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One-ended subgroups of mapping class groups

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Abstract.

Suppose we have a one-ended finitely presented group with a purely loxodromic action on a Gromov hyperbolic space satisfying an acylindricity condition. We show that, given a finite generating set, there is an automorphism of the group, and some point in the space which is moved a bounded distance by each of the images of the generators under the automorphism. Here the bound depends only on the group, generating set, and constants of hyperbolicity and acylindricity. With results from elsewhere, this implies that, up to conjugacy, there can only be finitely many purely pseudoanosov subgroups of a mapping class group that are isomorphic to a given one-ended finitely presented group.

§1. Introduction

Let Σ be a closed orientable surface, and write $\mathrm{Map}(\Sigma)$ for its mapping class group — the group of self-homeomorphisms up to homotopy. The Nielsen-Thurston classification partitions the non-trivial elements of $\mathrm{Map}(\Sigma)$ into finite order, reducible, and pseudoanosov, the last being the "generic" case. A subgroup of $\mathrm{Map}(\Sigma)$ is *purely pseudoanosov* if every non-trivial element is pseudoanosov. Such a subgroup is torsion-free. We shall say that a group is *indecomposable* if it does not split as a free product. It is a theorem of Stallings that a non-cyclic torsion-free finitely generated group is indecomposable if and only if it is one-ended. It is an open question as to whether $\mathrm{Map}(\Sigma)$ can contain any one-ended finitely generated purely pseudoanosov subgroup. Indeed the only purely pseudoanosov subgroups known at present are all free. (See the surveys [Re] and [Mo].)

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In this paper, we give an independent proof of the result in [DaF] that there can be only finitely many conjugacy classes of finitely presented one-ended subgroups of a given isomorphism type. More precisely:

Theorem 1.1. *Let* Σ *be a closed surface, and* $\mathrm{Map}(\Sigma)$ *its mapping class group. Let* Γ *be a finitely presented one-ended group. Then there is a finite collection,* $\Gamma_1, \ldots, \Gamma_n$*, of subgroups of* $\mathrm{Map}(\Sigma)$ *such that if* $\Gamma_0 \leq \mathrm{Map}(\Sigma)$ *is a purely pseudoanosov subgroup isomorphic to* Γ*, then there is some* $g \in \mathrm{Map}(\Sigma)$ *and some* $i \in \{1, \ldots, n\}$ *such that* $\Gamma_0 = g\Gamma_i g^{-1}$*.*

Both our proof, and that of [DaF], make use of Proposition 8.1 of [Bo3]. The aim is to show that the hypotheses of that result hold in general. The methods we employ here are rather different from those of [DaF]. They are based on a general result about acylindrical actions on a hyperbolic space, namely Theorem 1.2 below. Acylindrical actions have recently been much studied (see for example, [O]). We therefore hope that this result may be of some independent interest. First, we give a few more definitions.

We will write $\mathrm{Aut}(\Gamma)$, $\mathrm{Inn}(\Gamma)$ and $\mathrm{Out}(\Gamma)$, respectively for the groups of automorphisms, inner automorphisms and outer automorphisms of a group Γ. If Γ is one-ended and finitely presented, then one can define a normal subgroup, $\mathrm{Mod}(\Gamma) \lhd \mathrm{Out}(\Gamma)$, which we call the *modular group* (cf. [RiS1]). It can be defined as the subgroup of $\mathrm{Out}(\Gamma)$ generated by Dehn twists arising from splittings of Γ over infinite cyclic subgroups. (It is tied up with the JSJ splitting of Γ, and is discussed further in Section 4.) We remark that for a hyperbolic group, $\mathrm{Mod}(\Gamma)$ is finite index in $\mathrm{Out}(\Gamma)$, though this need not be the case in general. The preimage of $\mathrm{Mod}(\Gamma)$ in $\mathrm{Aut}(\Gamma)$ is called the "internal automorphism group", $\mathrm{Int}(\Gamma)$, in [RiS1]. In other words $\mathrm{Inn}(\Gamma) \lhd \mathrm{Int}(\Gamma) \lhd \mathrm{Aut}(\Gamma)$, and $\mathrm{Mod}(\Gamma) = \mathrm{Int}(\Gamma)/\mathrm{Inn}(\Gamma)$.

Suppose that $\mathcal{G}$ is a Gromov hyperbolic graph. The following notion is used in [Bo2] (in the context of curve graphs). It generalises Sela's notion of acylindrical actions on simplicial trees. The terminology arises from the theory of 3-manifolds.

Definition. We say that an action of a group Γ on $\mathcal{G}$ is *acylindrical* if, given any $r \geq 0$, there exist R, N such that if $x, y \in \mathcal{G}$ with $d(x, y) \geq R$, then there are at most N elements $g \in \Gamma$ with both $d(x, gx) \leq r$ and $d(y, gy) \leq r$.

Less formally, this says that only boundedly many elements of Γ move a long geodesic a short distance.

In this paper, we will show:

Theorem 1.2. *Let Γ be a finitely presented one-ended group, and let $A \subseteq \Gamma$ be a finite subset. Suppose that Γ admits a purely loxodromic acylindrical action on a k-hyperbolic graph $\mathcal{G}$. Then there is some $x \in \mathcal{G}$, and some $\theta \in \mathrm{Mod}(\Gamma)$, such that for all $g \in A$, $d(x, \theta(g)x) \leq K$, where K depends only on Γ, A, k and the acylindricity parameters.*

In fact, we could take $\mathcal{G}$ to be any Gromov hyperbolic space, though we will deal with a graph here to avoid technical details.

In the case where Γ is a closed orientable surface group, Theorem 1.2 is proven in [Ba], and used in [Bo3]. We shall elaborate on arguments from [Ba] for a proof in the general case. A similar statement to Theorem 1.1 for uniformly locally finite graphs is given in [RiS1]. Indeed the arguments as given there might be adaptable, though we shall phrase things a little differently here.

As we will explain in Section 2, this verifies the hypotheses of Proposition 8.1 of [Bo3], thereby proving Theorem 1.1. Indeed, we only need $\theta \in \mathrm{Out}(\Gamma)$ for this. Using the above, we can formulate a strengthening of Theorem 1.2. We define a *purely pseudoanosov* homomorphism from a group Γ to $\mathrm{Map}(\Sigma)$ as one which sends every non-trivial element to a pseudoanosov — i.e. injective with purely pseudoanosov image. Theorem 1.1 can then be rephrased by saying that there are only finitely many purely pseudoanosov homomorphisms up to precomposition by an element of $\mathrm{Out}(\Gamma)$ and postcompostion by an element of $\mathrm{Inn}(\mathrm{Map}(\Sigma))$. (Note that it makes sense to refer to $\mathrm{Out}(\Gamma)$ rather than $\mathrm{Aut}(\Gamma)$ here, since any inner automorphism of Γ goes over to an inner automorphism of $\mathrm{Map}(\Sigma)$.)

Given that we can always take $\theta \in \mathrm{Mod}(\Gamma)$, we can reduce from $\mathrm{Out}(\Gamma)$ to $\mathrm{Mod}(\Gamma)$.

Theorem 1.3. *If Γ is a finitely presented one-ended group, then there are only finitely many purely pseudoanosov homomorphisms from Γ to $\mathrm{Map}(\Sigma)$ up to precomposition in $\mathrm{Mod}(\Gamma)$ and postcomposition in $\mathrm{Inn}(\mathrm{Map}(\Sigma))$.*

We note the case of Theorem 1.1 where Γ is an orientable surface group Theorem 1.1 is equivalent to Theorem 1.3, and was proven in [Bo3]. In that case, where the conclusion of Theorem 1.2 was already known (see [Ba]). I have been informed by Daniel Groves that he has an alternative approach to Theorem 1.1, as well as some more general statements if one allows for pseudoanosovs.

We remark that analogues of Theorem 1.1 for hyperbolic and relatively hyperbolic groups are proven respectively in [De] and [Da].

We note that, in the case where Σ is the torus, the orientation preserving subgroup of $\mathrm{Map}(\Sigma)$ is $PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. In particular, Map($\Sigma$) is virtually free, and so any torsion-free subgroup is free. For this reason, we can assume that genus(Σ) ≥ 2.

