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# A few open problems on characteristic classes of foliations

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**Abstract.**

In this paper, we consider a few major open problems in the theory of characteristic classes of foliations. After recalling some history and background of each of these problems, we propose possible approaches to challenge them and present several "sub-problems".

## §1. Introduction

The theory of characteristic classes of foliations was initiated by the discovery, in the beginning of the 1970's, of the Godbillon–Vey class of codimension 1 foliations [14] and a ground-breaking work of Thurston [42] proving that it can vary continuously. Soon after this, Bott and Haefliger [5], and also Bernstein and Rozenfeld [3] presented a general framework for this theory and during the 1970's, it has been developed extensively by many people including Heitsch [20] and Hurder [22]. There also appeared the closely related theory of Gelfand and Fuks developed in their paper [11] on the cohomology of the Lie algebra of vector fields on manifolds and that of Chern and Simons [8] on the secondary characteristic classes of principal bundles. The notions of $\Gamma$-structures and their classifying spaces due to Haefliger [17] played a crucial role in this theory and Mather [30] and Thurston [43] obtained many fundamental results by using them.

However there remain numbers of important problems to be solved in future. In this paper, we focus on the following two major problems both of which turn out to be extremely difficult. One is the determination of the homotopy type of the classifying space $B\Gamma_1$ of $\Gamma_1$-structures

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in the $C^\infty$ category as well as the real analytic category. The other is development of characteristic classes of *transversely symplectic* foliations. As the most typical examples of these two problems, we discuss the cases of foliated $S^1$-bundles as well as foliated surface bundles in details.

## §2. Homotopy type of $\mathrm{B}\Gamma_1$

We denote by $\mathrm{B}\Gamma_n^\infty$ the Haefliger classifying space of $\Gamma_n$-structures in the $C^\infty$ category. Let $\mathrm{B}\overline{\Gamma}_n^\infty$ denote the homotopy fiber of the natural map $\mathrm{B}\Gamma_n^\infty \to \mathrm{BGL}(n,\mathbb{R})$. In this paper, we mainly consider the case $n = 1$. In this case, $\mathrm{B}\overline{\Gamma}_1^\infty$ is the homotopy fiber of the map $w_1 \colon \mathrm{B}\Gamma_1^\infty \to \mathrm{BGL}(1,\mathbb{R}) = K(\mathbb{Z}/2,1)$, corresponding to the first Stiefel–Whitney class. In other words, it is the classifying space of transversely oriented $\Gamma_1$-structures in the $C^\infty$ category. Haefliger [16] proved that $\mathrm{B}\overline{\Gamma}_n^\infty$ is $n$-connected for any $n$ and Mather proved that $\mathrm{B}\overline{\Gamma}_1^\infty$ is 2-connected. The Godbillon–Vey class can be considered as an element in $H^3(\mathrm{B}\Gamma_1^\infty;\mathbb{R})$ and Thurston [42] proved that it induces a surjection

$$\pi_3(\mathrm{B}\Gamma_1^\infty) \cong H_3(\mathrm{B}\Gamma_1^\infty;\mathbb{Z}) \twoheadrightarrow \mathbb{R}.$$

Although Tsuboi [46] obtained an interesting result about the kernel of the homomorphism $\pi_3(\mathrm{B}\Gamma_1^\infty) \twoheadrightarrow \mathbb{R}$, it remains unsettled whether $\pi_3(\mathrm{B}\Gamma_1^\infty) \cong \mathbb{R}$ or not. This is already a very difficult open problem. However, it is only a tiny part of the following major open problem in the theory of foliations.

**Problem 2.1.** Determine the homotopy type of $\mathrm{B}\Gamma_1^\infty$. More precisely, determine whether the classifying map

$$\mathrm{GV} \colon \mathrm{B}\overline{\Gamma}_1^\infty \to K(\mathbb{R},3),$$

induced by the Godbillon–Vey class, is a homotopy equivalence or not.

In [33], we introduced the concept of *discontinuous invariants* of foliations. One possible approach to attack the above problem would be to apply this general theory to the above particular case. In this case, it is based on the fact that the homology group $H_*(K(\mathbb{R},3);\mathbb{Z})$ is a huge group. More precisely, it can be described as

$$H_*(K(\mathbb{R},3);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & * = 0, \\ \bigwedge_{\mathbb{Z}}^k \mathbb{R} & * = 3k, \\ 0 & \text{otherwise.} \end{cases}$$

**Problem 2.2.** Determine whether the homomorphism

$$\mathrm{GV}_k\colon H_{3k}(\mathrm{B}\overline{\Gamma}_1^{\infty},\mathbb{Z}) \to \bigwedge_{\mathbb{Z}}^{k}\mathbb{R} \quad (\cong H_{3k}(K(\mathbb{R},3);\mathbb{Z})),$$

which is the homomorphism induced by the map GV on $H_{3k}$, is non-trivial or not.

