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Homomorphisms on groups of volume-preserving diffeomorphisms via fundamental groups

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Abstract.

Let M be a closed manifold. Polterovich constructed a linear map from the vector space of quasi-morphisms on the fundamental group $\pi_1(M)$ of M to the space of quasi-morphisms on the identity component $\mathrm{Diff}^\infty_\Omega(M)_0$ of the group of volume-preserving diffeomorphisms of M. In this paper, the restriction $H^1(\pi_1(M);\mathbb{R}) \to H^1(\mathrm{Diff}^\infty_\Omega(M)_0;\mathbb{R})$ of the linear map is studied and its relationship with the flux homomorphism is described.

§1. Introduction

Let M be a closed connected Riemannian manifold and Ω a volume form on M. We denote by $\mathrm{Diff}^\infty_\Omega(M)_0$ the identity component of the group of volume-preserving C^∞-diffeomorphisms of M. We assume that the center of the fundamental group $\pi_1(M)$ is finite. In [4], Gambaudo and Ghys constructed countably many quasi-morphisms on the group of area-preserving diffeomorphisms of the 2-disk from the signature quasi-morphism on the braid groups. After that Polterovich introduced in [6] a similar construction of quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(M)_0$ from quasi-morphisms on $\pi_1(M)$. Recently, Brandenbursky generalized these strategy and defined a linear map from the vector space of quasi-morphisms on the braid group or the fundamental group to the space of quasi-morphisms of area-preserving diffeomorphisms of surfaces [2], [3].

Polterovich's construction induces a linear map from the vector space of quasi-morphisms on $\pi_1(M)$ to the vector space of quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(M)_0$. By restricting it on $H^1(\pi_1(M);\mathbb{R})$, we have the linear map $\Gamma\colon H^1(\pi_1(M);\mathbb{R}) \to H^1(\mathrm{Diff}^\infty_\Omega(M)_0;\mathbb{R})$, which is defined in

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section 2 of this paper. Studying the linear map $\Gamma\colon H^1(\pi_1(M);\mathbb{R}) \to H^1(\mathrm{Diff}^\infty_\Omega(M)_0;\mathbb{R})$, we have a sufficient condition for vanishing of the volume flux group which is first obtained by Kędra–Kotschick–Morita in another way.

Theorem 1.1 (Kędra–Kotschick–Morita [5]). *If the center of* $\pi_1(M)$ *is finite, then the volume flux group of* M *is trivial.*

Let $\mathrm{Flux}\colon \mathrm{Diff}^\infty_\Omega(M)_0 \to H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})$ be the Ω-flux homomorphism. Let $I^k\colon H^k_{\mathrm{dR}}(M;\mathbb{R}) \to H^k(M;\mathbb{R})$ be the isomorphism which gives the identification of the de Rham cohomology and the singular cohomology defined by

$$I^k([\eta])(\sigma) = \int_\sigma \eta$$

for k dimensional closed differential form η and for singular k-chain σ. Let $PD\colon H^{n-1}(M;\mathbb{R}) \to H_1(M;\mathbb{R})$ be the Poincaré duality. Our main result is the following.

Theorem 1.2. *For any* $\phi \in H^1(\pi_1(M);\mathbb{R}) = H^1(M;\mathbb{R})$,

$$\Gamma(\phi) = \phi \circ PD \circ I^{n-1} \circ \mathrm{Flux}\colon \mathrm{Diff}^\infty_\Omega(M)_0 \to \mathbb{R}.$$

§2. Preliminaries

In this section, we define a linear map

$$\Gamma\colon H^1(\pi_1(M);\mathbb{R}) \to H^1(\mathrm{Diff}^\infty_\Omega(M)_0;\mathbb{R})$$

and recall a definition of the flux homomorphism.

Here and throughout this paper, we use functional notation. That is, for any homotopy classes γ_1 and γ_2 of loops with a fixed base point, the multiplication $\gamma_1\gamma_2$ means that γ_2 is applied first.

Choose a base point x^0 of M. For almost every $x \in M$, we choose the shortest geodesic $a_x\colon [0,1] \to M$ connecting x^0 with x if it is uniquely determined. For any $f \in \mathrm{Diff}^\infty_\Omega(M)_0$ and almost every $x \in M$ for which both the geodesics a_x and $a_{f(x)}$ are defined, we define the loop $l(f;x)\colon [0,1] \to M$ by

$$l(f;x)(t) = \begin{cases} a_x(3t) & (0 \le t \le \frac{1}{3}),\\ f_{3t-1}(x) & (\frac{1}{3} \le t \le \frac{2}{3}),\\ a_{f(x)}(3-3t) & (\frac{2}{3} \le t \le 1), \end{cases}$$