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§2. Hyperbolic spaces and graphs

In this section, we explain how to reduce the main results (Theorems 1.1 and 1.3) to Theorem 1.2. To apply this, we use the action of the mapping class on the curve graph, together with the result from [Bo3] alluded to in the introduction. The remainder of the paper will then be devoted to proving Theorem 1.2.

Let $\mathcal{G}$ be a k-hyperbolic graph in the sense of Gromov (see [Gr1, GhH]). (Note that we are not assuming $\mathcal{G}$ to be locally finite.) We write d for the combinatorial metric — so that each edge has length 1. We denote the vertex set by $V(\mathcal{G})$.

Suppose that a group Γ acts on $\mathcal{G}$. The *stable length*, $||g||$, of an element $g \in \Gamma$ is defined as $||g|| = \lim_{n\to\infty} \frac{1}{n} d(x, g^n x)$ for some, hence any, $x \in \mathcal{G}$. We say that g is *loxodromic* if $||g|| > 0$. (This is equivalent to saying that g moves some bi-infinite quasigeodesic a bounded Hausdorff distance — indeed the constants can be chosen to depend only on k.) We say that Γ is *purely loxodromic* if each non-trivial element is loxodromic. We write $\mathrm{inj}(\mathcal{G}, \Gamma) = \inf\{||g|| \mid g \in \Gamma \setminus \{1\}\}$ (cf. the injectivity radius of a negatively curved manifold).

The following is proven in [Bo2]:

Lemma 2.1. *Suppose that Γ acts acylindrically on $\mathcal{G}$, and that $g \in \Gamma$ is loxodromic. Then $||g|| \geq \eta$ where $\eta > 0$ depends only on the hyperbolicity constant, k, and the parameters of acylindricity.*

The following is also easily verified:

Lemma 2.2. *If Γ admits a purely loxodromic acylindrical action on a hyperbolic graph, then every infinite cyclic subgroup of Γ has finite index in its centraliser.*

This can be proven by a similar argument as that for subgroups of a hyperbolic group — only the acylindricity of the action on the Cayley graph is needed to make that work. We omit a proof here, since the conclusion is readily seen in the case we wish to apply it, namely purely pseudoanosov subgroups of the mapping class group.

As noted, the key new ingredient that we shall prove in this paper is Theorem 1.2 given in the previous section.

We remark that Theorem 1.2 could be applied in the case where $\mathcal{G}$ is a uniformly locally finite graph to give the result of [De] that there are only finitely many conjugacy classes of one-ended subgroups of any particular isomorphism type in a hyperbolic group. Of course, this approach would be a rather indirect and non-constructive route to that result. As noted in the introduction, a version of Theorem 1.2 specific to the curve complex (with $\mathrm{Out}(\Gamma)$ replacing $\mathrm{Mod}(\Gamma)$) has been independently proven by Dahmani and Fujiwara by different methods [DaF]. This can, of course, be substituted for Theorem 1.2 to deduce Theorem 1.1.

We will be applying the above results to the curve graph. Let Σ be a closed orientable surface. The *curve graph*, $\mathcal{G}(\Sigma)$, of Σ is the 1-skeleton of the curve complex as defined by Harvey [Ha]. That is to say, its vertex set, $V(\mathcal{G}(\Sigma))$, is the set of homotopy classes of essential simple closed curves in Σ, and two such vertices are adjacent if these curves can be homotoped to be disjoint in Σ. Note that $\mathrm{Map}(\Sigma)$ acts on $\mathcal{G}(\Sigma)$ with compact quotient.

We have:

Theorem 2.3. [MaM, Bo2] *$\mathcal{G}(\Sigma)$ is Gromov hyperbolic. The action of $\mathrm{Map}(\Sigma)$ on $\mathcal{G}(\Sigma)$ is acylindrical. The loxodromic elements of $\mathrm{Map}(\Sigma)$ are precisely the pseudoanosovs.*

The first and last statements are proven in [MaM] and the second statement (acylindricity) in [Bo2]. Of course, all the parameters involved depend only on $\mathrm{genus}(\Sigma)$. Note that there is some constant, $\eta > 0$, again depending only on $\mathrm{genus}(\Sigma)$, such that if $g \in \Gamma$ is pseudoanosov, then $||g|| \geq \eta$. A direct argument for this is given in [MaM], though it also follows from Lemma 2.1.

We remark that the curve graphs are now known to be uniformly hyperbolic [A, CRS, HePW, Bo4]. Also, simpler and constructive proofs of acylindricity (via the combinatorics of tight geodesics) are given in [We] and in [Wa].

We are now in a set up where we can apply Theorem 1.2. To explain how this implies Theorems 1.1 and 1.3, we need a result from [Bo3].

Suppose that Γ is a finitely presented group and that $A \subseteq \Gamma$ is a finite generating set. We can define its *complexity*, $c(A)$, as the minimal total length of a set of relators for Γ with respect to A. Here the "total length" is the sum of the lengths of the cyclic words in elements of A and their inverses representing the relators.

The following is proven in [Bo3]:

Theorem 2.4. *Suppose that Γ is a finitely presented and one-ended, and that $A \subseteq \Gamma$ is a finite generating set. Suppose that $\phi : \Gamma \longrightarrow \mathrm{Map}(\Sigma)$ is a purely pseudoanosov homomorphism, and that there is some $x \in V(\mathcal{G}(\Sigma))$ such that $d(x, \phi(g)x) \leq K$ for all $g \in A$. Then there is some $h \in \mathrm{Map}(\Sigma)$ such that the word length of $h\phi(g)h^{-1}$ in $\mathrm{Map}(\Sigma)$ is bounded above for all $g \in A$, in terms of K and $c(A)$.*

Here the word length in $\mathrm{Map}(\Sigma)$ is measured in terms of some fixed generating set of $\mathrm{Map}(\Sigma)$. The purpose of the result is to reduce these elements to some predetermined finite subset of $\mathrm{Map}(\Sigma)$.

We explain how Theorems 1.2 and Theorem 2.4 together imply Theorem 1.3, which of course, in turn implies Theorem 1.1.

Suppose that Γ is finitely presented and one-ended and that $\phi : \Gamma \longrightarrow \mathrm{Map}(\Sigma)$ is a purely pseudoanosov homomorphism. We fix some finite generating set, $A \subseteq \Gamma$, and consider the action of Γ on $\mathcal{G}(\Sigma)$. In view of Theorem 2.3, Theorem 1.2 tells us that, after precomposing by some element of $\mathrm{Mod}(\Gamma)$, there is some $x \in V(\mathcal{G}(\Sigma))$ such that $d(x, \phi(g)x) \leq K$ for all $g \in A$, where K depends only on Γ, A and $\mathrm{genus}(\Sigma)$. Applying an outer automorphism of Γ does not affect the complexity of A, and Theorem 2.4 tells us that there is some $h \in \mathrm{Map}(\Sigma)$ such that the word lengths of $h\phi(g)h^{-1}$ in $\mathrm{Map}(\Sigma)$ are bounded above in terms of $c(A)$ and K, and hence in terms of Γ, A and $\mathrm{genus}(\Sigma)$. In other words, after postcomposing with an inner automorphism of $\mathrm{Map}(\Sigma)$, $\phi(A)$ is contained in a finite subset of $\mathrm{Map}(\Sigma)$, predetermined by Γ and A. It follows that after these pre and post compositions, there are only finitely many possibilities for ϕ, as required for Theorem 1.3.

§3. Degeneration of hyperbolic structures

To prove Theorem 1.2, we will be arguing by contradiction, bringing the theory of actions on $\mathbb{R}$-trees into play. We begin by recalling some general facts about degenerating hyperbolic metrics. We shall use the formulation in terms of asymptotic cones [VW] (see also [Gr2]).

In this section, we fix a finitely generated group, Γ, and a finite generating set, $A \subseteq \Gamma$. Suppose that Γ acts on a hyperbolic graph $\mathcal{G}$. Given $a \in \mathcal{G}$, we write $D(\mathcal{G}, \Gamma, a) = \max\{d(a, ga) \mid g \in A\}$ and $D(\mathcal{G}, \Gamma) = \min\{D(\mathcal{G}, \Gamma, a) \mid a \in \mathcal{G}\}$.

