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On Thurston's construction of a surjective homomorphism $H_{2n+1}(B\Gamma_n, \mathbb{Z}) \to \mathbb{R}$

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§ Translator's remarks

This article is an English translation of notes by T. Mizutani on a theorem of Thurston [3]. The notes include a construction which seems not quite well-known, of a family of foliations of which the Godbillon–Vey class varies continuously. The contents are kept as it was. Some apparent errors are corrected, while historical comments are left original.

§1. Introduction

Thurston constructed codimension-one foliations of S^3 which are non-cobordant and showed that there exists a surjective homomorphism from $H_3(B\Gamma_1, \mathbb{Z})$ to $\mathbb{R}$ in [2]. The homomorphism is given by the integration of the Godbillon–Vey form of foliations over manifolds. The Godbillon–Vey forms are also defined for foliations of codimension greater than one, and it has been conjectured that an analogue also holds. A simple adaptation of constructions in codimension-one case does not work in higher codimensional case, however, there still exists a surjective homomorphism from $H_{2n+1}(B\Gamma_n, \mathbb{Z})$ to $\mathbb{R}$. Indeed, Thurston showed the following

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Theorem. *For any $r \in \mathbb{R}$, there exist a closed manifold W^{2n+1} of dimension $2n+1$ and a foliation $\mathcal{F}$ of W of codimension n such that*

$$\mathrm{gv}(W, \mathcal{F}) = r.$$

We give an outline of the proof after Thurston, omitting detailed calculations[1]. We remark that Heitsch recently extends Thurston's theorem to show the existence of surjective homomorphisms from $H_{2n+1}(B\Gamma_n, \mathbb{Z})$ to $\mathbb{R}^s$, where $s \geq 1$ is a certain integer, by using the Godbillon–Vey class as well as other exotic characteristic classes [7].

Finally we remark that this article is partly based on notes of Thurston's lectures taken by S. Morita[2] of Osaka City University.

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§2. Godbillon–Vey form

Let $(W^{n+p}, \mathcal{F})$ be a foliation of a smooth manifold W^{n+p} of codimension n. We assume that $\mathcal{F}$ is transversely orientable. If $\mathcal{F}$ is locally defined by a system of 1-forms $\{\omega_1, \dots, \omega_n\}$ with the equation $\omega_1 = \cdots = \omega_n = 0$, then there exists a global n-form Ω such that $\Omega = k\omega_1 \wedge \cdots \wedge \omega_n$ locally holds, where k is a positive function (it can be shown by partition of unity arguments). By the Frobenius theorem there exists a 1-form α such that

$$d\Omega = \alpha \wedge \Omega.$$

Note that the integrability of the distribution defined by $\omega_1 = \cdots = \omega_n = 0$ is equivalent to the existence of such a 1-form α as above also by the Frobenius theorem.

Definition 2.1. *The differential form $\gamma = \alpha \wedge (d\alpha)^n$ is called the Godbillon–Vey form. The cohomology class represented by γ is called the Godbillon–Vey class.*

It is indeed known that γ is a closed $(2n+1)$-form and that the cohomology class represented by γ depends only on $\mathcal{F}$ but not on the

[1] We slightly add some calculations for conveniences.

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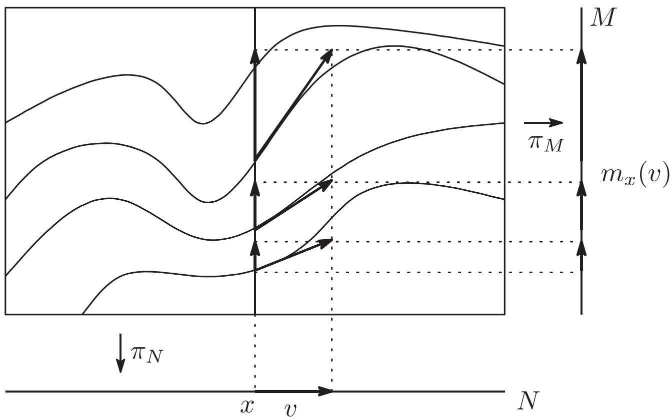

Fig. 1. The map m_x.

choice of Ω and α [1]. Therefore, if W is a closed manifold of dimension $2n+1$, then the integration of γ over W determines a real number, which we denote by $\mathrm{gv}(W, \mathcal{F})$ and call the Godbillon–Vey characteristic.

