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Brownian motion on foliated complex surfaces, Lyapunov exponents and applications

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§ Introduction

These lectures are motivated by the dynamical study of differential equations in the complex domain. Most of the topic will concern *holomorphic foliations on complex surfaces*, and their connections with the theory of *complex projective structures on curves*. In foliation theory, the interplay between geometry and dynamics is what makes the beauty of the subject. In these lectures, we will try to develop this relationship even more.

On the geometrical side, we have generalizations of the *foliation cycles* introduced by Sullivan, see [68]: namely the *foliated harmonic currents*, see e.g. [36, 4]. Those currents permit to think of the foliation as if it were a genuine algebraic curve. For instance, one can associate a homology class, compute intersections with divisors on the surface etc. These currents can often be viewed as limits of the (conveniently normalized) currents of integration on large leafwise domains defined via the uniformization of the leaves. This point of view, closely related to Nevanlinna theory, is very fruitful in the applications as we will see. See [5, 28].

On the dynamical side, the *leafwise Brownian motions* (w.r.t. to some hermitian metric on the tangent bundle to the foliation, e.g. coming from uniformization of leaves) generate a Markov process on the

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complex surface, whose study was begun by Garnett, see [34]. This Markov process seems to play a determinant role in the dynamics of foliated complex surfaces. One reason is that the Brownian motion in two dimensions is conformally invariant. Another reason is that leafwise Brownian trajectories equidistribute w.r.t. the product of a certain foliated harmonic current times the leafwise volume element. This makes the connection with the geometrical side mentioned above.

One of the main theme that will be developed in these lectures is the construction of numerical invariants that embrace these two aspects (dynamical and geometrical) of foliated complex surfaces. The discussion will emphasize on the definition and properties of the *foliated Lyapunov exponent* of a harmonic current, which heuristically measures the exponential rate of convergence of leaves toward each other along leafwise Brownian trajectories. A fruitful formula expresses this dynamical invariant in terms of the intersection of some foliated harmonic currents and the normal/canonical bundles of the foliation, see [16]. This formula is a good illustration of the interplay between geometry and dynamics in foliation theory. This will be developed in the first lecture.

In the second and third lectures, we will collect some applications of this formula in different contexts.

The first application concerns *Levi-flats* in complex algebraic surfaces. Those are (real) hypersurfaces that are foliated by holomorphic curves. Most examples occur as three (real) dimensional analytic invariant subsets of singular algebraic foliations. Foliations containing such subsets are analogous to *Fuchsian groups* (those having an invariant analytic circle in the Riemann sphere) in the context of Kleinian groups or to *Blashke products/Tchebychef polynomials* (having an invariant analytic circle/interval) in the context of iteration of rational functions. Very little is known about Levi-flats in algebraic surfaces. For instance, it is still unknown weather every algebraic surface contains a Levi-flat. A folklore conjecture predicts that the complex projective plane should not have any. Still, there exists a multitude of examples, e.g. in flat ruled bundles over curves, in singular holomorphic fibrations, in ramified covers of these etc. As we will see, some new restrictions concerning the topology of Levi-flats can be deduced from a detailed analysis of the foliated Lyapunov exponent and its relation to the geometry of the ambiant surface. For instance, we will prove that a Levi-flat hypersurface in a surface of general type is not diffeomorphic to the unitary tangent bundle of a two dimensional compact orbifold of negative curvature, nor to a hyperbolic torus bundle, and that its fundamental group has exponential growth. This will be explained in the second lecture, where we'll also construct many examples of Levi-flats, most notably we will realize all the models of Thurston's geometries as Levi-flats in algebraic surfaces apart the elliptic one. All this is based on a work in collaboration with Christophe Dupont, see [20].

The second application concerns *complex projective structures* on curves. These structures are of interest in various problems of uniformization in two or three dimensions. We will define some new invariants associated to complex projective structures: a *Lyapunov exponent*, a *degree*, and a family of *harmonic measures* (analogous to harmonic measure of a compact set in the complex line), and we will see how to relate these invariants. The connexion with foliation theory will be of utmost importance. It comes from the study of the particular class of *transversally holomorphic foliations*: any algebraic curve transverse to such a foliation inherits a complex projective structure by restricting the transverse projective structure of the foliation to the curve. As an illustration of this point of view, an algebraic curve in a Hilbert modular surface of the form $\Gamma\backslash\mathbb{H}\times\mathbb{H}$, where Γ is a cocompact lattice in $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$ inherits two (branched) complex projective structures from the two (horizontal and vertical) foliations. We will derive applications of these new invariants, most notably some estimates for the dimension of harmonic measures of complex projective structures. In particular, we will recover the Jones–Wolff and Makarov estimates for classical harmonic measures of limit sets of Kleinian groups. Another application will be to reinforce the analogy between complex projective structures and polynomial dynamics, that was brought to light by McMullen, see [58]. All these developments have been obtained in collaboration with Romain Dujardin, see [19].

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§1. Lecture 1—Lyapunov exponents associated to foliated complex surfaces

1.1. Basic definitions and examples

In this lecture, S will denote a complex surface. Recall that a holomorphic foliation on S is a maximal atlas $\mathcal{F}$ of holomorphic charts $(x,z)\colon U\to\mathbb{D}\times\mathbb{D}$ ($\mathbb{D}\subset\mathbb{C}$ is the unit disc) defined on open subsets U covering S, and overlapping as

$$(x',z')=(x'(x,z),z'(z)).$$

Hence the local fibrations $z = \mathrm{cst}$ are preserved by the change of coordinates. The fibers of these local fibrations, called the *plaques*, are glued together and define Riemann surfaces, called the leaves of the foliation. The sets $\{x\} \times \mathbb{D}$ are called transversal sets, and will be denoted $\mathbb{D}^{\pitchfork}$. We refer to the book [10] for the basics on foliation theory: most notably, the definition of holonomy maps, transverse invariant measures etc. The data of S and $\mathcal{F}$ will be referred to as a *foliated complex surface*. Given a foliated complex surface, a compact saturated subset is a compact set which is a union of leaves of $\mathcal{F}$. We have in mind various sources of examples.