Suppose that we have a sequence, $(\mathcal{G}_n)_n$, of k-hyperbolic graphs and points $a_n \in \mathcal{G}_n$, and a sequence of numbers $(L_n)_n$ with $L_n \to \infty$ and

$D(\mathcal{G}_n, \Gamma, a_n) \leq L_n$ for all n. We write $\frac{1}{L_n}\mathcal{G}_n$ for the graph $\mathcal{G}_n$ scaled by a factor $\frac{1}{L_n}$, i.e. so that each edge has length $\frac{1}{L_n}$. Choosing a non-principal ultrafilter on $\mathbb{N}$, the pointed spaces, $(\frac{1}{L_n}\mathcal{G}_n, a_n)$ converge to a space (T, a_∞). In this case, T is an $\mathbb{R}$-tree, admitting a limiting action of Γ.

In general, the limiting action might be *elliptic*, i.e. Γ fixes a point of T. However, in certain cases, this can be avoided. For example, suppose that we choose a_n so that $D(\mathcal{G}_n, \Gamma, a_n) = D(\mathcal{G}_n, \Gamma)$ and it happens that $D(\mathcal{G}_n, \Gamma)/L_n$ is bounded below by a positive constant. In this case, the limiting action will be non-elliptic.

For this to be useful, we need some more information. This may be provided by acylindricity:

Lemma 3.1. *Suppose that $\mathcal{G}_n$ is a sequence of k-hyperbolic spaces admitting purely loxodromic uniformly acylindrical actions of a finitely generated group, Γ, and let $A \subseteq \Gamma$ be a finite generating set. Suppose that $a_n \in \mathcal{G}_n$, and for all n, $D(\mathcal{G}_n, \Gamma, a_n) \leq L_n$, where $(L_n)_n$ is some sequence of numbers tending to ∞. Let T be the limiting $\mathbb{R}$-tree of $(\frac{1}{L_n}\mathcal{G}_n, a_n)$. Then each arc stabiliser of T is trivial or infinite cyclic.*

Here "uniformly acylindrical" means that the parameters of acylindricity are independent of n. An *arc stabiliser* is the pointwise stabiliser in Γ of a non-trivial arc in T. This is the same as the intersection of the stabilisers of its endpoints.

The proof of Lemma 3.1 will be an adaptation of the argument for a locally finite graph, that is frequently applied to hyperbolic groups (cf. [P, RiS1]).

We begin with the following general observation:

Lemma 3.2. *Let G be a group with only finitely many commutators. Then there is a finite index normal subgroup $N \lhd G$, and a finite (abelian) normal subgroup $F \lhd N$ such that N/F is abelian.*

Proof. Let $C \subseteq G$ be the set of all commutators, which we assumed finite. Now G acts on C by conjugation. Let $N \lhd G$ be the kernel of this action. Thus, $C \cap N$ is central in N. Moreover, each element of $C \cap N$ has finite order (since $[x, y]^n = [x^n, y] \in C \cap N$ for all $x, y \in N$ and $n \in \mathbb{N}$). Thus, the group generated by $C \cap N$ is finite abelian and normal in N, and N/F is abelian. Q.E.D.

Note that if G, or equivalently, N/F is finitely generated, then G is virtually abelian. It is well known that any torsion-free virtually cyclic group is cyclic, and so, in the case of interest to us where all abelian subgroups are trivial or infinite cyclic, we can deduce that G is trivial or infinite cyclic.

Now let Γ, $\mathcal{G}_n$, L_n, T be as in the hypotheses of Lemma 3.1. Let $p, q \in T$ be distinct points, and let $G \leq \Gamma$ be the intersection of their stabilisers. In view of Lemma 3.2, it is enough to show that G has only finitely many distinct commutators.

In fact, we show that G has at most N_0 commutators, where N_0 is determined by the hyperbolicity constant, k, and the fixed parameters of acylindricity, in the manner to be described below.

Let us assume, for contradiction, that G has more than N_0 commutators. This means that there is a finite symmetric subset $B \subseteq G$ with $|\{[g,h] \mid g,h \in B\}| > N_0$. Let $C \subseteq G$ be the finite set of elements of word length at most 4 in the elements of B. Now each $g \in C$ fixes $p, q \in T$. This means there are points $p_n, q_n \in \mathcal{G}_n$ so that $d(p_n, q_n)/L_n \to d(p,q) > 0$, whereas for each $g \in C$, $d(p_n, gp_n)/L_n$ and $d(q_n, gq_n)/L_n$ both tend to 0. Thus, for any $L \geq 0$, for all n sufficiently large we have $d(p_n, q_n) \geq 4L$, $d(p_n, gp_n) \leq L$ and $d(q_n, gq_n) \leq L$ for all $g \in C$. We fix some such n, given the choice of L to be determined below, and set $a = p_n$ and $b = q_n$.

We now bring uniform hyperbolicity of the graphs $\mathcal{G}_n$ into play. We will give a fairly informal account, since more precise definitions can be found in [Bo2], and the argument broadly follows that of [P].

Let α be any geodesic from a to b in $\mathcal{G}_n$. This has length at least $4L$. Let $\beta \subseteq \alpha$ be the middle segment of α of length $2L$. For each $g \in C$, β is k_0-closely translated some signed distance $\psi(g) \in [-L, L]$, where k_0 is some fixed multiple of k. (If $\psi(g) \geq 0$, this means that for all $t \in [0, 2L - \psi(g)]$, $d(g\beta(t), \beta(t + \psi(g)) \leq k_0$, where $\beta : [0, 2L] \longrightarrow \mathcal{G}$ is a parameterisation of β. This also has an obvious interpretration when $\psi(g) \leq 0$.) Now, it's not hard to see that if $g_1, g_2, g_3, g_4 \in B$, then $|\psi(g_1 g_2 g_3 g_4) - \psi(g_1) - \psi(g_2) - \psi(g_3) - \psi(g_4)|$ is bounded above by some fixed multiple of k_0. In particular, if $g, h \in B$, then we see that $|\psi([g,h])|$ is bounded in terms of k. Thus, for every $x \in \beta$, we have $d(x, [g,h]x) \leq r$, where r depends only on k.

The uniform acylindricity condition now gives us constants R, N such that if $x, y \in \mathcal{G}_n$ and $d(x,y) \geq R$, then at most N elements of Γ move both x and y a distance at most r. We now retrospectively set $N_0 = N$ giving rise to our sets B and C defined above. We choose any $L \geq R$. Now let n, α, β be as determined above and let $x, y \in \beta$ be any points with $d(x,y) \geq R$. If $g, h \in B$, then by construction, $d(x, [g,h]x) \leq r$ and $d(y, [g,h]y) \leq r$, and so acylindricity tells us that there are at most N_0 possibilities for $[g,h]$. But B was chosen precisely because there were strictly more than N_0 possibilities for the commutators $[g,h]$ for $g, h \in B$. This gives a contradiction.

This proves Lemma 3.1.

§4. JSJ splittings

In this section, we consider splittings of indecomposable groups over infinite cyclic subgroups. In particular, we are interested in variants of the JSJ splitting, which, in a certain sense describes all possible such splittings. These are motivated by the 3-manifold constructions of Waldhausen, Johanson, Jaco and Shalen, and were introduced by Sela in the context of hyperbolic groups. The basis of our account will be that of Rips and Sela [RiS2] for finitely presented groups. (See also [DuS] and [FP] for generalisations.) It seems that the version in [RiS2] is not quite in the form we need here, so we use it to construct a variant for a restricted class of groups, for which we can give a slightly stronger conclusion.

Our splitting will have two kinds of vertices: "taut" and "hanging", the latter being surface-type groups, and the underlying graph will be bipartite. We will allow the underlying surface of a hanging vertex group to be an annulus, so that the group is infinite cyclic. (This is disallowed in some formulations elsewhere.)