In the case of piecewise linear (*PL* for short) category, Greenberg [15] showed that there is a weak homotopy equivalence

$$\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}} \sim \mathrm{B}\mathbb{R}^{\delta} * \mathrm{B}\mathbb{R}^{\delta}$$

where the right hand side represents the join of two copies of $\mathrm{B}\mathbb{R}^{\delta}$, $\mathbb{R}^{\delta}$ being the abelian group $\mathbb{R}$ equipped with the *discrete* topology. It follows that $\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}}$ is 2-connected and he described the integral homology group of $\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}}$ completely. It also follows that the higher homotopy groups of this space is highly non-trivial. In particular

$$\pi_3(\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}}) \cong \mathbb{R}\otimes_{\mathbb{Z}}\mathbb{R}.$$

On the other hand, Ghys and Sergiescu [13] defined a cohomology class

$$\mathrm{GV}^{\mathrm{dis}} \in H^3(\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}};\mathbb{R})$$

which is called the *discrete* Godbillon–Vey class. By making use of the above result of Greenberg, Tsuboi [45] showed that the homomorphism

$$\mathrm{GV}_k^{\mathrm{dis}}\colon H_{3k}(\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}},\mathbb{Z}) \to \bigwedge_{\mathbb{Z}}^{k}\mathbb{R}$$

associated with the discrete Godbillon–Vey class $\mathrm{GV}^{\mathrm{dis}}$ is trivial for all $k \geq 2$.

Going back to the case of $C^\infty$ category, nothing is known about the non-triviality of the homomorphim

$$\mathrm{GV}_2\colon H_6(\mathrm{B}\overline{\Gamma}_1^{\infty},\mathbb{Z}) \to \bigwedge_{\mathbb{Z}}^{2}\mathbb{R}. \tag{1}$$

In the above cited paper [33], we related this problem with a certain property about the homology of the group $\mathrm{Diff}_K^{\delta}\,\mathbb{R}$ of $C^\infty$ diffeomorphisms of $\mathbb{R}$ with compact supports equipped with the discrete topology. Here we briefly recall this. Let us define an injective homomorphism

$$\mu\colon \mathrm{Diff}^{\delta}_{K}\,\mathbb{R} \times \mathrm{Diff}^{\delta}_{K}\,\mathbb{R} \to \mathrm{Diff}^{\delta}_{K}\,\mathbb{R}$$

as follows. Choose orientation preserving diffeomorphisms $\iota_-\colon \mathbb{R} \cong (-\infty, 0)$ and $\iota_+\colon \mathbb{R} \cong (0,\infty)$ and set

$$\mu(f,g)(t) = \begin{cases} \iota_- \circ f \circ \iota_-^{-1}(t) & t < 0, \\ 0 & t = 0, \\ \iota_+ \circ g \circ \iota_+^{-1}(t) & t > 0. \end{cases}$$

This homomorphism $\mu$ induces a certain product on the homology group $H_*(\mathrm{BDiff}^{\delta}_{K}\,\mathbb{R};\mathbb{Q})$ which we call the $*$-product. By making a crucial use of Mather's result [30] together with a classical theorem of Samelson in [40], we obtained the following result.

**Proposition 2.3** (see [33], somewhat rearranged)**.** (i) *The following two statements are equivalent.*

(a) *All the Whitehead products on* $\pi_*(\mathrm{B}\overline{\Gamma}_1^{\infty})$ *have finite orders.*

(b) *The* $*$*-product on* $H_*(\mathrm{BDiff}^{\delta}_{K}\,\mathbb{R};\mathbb{Q})$ *is graded commutative, namely the equality*

$$u * v = (-1)^{pq} v * u$$

*holds for any* $u \in H_p(\mathrm{BDiff}^{\delta}_{K}\,\mathbb{R};\mathbb{Q})$ *and* $v \in H_q(\mathrm{BDiff}^{\delta}_{K}\,\mathbb{R};\mathbb{Q})$*.*

(ii) *Assume that the statement* (i)-(b) *above concerning the* $*$*-product holds for* $p = q = 2$*. Then the homomorphism* (1) *is almost surjective in the sense that its cokernel is a torsion group.*

Observe here that if $\mathrm{B}\overline{\Gamma}_1^{\infty}$ is a $K(\mathbb{R},3)$, then all the Whitehead products on $\pi_*(\mathrm{B}\overline{\Gamma}_1^{\infty})$ clearly vanish.

**Problem 2.4.** Prove (or disprove) that the $*$-product on the homology group $H_*(\mathrm{BDiff}^{\delta}_{K}\,\mathbb{R};\mathbb{Q})$ is graded commutative. In particular, prove (or disprove) that the $*$-product on $H_2(\mathrm{BDiff}^{\delta}_{K}\,\mathbb{R};\mathbb{Q})$ is commutative.