Example 1.1 (Riemann–Hilbert correspondance)**.** Let C be an algebraic curve, and $\pi_1(C) \to \mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ be a representation. We define $S_\rho = C \ltimes_\rho \mathbb{P}^1(\mathbb{C})$ as the flat $\mathbb{P}^1(\mathbb{C})$ bundle over C with monodromy ρ. Recall that S_ρ is defined as the quotient of $\widetilde{C} \times \mathbb{P}^1(\mathbb{C})$ by the action of $\pi_1(C)$ given by

$$\gamma \cdot (x, z) = (\gamma \cdot x, \rho(\gamma) \cdot z),$$

for every $\gamma \in \pi_1(C)$ and $(x, z) \in \widetilde{C} \times \mathbb{P}^1(\mathbb{C})$. Here $\widetilde{C}$ denotes a universal cover of C, and $\pi_1(C)$ the covering group of this covering. The horizontal fibration on $\widetilde{C} \times \mathbb{P}^1(\mathbb{C})$ whose fibers are the subsets $\widetilde{C} \times z$ for $z \in \mathbb{P}^1(\mathbb{C})$, defines on S_ρ a non singular holomorphic foliation $\mathcal{F}_\rho$.

Other examples occurs as

Example 1.2 (Foliated 3-manifolds)**.** A 2-dimensional analytic foliation of a compact 3-manifold equipped with an analytic complex structure on its leaves can be embedded in a germ of foliated complex surface. Such a complex structure is built using a leafwise orientation plus an analytic metric on $T\mathcal{F}$, since Riemannian surfaces are conformally flat. In analytic regularity, this is a theorem of Gauss, see [15, Théorème I.2.1].

More generally, we will consider in a complex surface S smooth real hypersurfaces that are Levi-flat but not necessarily tangent to a (germ of) holomorphic foliation.

Definition 1.3 (Levi-flat)**.** *A hypersurface M of class C^2 in a complex surface S inherits a unique distribution by complex lines called the Cauchy–Riemann distribution. It is defined by the formula $TM \cap iTM$ where $i = \sqrt{-1}$. The hypersurface M is called Levi-flat iff the Cauchy–Riemann distribution integrates in a foliation, called the Cauchy–Riemann foliation and denoted by $\mathcal{F}$. If the hypersurface M is Levi-flat and analytic, then $\mathcal{F}$ can be extended in the neighborhood of M as a non singular holomorphic foliation.*

1.2. Foliated harmonic currents

Let M be a closed saturated subset of a foliated complex surface, or a Levi-flat hypersurface in a complex surface. We denote by $\mathcal{F}$ the holomorphic foliation to which M is tangent in the first case, and the Cauchy–Riemann foliation in the second one. Let $\mathcal{O}_{\mathcal{F}}$ the sheaf of continuous functions on M which are holomorphic along the leaves of $\mathcal{F}$, and by $C^\infty_{\mathcal{F}}$ the sheaf of functions f with compact support, that are smooth along the leaves and all whose leafwise derivatives $\frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial \overline{x}^\beta}$ in holomorphic foliated coordinates are continuous in (x, z). This definition is independent of the chosen foliated coordinate system. We also denote by $A^p_{\mathcal{F}}(M)$ (resp. $A^{(p,q)}_{\mathcal{F}}(M)$) the set of $C^\infty_{\mathcal{F}}$ forms with compact support of degree p (resp. bidegree (p, q)) on $T\mathcal{F}$, namely the set of smooth sections with compact support of the bundle $\Lambda^p(T^*\mathcal{F})$ (resp. $\Lambda^{p,q}(T^*\mathcal{F})$).

Definition-Proposition 1.4. *A foliated harmonic current is a* (*non vanishing*) *linear form* $T\colon A^{1,1}_{\mathcal{F}} \to \mathbb{C}$ *which verifies* $\partial\overline{\partial}_{\mathcal{F}} T = 0$ *in the weak sense* (*namely* $T(\partial\overline{\partial}_{\mathcal{F}} f) = 0$ *for any smooth function* $f\colon S \to \mathbb{R}$, *where* $\partial\overline{\partial}_{\mathcal{F}}$ *denotes the derivative along the leaves*) *and which is non negative on* $\mathcal{F}$ (*namely* $T(\eta) \geq 0$ *if* $\eta|_{\mathcal{F}} \geq 0$). *In foliated coordinates, a foliated harmonic current takes the form*

$$(1)\qquad T(\eta) = \int_{\mathbb{D}^{\pitchfork}} \left[\int_{\mathbb{D}\times z} \varphi(x, z)\eta \right] \nu(dz\, d\overline{z}), \quad \textit{if} \quad \operatorname{Supp}(\eta) \subset \mathbb{D} \times \mathbb{D},$$

where ν *is a Radon measure on the transversal* $\mathbb{D}^{\pitchfork}$ *and* $\varphi \in L^1(dx\, d\overline{x} \otimes \nu)$ *is positive and harmonic on* ν*-a.e. plaque* $\mathbb{D} \times z$.

The support of a harmonic current T is defined as usual: this is the set of points of M having a basis of neighborhoods V_i and forms $\eta_i \in A^{(1,1)}_{\mathcal{F}}(M)$ whose supports are contained in V_i and such that $T(\eta_i) \neq 0$.