We begin with some formal definitions. We start with an indecomposable torsion-free group, Γ. By a *cyclic splitting*, Υ, of Γ, we mean a representation of Γ as a finite graph of groups with every edge group infinite cyclic. We write $V(\Upsilon)$ for the vertex set, and if $v \in V(\Upsilon)$, we write $\Gamma(v)$ for the corresponding vertex group. We similarly write $\Gamma(e)$ for the group corresponding to an edge e. In general, such groups are only defined up to conjugacy in Γ, though some constructions are best viewed formally in terms of the action of Γ on the Bass-Serre tree, where one can talk about actual subgroups.

A vertex (or vertex group) is *trivial* if it has degree 1 in Υ and if the vertex group is equal to the incident edge group. We shall normally assume that there are no trivial vertices — such a vertex can simply be deleted along with the incident edge. Any group featuring as the edge group in some cyclic splitting will be referred to as a *(cyclic) splitting group*. Two such groups are *compatible* if they feature simultaneously in some common cyclic splitting.

We note that if Γ is finitely generated (respectively finitely presented) then each vertex group is finitely generated (respectively finitely presented). These facts appear to be have been known for some time. A proof in the finitely presented case can be found, for example, in [Bo1].

Definition. A vertex group is of *surface type* if it is the fundamental group of a compact surface, other than the Möbius band, and each incident edge group is a peripheral subgroup.

Note that we are allowing annuli here. (A Möbius band is best viewed as a non-surface type vertex attached by an edge group which we can subdivide to give an annular surface-type vertex group — effectively cutting the Möbius band along the core curve.)

Note that each peripheral subgroup must contain at least one incident edge group — otherwise, in the case of an annulus the vertex would be trivial, and in all other cases, we could obtain a splitting of Γ as a free product.

By a *two-sided* curve on a compact surface, we mean a locally separating homotopically non-trivial simple closed curve. (It might be peripheral.) Any two-sided curve in a surface-type vertex group gives rise to a splitting of Γ, namely with the edge group supported on the curve. Compatible splittings correspond to disjoint curves. Conversely, it's not hard to see that if $H \leq \Gamma(v)$ is a cyclic splitting group, then either H arises in this way, or else is a proper subgroup of one of the peripheral subgroups.

Elaborating on this, one can show that if $G \leq \Gamma(v)$ is itself a vertex group in some other cyclic splitting of Γ, then G is also of surface type.

Definition. We say that a vertex group is of *strong surface type* if the incident edge groups are precisely the peripheral subgroups.

In [RiS2] it was shown that any finitely presented group admits a cyclic splitting (possibly trivial) such that each cyclic splitting subgroup is either conjugate into a (non-annular) surface type subgroup, or else some finite index subgroup is conjugate into an edge group. (More is said in [RiS2], but the above is a consequence. See also the account of a related splitting in [DuS].) We shall see that if we make stronger assumptions on Γ, then we can draw stronger conclusions.

We shall assume henceforth that:

$(*)$ Γ is torsion-free and each infinite cyclic subgroup is finite index in its centraliser.

It follows that the centraliser is also the commensurator, and the unique maximal cyclic subgroup containing the original cyclic group.

By a *bipartite* splitting of Γ we mean a cyclic splitting, Υ, with $V(\Upsilon) = V_H(\Upsilon) \sqcup V_T(\Upsilon)$ so that each edge group has one vertex in each of $V_H(\Upsilon)$ and $V_T(\Upsilon)$, and each vertex in $V_H(\Upsilon)$ is of strong surface type. We refer to the vertices of $V_H(\Upsilon)$ as *hanging* and those in $V_T(\Upsilon)$ as *taut*. We write $V_C(\Upsilon) = \{v \in V_T(\Upsilon) \mid \Gamma(v) \cong \mathbb{Z}\}$ for the cyclic taut vertices.

By a *JSJ splitting* of Γ we mean a bipartite cyclic splitting with the following properties:

(J1) If $H \leq \Gamma$ is a cyclic splitting group, then $H \leq \Gamma(v)$ for some $v \in V_H(\Upsilon) \cup V_C(\Upsilon)$.

(J2) If $v \in V_C(\Upsilon)$ then $\Gamma(v)$ is a maximal cyclic subgroup of Υ, and if $v, w \in V_C(\Upsilon)$ with $\Gamma(v) = \Gamma(w)$, then $v = w$.

(J3) No subgroup of of the form $\Gamma(v)$ for $v \in V_T(\Upsilon)$ splits over $\mathbb{Z}$ relative to the incident edge groups.

Note that, in view of (J1), the only way that a taut vertex group might split relative to the incident edge groups is over a proper subgroup of one of the incident edge groups, and so (J3) is designed to rule out that possibility. It is, in fact, equivalent here to saying that there are no "unfoldings" of edge groups, as we discuss later. Also, note that non-cyclic taut surface group cannot be of surface type, since any group of surface type would admit a relative splitting. In other words, we see that if $v \in V(\Upsilon)$ and $\Gamma(v)$ is of surface type, then $v \in V_C(\Upsilon) \cup V_H(\Upsilon)$.

We claim:

Proposition 4.1. *Let Γ be a one-ended finitely presented torsion-free group such that each cyclic subgroup is finite index in its centraliser. Then Γ admits a JSJ splitting (of the type described above).*

(It is not clear if this splitting is unique, though the underlying graph is canonically determined.)

To prove Proposition 4.1, we begin with a more general cyclic splitting of the type described in [RiS2], and proceed by series of simple modification to obtain one of the type desired. We begin by describing a few general principles.

Suppose first that Υ is any cyclic splitting of Γ. Suppose that $v, w \in V(\Upsilon)$ are adjacent vertices with $\Gamma(v)$ and $\Gamma(w)$ cyclic. They are therefore commensurable, and generate a cyclic subgroup, H. We can therefore collapse the edge, replacing it by one vertex with vertex group H. (In fact, v and w must be distinct vertices.)

Suppose that $v \in V(\Upsilon)$ and $e_1, \ldots, e_n$ are incident edges, all of whose groups are commensurable. Then they generate a cyclic subgroup, $H \leq \Gamma(v)$. We can "pull out" the subgroup H from $\Gamma(v)$. That is, we produce a new vertex w, with $\Gamma(w) = H$, incident on each of $e_1, \ldots, e_n$, and a new edge e_0, connecting w to v, with edge group H. In fact, we could instead, pull out any cyclic subgroup, H, with $\Gamma(e_i) \subseteq H \subseteq \Gamma(v)$ for all i.

Let us now go back to the splitting, Υ, given by [RiS2]. We now modify this in a series of steps. (We do not, for the moment, assume our splittings to be bipartite.)

First note that if $\Gamma(v)$ is of surface type, we can assume it to be of strong surface type, after pulling out the peripheral subgroups if necessary. (We have already observed that each peripheral subgroup contains at least one edge group.) By adding degree-2 vertex groups, we can assume that each edge group is commensurable with a cyclic vertex group.

Suppose now that $H \leq \Gamma$ s a maximal cyclic subgroup. Let $W = \{v \in V(\Upsilon) \mid \Gamma(v) \leq H\}$. (Formally this construction is best viewed in terms of the Bass-Serre tree where we don't need to worry about conjugacy classes, though we shall describe it less formally in terms of the graph Υ.) Now we can assume that there is a connected graph, $\Psi \subseteq \Upsilon$, with vertex set W. To see this, let Ψ_0 be the union of all edges with both endpoints in W. If Ψ_0 is not connected, let α be an arc in Υ connecting two components of Ψ_0, and with each edge group contained in H. Now by pulling out subgroups of H from the corresponding vertex groups along α, we can assume that these vertex groups are all contained in H, so that, after this process is completed all the vertices of α will be contained in W. This reduces the number of components of Ψ_0, and so we eventually end up with a connected graph, Ψ, as claimed. We can now collapse this graph to a single vertex, whose vertex group is a subgroup of H. (In fact Ψ has to be a tree, since any circuit would give rise to a subgroup containing an infinite-index central cyclic subgroup.)

After performing this construction for all maximal cyclic subgroups, we can arrange that if $v, w \in V(\Upsilon)$ with $\Gamma(v)$ and $\Gamma(w)$ cyclic and commensurable, then $v = w$.

