in this section we will use our inflexibility theorems to control the size of the Schwarzian derivative.

We very briefly discuss projective structures and the Schwarzian derivatives. For more detail see Section 6 of [BB2]. A projective structure on a surface is $(\mathrm{PSL}_2(\mathbb{C}), \widehat{\mathbb{C}})$-structure; each projective structure has an underlying conformal structure and we let $P(X)$ denote the space of projective structures with conformal structure X. If Σ_0 and Σ_1 are two projective structures in $P(X)$ we let $f\colon \Sigma_0 \to \Sigma_1$ be the conformal map between them. In projective charts, the *Schwarzian derivative* $S(f)$ of f, naturally a quadratic differential on Σ_0, measures the deviation of f from being a Möbius transformation. We use this to define a distance on $P(X)$ by setting

$$d(\Sigma_0, \Sigma_1) = \|S(f)\|_\infty.$$

There is also an infinitesimal version of the Schwarzian derivative: in a projective chart the derivative of a smooth 1-parameter family of projective structures is a conformal vector field. Using the chart this is a vector field v on a domain in $\widehat{\mathbb{C}}$. At each point there is a unique projective vector field that best approximates v. In such a way v defines a map from the domain in $\widehat{\mathbb{C}}$ to $sl_2\mathbb{C}$ the Lie algebra of projective vector fields. The derivative of this map is the *Schwarzian derivative* of the deformation and it naturally identified with a holomorphic quadratic differential on the conformal structure.

Given two projective structures we define the notion of a *projective map* between them in the usual way via charts. For example a round disk in $\widehat{\mathbb{C}}$ inherits a projective structure as a subspace of $\widehat{\mathbb{C}}$. On a arbitrary projective structure Σ a *round disk* is a projective map from a round disk to Σ. Note that we don't assume that this map is an embedding. Every round disk in $\widehat{\mathbb{C}}$ bounds a half space $\mathbb{H}^3$. If Σ is the projective boundary of a hyperbolic 3-manifold M then a round disk in Σ bounds a half space in M if there is an isometry from a half space in $\mathbb{H}^3$ into M that extends to a projective map on the boundary round disk. We will need the following lemma about round disks.

Lemma 4.1. *Let Σ be projective structure with trivial holonomy. The every round disk is embedded.*

Proof. Let $\tilde{\Sigma}$ be the universal cover of Σ. Recall that there is a projective developing map $D : \tilde{\Sigma} \to \widehat{\mathbb{C}}$ and a representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ such that $D \circ \gamma = \rho(\gamma) \circ D$ where the action of γ in the left side of the inequality is by deck transformations. By assumption the holonomy representation ρ is the trivial representation.

Let U be a round disk in $\widehat{\mathbb{C}}$ and $\phi : U \to \Sigma$ projective map. Let $\tilde{\phi} : U \to \tilde{\Sigma}$ be the lift of ϕ. Then $D \circ \tilde{\phi}$ is a projective map of $U \subset \widehat{\mathbb{C}}$ into $\widehat{\mathbb{C}}$. Since $D \circ \tilde{\phi}$ is the restriction of an element of $\mathrm{PSL}_2(\mathbb{C})$ it is an embedding and hence $\tilde{\phi}$ is an embedding. If ϕ is not an embedding then there exists $x, y \in U$ such that $\phi(x) = \phi(y)$. Since $\tilde{\phi}(x) \neq \tilde{\phi}(y)$ there must be a $\gamma \in \pi_1(\Sigma)$ such that $\gamma(\tilde{\phi}(x)) = \tilde{\phi}(y)$. Since $D \circ \gamma(\tilde{\phi}(y)) = \rho(\gamma) \circ D(\tilde{\phi}(y))$ and $\rho(\gamma)$ is the identity we have $D(\tilde{\phi}(x)) = D(\tilde{\phi}(y))$. Since $D \circ \tilde{\phi}$ is injective this is a contradiction and hence ϕ is injective. Q.E.D.