Proposition 1.5. *A compact saturated subset M supports a foliated harmonic current.*

Proof. The following proof is due to Ghys, see [36], following ideas of Sullivan, see [68]. Let $A^{1,1}_c(M, \mathcal{F})$ be the Banach space of continuous $(1, 1)$-forms on $T\mathcal{F}|_M$, $\mathcal{P} \subset A^{1,1}_c(M, \mathcal{F})$ denotes the open convex cone of positive ones, and E be the set of uniform limits of forms of the type $\partial\overline{\partial}_{\mathcal{F}} f$ with $f \in C^\infty(\mathcal{F})$. By the maximal principle, $\mathcal{P} \cap E = \emptyset$, hence the Hahn–Banach separation theorem concludes. Q.E.D.

Remark 1.6. The existence of foliated harmonic current has been generalized to singular holomorphic foliations by Berndtsson and Sibony. We refer to [4, Theorem 1.4].

Definition-Proposition 1.7 (Foliation cycles)**.** *A foliation cycle is a foliated harmonic current which is $d_{\mathcal{F}}$-closed, namely it satisfies $T(d_{\mathcal{F}}\eta) = 0$ for every $\eta \in A^1_{\mathcal{F}}$ where $d_{\mathcal{F}}$ is the derivative along the leaves. A foliation cycle is expressed locally as*

$$T(\eta) = \int_{\mathbb{D}^{\pitchfork}} \left[\int_{\mathbb{D}\times z} \eta \right] \nu(dz\, d\overline{z}) \tag{2}$$

where ν is a Radon measure. The family of measures ν defines a transverse invariant measure for the foliation $(M, \mathcal{F})$.

Example 1.8 (Leaf closed at infinity)**.** The basic example of foliation cycle is the integration current on a leaf. A generalization of this is due to Plante, see [62, Theorem 3.1]. Assume that M is compact and that $A_n \subset L_n$ is a sequence of compact domains contained in leaves L_n of M, and that we have

$$\frac{\operatorname{length}(\partial A_n)}{\operatorname{area}(A_n)} \to_{n\to\infty} 0 \tag{3}$$

where the length and area are measured w.r.t. to a hermitian metric along the leaves. Then the family of currents $T_n := \frac{1}{\operatorname{area}(A_n)}[A_n]$ is relatively compact in the weak* topology, and moreover any limit $\lim_{n_k\to\infty} T_{n_k}$ is a foliation cycle. Sullivan generalized this construction, see [68, Theorem II.8].

1.3. Uniformization

Other examples of foliation cycles or foliated harmonic currents come from the uniformization of Riemann surfaces, which is stated as follows.

Theorem 1.9 (Poincaré–Koebe)**.** *Every Riemann surface is covered resp. by $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or $\mathbb{D}$. This trichotomy is exclusive. The Riemann surface is resp. called elliptic, parabolic or hyperbolic.*

We refer to the book [15] for the history and the various proofs of this theorem.

Example 1.10 (Ahlfors)**.** If M is compact and L is a parabolic leaf contained in M, and $f\colon \mathbb{C} \to L$ a uniformization of L, and g a hermitian metric along the leaves, one can extract from the family of currents

$$\forall \eta \in A^{1,1}_{\mathcal{F}}(M), \quad T_r(\eta) := \frac{1}{\operatorname{area}_{f^*g}(\mathbb{D}_r)} \int_{\mathbb{D}_r} f^*\eta \tag{4}$$

a subsequence converging in the weak*-topology towards a foliation cycle. Here $\mathbb{D}_r := \{x \in \mathbb{C} \mid |x| < r\}$. We refer to [1] and [7, Lemme 0] for a proof of this fact.

Let us now review what happens if the leaves are hyperbolic. We begin by the following theorem of Verjovsky, generalized by Candel in the context of general Riemann surface laminations. Recall that the unit disc has a unique complete conformal metric of curvature -1, given by

$$g_P = 4\frac{|dx|^2}{(1-|x|^2)^2}. \tag{5}$$

This metric is invariant under the group $\mathrm{Aut}(\mathbb{D})$ of automorphisms of the unit disc, hence it defines a conformal metric on any hyperbolic Riemann surface. We have

Theorem 1.11 (Verjovsky–Candel)**.** *Assume that M is compact and that all the leaves of M are hyperbolic. Then the Poincaré metric on each of these leaves defines a continuous metric on $T\mathcal{F}|_M$.*

Example 1.12 (Fornaess–Sibony)**.** Assume that all the leaves of M are hyperbolic Riemann surfaces. Let $f\colon \mathbb{D} \to L$ be the uniformization of one leaf of M. Then the family of currents

$$\forall \eta \in A^{1,1}_{\mathcal{F}}(M), \quad T_r(\eta) = \frac{\int_{\mathbb{D}_r} \log\left(\frac{r}{|x|}\right) f^*\eta}{\int_{\mathbb{D}_r} \log\left(\frac{r}{|x|}\right) v_P} \tag{6}$$

is relatively compact in the weak*-topology and the limit of any convergent subsequence T_{r_n} with $r_n \to 1$ is foliated harmonic. Here v_P refers to the volume element of the Poincaré metric.

1.4. Homology, intersection, and Chern–Candel classes

Recall that there is a restriction map $r\colon A^2(S) \to A^{1,1}_{\mathcal{F}}(M)$ which satisfies $d_{\mathcal{F}} r = rd$. Thus, a foliation cycle can be viewed as a current $T\colon A^2(S) \to \mathbb{C}$ which is d-closed. Therefore, it naturally defines a homology class $[T] \in H_2(S,\mathbb{C})$ (by duality) by the formula

$$[T]\cdot[\eta] = T(\eta),$$

for every closed 2-form η. Notice that the positivity of T implies that $[T] \in H_2(S,\mathbb{R})$. In particular, one can consider the intersection product $[T]\cdot c_1(E)$ if $E \to S$ is any complex line bundle over S, and $c_1(E)$ denotes the first Chern class of E. We will denote it succinctly by $T \cdot E$. One can compute this intersection by using differential geometry, namely

$$T \cdot E = \frac{1}{2\pi} T(\omega) \tag{7}$$

where ω is the curvature form of *any* connexion ∇ on E. In fact, it is sufficient to have a smooth connexion which is only defined along every leaf of $\mathcal{F}$, but we will not verify this here. All this makes sense since the curvature forms of two different connexions on E differ by an exact 2-form.