where $\{f_t\}_{t\in[0,1]}$ is an isotopy such that f_0 is the identity and $f_1 = f$. Of course for some $x \in M$ there exist two or more shortest geodesics connecting x^0 with x. However for almost every $x \in M$ the loop $l(f;x)$ is well-defined. We denote by $\gamma(f;x)$ the homotopy class represented by the loop $l(f;x)$. For a homomorphism $\phi \in H^1(\pi_1(M);\mathbb{R})$, we define the homomorphism $\Gamma(\phi) \in H^1(\mathrm{Diff}^\infty_\Omega(M)_0;\mathbb{R})$ by

$$\Gamma(\phi)(f) = \int_{x\in M} \phi(\gamma(f;x))\Omega.$$

For almost every $x \in M$, the homotopy class $\gamma(f;x)$ is well-defined and is unique up to elements of the center of $\pi_1(M)$ [6]. Since the center of $\pi_1(M)$ is finite, the image of $\gamma(f;x)$ by the homomorphism $\phi\colon \pi_1(M;x^0) \to \mathbb{R}$ is independent of the choice of the flow $\{f_t\}_{t\in[0,1]}$. Since the manifold M is compact, the loops $l(f;x)$ have uniformly bounded lengths for fixed $\{f_t\}_{t\in[0,1]}$. Hence the map $\gamma(f;\ \cdot\ )\colon M \to \pi_1(M;x^0)$ has a finite image and the value $\Gamma(\phi)(f)$ is well-defined.

Let $\widetilde{\mathrm{Diff}^\infty_\Omega}(M)_0$ be the universal cover of $\mathrm{Diff}^\infty_\Omega(M)_0$. Consider a path $\{f_t\}_{t\in[0,1]}$ in $\mathrm{Diff}^\infty_\Omega(M)_0$ such that f_0 is the identity. Let X_t be the corresponding vector field. Then the map $\widetilde{\mathrm{Flux}}\colon \widetilde{\mathrm{Diff}^\infty_\Omega}(M)_0 \to H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})$ is defined by

$$\widetilde{\mathrm{Flux}}(\{f_t\}) = \left[\int_0^1 \iota_{X_t}(\Omega)\,dt\right],$$

where ι_{X_t} is the interior product by X_t. The map $\widetilde{\mathrm{Flux}}\colon \widetilde{\mathrm{Diff}^\infty_\Omega}(M)_0 \to H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})$ is a well-defined homomorphism and called the Ω-*flux homomorphism*. The fundamental group $\pi_1(\mathrm{Diff}^\infty_\Omega(M)_0)$ is contained in $\widetilde{\mathrm{Diff}^\infty_\Omega}(M)_0$ as a subgroup of deck transformations. The image $G_\Omega = \widetilde{\mathrm{Flux}}(\pi_1(\mathrm{Diff}^\infty_\Omega(M)_0))$ of $\pi_1(\mathrm{Diff}^\infty_\Omega(M)_0)$ by the Ω-flux homomorphism $\widetilde{\mathrm{Flux}}\colon \widetilde{\mathrm{Diff}^\infty_\Omega}(M)_0 \to H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})$ is called the *volume flux group* of M and the homomorphism $\widetilde{\mathrm{Flux}}\colon \widetilde{\mathrm{Diff}^\infty_\Omega}(M)_0 \to H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})$ descends to the homomorphism $\mathrm{Flux}\colon \mathrm{Diff}^\infty_\Omega(M)_0 \to H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})/G_\Omega$, which is also called the Ω-flux homomorphism.

§3. Proofs

In this section, we give proofs of Theorems 1.1 and 1.2. The following theorem is mentioned in [6] without proof.

Theorem 3.1. *The linear map*

$$\Gamma\colon H^1(\pi_1(M);\mathbb{R}) \to H^1(\mathrm{Diff}^\infty_\Omega(M)_0;\mathbb{R})$$

is injective.

We give a proof of Theorem 3.1. Let $\beta \in \pi_1(M; x^0)$. Then we can choose a loop l representing β without self-intersection. Choose a tubular neighborhood $N \subset M$ of l and a diffeomorphism $\varphi\colon N \to D^{n-1} \times S^1$. Let (z, s) be the coordinate on $D^{n-1} \times S^1$. We may assume that there exists $\Omega' \in A^{n-1}(D^{n-1}; \mathbb{R})$ such that $\varphi^*(\Omega' ds) = \Omega|_N$ by changing the neighborhood N and the diffeomorphism φ if necessary. Let $\omega\colon D^{n-1} \to \mathbb{R}$ be a function such that $\omega(z) = 0$ in a neighborhood of the boundary. We define the volume-preserving diffeomorphism f_ω of $D^{n-1} \times S^1$ by

$$f_\omega(z, s) = (z, s + \omega(z)).$$

and define $F_\omega \in \mathrm{Diff}^\infty_\Omega(M)_0$ to be the identity outside N and $F_\omega = \varphi^{-1} f_\omega \varphi$ in N.