We now state the main inflexibility theorem for Schwarzian derivatives from [BB2].

As the projective structure is at infinity we cannot measure its distance from the cone singularity. Instead we assume that each round disk in the projective structure bounds a half space in the manifold and then measure the distance to the half space.

Theorem 4.2. *Let g_t, $t \in [a, b]$, be a one-parameter family of hyperbolic metrics on the interior of a 3-manifold M with boundary. Let η_t be the time t derivative of the metrics g_t and let N_t be a family of submanifolds of M with compact boundary such that η_t is a harmonic strain field on N_t. Also assume that*

$$\int_{N_t} \|\eta_t\|^2 + \|D_t \eta_t\|^2 \leq K^2$$

for some $K > 0$. Let S be a component of ∂M such that each hyperbolic metric g_t extends to a fixed conformal structure X on S and a family of projective structures Σ_t on S. Assume that at every embedded round disk in Σ_t bounds an embedded half space H in N_t and that

$$d_{M_t}(H, M \backslash N_t) \geq d$$

for some $d > 0$. Then

$$d(\Sigma_a, \Sigma_b) \leq C K e^{-d}$$

where C is a constant depending on the sup-norm of the Schwarzian derivative of the quadratic differential from the unique Fuchsian projective structure with conformal structure X and the injectivity radius of X.

To apply this result we need to know that round disks in the projective boundary of a hyperbolic cone-manifold bound half spaces.

Lemma 4.3. *Let M be the non-singular part of a 3-dimensional hyperbolic cone-manifold. Then every round disk on the projective boundary of M extends to a half-space in M, and if the disk is embedded the half space is embedded.*

Proof. In Lemma 3.3 of [Brm1] it is shown that every embedded round disk extends to an embedded half space so we only need to show that every (possibly immersed) round disk extends to a half space. To do this we would like to apply the lemma to the universal cover $\tilde{M}$ of the non-singular part of the hyperbolic cone-manifold. We first observe that if $\tilde{\Sigma}$ is a component of the projective boundary then its holonomy representation will be trivial so by Lemma 4.1 every round disk in $\tilde{\Sigma}$ will be embedded. On the other hand, $\tilde{M}$ is not a hyperbolic cone-manifold in the sense that is used in the proof of Lemma 3.3 of [Brm1] so we will briefly review the proof to see that it applies in our situation.

A hyperbolic half space $H \subset \mathbb{H}^3$ is foliated by constant curvature planes P_d where P_d is the locus of points distance d from the hyperbolic plane that bounds H. Let $H_d \subset H$ be the union of the P_t with $t > d$. Let U be a round disk in $\tilde{\Sigma}$ whose closure is compact. Using a compactness argument we can extend the round disk to H_d for some large d. We identify H_d with its image in $\tilde{M}$. When $d > 0$ the boundary of H_d is strictly concave so $\tilde{M} \backslash H_d$ is strictly convex and therefore the closure of H_d is embedded in $\tilde{M}$ if $d > 0$. This implies that we can extend the round disk to H_0. Q.E.D.

If Σ is the projective boundary of a hyperbolic cone-manifold M we define its neighborhood $\mathcal{N}(\Sigma)$ to be the union of all half-spaces that are bounded by round disks in Σ. Since two half-spaces in M will intersect if and only if their boundary round disks intersect, disjoint components of the projective boundary will determine disjoint neighborhoods.

Thurston parameterized the space of projective structures on a surface S by the product of the Teichmüller space and the space of measured laminations. In his proof he extends a projective structure to a hyperbolic structure on $\Sigma \times [0, \infty)$ where the boundary is a locally concave pleated surface (or a locally convex pleated surface if it is embedded in a larger manifold). Lemma 4.3 essentially shows that this hyperbolic structure constructed by Thurston is our neighborhood $\mathcal{N}(\Sigma)$. We now state Thurston's result in a form that will be useful to us. For a proof see [KT].