This does not work this way if T is only assumed to be harmonic, since in this case we only get a homology class in the dual of the Bott–Chern cohomology group

$$H^{1,1}_{\partial\overline{\partial}}(S,\mathbb{C}) = \{\text{closed } (1,1)\text{-forms}\}/\partial\overline{\partial}C^\infty(S). \tag{8}$$

Nevertheless, following an observation of Candel, one can define the intersection product of T with E when E is any holomorphic line bundle along the leaves of M (namely every element of $H^1(M,\mathcal{O}^*_{\mathcal{F}})$). This can be achieved by the use of the Chern connexion of a hermitian metric on E, whose expression is given locally by

$$\omega_{\|\cdot\|} = \frac{1}{i}\partial\overline{\partial}\log\|s\|^2, \tag{9}$$

where s is any non vanishing local holomorphic section of E. Notice that $\omega_{\|\cdot\|}$ does not depend on the choice of s, and therefore it is defined globally: $\omega_{\|\cdot\|} \in A^{1,1}_{\mathcal{F}}$. One then defines

$$T \cdot E := \frac{1}{2\pi}T(\omega_{\|\cdot\|}), \tag{10}$$

where $\|\cdot\|$ is any hermitian metric on E. Since T is harmonic, the definition does not depend on the chosen hermitian metric $\|\cdot\|$.

This formula permits to define an important invariant of a harmonic current: its Euler characteristic. This is the intersection of the harmonic current with the tangent bundle of the foliation $\mathcal{F}$. In what follows, we will be more interested in the opposite of this number, namely the intersection of T with the canonical bundle of $\mathcal{F}$ being defined by $K_{\mathcal{F}} := T^*\mathcal{F}$.

An interesting case is where S is a compact Kähler surface, since under this assumption one knows that the group (8) is isomorphic to the Dolbeault cohomology group $H^{1,1}_{\overline{\partial}}(S,\mathbb{C}) \subset H^2(S,\mathbb{C})$, by the $\partial\overline{\partial}$-lemma. Thus we can define a homology class $[T]$ of T belonging to $H_2(S,\mathbb{C})$ (by duality) in that case. Observe that if $E \to S$ is a holomorphic line bundle, the number $T \cdot E$ defined by (10) computes the cohomological intersection $[T] \cdot c_1(E)$, where $c_1(E)$ is the Chern class of E.

1.5. Garnett's theory

Here is the basic ingredient that will be needed in this lecture. Let (L,g) be a complete Riemannian manifold with bounded curvature, and $x \in L$ be a point. Then there exists a unique measure W^x, called the *Wiener measure*, on the set Ω_x of continuous paths $\omega\colon [0,\infty) \to L$ starting at $\omega(0) = x$, satisfying the following

$$\begin{aligned} & W^x(\{\omega \mid \omega(t_i) \in B_i\}) \\ & = \int_{B_1\times\cdots\times B_k} \prod_{j=1}^{k} p(x_{j-1}, x_j, t_j - t_{j-1}) v_g(dx_1)\cdots v_g(dx_k) \end{aligned} \tag{11}$$

for every $k \in \mathbb{N}^*$, every non decreasing sequence $t_0 = 0 \leq t_1 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k$, every family $\{B_j\}_j$ of Borel subsets of L, and the convention $x_0 = x$. Here, v_g denotes the volume element, and $p(x, y, t)$ is the heat kernel on L (namely $p(x, \cdot\,, \cdot\,)$ satisfies the heat equation $\frac{\partial u}{\partial t} = \Delta u$ and $p(x, y, t)\, dy$ weakly tends to the Dirac mass δ_x at x). We refer to [13, Chapter VI].

Let now M be a compact saturated subset of a foliated complex surface $(S, \mathcal{F})$, or a Levi-flat of a complex surface S whose CR foliation is $\mathcal{F}$. Let g be a smooth hermitian metric on $T\mathcal{F}$, defined in a neighborhood of M, and $\Delta_{\mathcal{F}}$ the leafwise Laplacian associated to this metric. A *foliated harmonic measure* on M is a probability measure which satisfies in the weak sense the equation $\Delta_{\mathcal{F}}\mu = 0$. Those are the measures

$$\mu := T \wedge v_g \tag{12}$$

where T is a (conveniently normalized) foliated harmonic current and v_g is the leafwise volume element of the Riemannian tensor g. Precisely, $\mu(f) = T(f v_g)$ for every $f \in C^\infty_{\mathcal{F}}(M)$. Notice that $(\Delta_{\mathcal{F}} f) v_g = 2i\partial\overline{\partial}_{\mathcal{F}} f$ implies $\Delta_{\mathcal{F}}\mu = 0$. In particular, a foliated harmonic measure always exists, by Proposition 1.5.

Let Ω be the set of continuous paths $\omega\colon [0,\infty) \to M$ which are contained in a leaf of M, and Ω^w those conditioned to begin at $\omega(0) = w$. Shifting the time defines a semi-group $\sigma = \{\sigma_t\}_{t\geq 0}$ of transformations acting on Ω by the formula $\sigma_t(\omega)(\cdot) := \omega(t+\cdot)$. Given a probability measure μ on M, let $\overline{\mu}$ be the measure on Ω defined by $\overline{\mu} := \int_M W^w \mu(dw)$. An easy observation shows that μ is harmonic iff $\overline{\mu}$ is σ-invariant. We can then apply ergodic theory to the system $(\Omega, \sigma, \overline{\mu})$. Garnett proved the following version of the random ergodic theorem in this context:

Theorem 1.13 (Random ergodic theorem)**.** *The foliated harmonic measure μ is extremal in the compact convex set of harmonic measures iff the system $(\Omega, \sigma, \overline{\mu})$ is ergodic.*

We refer to [34] and to the survey paper by Candel [12]. A foliated harmonic measure satisfying the assumptions of the theorem will be called *ergodic*. Observe that in particular, for a.e. point w w.r.t. a foliated ergodic harmonic measure, W^w-a.e. Brownian path ω starting at x equidistributes w.r.t. μ, namely $\frac{1}{t}\int_0^t \delta_{\omega(s)}\,ds$ tends to μ as t tends to $+\infty$.