In a certain case of low differentiability (Lipschitz with bounded variation of derivatives), Tsuboi [47] settled the above problem for this group affirmatively, thereby proved that the second discontinuous invariant

$$\mathrm{GV}_2^{\mathrm{Lip,bdd}}\colon H_6(\mathrm{B}\overline{\Gamma}_1^{\mathrm{Lip,bdd}},\mathbb{Z}) \to \bigwedge_{\mathbb{Z}}^{2}\mathbb{R}$$

is almost surjective where $\mathrm{GV}^{\mathrm{Lip,bdd}}$ is the one he extended to this case.

The Godbillon–Vey class can be defined for transversely holomorphic foliations with trivialized normal bundles as well. More precisely, it is defined as an element

$$\mathrm{GV}^{\mathbb{C}} \in H^3(\mathrm{B}\overline{\Gamma}_1^{\mathbb{C}};\mathbb{C})$$

where $\mathrm{B}\overline{\Gamma}_1^{\mathbb{C}}$ denotes the Haefliger classifying space of transversely holomorphic $\Gamma_1$-structures with trivialized normal bundles. Bott [4] proved that the associated homomorphism

$$\mathrm{GV}^{\mathbb{C}}\colon \pi_3(\mathrm{B}\overline{\Gamma}_1^{\mathbb{C}}) \to \mathbb{C}$$

is surjective. Also it is known that $\mathrm{B}\overline{\Gamma}_1^{\mathbb{C}}$ is 2-conneced (see Haefliger and Sithanantham [18]).

**Problem 2.5.** Determine the homotopy type of $\mathrm{B}\Gamma_1^{\mathbb{C}}$. More precisely, determine whether the classifying map

$$\mathrm{GV}^{\mathbb{C}}\colon \mathrm{B}\overline{\Gamma}_1^{\mathbb{C}} \to K(\mathbb{C},3)$$

induced by the complex Godbillon–Vey class, is a homotopy equivalence or not.

We refer to a book [1] by Asuke for a recent study of $\mathrm{GV}^{\mathbb{C}}$.

Finally we recall a closely related problem. Let $\mathcal{M}^h(3)$ denote the set of orientation preserving diffeomorphism classes of closed oriented *hyperbolic* 3-manifolds. For any such manifold $M$, we have its volume $\mathrm{vol}(M)$ and the $\eta$-invariant $\eta(M)$ of Atiyah–Patodi–Singer [2]. The combination $\eta + i\,\mathrm{vol}$ gives rise to a mapping

$$\eta + i\,\mathrm{vol}\colon \mathcal{M}^h(3) \to \mathbb{C}.$$

**Problem 2.6** (Thurston ([44], Questions 22, 23)**.** Study the above map. In particular, determine whether the dimension over $\mathbb{Q}$ of the $\mathbb{Q}$-subspace of $i\mathbb{R}$ generated by the second component of the image of the above map is infinite or not.

Recall that any such $M$ defines a homology class

$$[M] \in H_3(\mathrm{PSL}(2,\mathbb{C})^{\delta};\mathbb{Z})$$

and we have the following closely related problems.

**Problem 2.7.** Determine the image of the map

$$\mathcal{M}^h(3) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z}) \xrightarrow{(\mathrm{CS}, i\,\mathrm{vol})} \mathbb{C}/\mathbb{Z}$$

where CS denotes the Chern Simons invariant. Recall that CS is an invariant defined modulo integers and $\eta$ is an integral lift of it.

**Problem 2.8.** Study the discontinuous invariants of the group $\mathrm{PSL}(2,\mathbb{C})^\delta$ associated with the above classes. In particular, determine the value of the *total Chern Simons invariant* introduced in Dupont [9].

## §3. Real analytic foliations of codimension 1

Let $\mathrm{B}\overline{\Gamma}_1^\omega$ denote the classifying space of transversely oriented *real analytic* $\Gamma_1$-structures. Haefliger proved the following theorem.