Theorem 4.4 (Thurston)**.** *Each neighborhood $\mathcal{N}(\Sigma)$ is homeomorphic to $\Sigma \times (0, \infty)$. If the singular locus does not intersect the boundary of $\mathcal{N}(\Sigma)$ then the boundary is a locally convex pleated surface.*

Our inflexibility theorems will be vacuous if the singular locus is on the boundary of $\mathcal{N}(\Sigma)$ so we can effectively assume that this is not the case and that the boundary of $\mathcal{N}(\Sigma)$ is a locally convex pleated surface.

The *convex core* of a complete manifold of pinched negative curvature is the smallest convex subset whose inclusion is a homotopy equivalence. As the non-singular part of a cone-manifold is not complete we need to be more careful in how we define the convex core. The following lemma will be essential.

Lemma 4.5. *Let (M, g) be the non-singular part of a 3-dimensional hyperbolic cone-manifold such that the singular locus is contained in $M \backslash \mathcal{N}(\Sigma)$ and let (M, g') be a complete Riemannian metric on M with pinched negative curvature such that $g = g'$ on $\mathcal{N}(\Sigma)$. Then $M \backslash \mathcal{N}(\Sigma)$ is the convex core of (M, g').*

Proof. By Theorem 4.4 the manifold M deformation retracts onto $M \backslash \mathcal{N}(\Sigma)$ so the inclusion of $M \backslash \mathcal{N}(\Sigma)$ into M will be a homotopy equivalence. The boundary of $M \backslash \mathcal{N}(\Sigma)$ will be locally convex in (M, g) and therefore also in (M, g'). This implies that $M \backslash \mathcal{N}(\Sigma)$ is a convex submanifold in (M, g') whose inclusion is a homotopy equivalence and therefore the convex core is contained in $M \backslash \mathcal{N}(\Sigma)$.

Next we show that the pleating locus of the pleated surfaces bounding $M \backslash \mathcal{N}(\Sigma)$ must be contained in the convex core. To see this we first note that any closed geodesic is in the convex core. The pleating locus can be approximated by closed geodesics so it must also be in the convex core.

Finally the join of anything in the convex core will also be in the convex core. Since the join of the pleating locus will contain the pleated surface we have that $\partial(M \backslash \mathcal{N}(\Sigma))$ lies in the convex core so $M \backslash \mathcal{N}(\Sigma)$ lies in the convex core. Q.E.D.

Given this lemma, it is natural to define the convex core of a hyperbolic cone-manifold by $C(M) = M \backslash \mathcal{N}(\Sigma)$. It is possible that the singular locus lies on the boundary of convex core, in which case the above lemma doesn't apply. However, when the singular locus is not deep in the convex core our main result reduces to Theorem 1.3 in [Brm1]. For this definition to be useful we need to know that the image of the convex core under a bi-Lipschitz map will be uniformly close in the Hausdorff metric to the convex core of the image manifold. This will follow from the following proposition which is due to McMullen when the manifold is hyperbolic. The general case requires work of Anderson and Bowditch.

Proposition 4.6. *Given $B > 1$ and $\epsilon \in (0,1)$ there exists $d > 0$ such that the following holds. Let g_0 and g_1 be complete Riemannian metrics on a manifold M with sectional curvatures in $(-1-\epsilon, -1+\epsilon)$ and let $\phi : (M, g_0) \to (M, g_1)$ be B-bi-Lipschitz. Then then Hausdorff distance between $C(M, g_1)$ and $\phi(C(M, g_0))$ is less than d.*

The final piece we need to prove our Schwarzian inflexibility theorem is a version of the deformation theorem for cone-manifolds that controls the distance from the standard neighborhood of the singular locus to the convex core boundary. It will be convenient to restate part of the original deformation theorem, Theorem 3.1.