Now suppose that $\Gamma(v)$ is cyclic. Let $H \leq \Gamma$ be its centraliser. (This is a maximal cyclic subgroup of Γ.) Now either $H = \Gamma(v)$ or else $H \leq \Gamma(w)$ for some adjacent $w \in V(\Upsilon)$. By pulling out H from $\Gamma(w)$, and collapsing the new vertex with v, we can arrange that $\Gamma(v) = H$. In other words, we can now assume that all cyclic vertex groups are maximal cyclic subgroups of Γ, and that any two such subgroups are distinct.

In the original splitting we started with, any cyclic splitting group, H, was contained in a vertex group of non-annular surface-type, or else commensurable with an edge group. After the above tinkering, we can now assume that any such group is contained in a vertex group of non-annular strong surface type, or else contained in a cyclic vertex group.

Next, we can make the graph Υ bipartite. Suppose that two strong surface type groups are adjacent. They cannot both be annular, and collapsing the edge gives another such group. (In fact, at least one of the original surfaces must be an annulus, otherwise there would be a splitting corresponding to a curve crossing the edge group. Indeed, this reasoning shows that this edge cannot be a loop.) We can therefore eliminate all edges of this type. If two non surface type vertices are adjacent, we just introduce a degree-2 vertex in the middle of the edge. This is of annular strong surface type.

We now write $V_H(\Upsilon)$ for the set of all surface type vertices, and $V_T(\Upsilon) = V(\Upsilon) \setminus V_H(\Upsilon)$. We refer to these as "hanging" and "taut" respectively.

There is one remaining complication. Suppose $v \in V_T(\Upsilon)$. It is still possible that $\Gamma(v)$ might split relative to the incident edge groups over a proper subgroup, H_1, of one of these edge groups, H. We can now pull out the subgroup H_1 from from $\Gamma(v)$ and collapse the original edge. This does not change the graph, Υ, or the vertex group, but replaces the edge group, H, by the smaller group H_1. This is an "unfolding". It may be that $\Gamma(v)$ still splits over a proper subgroup $H_2 \leq H_1$, and we unfold again to H_2. We thus get a sequence of such unfoldings, $H_1 \geq H_2 \geq H_3 \geq \cdots$. Now it is a theorem of Sela (see [RiS2]) that since Γ is finitely generated, any such sequence must terminate after a finite number of steps on some subgroup, $H_n \leq H$. We thus replace H with H_n. We do this whenever such an unfolding exists.

The resulting graph now has all the properties (J1)–(J3) listed above, as required by Proposition 4.1.

We can give a more geometric picture by constructing a complex, Δ, with $\pi_1(\Delta) \cong \Gamma$. For each $v \in V(\Upsilon)$, let $\Delta(v)$ be a complex with $\pi_1(\Delta(v)) \cong \Gamma(v)$. If $v \in V_C(\Upsilon)$, we take $\Delta(v)$ to be a circle, and if $v \in V_H(\Upsilon)$, we take $\Delta(v)$ to be a compact surface. We refer to $\Delta(v)$ as "hanging" or "taut" depending on whether v lies in $V_H(\Upsilon)$ or $V_T(\Upsilon)$. We now glue the hanging surfaces to the taut complexes by gluing their boundary components to corresponding closed curves in the adjacent taut complexes. We can assume these curves lie in the 1-skeletons, so that the complex Δ is simplicial. We note that if Γ is finitely presented, then each $\Gamma(v)$ is finitely presented, so we can take each $\Delta(v)$ hence Δ to be finite.

Suppose that Π is a set of embedded closed curves in Δ such that for each $v \in V_T(\Upsilon)$, $\Pi \cap \Delta(v) = \emptyset$, and for each $v \in V_H(\Upsilon)$, $\Pi \cap \Delta(v)$ is a (possibly empty) set of two-sided curves, no two of which are homotopic in $\Delta(v)$. (We are allowing peripheral curves.) Now Π gives rise to a splitting, Ψ, of Γ where the edge groups correspond to the components of

Π, and the vertex groups are the fundamental groups of the components of $\Delta \setminus \Pi$.

Note that, by construction, each taut vertex group, G, of Υ is conjugate into a vertex group, K, of Ψ. Indeed, if $G \not\cong \mathbb{Z}$, then K cannot be of surface type in Ψ, since we have observed that, in such a case, G would have to be of surface type in Υ, and hence hanging, by the observations after the statements of properties (J1)–(J3).

We will write $\mathrm{Mod}_\Upsilon(\Gamma)$ for the subgroup of $\mathrm{Out}(\Gamma)$ which respects the splitting Υ (fixing the conjugacy classes of vertex groups). Any such outer automorphism is supported on the hanging and cyclic vertices. In other words it is a product of outer automorphisms of groups of the form $\Gamma(v)$ for $v \in V_H(\Upsilon) \cup V_C(\Upsilon)$, which preserve the incident edge groups. Such automorphisms are induced by mapping classes of the corresponding surfaces which fix the boundary.

Before finishing this section, we say a few words about the modular group, $\mathrm{Mod}(\Gamma)$. This makes sense for any group, Γ, though we shall restrict our attention to groups satisfying $(*)$ as in the hypotheses of Proposition 4.1.

Suppose that Γ splits as an amalgamated free product or HNN extension over an infinite cyclic subgroup, $C \leq \Gamma$. We can perform a "Dehn twist" along H, giving us an element of $\mathrm{Out}(\Gamma)$. This is best seen by representing it as a complex where the edge group is supported on an annulus, and then twisting this annulus. Thus, for example, if $\Gamma = A *_C B$, this has the effect (up to inner automorphism) of holding A fixed while conjugating B by an element of C. We write $\mathrm{Mod}(\Gamma) \lhd \mathrm{Out}(\Gamma)$ for the subgroup of $\mathrm{Out}(\Gamma)$ generated by all Dehn twists for all such splittings.

In terms of the complex Δ described above, any Dehn twist about any two-sided curve in any hanging surface will be a Dehn twist of this type. There may be other Dehn twists arising from splitting groups lying inside taut cyclic vertex groups.

We claim that $\mathrm{Mod}(\Gamma)$ and $\mathrm{Mod}_\Upsilon(\Gamma)$ are commensurable subgroups of $\mathrm{Out}(\Gamma)$. The fact that $\mathrm{Mod}(\Gamma) \cap \mathrm{Mod}_\Upsilon(\Gamma)$ has finite index in $\mathrm{Mod}(\Gamma)$ follows from the description of the JSJ splitting together with the fact that every cyclic splitting of a surface group is geometric. We will not elaborate on that here since it is not logically required for our main result. The fact that it has finite index in $\mathrm{Mod}_\Upsilon(\Gamma)$ is implicit in [RiS1] by an indirect non-constructive argument. It also follows from the fact that mapping class groups are virtually generated by Dehn twists.

To be more precise, let D be a compact surface with (possibly empty) boundary, ∂D, and let M be the relative mapping class group, i.e. homeomorphisms up to homotopy fixing ∂D pointwise. Let $T \lhd M$

be the subgroup generated by Dehn twists along two-sided curves (including peripheral curves). Then:

Proposition 4.2. *T has finite index in M.*

For closed surfaces, it was shown by Lickorish that $T = M$ in the orientable case and that that T has index 2 in M in the non-orientable case (see [L]). For compact orientable surfaces one can readily deduce that $T = M$ also, though the non-orientable case is more subtle. The general result for non-orientable surfaces is implicit in [Ko] and also in [S] though only explicitly stated for large genera (at least 7 and 3 respectively). In fact, M/T is an abelian 2-group (quite possibly always $\mathbb{Z}_2$). I am indebted to Mustafa Korkmaz for explaining the following argument to me.

First factor out both M and T by peripheral Dehn twists, and collapse each boundary component of D to a point to give us a closed surface, D_0, with a finite set of preferred points or "punctures". We can then identify (the quotient of) M with the subgroup, P_0, of the pure mapping class group, P, of D_0 ("$\mathcal{PM}$" in the notation of [Ko]), where P_0 preserves the local orientation at each puncture. We claim that P/T is a finite abelian 2-group. This is stated explicitly in [Ko] for genus at least 7 (see Corollary 6.2 thereof), though the argument works in general. The following refers to notation and results of that paper.