**Theorem 3.1** (Haefliger [17]). $\mathrm{B}\overline{\Gamma}_1^\omega$ *is a* $K(\pi,1)$ *space for certain group* $\Gamma_H$ *which has the following three properties.*

(i) *All the non-trivial elements of* $\Gamma_H$ *are of infinite order,*
(ii) $\Gamma_H$ *is perfect, namely it is equal to its commutator subgroup,*
(iii) $\Gamma_H$ *is uncountable.*

We may call $\Gamma_H$ the *Haefliger group*. It may be said that almost nothing is known about this group beyond the above properties proved by Haefliger. There exists a natural forgetful map

$$\mathrm{B}\overline{\Gamma}_1^\omega \to \mathrm{B}\overline{\Gamma}_1^\infty \tag{2}$$

which is of course far from being a homotopy equivalence. However, since $\Gamma_H$ is a perfect group, we can apply the Quillen's plus construction (see [39]) on $\mathrm{B}\overline{\Gamma}_1^\omega$ to obtain a map

$$\mathrm{B}\overline{\Gamma}_1^\omega = K(\Gamma_H,1) \to (\mathrm{B}\overline{\Gamma}_1^\omega)^+ \tag{3}$$

which induces isomorphism on homology. Furthermore, it is known that the above map (2) factors through (3) so that there exists an induced mapping

$$(\mathrm{B}\overline{\Gamma}_1^\omega)^+ \to \mathrm{B}\overline{\Gamma}_1^\infty.$$

**Problem 3.2.** Determine whether the natural map

$$(\mathrm{B}\overline{\Gamma}_1^\omega)^+ \to \mathrm{B}\overline{\Gamma}_1^\infty$$

is a homotopy equivalence or not.

Both of $(\mathrm{B}\overline{\Gamma}_1^\omega)^+$ and $\mathrm{B}\overline{\Gamma}_1^\infty$ are simply connected and the latter space is known to be 2-connected as mentioned in the previous section. Thus the first problem is to decide whether $\pi_2(\mathrm{B}\overline{\Gamma}_1^\omega)^+ \cong H_2(\Gamma_H;\mathbb{Z})$ is trivial or not. This is already a very difficult problem.

As for the 3 dimensional homology group of $\mathrm{B}\overline{\Gamma}_1^\omega$, let us recall here the following result of Thurston. He constructed a family of real analytic codimension 1 foliations on a certain 3-manifold by making use of the group

$$\mathrm{SL}(2,\mathbb{R}) *_{\mathrm{SO}(2)} \widetilde{\mathrm{SL}}(2,\mathbb{R})_n \subset \mathrm{Diff}_+^\omega S^1$$

thereby proving that the homomorphism

$$\mathrm{GV}\colon H_3(\mathrm{B}\overline{\Gamma}_1^\omega;\mathbb{Z}) \cong H_3(\Gamma_H;\mathbb{Z}) \to \mathbb{R}$$

is *surjective*. Here $\widetilde{\mathrm{SL}}(2,\mathbb{R})_n$ denotes the $n$-fold covering group of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{Diff}_+^\omega S^1$ the group of orientation preserving real analytic diffeomorphisms of $S^1$.

The following problem is the real analytic version of Problem 2.2 and it should be a very deep and challenging problem to determine whether these two problems have the same answer or not.

**Problem 3.3.** Determine whether the homomorphism

$$\mathrm{GV}_k\colon H_{3k}(\mathrm{B}\overline{\Gamma}_1^\omega;\mathbb{Z}) \cong H_{3k}(\Gamma_H;\mathbb{Z}) \to \bigwedge_{\mathbb{Z}}^{k} \mathbb{R}$$

induced by the discontinuous invariants associated with the Godbillon–Vey class is non-trivial or not.

Next let $\mathrm{Diff}_+^\omega \mathbb{R}$ denote the group consisting of all the orientation preserving real analytic diffeomorphisms of $\mathbb{R}$ and let $\mathrm{Diff}_+^{\omega,\delta} \mathbb{R}$ denote the same group with the discrete topology. Then, there exists a natural mapping

$$\mathrm{BDiff}_+^{\omega,\delta} \mathbb{R} \to \mathrm{B}\overline{\Gamma}_1^\omega$$

which classifies the universal foliated $\mathbb{R}$-bundle over the classifying space $\mathrm{BDiff}_+^{\omega,\delta} \mathbb{R}$. This induces a homomorhism

$$\rho\colon \mathrm{Diff}_+^\omega \mathbb{R} \to \Gamma_H. \tag{4}$$

**Problem 3.4.** (i) Determine the kernel of the above homomorphism $\rho\colon \mathrm{Diff}_+^\omega \mathbb{R} \to \Gamma_H$.

(ii) Study general property of elements of $\Gamma_H$ which are outside of the image of the above homomorphism.

Here we would like to make a few remarks for the former part of the above problem. Let $T \in \mathrm{Diff}^{\omega}_{+}\,\mathbb{R}$ be the element defined by $T(x) = x+1$ $(x \in \mathbb{R})$. Then it is easy to see that the element $\rho(T) \in \Gamma_H$ classifies the $\overline{\Gamma}^{\omega}_{1}$ structure on $S^1$ induced by its natural structure of a real analytic 1 dimensional manifold.