1.6. The foliated Lyapunov exponent

In this section, we endow the tangent bundle $T_{\mathcal{F}}$, resp. the normal bundle $N_{\mathcal{F}}$, with smooth hermitian metrics. Recall that if $\omega\colon [0,t]\to L$ is a continuous path in a leaf of L, there is a holonomy map $h_\omega\colon \tau_{\omega(0)}\to \tau_{\omega(t)}$ from a transversal $\tau_{\omega(0)}$ at $\omega(0)$ to a transversal $\tau_{\omega(t)}$ at $\omega(t)$. See the book [10] for the definition of holonomy map. The derivative of h_ω at $\omega(0)\in\tau_{\omega(0)}$ will be denoted $Dh_\omega(\omega(0))$.

Denote by $v_g\in A^{1,1}_{\mathcal{F}}(M)$ the volume form along the leaves defined by the hermitian metric g on $T\mathcal{F}$. Let T be a harmonic current on M, and choose a such that $\mu = av_g\wedge T$ is a probability measure. We call T ergodic if μ is ergodic. For any $t\geq 0$, define $H_t\colon \Omega\to\mathbb{R}$ by

$$H_t(\omega) := \log\|Dh_{\omega|_{[0,t]}}(\omega(0))\|.$$

It satisfies $H_{t+s} = H_t + H_s\circ\sigma_t$ for every $s,t\geq 0$.

Definition-Proposition 1.14. *There exists a number $\lambda=\lambda(T)$ such that*

$$\int_\Omega H_t\,d\overline{\mu} = \lambda t \quad \textit{for every} \quad t\geq 0.$$

Then the ergodicity of T implies that for μ-a.e. point $w\in M$, and W^w-almost every path $\omega\colon [0,\infty)\to L_w$ starting at $\omega(0)=w$, we have

$$\frac{1}{t}\log\|Dh_{\omega|_{[0,t]}}(\omega(0))\| \to_{t\to+\infty} \lambda. \tag{13}$$

To get the result we apply the ergodic theorem to the cocycle

$$H_t(\omega) := \log\|Dh_{\omega|_{[0,t]}}(\omega(0))\|, \tag{14}$$

which satisfies the relation $H_{t+s}(\omega) = H_t(\omega)+H_s(\sigma_t(\omega))$ for every $\omega\in\Omega$ and every $s,t\geq 0$. To get the result one needs to verify that H_t is $\overline{\mu}$-integrable. This relies on Cheng–Li–Yau estimates for the heat kernel:

$$p(x,y,t)\leq C\exp(-\alpha d(x,y)^2),$$

where $C,\alpha>0$ are constant depending only on t and the local geometry of the manifold. See [14].

In the case all the leaves of $\mathcal{F}$ are hyperbolic Riemann surfaces, one can parametrize Brownian motions using the Poincaré metric. In this case, the Lyapunov exponent depends on cohomological quantities.

Proposition 1.15 (Cohomological formula for the Lyapunov exponent)**.** *Let $(S, \mathcal{F})$ be a foliated complex surface and M be a minimal set. Assume that the leaves of M are hyperbolic Riemann surfaces. We endow its tangent bundle with the Poincaré metric. Then for every foliated harmonic current T on M, we have*

$$\lambda(T) = -\frac{T \cdot N_{\mathcal{F}}}{T \cdot K_{\mathcal{F}}}.$$

In this formula, $N_{\mathcal{F}} = TS/T\mathcal{F}$ and $K_{\mathcal{F}} = T^*\mathcal{F}$ stand for the normal bundle and the canonical bundle of $\mathcal{F}$.

Proof. We reproduce here the proof given in [16, Appendice A]. Observe that the formula depends only on T modulo multiplication by a positive constant, so we can assume that the measure $\mu := T \wedge v_g$ has mass one, namely $T(v_g) = 1$. Introduce some coordinates (x, z) where the foliation is defined by $dz = 0$, and consider the infinitesimal distance between leaves, namely the function $\left\|\frac{\partial}{\partial z}\right\|$, where $\|\cdot\|$ is a hermitian metric on $N_{\mathcal{F}}$. This function depends on the foliated coordinates, but when changing coordinates, it is multiplied by a positive function which is constant on the leaves. In particular, the function $\Delta_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\|$ is well-defined on M. Similarly $d_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\|$ is a well-defined 1-form along the leaves of $\mathcal{F}$.

Lemma 1.16. $\lambda = \int_M \Delta_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\| d\mu.$

Proof. The starting point of the proof relies on the fact that $\int H_t \, d\overline{\mu} = \lambda t$, hence

$$\lambda = \frac{d}{dt}\bigg|_{t=0} \int H_t \, d\overline{\mu}.$$

Now, we have

$$\int H_t d\overline{\mu} = \int_X \left[\int_{\Omega^w} H_t \, dW^w \right] \mu(dw).$$

So we deduce

$$\lambda = \int_X \left[\frac{d}{dt}\bigg|_{t=0} \int_{\Omega^w} H_t \, dW^w \right] \mu(dw).$$

Fix w and introduce the universal covering $\widetilde{L_w}$ of L_w, viewed as the set of homotopy classes of paths $\omega\colon [0,1] \to L_w$ starting at w with fixed extremities. Let φ be a primitive of the form $d_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\|$ which vanishes at w. The Laplacian of φ is invariant by the covering group and gives the function $\Delta_{\mathcal{F}} \log\|\frac{\partial}{\partial z}\|$ on the quotient. Moreover, we have $H_t(\omega) = \varphi(\omega(t))$. Hence we get

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \int_{\Omega^w} H_t \, dW^w &= \frac{d}{dt}\Big|_{t=0} \mathbb{E}^w(\varphi(\omega(t))) \\ &= \Delta_{\mathcal{F}}\varphi(w) = \Delta_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\|(w). \end{aligned}$$

This proves the formula. Q.E.D.