§3. A formula for foliated M-products

Let N and M be closed manifolds of dimension $n+1$ and n, respectively. Suppose that W is a fiber bundle over N with fibers M. A foliation $\mathcal{F}$ of W of codimension n which is transverse to fibers is called a foliated bundle. In particular if W is a trivial bundle, then we call $(W, \mathcal{F})$ a foliated M-product.

Let $(W, \mathcal{F})$ be a foliated M-product. We denote by $\mathcal{L}(M)$ the Lie algebra of smooth (of class C^∞) vector fields on M. For $x \in N$, we will define a linear map $m_x \colon T_xN \to \mathcal{L}(M)$ as follows. Let $\pi_N \colon W = N \times M \to N$ and $\pi_M \colon W = N \times M \to M$ be the projections. Given $v \in T_xN$ and $y \in \pi_N^{-1}(x)$ ($\cong M$), let $\widetilde{v}_y$ be the unique element of $T_y\mathcal{F}$ such that $\pi_{N*}(\widetilde{v}_y) = v$. We set then $m_x(v)(y) = \pi_{M*}(\widetilde{v}_y)$. It is easy to see that $m_x(v)$ is smooth if $\mathcal{F}$ is smooth. Next we introduce a Gel'fand–Fuchs cocycle which we denote by β. We fix a Riemannian metric on M and let ω be the volume form. Let $X \in \mathcal{L}(M)$ and denote by L_X the Lie derivative with respect to X. Then the function $\operatorname{div} X$ is defined by the equality

$$L_X\omega = (\operatorname{div} X)\omega.$$

We define β by the formula

$$\beta(X_1, X_2, \dots, X_{n+1}) = \int_M (\operatorname{div} X_1)\, d(\operatorname{div} X_2) \wedge \cdots \wedge d(\operatorname{div} X_{n+1}).$$

The cocycle β, homomorphism m_x and the Godbillon–Vey characteristic are related as follows.

Lemma 3.1 (Thurston, cf. [4], [5], [6], [8]). *Let $(N^{n+1} \times M^n, \mathcal{F})$ be a foliated M-product. Then, we have*

$$\begin{aligned} &\operatorname{gv}(N \times M, \mathcal{F}) \\ &= \int_N \beta\left(m_x\left(\frac{\partial}{\partial x^1}\right), \dots, m_x\left(\frac{\partial}{\partial x^{n+1}}\right)\right) dx^1 \wedge \cdots \wedge dx^{n+1} \\ &= \int_N (m_x^* \beta)\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n+1}}\right) dx^1 \wedge \cdots \wedge dx^{n+1}, \end{aligned}$$

where $x = (x^1, \dots, x^{n+1})$ is a system of local coordinates on N.

§4. Proof of Theorem and construction of foliations

We will show the following theorem of Thurston.

Theorem 4.1 (Thurston). *For any $r \in \mathbb{R}$, there exist a closed manifold W^{2n+1} of dimension $2n+1$ and a foliation $\mathcal{F}$ of W of codimension n such that*

$$\operatorname{gv}(W, \mathcal{F}) = r.$$

Corollary 4.2. *There exists a surjective homomorphism from $H_{2n+1}(B\Gamma_n, \mathbb{Z})$ to $\mathbb{R}$.*