The group P is generated by finite sets $\{v_i\}$ in genus 1 (Theorem 4.3), $\{t_b, y, v_i, w_i\}$ in genus 2 (Theorem 4.9), and $\{t_l, y, v_i\}$ or $\{t_l, y, v_i, w_i\}$ in genus at least 3 (Theorem 4.13). Here the t's are Dehn twists (in T), the y's slide cross caps around one-sided curves, and v's and w's slide punctures around one-sided curves. The squares of all of these generating elements lie in T. Moreover, using Lemma 4.2, we have $(v_i v_j)^2 \in T$ for all i, j. As discussed in the proof of Theorem 6.1, $v_i w_i \in T$ and $(y v_i)^2 \in T$ for all i. In other words, $T v_i = T w_i$, and the $T v_i$ and $T y$ all commute. It follows that P/T is abelian.

§5. Actions on $\mathbb{R}$-trees

In this section we prove a lemma about groups with JSJ splittings acting on $\mathbb{R}$-trees. We will make use of the Rips theory of such actions, in particular, a result originating in the work of Bestvina and Feighn [BeF]. We make specific reference to the account given in [Ka], in particular, Theorem 12.72, which refers to stable minimal actions of finitely presented groups, and classifies such actions into four types: "non-pure", "surface type", "toral type" and "thin type", which will feature in the proof.

We show:

Lemma 5.1. *Let* Γ *be a torsion-free finitely presented one-ended group where each infinite cyclic subgroup is finite index in its centraliser. Let* Υ *be a JSJ splitting of* Γ *(as given by Proposition 4.1). Suppose that* Γ *acts by isometry on an* $\mathbb{R}$*-tree,* T*, with trivial or cyclic arc stabilisers. Then any taut vertex group of the splitting* Υ *is elliptic (i.e. fixes a point of* T*).*

Proof. We begin by noting that the condition on centralisers implies that the action on T is stable, which allows us to bring the Rips theory into play.

Let Δ be the complex associated to Υ, as described in Section 4. For each $v \in V_H(\Upsilon)$, we take a maximal set, $\Pi(v)$, of curves in $\Delta(v) \setminus \partial\Delta(v)$ in distinct homotopy classes such that the subgroups of Γ supported on each of these curves is elliptic. We let $\Pi = \bigcup_{v \in V_H(\Upsilon)} \Pi(v)$. As discussed in Section 4, Π gives rise to a splitting, Ψ, of Γ. By construction all the edge groups of Ψ are elliptic in T. Note that if $v \in V_C(\Upsilon)$ and $\Gamma(v)$ is elliptic, then $\Gamma(v)$ is also a vertex group of Ψ. To see this, suppose that $v' \in V_H(\Upsilon)$ is an adjacent vertex. Now, $\Delta(v')$ is attached to $\Delta(v)$ along a boundary curve (possibly wrapping several times around $\Delta(v')$). Since the boundary curve is elliptic in T, a homotopic peripheral curve in $\Delta(v') \setminus \partial\Delta(v')$ is necessarily included in Π. (This will be a core curve if $\Delta(v')$ happens to be an annulus.) As v' varies of the set of adjacent vertices, the set of such curves bound a component of $\Delta \setminus \Pi$, and the inclusion of $\Delta(v)$ into this component will be a homotopy equivalence. Thus $\Gamma(v)$ is a vertex group of the splitting Ψ as claimed.

Let $w \in V(\Psi)$, and let $\Gamma(w)$ be the corresponding vertex group of Ψ. We claim that $\Gamma(w)$ does not split over any elliptic cyclic subgroup relative to the incident edge groups in Ψ. For suppose that $\Gamma(w)$ did split over some such subgroup $H \leq \Gamma(w)$. Now H is a splitting group of Γ, and so is conjugate into a vertex group, $\Gamma(v)$, of the splitting Υ, where $v \in V_C(\Upsilon) \cup V_H(\Upsilon)$. Now if $v \in V_H(\Upsilon)$, then H corresponds to a curve in $\Delta(v)$. By maximality such a curve must already be contained in Π, and so the relative splitting would be trivial. So we can assume that $v \in V_C(\Upsilon)$. As observed above, $\Gamma(v)$ is also a vertex group of Ψ, and the components of $\Delta \setminus \Pi$ corresponding to $\Gamma(v)$ and to $\Gamma(w)$ meet along a component, π, of Π which is peripheral in some hanging surface $\Delta(u)$ of the splitting Υ. This π corresponds to an edge group of both splittings. But now the fact that $\Gamma(w)$ splits means that $\Gamma(u)$ also splits relative to its incident edge groups. (That is, there is an unfolding along the edge in each splitting.) This gives a contradiction.

Now, we also noted in Section 4 that each taut vertex group of Υ is contained in a vertex group of Ψ that is not of surface type. To prove the lemma it therefore suffices to show that if $G \leq \Gamma$ is a non-cyclic vertex group of Ψ, then either G is elliptic or of surface type.

Suppose that G is not elliptic. Let T' be the minimal G-invariant subtree of T. Let $H_1, \ldots, H_n$ be the incident edge groups of Ψ. By construction, each edge group, H_i, is elliptic in T, and hence also in T'. Let $N \lhd G$ be the kernel of the action of G on T'. As in [Ka], we distinguish four cases:

(1) "non-pure":

In this case, G splits relative to $\{H_i\}$ over an elliptic subgroup $J \leq G$, such that J contains a normal subgroup, $I \lhd J$, with J/I cyclic, and such that I fixes an arc of J. Thus I is also cyclic. Now since every abelian subgroup of G is cyclic, we can deduce that J is cyclic, contradicting the statement made earlier that no vertex group of Ψ splits of a cyclic group relative to the incident edge groups.

(2) "orbifold case":

G/N is a 2-orbifold group, and each subgroup NH_i/N is peripheral. Since every abelian subgroup of G is cyclic, it follows that N is trivial in this case, and since G is torsion-free, it must be a surface group. In other words, G is of surface type.

(3) "toral case":

G/N admits an action by isometry on the real line. We deduce that $G \cong \mathbb{Z}$ contrary to our assumption.

(4) "thin case":

G splits relative to $\{H_i\}$ over a subgroup J that fixes an arc of T'. Thus J is cyclic and elliptic, again giving a contradiction as in (1).

This proves the claim and hence the lemma. Q.E.D.

§6. Carrying graphs

In this section we discuss some constructions involving carrying graphs. Most of this elaborates on ideas in [Ba].

Suppose that Γ is finitely presented, and let Δ be a finite simplicial complex with $\Gamma \cong \pi_1(\Delta)$. A *carrying graph*, Ω, for Γ is an embedded connected graph $\Omega \subseteq \Delta$ carrying all of $\pi_1(\Delta)$. (We can assume that Ω lies in the 1-skeleton of some subdivision of Δ.) We write $\tilde{\Omega} \subseteq \tilde{\Delta}$ for the lift of Ω to the universal cover, $\tilde{\Delta}$, of Δ. Thus, Γ acts freely on $\tilde{\Omega}$ with $\tilde{\Omega}/\Gamma = \Omega$.

Suppose that Γ acts on a (hyperbolic) graph $\mathcal{G}$. A *realisation* of Ω is a Γ-equivariant map, $f : \tilde{\Omega} \longrightarrow \mathcal{G}$, sending each edge of Ω to a geodesic segment. The *length*, $L(\Omega, f)$, of f can be defined as the sum of the lengths of the edges in some Γ-transversal to the action of Γ on $\tilde{\Omega}$. (Informally, this can be thought of as the length of the image of Ω in $\mathcal{G}/\Gamma$, descending to the quotients.) If $\Omega_0 \subseteq \Omega$ we write $L(\Omega_0, f)$ for the length of Ω_0 under this realisation. Note that if $x \in \Omega$, then $D(\mathcal{G}, \Gamma, f(x)) \leq L(\Omega, f)$, so certainly $D(\mathcal{G}, \Gamma) \leq L(\Omega, f)$. Note conversely, that if Ω is a wedge of circles, then this determines a generating set, A, of Γ. In this case, $L(\Omega, f) = \sum_{g \in A} d(fx, gfx)$ for a suitable lift, $x \in \tilde{\Omega}$, of the unique vertex of Ω. In particular, we have $L(\Omega, f) \leq |A| D(\mathcal{G}, \Gamma, f(x))$.