Lemma 3.2. *For any* $\phi \in H^1(\pi_1(M); \mathbb{R})$,

$$\Gamma(\phi)(F_\omega) = \phi(\beta) \int_{z \in D^{n-1}} \omega(z)\Omega'.$$

Proof. Note that the base point x^0 of M is in N. Let us denote $\varphi(x^0)$ by (z^0, s^0) and $\varphi(x)$ by (z^1, s^1). Let v be the smallest non-negative number such that $s^1 + v = s^0$. For each $x \in N$ we define the paths $l_1, l_2, l_3\colon [0, 1] \to D^{n-1} \times S^1$ by

$$\begin{aligned} l_1(t) &= (tz^0 + (1 - t)z^1, s^1), \\ l_2(t) &= (z^0, s^1 + tv), \\ l_3(t) &= (z^1, s^1 + t(\omega(z^1) - [\omega(z^1)])). \end{aligned}$$

We define the homotopy classes ζ_x, η_x of loops in M by

$$\zeta_x = [(\varphi^{-1})_*(l_2 l_1) a_x], \quad \eta_x = [a^{-1}_{F_\omega(x)} (\varphi^{-1})_*(l_3) a_x].$$

Since the path $\{F_{t\omega}\}_{t \in [0,1]}$ connects the identity and F_ω in $\mathrm{Diff}^\infty_\Omega(M)_0$, the homotopy class $\gamma(F_\omega; x)$ can be written as

$$\gamma(F_\omega; x) = \eta_x \zeta_x^{-1} \beta^{[\omega(z')]} \zeta_x$$

if $x \in N$. On the other hand, the homotopy class $\gamma(F_\omega; x)$ is trivial if $x \notin N$. Therefore,

$$\begin{aligned} \Gamma(\phi)(F_\omega) &= \int_{x \in N} \phi(\gamma(F_\omega; x))\Omega \\ &= \phi(\beta) \int_{x \in N} [\omega(z')]\Omega + \int_{x \in N} \phi(\eta_x)\Omega. \end{aligned}$$

Since $F_\omega^k = F_{k\omega}$ for any $k \in \mathbb{Z}$,

$$\Gamma(\phi)(F_\omega) = \lim_{k\to\infty} \frac{1}{k}\Gamma(\phi)(\gamma(F_{k\omega};x))\Omega.$$

Since the domain N is compact, the value $\phi(\eta_x)$ is bounded and thus we have

$$\begin{aligned}\Gamma(\phi)(F_\omega) &= \phi(\beta)\int_{x\in N} \omega(x)\Omega \\ &= \phi(\beta)\int_{z\in D^{n-1}} \omega(z)\Omega'.\end{aligned}$$

Q.E.D.

Proof of Theorem 3.1. Suppose a homomorphism

$$\phi \in H^1(\pi_1(M);\mathbb{R})$$

is non-trivial. Then there exists a homotopy class β of a loop without self-intersection in M such that $\phi(\beta) \neq 0$. It is sufficient to prove that there exists $g \in \mathrm{Diff}^\infty_\Omega(M)_0$ such that $\Gamma(\phi)(g) \neq 0$. If we choose a function $\omega\colon D^{n-1} \to \mathbb{R}$ such that

$$\int_{z\in D^{n-1}} \omega(z)\Omega' \neq 0,$$

then by Lemma 3.2 we have $\Gamma(\phi)(F_\omega) \neq 0$. Q.E.D.

Proof of Theorem 1.1. It is known that the flux homomorphism gives the abelianization of the group $\mathrm{Diff}^\infty_\Omega(M)_0$ [1]. Hence for any homomorphism $\phi \in H^1(\pi_1(M);\mathbb{R})$ there exists a homomorphism

$$A_\phi\colon H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})/G_\Omega \to \mathbb{R}$$

such that the homomorphism $\Gamma(\phi) \in H^1(\mathrm{Diff}^\infty_\Omega(M)_0;\mathbb{R})$ can be represented by the composition of homomorphisms $\mathrm{Flux}\colon \mathrm{Diff}^\infty_\Omega(M)_0 \to H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})/G_\Omega$ and $A_\phi\colon H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})/G_\Omega \to \mathbb{R}$. That is,

$$\Gamma(\phi) = A_\phi \circ \mathrm{Flux}\colon \mathrm{Diff}^\infty_\Omega(M)_0 \to \mathbb{R}.$$