Theorem 4.7. *Given $\alpha_0, L, K > 0$ and $B > 1$ there exists an $R > 0$ such that the following holds. Let (M, g_α) be a geometrically finite hyperbolic cone-manifold with all cone-angles $\alpha < \alpha_0$ and with singular locus of length at most L. Then there exists a one-parameter family of geometrically finite hyperbolic cone-manifolds (M, g_t) defined for $t \in [0, \alpha]$ with the following properties:*

(1) *All cone angles of (M, g_t) are t and the conformal boundary is the same as the conformal boundary of (M, g_α).*

(2) *The derivative η_t of g_t is a family of harmonic strain fields outside of a radius $\sinh^{-1} 1/\sqrt{2}$ neighborhood of the singular locus.*

(3) *Let U_α be the R-tubular neighborhood of the singular locus in (M, g_α) and let U_t be a tubular neighborhood of the singular locus in (M, g_t) such that $\operatorname{area}(\partial U_t) = \operatorname{area}(\partial U_\alpha)$. Then*

$$\int_{M_t \setminus U_t} \|\eta_t\|^2 + \|D_t \eta_t\|^2 \leq K.$$

(4) *Let X be a component of the conformal boundary and Σ_t the projective structure on X induced by (M, g_t). Then*

$$d(U_t, \mathcal{N}(\Sigma_t)) \geq d(U_\alpha, \mathcal{N}(\Sigma_\alpha))/B - d.$$

Proof. Except for (4) this is exactly the same Theorem 3.1. To prove (4) we would like to apply Proposition 4.6 but since our metrics are incomplete we cannot do so directly. We will use the same trick that we used in the proof of Theorem 3.1 and replace the metrics g_α and g_t with complete metrics of pinched negative curvature, g'_α and g'_t and then use the extended B-bi-Lipschitz diffeomorphism ϕ'_t from (M, g'_α) to (M, g'_t). We then apply Proposition 4.6 which shows that

$$Bd(U_t, M\backslash C(M, g'_t)) + d \geq d(U_\alpha, M\backslash C(M, g'_\alpha)).$$

Note that we can assume that U_α is contained in $C(M, g_\alpha)$ for otherwise (4) is vacuous. The inequality then follows from Lemma 4.5. Q.E.D.

We can now apply Theorems 4.2 and 4.7 to get our Schwarzian inflexibility theorem for cone-manifolds.

Theorem 4.8. *Given $\alpha_0, L, K > 0$ and $B > 1$ there exists an $R > 0$ such that the following holds. Let (M, g_α) be a geometrically finite hyperbolic cone-manifold with all cone-angles $\alpha < \alpha_0$, singular locus of length at most L and tube radius of the singular locus at least R. Let $M_t = (M, g_t)$ be the one-parameter family of geometrically finite cone-manifolds given by Theorem 4.7. Let Σ_t be a component of the projective boundary of the M_t with underlying conformal structure X. Then*

$$d(\Sigma_\alpha, \Sigma_t) \leq CKe^{-d(U_\alpha, \mathcal{N}(\Sigma_\alpha))/B - d}$$

where U_α is the tubular neighborhood of the singular locus of radius R_0 and C is a constant depending on $\|\Sigma_\alpha\|_F = d(\Sigma_\alpha, \Sigma_F)$ and the injectivity radius of X, where Σ_F is the unique Fuchsian projective structure with underlying conformal structure X.

Proof. We apply Theorem 4.2 to M_t where the convex cores $C(M_t)$ play the role of the submanifolds N_t. Every half space H bounding a round disk in Σ_t will be contained in $\mathcal{N}(\Sigma_t)$ so by (4) of Theorem 4.7 there exists $d > 0$ such that

$$d(U_t, \mathcal{N}(\Sigma_t)) \geq d(U_\alpha, \mathcal{N}(\Sigma_\alpha))/B - d.$$

The theorem then follows from (3) of Theorem 4.7 and Theorem 4.2. Q.E.D.

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