Proposition 1.15 follows from Lemma 1.16, since the following elementary identity $2i\partial\overline{\partial} = \Delta_g \cdot v_g$ implies

$$T \cdot N_{\mathcal{F}} = -\frac{1}{2\pi} \int_M \Delta_g \log\left\|\frac{\partial}{\partial z}\right\| d\mu = -\frac{1}{2\pi}\lambda(T) = -(T \cdot K_{\mathcal{F}})\lambda(T).$$

Q.E.D.

Remark 1.17. The existence of an analogous Lyapunov exponent for singular holomorphic foliations (say on algebraic surfaces) is not obvious at all. Assume for instance we are in the following situation. Let $(S, \mathcal{F})$ be a singular holomorphic foliation of a compact complex surface, whose leaves are hyperbolic Riemann surfaces, and whose singularities are linearizable. Then the product $T \wedge v_P$ is finite, see [22], and Garnett's theory can be extended almost line by line, by using the fact that the Poincaré metric is continuous in that case. The only problem to define the Lyapunov exponent in this context is the integrability of the cocycle (14). The integrability can be proved when the singularities are in the Siegel domain, namely conjugate to ones of the form $x\,dy - \alpha y\,dx$ where $\alpha \in \mathbb{R}$. Then formula Proposition 1.15 holds with a correction term involving some indices defined at each singularity. However, in the hyperbolic case $\Im\alpha \neq 0$, the integrability remains an open problem.

1.7. Unique ergodicity

A general principle is that foliated harmonic currents associated to minimal sets are unique. This fact was already observed in the work of Garnett (unique ergodicity of the weak stable foliation of the geodesic flow of a compact surface of constant curvature -1, see [34, Proposition 5]). Here is a result in that direction that we obtained in collaboration with Victor Kleptsyn:

Theorem 1.18 (Unique ergodicity)**.** *Let M be a compact minimal set of either*

- *a complex foliated surface or*

- *a Levi-flat of class C^2.*

We denote $\mathcal{F}$ the holomorphic foliation in the first case and the Cauchy–Riemann foliation in the second case. Assume that $\mathcal{F}$ does not support any foliation cycle on M. Then there exists a unique harmonic current on M up to multiplication by a constant. Moreover, given a hermitian metric on $N\mathcal{F}$, there exists a number $\lambda < 0$ such that for every point $w \in M$, and W^w-a.e. leafwise Brownian path ω starting at w, the limit (13) *exists and equal λ.*

We refer to [21] for the proof of this result, the main step being the existence of at least one harmonic current whose associated Lyapunov exponent is negative. This being done, a second step (the similarities between Brownian motions on different leaves) permits to infer unique ergodicity. A weak version of the contraction statement was used by Thurston for the construction of his universal circle theorem, see [70].

Observe that under the assumption of Theorem 1.18, the leaves of M are hyperbolic Riemann surfaces since otherwise there would exist a foliation cycle. In particular, for every uniformization $f\colon \mathbb{D} \to L$ of a leaf, the family of currents T_r defined by (6) converge to a certain harmonic current T. In the context of flat $\mathbb{P}^1$-bundles over a curve of finite type C, Bonatti and Gomez-Mont have obtained a much more precise equidistribution statement, namely that of large leafwise discs. See [5]. Recall that a representation from an abstract group to $\mathrm{PSL}(2,\mathbb{C})$ is *non elementary* iff it does not preserve any probability measure on $\mathbb{P}^1(\mathbb{C})$.

Theorem 1.19 (Equidistribution of large leafwise discs)**.** *Let C be a Riemann surface of finite type and $\rho\colon \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation sending the peripheral curves to parabolic transformations. Assume that ρ is non elementary. Then for every sequence of points $w_n \in S_\rho = C \ltimes_\rho \mathbb{P}^1$ whose projections to C stay in a compact set, and every sequence of positive numbers R_n tending to infinity, we have the following*

$$\frac{1}{V(R_n)}[B_{\mathcal{F}}(w_n, R_n)] \to_{n\to\infty} T, \tag{15}$$

where $V(R)$ is the volume of a ball of radius R in hyperbolic plane, and T is the unique harmonic current normalized so that $\int T \wedge v_P = 1$.

We end this lecture by insisting on the fact that the dynamical method based on the Lyapunov exponent does not work to prove unique ergodicity in the context of singular holomorphic foliations on compact complex surfaces since, as was already mentioned, the definition of the

Lyapunov exponent is unclear in this case. Fornaess and Sibony succeeded proving a similar unique ergodicity statement for generic singular holomorphic foliations of the complex projective plane, see [27, 28]. Their proof is based on a completely different approach (a computation of the self-intersection of a foliated harmonic current together with Hodge index theorem), which nevertheless does not extend to all compact complex surfaces: it necessitates a non trivial automorphism group of the ambiant surface.

§2. Lecture 2—Topology of Levi-flats in algebraic surfaces

2.1. A rough guide to complex algebraic surfaces

A smooth complex algebraic manifold is a compact complex manifold which embeds holomorphically in a complex projective space $\mathbb{P}^N(\mathbb{C})$ for some $N \geq 1$. By the GAGA principle, such a compact complex submanifold is defined by algebraic homogeneous equations.