On the other hand, the universal covering group, denoted by $\widetilde{\mathrm{Diff}}^{\omega}_{+}S^1$, of $\mathrm{Diff}^{\omega}_{+}\,S^1$ can be described by the central extension

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{Diff}}^{\omega}_{+}S^1 \to \mathrm{Diff}^{\omega}_{+}\,S^1 \to 1.$$

It follows that there exists a natural isomorphism

$$\widetilde{\mathrm{Diff}}^{\omega}_{+}S^1 \cong \{\varphi \in \mathrm{Diff}^{\omega}_{+}\,\mathbb{R}; f \circ T = T \circ f\}$$

where $1 \in \mathbb{Z}$ corresponds to $T$. In other words, the group $\widetilde{\mathrm{Diff}}^{\omega}_{+}S^1$ can be identified with the centralizer of $T$ in $\mathrm{Diff}^{\omega}_{+}\,\mathbb{R}$. Now Herman [21] proved that $\mathrm{Diff}^{\omega}_{+}\,S^1$ is a *simple* group. Therefore the non trivial normal subgroups of the group $\widetilde{\mathrm{Diff}}^{\omega}_{+}S^1$ are the center $\mathbb{Z}$ and its subgroups. We know already that $\rho(T) \neq 1 \in \Gamma_H$ which has an infinite order. We can now conclude that the homomorphism $\rho$ (see (4)) is *injective* on the subgroup $\widetilde{\mathrm{Diff}}^{\omega}_{+}S^1 \subset \mathrm{Diff}^{\omega}_{+}\,\mathbb{R}$. Summarizing, we can say that

The Haefliger group $\Gamma_H$ contains a subgroup isomorphic to $\widetilde{\mathrm{Diff}}^{\omega}_{+}S^1$.

It seems to be unkown whether the group $\mathrm{Diff}^{\omega}_{+}\,\mathbb{R}$ is simple or not. We put this as a problem.

**Problem 3.5.** Determine whether the group $\mathrm{Diff}^{\omega}_{+}\,\mathbb{R}$ is simple or not.

## §4. Characteristic classes of transversely symplectic foliations

One surprising feature of the Gelfand–Fuks cohomology theory (see [11]) was that

$$\dim H^*_c(\mathfrak{a}_n) < \infty$$

where $\mathfrak{a}_n$ denotes the Lie algebra consisting of all the formal vector fields on $\mathbb{R}^n$. The associated characteristic homomorphism

$$\Phi\colon H^*_c(\mathfrak{a}_n) \to H^*(\mathrm{B}\overline{\Gamma}^{\infty}_{n}; \mathbb{R})$$

is now relatively well understood, although there still remain unsolved problems related to rigid classes. In contrast with this, the case of all the *volume preserving* formal vector fields $\mathfrak{v}_n \subset \mathfrak{a}_n$ and that of all the

*Hamiltonian* formal vector fields $\mathfrak{ham}_{2n} \subset \mathfrak{a}_{2n}$ are both far from being understood.

**Problem 4.1.** Compute

$$H_c^*(\mathfrak{v}_n), \quad H_c^*(\mathfrak{v}_n, \mathrm{O}(n)), \quad H_c^*(\mathfrak{ham}_{2n}), \quad H_c^*(\mathfrak{ham}_{2n}, \mathrm{U}(n)),$$

where we may replace the last one by $H_c^*(\mathfrak{ham}_{2n}, \mathfrak{Sp}(2n, \mathbb{R}))$ which is a little bit easier to compute. In particular, prove (or disprove) that

$$\dim H_c^*(\mathfrak{v}_n) = \infty, \quad \dim H_c^*(\mathfrak{ham}_{2n}) = \infty.$$

Recall here that there are very few known results concerning this problem. First, Gelfand, Kalinin and Fuks [12] found an *exotic* class

$$\text{GKF class} \in H_c^7(\mathfrak{ham}_2, \mathfrak{Sp}(2, \mathbb{R}))_8$$

and later Metoki [31] found another exotic class

$$\text{Metoki class} \in H_c^9(\mathfrak{ham}_2, \mathfrak{Sp}(2, \mathbb{R}))_{14}.$$

On the other hand, Perchik [38] obtained a formula for the Euler characteristic and computed it up to certain degree. It suggests strongly that the cohomology would be infinite dimensional.

Let $\mathrm{B}\Gamma_{2n}^{\mathrm{symp}}$ denote the Haefliger classifying space of transversely symplectic foliations of codimension $2n$. Then there exists a homomorphism

$$\Phi\colon H_c^*(\mathfrak{ham}_{2n}, \mathfrak{Sp}(2n, \mathbb{R})) \to H^*(\mathrm{B}\Gamma_{2n}^{\mathrm{symp}}; \mathbb{R}).$$

**Problem 4.2.** Prove that, under the homomorphism

$$\Phi\colon H_c^*(\mathfrak{ham}_2, \mathfrak{Sp}(2, \mathbb{R})) \to H^*(\mathrm{B}\Gamma_2^{\mathrm{symp}}; \mathbb{R}),$$

the GKF class and the Metoki class survive as non-trivial characteristic classes.