Let us first consider the case where Δ is a compact surface, with boundary $\partial\Delta$. We write $G = \pi_1(\Delta)$. We say that a carrying graph $\Omega \subseteq \Delta$ is *simple* if $\partial\Delta \subseteq \Omega$, if Ω has no degree-1 vertices and every degree-2 vertex lies in $\partial\Delta$, and if $\Delta \setminus \Omega$ is connected (hence a single disc). Note that, in this case, if the structure of Ω on $\partial\Delta$ is determined, then there are only finitely many possibilities for Ω up to homeomorphism of Δ fixing $\partial\Delta$. By slight abuse of notation, we will use $\Omega \setminus \partial\Delta$ for the subgraph of Ω where we omit all edges and degree-2 vertices in $\partial\Delta$. (Note that $\Omega \setminus \partial\Delta$ need not be connected.)

Suppose that $f : \tilde{\Omega} \longrightarrow \mathcal{G}$ is a realisation of Ω. If α is a component of $\partial\Delta$, then $f(\tilde{\alpha})$ is a piecewise geodesic bi-infinite path, invariant under the associated cyclic peripheral subgroup, $H \leq G$. The break points will all be vertices of $\tilde{\Omega}$, though it is possible that it may have fewer genuine geometric breakpoints. (For example, if it happens to be geodesic, it will have none.) We write $b(\alpha, f)$ for the number of geometric break points of α, and write $b(\partial\Delta, f)$ for the sum over all boundary components, α, of $\partial\Delta$.

Proposition 6.1. *Let $n \in \mathbb{N}$. Let Δ be a compact surface and $G = \pi_1(\Delta)$. Suppose that G acts as a purely loxodromic group on a k-hyperbolic graph $\mathcal{G}$ with* $\mathrm{inj}(\Gamma, G) \geq \eta > 0$. *Suppose that $f : \partial\tilde{\Delta} \longrightarrow \mathcal{G}$ is a G-equivariant map sending each boundary component to a broken geodesic, with $b(\partial\Delta, f) \leq n$. Then there is a simple carrying graph, $\Omega \subseteq \Delta$, and a realisation $f : \tilde{\Omega} \longrightarrow \mathcal{G}$, extending $f|\partial\tilde{\Delta}$ and with $L(\Omega \setminus \partial\Delta, f) \leq l$ where l depends only on k, η, n and the topological type of Δ.*

Here, each component of $\partial\Delta$ may contain degree-2 vertices, as required by the original break points. A bounded number of higher degree vertices may have been introduced in the construction of Ω.

Proof. The case when Δ is orientable, and $\partial\Delta = \emptyset$ is proven in [Ba]. Essentially the same argument works here, and it only requires a few comments to explain why. Let us first consider the orientable case.

We choose a simple carrying graph, Ω, and a realisation, $f : \tilde{\Omega} \longrightarrow \mathcal{G}$, extending f as given on $\partial\tilde{\Delta}$, such that $L(\Omega, f)$ (hence $L(\Omega \setminus \partial\Delta, f)$) is minimal. Since Ω has a bounded number of edges, it suffices to bound $L(\alpha, f)$ for each edge, α, of $\Omega \setminus \partial\Delta$.

Suppose that α is an edge of $\Omega \setminus \partial\Delta$, with $L(\alpha, f)$ very large. The complement of Ω in Δ is a disc, P, with a bounded number of edges, so that Δ is recovered by carrying out certain edge identifications around ∂P. The idea is to carry out a modification of Ω by adding an edge that crosses P from α to another edge of P, and then deleting part of the edge α. The effect will be to reduce $L(\Omega, f)$, giving a contradiction.

The geometric construction is best described in the universal cover, $\tilde{\Delta}$. Let $\tilde{P}$ be a lift of P, and let $\tilde{\alpha}$ be a lift of α. Now $f(\partial\tilde{P})$ is a closed curve in $\mathcal{G}$, consisting of a bounded number of geodesic segments. By hyperbolicity, there is another edge, $\tilde{\beta}$, of $\partial\tilde{P}$, so that $f(\tilde{\alpha})$ and $f(\tilde{\beta})$ remain close over a large distance. More precisely, there are segments $\alpha' \subseteq \tilde{\alpha}$ and $\beta' \subseteq \tilde{\beta}$, such that $f(\alpha')$ and $f(\beta')$ are long and a bounded Hausdorff distance apart (depending on k). We write $\beta \subseteq \Omega$ for the projection of $\tilde{\beta}$.

Now if β does not lie in $\partial\Delta$, we can perform a shortening operation exactly as in [Ba] so as to give a contradiction. (This requires the lower bound on $\mathrm{inj}(\mathcal{G}, G)$ if it happens that $\beta = \alpha$.)

If $\beta \subseteq \partial\Delta$, then a similar construction works. Let $\tilde{x} \in \alpha'$ be such that $f(\tilde{x})$ is the midpoint of the segment $f(\alpha')$. There is some $\tilde{y} \in \beta'$ such that $d(f(\tilde{x}), f(\tilde{y}))$ is bounded (in terms of k). We let $x \in \alpha$, $y \in \beta$ be the projections to Ω. We now connect x to y by an arc in P, and remove one half of the edge α, starting at x, to give us a new graph, Ω' (deleting degree 2 vertices if necessary). By removing the correct half of α, the complement of Ω' remains connected, and so is also a simple carrying graph. In terms of the realisation we have effectively replaced a long edge by one of bounded length, and so after straightening, we get a realisation f' of Ω' with $L(\Omega', f') < L(\Omega, f)$, contradicting minimality.

Finally we need to say something about the non-orientable case. The only essentially new situation that may arise is when $\alpha = \beta$. Let γ be a simple closed curve containing $\alpha = \beta$, and with $\gamma \subseteq \Omega$. If γ is two-sided, we can perform a shortening as usual. If it is one-sided, then it becomes more complicated to arrange that the supporting graph remains embedded in this process. We can get around this as follows.

Let $g \in \Gamma$ be the element corresponding to γ, so that $\tilde{\beta} = g\tilde{\alpha}$. We can assume that each component of $\tilde{\alpha} \setminus \alpha'$ has bounded f-image in $\mathcal{G}$, otherwise we could find a segment in a different edge of $\tilde{P}$ whose f-image is long and close to $f(\tilde{\alpha})$ and so we reduce to one of the earlier cases. The same must also be true of β' in $\tilde{\beta} = g\tilde{\alpha}$. It now follows easily that g must translate some point, z, of $f(\alpha')$ a bounded distance. We can define a $\langle g \rangle$-equivariant map, $f' : \tilde{\gamma} \longrightarrow \mathcal{G}$, mapping to the broken geodesic $\bigcup_{n \in \mathbb{Z}} [g^n z, g^{n+1} z]$. Thus, $b(\gamma, f') = 1$ and $L(\gamma, f')$ is bounded. We now cut Δ along γ to give us a simpler surface Δ_1 with new boundary component, δ, which doubly covers γ. We get a realisation, $f : \partial\tilde{\Delta}_1 \longrightarrow \mathcal{G}$, with $b(\delta, f) = 2$ and $L(\delta, f)$ bounded. We can now proceed by induction on genus. We eventually end up with a carrying graph, Ω, with $L(\Omega \setminus \partial\Delta, f)$ bounded. After deleting some edges, we can assume it to be simple. Q.E.D.

We can apply this to the complex associated to a JSJ splitting. Let Γ be as in Proposition 4.1, and let Υ be the JSJ splitting, and Δ an associated complex with $\Gamma = \pi_1(\Delta)$.