An important character in the understanding of an algebraic manifold X is its canonical bundle, namely the bundle $K_X := \bigwedge^d T^*X$, where d is the dimension of X. The plurigenera of X are defined by the dimensions $P_n(X) = h^0(X, nK_X)$ of the spaces of holomorphic sections of the powers nK_X of the canonical bundle (the tensor product of line bundles is denoted additively in the sequel). Their asymptotics when n tends to $+\infty$ is governed by the Kodaira dimension $k(X)$, which is defined by $k(X) := \lim_{n\to\infty} \frac{\log P_n}{\log n}$. The Kodaira dimension can assume any value $k \in \{-\infty, 0, 1, \dots, d\}$, where by convention $k(X) = -\infty$ means that the plurigenera vanishes for every n.

As we have seen, algebraic curves can be classified into three classes, depending upon the type of their universal covering: $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{D}$. This trichotomy can be detected by the Kodaira dimension, being respectively equal to $-\infty$, 0 or 1.

Algebraic surfaces are more difficult to classify. The surfaces with Kodaira dimension being $-\infty, 0, 1$ are relatively well understood, thanks to the classification of Enriques–Kodaira, and fall into eight classes: rational, ruled, $K3$, Enriques, Kodaira, toric, hyperelliptic, and properly quasi-elliptic. We refer to [3] for a complete treatment of this topic. Concerning the class of surfaces with Kodaira dimension 2, not much is known about their classification, though many examples have been found. These surfaces are called *surfaces of general type*, and in a sense, are the most common surfaces.

Examples of general type surfaces are smooth hypersurfaces of degree $d \geq 5$ in $\mathbb{P}^3(\mathbb{C})$, quotients of bounded domains in $\mathbb{C}^2$, double covers of $\mathbb{P}^2(\mathbb{C})$ ramified along a non singular curve of even degree ≥ 8 etc.

Surfaces with a hermitian metric of negative holomorphic curvature are of general type. There is a weak converse to this statement: a theorem of Mumford states that the canonical bundle of a (minimal) surface of general type admits a metric whose curvature is positive on all complex directions apart from a finite union of (-2)-rational curves.

2.2. Thurston's eight geometries as Levi-flats in algebraic surfaces

We will say that a 3-manifold possesses a *geometry* if it admits a complete locally homogeneous metric (locally homogeneous meaning that two different points admit isometric neighborhoods). Thurston classified in eight classes the compact 3-manifolds possessing a geometry, depending on the isometric class of their universal cover among:

$$\mathbb{S}^3, \quad \mathbb{R}^3, \quad \mathbb{H}^3, \quad \mathbb{S}^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}, \quad \mathrm{Nil}, \quad \widetilde{\mathrm{SL}(2,\mathbb{R})}, \quad \mathrm{Sol}. \tag{16}$$

The spaces $\mathbb{S}^p$, $\mathbb{R}^p$ and $\mathbb{H}^p$ for $p \in \{2,3\}$ stand for the complete simply connected Riemannian manifolds of dimension p of constant sectional curvature, resp. 1, 0, -1. The last three models are Lie groups equipped with left invariant metrics. We refer to the article of Scott [65] for a more complete treatment. Let $\mathcal{M}$ be one of the eight simply connected manifolds in the list (16). We say that a compact 3-manifold M *carries the geometry* of $\mathcal{M}$ if M is the quotient of $\mathcal{M}$ by a discrete group of isometries of $\mathcal{M}$.

All the geometries (16) are carried by Levi-flats in algebraic complex surfaces, apart $\mathbb{S}^3$. The fact that $\mathbb{S}^3$ does not appear is an observation by Inaba and Michshenko, see [46, Theorem 1], which relies on the Kähler property for algebraic surfaces, together with the famous theorem of Novikov on existence of Reeb components, see Theorem 2.4.

Proposition 2.1 (Inaba–Michshenko)**.** *A Levi-flat in a Kähler surface has an infinite fundamental group. In particular, such a Levi-flat does not carry the geometry* $\mathbb{S}^3$.

Let us review the argument. We adopt the following definition:

Definition 2.2 (Reeb component)**.** *A Reeb component is a saturated set homeomorphic to the solid torus which admits no compact leaf in the interior.*

Recall that a Kähler form on a surface S is a closed $(1,1)$-form ω which is positive on complex lines of the tangent bundle, namely $\omega(u, iu) > 0$ for every $u \neq 0 \in TS$. A complex surface is called Kähler iff it admits a Kähler form.

Lemma 2.3. *The Cauchy–Riemann foliation of a Levi-flat in a Kähler surface does not have any Reeb component.*

Proof. By contradiction, the integral of ω on the boundary would both be positive (by Kähler property) and zero (by Stokes formula). Q.E.D.

Hence, Proposition 2.1 is a consequence of Lemma 2.3 and of the following result:

Theorem 2.4 (Novikov)**.** *Let M be a compact orientable 3-manifold endowed with a transversally orientable 2-dimensional foliation $\mathcal{F}$ of class C^2. The following assertions are equivalent*

(1) *The foliation $\mathcal{F}$ contains a Reeb component.*

(2) *There exists a leaf $L \in \mathcal{F}$ such that the inclusion map $\pi_1(L) \to \pi_1(M)$ between the fundamental groups has a non-trivial kernel.*

Moreover, if there exists a closed and homotopically trivial loop transverse to $\mathcal{F}$, then the foliation $\mathcal{F}$ contains a Reeb component. This occurs in particular when the fundamental group of M is finite.

We now review examples showing that all of the geometries (16) except $\mathbb{S}^3$ are carried by Levi-flats in algebraic surfaces. First we recall that the geometries Nil, Sol and $\mathbb{H}^3$ are supported by non trivial surface bundles. A *surface bundle* is the quotient of $[0,1] \times \Sigma$ by the relation $(0,x) \sim (1,\Phi(x))$, where Σ is a compact oriented surface and Φ is a diffeomorphism of Σ preserving the orientation.