Recall here that the geometric non-triviality of the Godbillon–Vey class was proved almost immediately after its definition appeared while that of the GKF class remain unanswered for more than 40 years after its discovery to the present.

Meanwhile Kontsevich [26] introduced a new viewpoint in this situation. He considered two Lie subalgebras

$$\mathfrak{ham}_{2g}^1 \subset \mathfrak{ham}_{2g}^0 \subset \mathfrak{ham}_{2g}$$

consisting of Hamiltonian formal vector fields without constant terms and without constant as well as linear terms, respectively. Then he constructed a homomorphism

$$\Psi\colon H^*_c(\mathfrak{ham}^0_{2g}, \mathfrak{Sp}(2g,\mathbb{R})) \cong H^*_c(\mathfrak{ham}^1_{2g})^{\mathfrak{Sp}} \to H^*_{\mathcal{F}}(M)$$

for any transversely symplectic foliation $\mathcal{F}$ on a smooth manifold $M$ of codimension $2n$, where $H^*_{\mathcal{F}}(M)$ denotes the foliated cohomology group. By using this viewpoint, in a joint work with Kotschick [28] we proved that the Gelfand-Kalinin-Fuks class can be expressed, cohomologically *uniquely*, as a product

$$\text{GKF class} = \eta \wedge \omega$$

where $\eta \in H^5_c(\mathfrak{ham}^0_2, \mathfrak{Sp}(2,\mathbb{R}))_{10}$ is a certain leaf cohomology class and $\omega$ denotes the transverse symplectic form. In that paper, we made a conjecture that the Metoki class has a similar decomposition. Recently, Mikami [32] solved it affirmatively. Namely he proved that the Metoki class can also be expressed, cohomologically *uniquely*, as a product

$$\text{Metoki class} = \eta' \wedge \omega$$

for some $\eta' \in H^7_c(\mathfrak{ham}^0_2, \mathfrak{Sp}(2,\mathbb{R}))_{16}$.

Let $\mathrm{Symp}^\delta(\mathbb{R}^2, o)$ denote the group of area-preserving diffeomorphisms of the plane $\mathbb{R}^2$ which fix the origin $o$, equipped with the discrete topology. Then we have a natural homomorphism

$$\Psi\colon H^*_c(\mathfrak{ham}^0_2, \mathfrak{Sp}(2,\mathbb{R})) \to H^*(\mathrm{BSymp}^\delta(\mathbb{R}^2, o);\mathbb{R}).$$

The following problem would be easier than Problem 4.2.

**Problem 4.3.** Prove that, under the homomorphism

$$\Psi\colon H^*_c(\mathfrak{ham}^0_2, \mathfrak{Sp}(2,\mathbb{R})) \to H^*(\mathrm{BSymp}^\delta(\mathbb{R}^2, o);\mathbb{R}),$$

the leaf cohomology classes $\eta, \eta' \in H^*_c(\mathfrak{ham}^0_2, \mathfrak{Sp}(2,\mathbb{R}))$ survive as non-trivial cohomology classes in the target.

If we replace $\mathrm{Symp}^\delta(\mathbb{R}^2, o)$ with the group of germs, at the origin, of its elements, then we obtain slightly easier problem. Here the work of Ishida in [23] would be suggestive in challenging the above problem.

On the other hand, $\mathfrak{ham}^0_{2g}, \mathfrak{ham}^1_{2g}$ can be described as

$$\mathfrak{ham}^0_{2n} = \widehat{\mathfrak{c}}_n \otimes \mathbb{R}, \quad \mathfrak{ham}^1_{2n} = \widehat{\mathfrak{c}}^+_n \otimes \mathbb{R}$$

where $\mathfrak{c}_n$ denotes one of the three Lie algebras (commutative one) in Kontsevich's theory [24], [25] of graph homology and $\widehat{\mathfrak{c}}_n$ denotes its completion. Thus the above homomorphim $\Psi$ can be written as

$$\Psi\colon H_c^*(\widehat{\mathfrak{c}}_n^+)^{\mathfrak{Sp}} \otimes \mathbb{R} \cong H_c^*(\mathfrak{ham}_{2n}^1)^{\mathfrak{Sp}} \to H_{\mathcal{F}}^*(M).$$

Besides the theory of transversely symplectic foliations as above, the graph homology of $\mathfrak{c}_n$ has another deep connection with the theory of *finite type* invariants for homology 3-spheres initiated by Ohtsuki [37] which we briefly recall. Let $\mathcal{A}(\phi)$ denote the commutative algebra generated by vertex oriented connected trivalent graphs modulo the (AS) relation together with the (IHX) relation. This algebra plays a fundamental role in this theory. In fact, the completion $\widehat{\mathcal{A}}(\phi)$ of $\mathcal{A}(\phi)$ with respect to its gradings is the target of the LMO invariant [29].