Thurston's proof in the case where $n = 1$ appeared in [2]. We will explain an outline of the proof in the case where $n > 1$ after Thurston. In the arguments, W will be an S^n-bundle over $\Sigma \times T^{n-1}$, where Σ is a closed hyperbolic surface and $(W, \mathcal{F})$ will be a foliated bundle. The strategy is as follows: we will construct enough number of representations from $\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ to $\mathrm{Diff}(S^n)$, namely, actions of $\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ on S^n. Then construct $\mathcal{F}$ on $\Gamma \times \mathbb{Z}^{n-1} \backslash (\mathbb{H} \times \mathbb{R}^{n-1} \times S^n)$, where $\mathbb{H} = \{z = x + \sqrt{-1}y \mid x, y \in \mathbb{R},\ y > 0\}$ is the Poincaré upper half plane and Γ is a cocompact lattice of $\mathrm{SL}(2;\mathbb{R})/\mathrm{SO}(2)$ such that $\Sigma = \Gamma \backslash \mathbb{H}$. Let $\mathfrak{sl}(2;\mathbb{R})$ be the Lie algebra of $\mathrm{SL}(2;\mathbb{R})$. We consider an action of $\mathrm{SL}(2;\mathbb{R})$ on $\mathbb{R}^{n+1} = \mathbb{R}^2 \times \mathbb{R}^{n-1}$ such that the action on the $\mathbb{R}^2$ is the linear one and the one on $\mathbb{R}^{n-1}$ is trivial. Then, there is a homomorphism of Lie algebras

$$\lambda_{n+1} \colon \mathfrak{sl}(2;\mathbb{R}) \to \mathcal{L}(\mathbb{R}^{n+1}).$$

Let (x^1, x^2) be the standard coordinates on $\mathbb{R}^2$ and e_2 the Euler vector field. If we introduce the polar coordinates (r, θ) on $\mathbb{R}^2 \setminus \{o\}$, then $e_2 = r\frac{\partial}{\partial r}$. We trivialize $T(\mathbb{R}^2 \setminus \{o\})$ by $\left\{r\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$. We will extend $r\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ to the whole $\mathbb{R}^2$ by the formulas $e_2 = r\frac{\partial}{\partial r} = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial \theta} = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}$. Let $a = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix} \in \mathfrak{sl}(2;\mathbb{R})$. If we set $b = \binom{b^1}{b^2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} a \binom{\cos\theta}{\sin\theta}$, then we can represent

$$\begin{aligned} \lambda_2(a) &= (a^1_1 x^1 + a^1_2 x^2)\frac{\partial}{\partial x^1} + (a^2_1 x^1 + a^2_2 x^2)\frac{\partial}{\partial x^2} \\ &= b^1 r\frac{\partial}{\partial r} + b^2\frac{\partial}{\partial \theta} \\ &= k(\theta)e_2 + \rho_2(a) \end{aligned}$$

on $\mathbb{R}^2 \setminus \{o\}$. Note that $\rho_2(a)$ is the projectivization of λ_2. Indeed, by regarding S^1 as the set of oriented lines in $\mathbb{R}^2$ which pass through the origin, we obtain ρ_2 from λ_2. Note also that $\rho_2(a)$ is parallel to $\frac{\partial}{\partial \theta}$ and depends only on θ. We consider the standard metric on $\mathbb{R}^2$. Then, $\operatorname{div}\lambda_2(a) = 0$ because $a \in \mathfrak{sl}(2;\mathbb{R})$, and we have $k(\theta) = -\frac{1}{2}\operatorname{div}\rho_2(a)$. Therefore

$$\lambda_2(a) = -\frac{1}{2}\operatorname{div}\rho_2(a)e_2 + \rho_2(a).$$

Assume that $n \geq 2$ and introduce the polar coordinates on the first factor of $(\mathbb{R}^2 \setminus \{o\}) \times \mathbb{R}^{n-1}$. Let $(r, \theta, x^3, \ldots, x^{n+1})$ be the natural coordinates and $e_{n+1} = r\frac{\partial}{\partial r}$. We trivialize $T((\mathbb{R}^2 \setminus \{o\}) \times \mathbb{R}^{n-1})$ by $\left\{e_{n+1}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial x^3}, \ldots, \frac{\partial}{\partial x^{n+1}}\right\}$. Then we can represent $\lambda_{n+1}(a)$ as