For each $w \in V_T(\Upsilon)$, we take a carrying graph, $\Omega(w) \subseteq \Delta(w)$ for $\Gamma(w) \cong \pi_1(\Delta(w))$. We can assume $\Omega(w)$ to be a wedge of circles. (This is not essential, but it will make certain arguments easier to see.) We write $\Omega_T = \bigcup_{w \in V_T(\Upsilon)} \Omega(w)$. We can assume that we constructed Δ so that for all $v \in V_H(\Upsilon)$, $\partial\Delta(v) \subseteq \Omega_T$. We regard Δ and $\Omega_T \subseteq \Delta$ as fixed once and for all. We write $\tilde{\Omega}_T \subseteq \tilde{\Omega}$ for its preimage in $\tilde{\Delta}$. For each $v \in V_H(\Upsilon)$, we can choose a simple carrying graph $\Omega(v) \subseteq \Delta(v)$, and let $\Omega = \Omega_T \cup \bigcup_{v \in V_H(\Upsilon)} \Omega(v)$. Now if $v \in V_H(\Upsilon)$, we have $\partial\Delta(v) \subseteq \Omega(v)$ (by definition of "simple"). If $w \in V_T(\Upsilon)$ is adjacent, then the corresponding component of $\partial\Delta(v)$ is attached to $\Delta(w)$ along a curve in $\Omega(w)$. In particular, $\Omega(v) \cap \Omega(w) \neq \emptyset$. Since Υ is connected and bipartite, it follows that Ω is connected. One can also easily see that any closed curve in Δ can be homotoped into Ω. Thus $\Omega \subseteq \Delta$ is a carrying graph for Γ. We refer to a graph, Ω, constructed in this way as a "simple extension" of Ω_T. Note that any element of $g \operatorname{Mod}_\Upsilon \Gamma$ gives rise to another carrying graph, $g\Gamma$, with $g\Omega(v) = \Omega(v)$ for all $v \in V_T(\Upsilon)$ (in particular $g\Omega_T = \Omega_T$). We note that there are only finitely many combinatorial possibilities for a simple extension, modulo the action of $\operatorname{Mod}_\Upsilon(\Gamma)$, as defined in Section 4.

Lemma 6.2. *Let Γ, Δ and Ω_T be as described above. Suppose Γ has a purely loxodromic action on a k-hyperbolic graph, $\mathcal{G}$, with* $\operatorname{inj}(\mathcal{G}, \Gamma) \geq \eta > 0$. *Given any realisation, $f : \tilde{\Omega}_T \longrightarrow \mathcal{G}$, there is a simple extension,*

Ω, of Ω_T, and a realisation $f : \tilde{\Omega} \longrightarrow \mathcal{G}$ extending $f|\tilde{\Omega}_T$, with $L(\Omega \setminus \Omega_T, f) \leq l$, where l depends only on k, η, Δ and Ω_T.

Proof. If $v \in V_H(\Upsilon)$, then $\partial\Delta(v) \subseteq \Omega_T$, so $f|\partial\tilde{\Delta}(v)$ is determined, and $b(\partial\Delta, f)$ is bounded in terms of the number of edges in Ω_T. We can thus apply Lemma 6.1, to give $\Omega(v)$ and $f : \tilde{\Omega}(v) \longrightarrow \mathcal{G}$. We do this for all such v. Q.E.D.

§7. Proof of Theorem 1.2

In this section, we prove Theorem 1.2, thereby concluding the proofs of Theorems 1.1 and 1.3, as described in Section 2.

Let Γ be a one-ended finitely presented group, and suppose $A \subseteq \Gamma$ is a finite subset, which we can assume to be a generating set. Suppose that Γ has a purely loxodromic acylindrical action on a k-hyperbolic space, $\mathcal{G}$. In the notation of Section 3, we want to show that $D(\mathcal{G}, \Gamma)$ is bounded above in terms of Γ, A, k and the acylindricity parameters, modulo precomposition with $\mathrm{Mod}(\Gamma)$.

Note first that the existence of such an action implies that every cyclic subgroup of Γ is finite index in its centraliser. Let Υ be a JSJ splitting of Γ, as given by Proposition 4.1, and let $\Omega_T \subseteq \Delta$ be as constructed in Section 6. For each $v \in V_T(\Upsilon)$, $\Omega(v)$ is a wedge of circles, determining a generating set, $A(v)$, for $\Gamma(v)$.

Recall that $\mathrm{Mod}(\Gamma)$ and $\mathrm{Mod}_\Upsilon(\Gamma)$ are commensurable. Thus, if Theorem 1.2 fails, we can construct a sequence of actions of Γ on k-hyperbolic spaces, $\mathcal{G}_n$, which are purely loxodromic and uniformly acylindrical, and such that $D_n \to \infty$, where D_n is the minimal value of $D(\mathcal{G}_n, \Gamma)$ among all precompositions of the action of Γ on $\mathcal{G}_n$ by an element of $\mathrm{Mod}_\Upsilon(\Gamma)$. In fact, we can suppose we have realised this minimum, so that $D(\mathcal{G}_n, \Gamma) = D_n$.

Now, for each n, and each $v \in V_T(\Upsilon)$, we choose some $a_n(v) \in \mathcal{G}_n$, so that $D(\mathcal{G}_n, \Gamma(v), a_n(v)) = D(\mathcal{G}_n, \Gamma(v))$ (i.e. to minimise the maximum displacement by elements of the generating set, $A(v)$, of $\Gamma(v)$). We can now take a realisation $f_n : \tilde{\Omega}(v) \longrightarrow \mathcal{G}_n$, so that $L(\Omega(v), f_n)$ is bounded above by some fixed multiple of $D(\mathcal{G}_n, \Gamma(v))$ (namely, $L(\Omega(v), f_n) \leq |A(v)| D(\mathcal{G}_n, \Gamma(v))$). We perform this construction for each $v \in V_T(\Upsilon)$, to give us a realisation, $f_n : \tilde{\Omega}_T \longrightarrow \mathcal{G}_n$.

Note that $\mathrm{inj}(\mathcal{G}_n, \Gamma)$ is bounded below by some positive constant depending only on k and the acylindricity parameters (Lemma 2.1). We can thus apply Lemma 6.2 to get a simple extension, Ω, of Ω_T, together with a realisation $f_n : \tilde{\Omega}_n \longrightarrow \mathcal{G}_n$, extending $f_n|\tilde{\Omega}_T$, with $L(\Omega_n \setminus \Omega_T, f_n)$ uniformly bounded.

Now there are only finitely many combinatorial possibilities for Ω_n up to the action of $\mathrm{Mod}_\Upsilon(\Gamma)$ (which fixes Ω_T). After passing to a subsequence, and precomposing the actions, we can suppose that $\Omega_n = \Omega$ is fixed. The combined effect of applying $\mathrm{Mod}_\Upsilon(\Gamma)$ to the domain and conjugating the action means that the image of f_n, and hence $L(\Omega, f_n)$, remains unchanged. Also, $D(\mathcal{G}_n, \Gamma)$ can only increase, so we still have $D(\mathcal{G}_n, \Gamma) \to \infty$. Setting $L_n = L(\Omega, f_n)$, this implies that $L_n \to \infty$. It remains true that $L(\Omega(v), f_n)$ is bounded above by some multiple of $D(\mathcal{G}_n, \Gamma(v))$ for each $v \in V_T(\Upsilon)$.

But now, $L_n = L(\Omega, f_n) = L(\Omega \setminus \Omega_T, f_n) + \sum_{v \in V_T(\Upsilon)} L(\Omega(v), f_n)$, and so, again after passing to a subsequence, there must be some $v_0 \in V_T(\Upsilon)$, so that $D(\mathcal{G}_n, \Gamma(v_0), a_n(v_0))/L_n$ is bounded below by a positive constant.

We now obtain a limiting action of Γ on a $\mathbb{R}$-tree, T, where

$$\left(\frac{a}{L_n}\mathcal{G}_n, a_n(v_0)\right) \to (T, a_\infty).$$

Since $D(\mathcal{G}_n, \Gamma(v_0)) = D(\mathcal{G}_n, \Gamma(v_0), a_n(v_0)) = L(\Omega(v_0), f_n)$,

$$\frac{L(\Omega(v_0), f_n)}{L_n}$$

is bounded below by a positive constant, and so the action of $\Gamma(v_0)$ on T is not elliptic. Moreover, by Lemma 3.1, each arc stabiliser of T in Γ is trivial or infinite cyclic.

We can now apply Lemma 5.1, to tell us that $\Gamma(v)$ is elliptic for all $v \in V_T(\Upsilon)$. In particular, $\Gamma(v_0)$ is elliptic, contradicting the statement above.

This proves Theorem 1.2.

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