By using a result of Garoufalidis and Nakamura [10], in a joint work with Sakasai and Suzuki [35] we constructed an injection

$$\mathcal{A}(\phi) \to H_*(\mathfrak{c}_\infty^+)^{\mathfrak{Sp}}$$

and defined the "complementary" algebra $\mathcal{E}$ so as to obtain an isomorphism

$$H_*(\mathfrak{c}_\infty^+)^{\mathfrak{Sp}} \cong \mathcal{A}(\phi) \otimes \mathcal{E}$$

of bigraded algebras. $\mathcal{E}$ can be interpreted as the dual of the space of all the *exotic* stable leaf cohomology classes for transversely symplectic foliations.

**Problem 4.4** (cf. Sakasai–Suzuki–Morita. [35])**.** Study the structure of $\mathcal{E}$.

## §5. Homology of $\mathrm{Diff}^\delta M$ and $\mathrm{Symp}^\delta(M,\omega)$

In general, homology group of the diffeomorphism group $\mathrm{Diff}^\delta M$ of a closed $C^\infty$ manifold $M$, considered as a discrete group, or that of the symplectomorphism group $\mathrm{Symp}^\delta(M,\omega)$ of a closed symplectic manifold $(M,\omega)$, again with the discrete topology, is a widely open research area. One can also consider the real analytic case. Here we present a few problems in the cases of the circle $S^1$ and closed surfaces.

It was proved in our paper [34] that the natural homomorphism

$$\Phi\colon H_c^*(\mathcal{X}(S^1), \mathrm{SO}(2))) \cong \mathbb{R}[\alpha,\chi]/(\alpha\chi) \to H^*(\mathrm{BDiff}_+^\delta S^1;\mathbb{R})$$

from the Gelfand–Fuks cohomology of $S^1$, relative to $\mathrm{SO}(2) \subset \mathrm{Diff}_+ S^1$, to the cohomology of $\mathrm{Diff}_+^\delta S^1$, is injective. There were also given certain non-triviality results for the associated discontinuous invariants. We mention that Ghys and Sergiescu [13] gives a different proof of the non-triviality of the powers of the Euler class.

In particular, our proof of non-triviality of any power $\chi^n \neq 0 \in H^{2n}(\mathrm{BDiff}^{\delta}_{+} S^1;\mathbb{R})$ crucially depends on the following result of Thurston which is a special case of a general theorem (Theorem 5) given in [43].

**Theorem 5.1** (Thurston)**.** *Let*

$$\mathrm{B}\widetilde{\mathrm{Diff}}^{\delta}_{+} S^1 \times S^1 \to \mathrm{B}\overline{\Gamma}_1^{\infty}$$

*be the classifying map for the universal foliated $S^1$-bundle over $\mathrm{B}\widetilde{\mathrm{Diff}}^{\delta}_{+} S^1$ and let*

$$\mathcal{H}\colon \mathrm{B}\widetilde{\mathrm{Diff}}^{\delta}_{+} S^1 \to \Lambda\mathrm{B}\overline{\Gamma}_1^{\infty} \tag{5}$$

*be its adjoint map where $\Lambda\mathrm{B}\overline{\Gamma}_1^{\infty}$ denotes the free loop space of $\mathrm{B}\overline{\Gamma}_1^{\infty}$. Then $\mathcal{H}$ induces an isomorphism on homology.*

Now for each $k = 1, 2, \ldots$, we define

$$\varphi_k\colon \widetilde{\mathrm{Diff}}_{+} S^1 \to \widetilde{\mathrm{Diff}}_{+} S^1$$

to be the homomorphism given by

$$\varphi_k(f)(x) = \frac{1}{k} f(kx).$$

Then we proved in [34] that the induced homomorphism

$$(\varphi_k)_*\colon H_m(\mathrm{B}\widetilde{\mathrm{Diff}}^{\delta}_{+} S^1;\mathbb{Q}) \to H_m(\widetilde{\mathrm{Diff}}^{\delta}_{+} S^1;\mathbb{Q})$$

is *diagonalizable* in the sense that for any homology class

$$u \in H_m(\mathrm{B}\widetilde{\mathrm{Diff}}^{\delta}_{+} S^1;\mathbb{Q}),$$

there exists a finite dimensional linear subspace $V \subset H_m(\mathrm{B}\widetilde{\mathrm{Diff}}^{\delta}_{+} S^1;\mathbb{Q})$ which contains $u$ such that $V$ is $(\varphi_k)_*$-invariant and the homomorphism $(\varphi_k)_*\colon V \to V$ is diagonalizable with eigenvalues of the forms $k^e$ ($e = 0, 1, 2, \ldots$). Our proof uses the following two facts. One is that under the homology equivalence (5), the operation $\mathrm{B}\varphi_k$ on the left hand side corresponds to the operation

$$\psi_k\colon \Lambda\mathrm{B}\overline{\Gamma}_1^{\infty} \to \Lambda\mathrm{B}\overline{\Gamma}_1^{\infty}$$

on the right hand side which is given by $\psi_k(l)(t) = l(t^k)$ $(t \in S^1)$ for each $l\colon S^1 \to \overline{\Gamma}_1^\infty$. The other is a general method of describing the homology of free loop spaces due to Sullivan [41].