$$\lambda_{n+1}(a) = k(\theta)e_{n+1} + \widetilde{\rho}_2(a),$$

where $\widetilde{\rho}_2(a)$ is parallel to $\frac{\partial}{\partial \theta}$ and depends only on θ. By the same reason as above, $k(\theta) = -\frac{1}{2}\operatorname{div}\widetilde{\rho}_2(a)$. Therefore,

$$\lambda_{n+1}(a) = -\frac{1}{2}\operatorname{div}\widetilde{\rho}_2(a)e_{n+1} + \widetilde{\rho}_2(a)$$

on $(\mathbb{R}^2 \setminus \{o\}) \times \mathbb{R}^{n-1}$. Note that

$$(1) \qquad \begin{cases} 1) & \widetilde{\rho}_2(a) \text{ is parallel to } \frac{\partial}{\partial \theta} \text{ and depends only on } \theta. \\ 2) & \operatorname{div}\widetilde{\rho}_2(a) = \operatorname{div}\rho_2(a) \text{ and it depends only on } \theta. \end{cases}$$

We remark for later use that $\operatorname{div}\rho_2(Y) = -2\sin\theta\cos\theta$ and $\operatorname{div}\rho_2(Z) = -\cos^2\theta + \sin^2\theta$, where $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We denote by

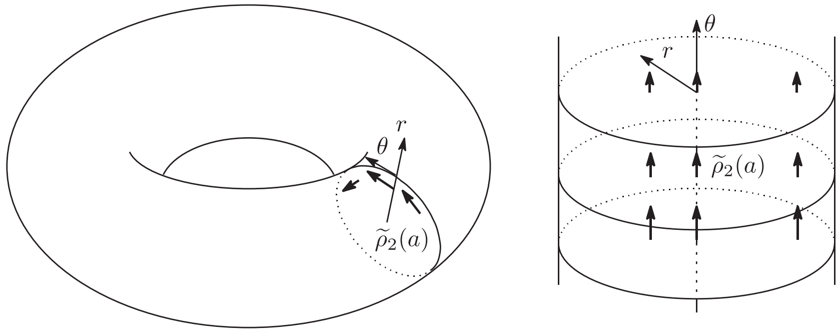

Fig. 2. Extension of $\sigma_{n+1}(a)$.

D^l_t the round open ball of radius t in $\mathbb{R}^l$. Let $\epsilon \in (0, 1/2)$ and regard[3] $S^n = (D^2_{1+\epsilon} \times S^{n-2}) \cup (S^1 \times D^{n-1}_{1+\epsilon})$, where $(r, \theta, p) \in D^2_{1+\epsilon} \times S^{n-2}$ is identified with $(\theta, p/r) \in S^1 \times D^{n-1}_{1+\epsilon}$ if $|r-1| < \epsilon$. Let $f^i\colon S^{n-2} \to \mathbb{R}$ be any C^∞-functions, where $3 \leq i \leq n+1$, and let g be a function on $\mathbb{R}$ such that $g(r) = 0$ if $r > 1-\epsilon$ and $g(r) = 1$ if $r < \epsilon$. We will define $\sigma_{n+1}\colon \mathfrak{sl}(2;\mathbb{R}) \times \mathbb{R}^{n-1} \to \mathcal{L}(S^n)$ as follows. First let

$$U_0 = D^2_{\epsilon/2} \times S^{n-2},$$
$$U_1 = \{(r, \theta, p) \in D^2_{1+\epsilon} \times S^{n-2} \mid r > \epsilon/3\}.$$

We then define $\sigma_{n+1}\colon \mathfrak{sl}(2;\mathbb{R}) \times \mathbb{R}^{n-1} \to \mathcal{L}(D^2_{1+\epsilon} \times S^{n-2})$ by

$$\sigma_{n+1}(a) = \begin{cases} \lambda_{n+1}(a), & \text{on} \quad U_0, \\ -\frac{1}{2}(\operatorname{div}\rho_2(a))g \cdot r\frac{\partial}{\partial r} + \widetilde{\rho}_2(a), & \text{on} \quad U_1, \end{cases} \quad a \in \mathfrak{sl}(2;\mathbb{R}),$$
$$\sigma_{n+1}(t_i) = f^i g \cdot r\frac{\partial}{\partial r}, \quad 3 \leq i \leq n+1,$$