Since the diffeomorphism F_ω is the time 1-map of the time independent vector field

$$X_x = \begin{cases} (\varphi^{-1})_*\big(\omega(z)\frac{d}{ds}\big) & \text{if} \quad x \in N, \\ 0 & \text{if} \quad x \notin N, \end{cases}$$

we have

$$\mathrm{Flux}(F_\omega) = \iota_X\Omega = \varphi^*[\omega(z)\Omega'].$$

In particular,

$$\mathrm{Flux}(F_{t\omega}) = t\,\mathrm{Flux}(F_\omega)$$

for any $\beta \in \pi_1(M)$, any function $\omega\colon D^{n-1} \to \mathbb{R}$ and any $t \in \mathbb{R}$. On the other hand by Lemma 3.2

$$\Gamma(\phi)(F_{t\omega}) = t\Gamma(\phi)(F_\omega)$$

for any $t \in \mathbb{R}$. Choose elements $\beta_1, \dots, \beta_m \in \pi_1(M, x^0)$ whose images by the projection $\pi_1(M, x^0) \to H_1(M;\mathbb{Z})$ form a basis of $H_1(M;\mathbb{R})$. If we replace β with $\beta_1, \dots, \beta_m$, then $(n-1)$-classes $\varphi^*[\omega(z)\Omega']$'s form a basis of $H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})$. Hence if there exists a non-trivial element $\xi \in G_\Omega$, then $A_\phi(t\xi) = 0$ for any $t \in \mathbb{R}$. The map A_ϕ descends to the $\mathbb{R}$-linear map $A'_\phi\colon H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})/\langle G_\Omega\rangle \to \mathbb{R}$, where $\langle G_\Omega\rangle$ means the vector subspace of $H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})$ spanned by elements of G_Ω.

By Theorem 3.1,

$$\mathrm{rank}_{\mathbb{R}}\, H^1(M;\mathbb{R}) = \mathrm{rank}_{\mathbb{R}}\, \mathrm{Im}\,\Gamma \le \mathrm{rank}_{\mathbb{R}}\, \mathrm{Hom}(H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})/\langle G_\Omega\rangle, \mathbb{R}).$$

If there exists a non-trivial element $\xi \in G_\Omega$, then

$$\mathrm{rank}_{\mathbb{R}}\, \mathrm{Hom}(H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})/\langle G_\Omega\rangle, \mathbb{R}) < \mathrm{rank}_{\mathbb{R}}\, H^{n-1}(M;\mathbb{R})$$

while by the Poincaré duality

$$\mathrm{rank}_{\mathbb{R}}\, H^1(M;\mathbb{R}) = \mathrm{rank}_{\mathbb{R}}\, H^{n-1}(M;\mathbb{R}).$$

This contradiction shows that there are no non-trivial elements in G_Ω.

Q.E.D.

Proof of Theorem 1.2. The statement is that

$$A_\phi = \phi \circ PD \circ I^{n-1}\colon H^{n-1}_{\mathrm{dR}}(M;\mathbb{R}) \to \mathbb{R}.$$

Since $A_\phi\colon H^{n-1}_{\mathrm{dR}}(M;\mathbb{R}) \to \mathbb{R}$ is an $\mathbb{R}$-linear map, it is sufficient to choose $\eta_1, \dots, \eta_m$ generating $H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})$ and prove that $A_\phi(\eta_i) = \phi \circ PD \circ I^{n-1}(\eta_i)$ for $1 \le i \le m$.

Since

$$\mathrm{Flux}(F_\omega) = \iota_X\Omega = \varphi^*[\omega(z)\Omega'],$$

we have

$$I^{n-1} \circ \mathrm{Flux}(F_\omega)(\sigma) = \int_{\varphi_*\sigma} \omega(z)\Omega'.$$

Therefore,

$$PD \circ I^{n-1} \circ \mathrm{Flux}(F_\omega) = \left(\int_{z\in D^{n-1}} \omega(z)\Omega'\right)\beta.$$

Comparing this equation with Lemma 3.2, we have

$$\Gamma(\phi)(F_\omega) = \phi \circ PD \circ I^{n-1} \circ \mathrm{Flux}(F_\omega)$$

for any $\phi \in H^1(M;\mathbb{R})$.

As in the proof of Theorem 1.1, choose homotopy classes $\beta_1, \ldots,$ $\beta_m \in \pi_1(M, x^0)$ whose images by the projection $\pi_1(M, x^0) \to H_1(M;\mathbb{Z})$ form a basis of $H_1(M;\mathbb{R})$. If we replace β with $\beta_1, \ldots, \beta_m$, then $\mathrm{Flux}(F_\omega)$'s form a basis of $H^{n-1}_{\mathrm{dR}}(M;\mathbb{R})$ and hence this completes the proof. Q.E.D.

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