In these arguments, if we replace $\widetilde{\mathrm{Diff}}_+^\delta S^1$ and $\mathrm{B}\overline{\Gamma}_1^\infty$ with $\widetilde{\mathrm{Diff}}_+^{\omega,\delta} S^1$ and $\mathrm{B}\overline{\Gamma}_1^\omega$, respectively, we still have the mapping

$$\mathcal{H}^\omega\colon \mathrm{B}\widetilde{\mathrm{Diff}}_+^{\omega,\delta} S^1 \to \Lambda \mathrm{B}\overline{\Gamma}_1^\omega = \Lambda K(\Gamma_H, 1).$$

However, the space $\Lambda K(\Gamma_H, 1)$ is *not* connected and the above theorem of Thurston does not apply to this case. It follows that our proof does not work either and we have the following very important and difficult problem.

**Problem 5.2.** Prove (or disprove) that the homomorphism

$$\Phi\colon H_c^*(\mathcal{X}(S^1), \mathrm{SO}(2)) \cong \mathbb{R}[\alpha, \chi]/(\alpha\chi) \to H^*(\mathrm{BDiff}_+^{\omega,\delta} S^1; \mathbb{R})$$

is injective, where $\mathrm{Diff}_+^{\omega,\delta} S^1$ denotes the real analytic diffeomorphism group of $S^1$ equipped with the discrete topology. In particular, determine the maximum power $\chi^{n_0} \in H^{2n_0}(\mathrm{BDiff}_+^{\omega,\delta} S^1; \mathbb{R})$ of the Euler class which is non-trivial. Is $n_0$ finite or infinity?

As for the last part of the above problem, we propose the following.

**Problem 5.3.** Prove that the homomorphism

$$(\varphi_k)_*\colon H_m(\mathrm{B}\widetilde{\mathrm{Diff}}_+^{\omega,\delta} S^1; \mathbb{Q}) \to H_m(\widetilde{\mathrm{Diff}}_+^{\omega,\delta} S^1; \mathbb{Q})$$

is diagonalizable.

We mention that an affirmative answer to the above problem for the case $m = 2n - 1$ (with any $k \geq 2$) implies the non-triviality $\chi^n \neq 0 \in H^{2n}(\mathrm{B}\widetilde{\mathrm{Diff}}_+^{\omega,\delta} S^1; \mathbb{Q})$.

The following is a closely related general problem.

**Problem 5.4.** Determine whether the natural inclusion

$$\mathrm{Diff}_+^{\omega,\delta} S^1 \to \mathrm{Diff}_+^\delta S^1$$

induces an isomorphism on homology or not.

Of course one can consider the above problem for any closed manifold $M$.

Let $\Sigma_g$ denote a closed oriented surface of genus $g$. Harer stability theorem [19] states that the homology group $H_k(\mathrm{BDiff}_+ \Sigma_g)$ is independent of $g$ in a certain stable range $k \ll g$ (see a survey paper [48] by Wahl for more details).

By applying a general method, we can define certain characteristic classes for foliated $\Sigma_g$-bundles, namely elements of $H^*(\mathrm{BDiff}^\delta_+ \Sigma_g; \mathbb{R})$. Also, in [27] certain characteristic classes for foliated $\Sigma_g$-bundles with area-preserving holonomy, namely elements of $H^*(\mathrm{BSymp}^\delta_+ \Sigma_g; \mathbb{R})$, were defined by making use of the notion of the flux homomorphism. These classes are all *stable* with respect to the genus $g$ and it seems reasonable to propose the following.

**Problem 5.5.** Determine whether certain analogue of Harer stability theorem holds for the group $\mathrm{Diff}^\delta \Sigma_g$ and/or $\mathrm{Symp}^\delta \Sigma_g$.

We mention that Bowden [6], [7] obtained some interesting results related to this problem. See also Nariman [36] where the author gives an affirmative solution to a similar problem for the cases of certain higher dimensional manifolds.

*Added in proof.* Recently Nariman solved the above problem for the case $\mathrm{Diff}^\delta(\Sigma_g, D^2)$ affirmatively in his thesis, Stanford University, 2015.

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