where $\mathbb{R}^{n-1}$ is regarded as the Lie algebra of $\mathbb{R}^{n-1}$ and $\{t_3, \ldots, t_{n+1}\}$ is the standard basis for $\mathbb{R}^{n-1}$, and the natural images of elements of $\mathfrak{sl}(2;\mathbb{R})$ and $\mathbb{R}^{n-1}$ in $\mathfrak{sl}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ are denoted by the same symbols by abuse of notation. Note that $\sigma_{n+1}(a)$ and $\sigma_{n+1}(t_i)$ are indeed tangent to $D^2_{1+\epsilon} \times S^{n-2}$. Since $\sigma_{n+1}(a)$ depends only on θ and parallel to $\frac{\partial}{\partial\theta}$ on a neighborhood of $\partial(D^2_{1+\epsilon} \times S^{n-2})$, and since $\sigma_{n+1}(t_i)$ is independent of θ and vanishes outside $D^2_1 \times S^{n-2}$, these vector fields naturally extends to S^n. By abuse of notations, we denote thus obtained mapping from

[3]The original construction makes use of joins instead of decomposing S^n. We modified the construction for clarity.

$\mathfrak{sl}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ to $\mathcal{L}(S^n)$ again by σ_{n+1}. Then, by the property (1), σ_{n+1} is indeed a morphism of Lie algebras. Moreover, if $a = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then $\sigma_{n+1}(a) = \widetilde{\rho}_2(a) = -\frac{\partial}{\partial\theta}$. Therefore, the $\mathbb{R}$-action generated by a is periodic and σ_{n+1} induces a group action of $\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ on S^n which we denote by $\widetilde{\sigma}_{n+1}$. We will equip the trivial bundle $\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1} \times S^n \to \mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ with a foliation[4] such that the leaf $\widetilde{L}_{(g,u,w)}$ which passes $(g,u,w) \in \mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1} \times S^n$ is given by

$$\widetilde{L}_{(g,u,w)} = \{(gh, u+v, \widetilde{\sigma}_{n+1}(h,v)^{-1}w) \mid (h,v) \in \mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}\}.$$

Note that $\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ acts on $\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1} \times S^n$ on the right by

$$(g,u,w)(h,v) = (gh, u+v, \widetilde{\sigma}_{n+1}(h,v)^{-1}w)$$

and on the left by

$$(h,v)(g,u,w) = (hg, v+u, w),$$

respectively. The foliation $\{\widetilde{L}_{(g,u,w)}\}$ is invariant under the both actions. Therefore, by first taking the quotient by $\mathrm{SO}(2)$ on the right, we obtain a foliated S^n-bundle over $\mathbb{H} \times \mathbb{R}^{n-1}$ which is in fact a foliated product as we will explain below. Now let Γ be a cocompact lattice of $\mathrm{SL}(2;\mathbb{R})/\mathrm{SO}(2)$, and take the quotient of $(\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}) \times_{\mathrm{SO}(2)} S^n \cong \mathbb{H} \times \mathbb{R}^{n-1} \times S^n$ by $\Gamma \times \mathbb{Z}^{n-1}$ on the left. Then we obtain a foliated S^n-bundle over $\Gamma\backslash\mathbb{H} \times T^{n-1}$ of which the total space is $\Gamma\backslash(\mathrm{SL}(2;\mathbb{R}) \times T^{n-1}) \times_{\mathrm{SO}(2)} S^n$. We denote by $\mathcal{F}$ thus obtained foliation.

A trivialization of the foliated S^n-bundle over $\mathbb{H} \times \mathbb{R}^{n-1}$ is given as follows. We denote by $[g,u,w]$ the equivalence class represented by $(g,u,w) \in \mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1} \times S^n$. Let ι be an embedding of $\mathbb{H}$ into $\mathrm{SL}(2;\mathbb{R})$ given by $\iota(x+\sqrt{-1}y) = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$. We define $F\colon \mathbb{H} \times \mathbb{R}^{n-1} \times S^n \to (\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}) \times_{\mathrm{SO}(2)} S^n$ by $F(z,u,w) = [\iota(z), u, w]$. Then, F is a diffeomorphism and the leaf L_w of $\mathcal{F}$ which passes $(\sqrt{-1}, 0, w) \in \mathbb{H} \times \mathbb{R}^{n-1} \times S^n$ is given by

$$L_w = \{(z, u, \widetilde{\sigma}_{n+1}(\iota(z), u)^{-1}w) \mid (z,u) \in \mathbb{H} \times \mathbb{R}^{n-1}\}.$$

[4] We slightly modified the construction in view of [7], §5.

Let $(z,u)=(x,y,u^3,\dots,u^{n+1})$ be the natural coordinates on $\mathbb{H}\times\mathbb{R}^{n-1}$. Then,

$$m_{(\sqrt{-1},0)}\left(\frac{\partial}{\partial x}\right)=-\sigma_{n+1}(Y),$$
$$m_{(\sqrt{-1},0)}\left(\frac{\partial}{\partial y}\right)=-\sigma_{n+1}(Z),$$
$$m_{(\sqrt{-1},0)}\left(\frac{\partial}{\partial u^i}\right)=-\sigma_{n+1}(t_i),$$

where $3\le i\le n+1$. In general, $m_{(z,u)}=\widetilde{\sigma}_{n+1}(\iota(z),u)_*\circ m_{(\sqrt{-1},0)}$. On the other hand, if we set $h=\operatorname{div}\big(g\cdot r\frac{\partial}{\partial r}\big)=r\frac{dg}{dr}+2g$ then

1) $h=2$ on the image of $S^{n-2}=\{o\}\times S^{n-2}$ in $S^n=(D^2_{1+\epsilon}\times S^{n-2})\cup(S^1\times D^{n-1}_{1+\epsilon})$.
2) $h=0$ on $S^1\times D^{n-1}_{1+\epsilon}\subset S^n$.

Therefore,

$$\begin{aligned}
&(m^*_{(\sqrt{-1},0)}\beta)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial u^3},\dots,\frac{\partial}{\partial u^{n+1}}\right)\\
&=(-1)^n\left(\int_r\left(1-\frac{1}{2}h\right)^2h^{n-2}\,dh\right)\left(\int_\theta \operatorname{div}\rho_2(Y)\,d(\operatorname{div}\rho_2(Z))\right)\\
&\quad\cdot\left(\int_{S^{n-2}}\sum_{i=3}^{n+1}(-1)^{i-3}f^i df^3\wedge\cdots\wedge\widehat{df^i}\wedge\cdots\wedge df^{n+1}\right)\\
&=(-1)^n\frac{2^{n+1}\pi}{n(n^2-1)}\int_{S^{n-2}}\widetilde{f}^*\omega_{n-1},
\end{aligned}$$

where $\widetilde{f}=(f^3,\dots,f^{n+1})\colon S^{n-2}\to\mathbb{R}^{n-1}$, $\omega_{n-1}=\sum_{i=1}^{n-1}(-1)^{i+1}x^i\,dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^{n-1}$ and the symbol '$\widehat{\ }$' means omission. Note that if we set

$$V=\int_{S^{n-2}}\widetilde{f}^*\omega_{n-1},$$

then V is a generalization of the volume of the region bounded by $\widetilde{f}(S^{n-2})$. We have

$$\operatorname{gv}(W,\mathcal{F})=(-1)^n\frac{2^{n+1}\pi V}{n(n^2-1)}\int_N \operatorname{vol}_N,$$

where $N=(\Gamma\backslash\mathrm{SL}(2;\mathbb{R})/\mathrm{SO}(2))\times T^{n-1}=\Sigma\times T^{n-1}$ and vol_N denotes the volume form of N, so that $\operatorname{gv}(W,\mathcal{F})$ attains any value in $\mathbb{R}$ as f_i's vary.

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Remarks on References. The paper [3] is the original of this translation. The papers [1] and [2] are cited in [3]. The rest is added by the translator.

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