We shortly denote a surface bundle $\mathbb{S}^1 \ltimes_\Phi \Sigma$. Its *monodromy* is the projection $[\Phi]$ of Φ in the mapping class group $\mathrm{MCG}(\Sigma)$. An element $[\Phi] \in \mathrm{MCG}(\Sigma)$ is called *elliptic* if its order is finite, *reducible* if there is a finite collection of pairwise disjoint simple closed curves in Σ whose union is invariant by a diffeomorphism in $[\Phi]$, and *pseudo-Anosov* in the other cases, see [69, Section 2].

If Σ has genus 1, the surface bundle is called a *torus bundle*. The group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\Sigma \simeq \mathbb{R}^2/\mathbb{Z}^2$ by linear transformations and captures all the classes of $\mathrm{MCG}(\Sigma)$. A *unipotent torus bundle* is a torus bundle whose monodromy comes from a unipotent matrix in $\mathrm{SL}(2,\mathbb{Z})$ (reducible monodromy), it carries the Nil geometry. A *hyperbolic torus bundle* is a torus bundle whose monodromy comes from a hyperbolic matrix in $\mathrm{SL}(2,\mathbb{Z})$ (pseudo-Anosov monodromy), it carries the Sol geometry.

We shall realize such surface bundles in *singular holomorphic fibrations*. Such a fibration stands for a holomorphic map $f\colon S \to B$ where S is a complex surface and B is a compact Riemann surface, see [3, Chapter V]. Let $p_1, \dots, p_n$ be the singular values of f (it may be empty). A *fibered Levi-flat hypersurface* is a Levi-flat hypersurface of the form $f^{-1}(\gamma)$, where $f\colon S \to B$ is a singular holomorphic fibration and $\gamma \subset B \setminus \{p_1, \dots, p_n\}$ is a simple closed path. Such hypersurfaces were already considered by Poincaré in his study of cycles on algebraic surfaces, see [63].

Proposition 2.5. *Every geometry* $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil *or* Sol *is carried by a fibered Levi-flat hypersurface. Moreover,* $\mathbb{H}^3$ *and* $\mathbb{H}^2 \times \mathbb{R}$ *are carried by fibered Levi-flat hypersurfaces in surfaces of general type.*

We give the sketch of proof of this fact. It is easy to realize $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ by using products of compact Riemann surfaces $S = \Sigma \times B$. To exhibit fibered Levi-flat hypersurfaces with the geometries Nil and Sol, we use the following classical proposition, see [31, Chapter II, Section 2.3]. Here the complex surface S comes from a singular holomorphic fibration by elliptic curves over the Riemann sphere.

Proposition 2.6. *Let* $f\colon S \to \mathbb{P}^1(\mathbb{C})$ *be a singular elliptic fibration. Let* $p_1, \dots, p_n$ *be the singular values of* f, *assume that this set has cardinality* ≥ 3. *Then the monodromy representation from the fundamental group of* $\mathbb{P}^1(\mathbb{C}) \setminus \{p_1, \dots, p_n\}$ *to* $\mathrm{SL}(2, \mathbb{Z})$ *is surjective.*

Using this proposition, one easily constructs Levi-flat hypersurfaces of the form $f^{-1}(\gamma)$ (up to finite coverings of f) carrying the geometries Nil or Sol. We refer to [20]. To realize $\mathbb{H}^3$ we use Thurston's theorem, see [69, Theorem 0.1].

Theorem 2.7 (Thurston)**.** *Let* Σ *be a compact oriented surface of genus* $g \geq 2$. *A surface bundle* $\mathbb{S}^1 \ltimes_\Phi \Sigma$ *carries the geometry* $\mathbb{H}^3$ *if and only if its monodromy* $[\Phi]$ *is pseudo-Anosov.*

By using the same arguments as before, the following theorem provides fibered Levi-flat hypersurfaces modelled on $\mathbb{H}^3$, see [67, Corollary 1].

Theorem 2.8 (Shiga)**.** *Let* B *be a compact Riemann surface with genus larger than or equal to* 2. *Let* $f\colon S \to B$ *be a singular holomorphic fibration, such that the generic fiber has genus* ≥ 2 *and its modulus is not locally constant* (e.g. *a Kodaira fibration*). *Let* $p_1, \dots, p_n$ *be the critical values of* f. *Then there exists an immersed simple closed curve* γ *in* $B \setminus \{p_1, \dots, p_n\}$ *whose monodromy is pseudo-Anosov.*

Note that the surface S in this theorem is of general type, since the genus of the base and the fibers of f is larger than 1, see [3, Chapter 3, Theorem 18.4]. Passing to a finite cover if necessary, this completes the proof of Proposition 2.5.

It remains to treat the geometry of $\widetilde{\mathrm{SL}(2, \mathbb{R})}$. This geometry is supported for instance by non-trivial circle bundles over compact oriented surfaces of genus $g \geq 2$, see [65, Theorem 5.3]. There exists Levi-flat hypersurfaces with this topology in flat $\mathbb{P}^1(\mathbb{C})$-bundles over compact Riemann surfaces. Namely, we consider a representation $\rho\colon \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$, and the flat $\mathbb{P}^1(\mathbb{C})$ bundle $S_\rho = C \ltimes_\rho \mathbb{P}^1(C)$, see Example 1.1; the subset $M_\rho := C \ltimes_\rho \mathbb{P}^1(\mathbb{R}) \subset S_\rho$ is an analytic Levi-flat hypersurface, having the structure of an oriented circle bundle over C. We denote e the Euler class of M_ρ. We recall that this invariant belongs to $H^2(C,\mathbb{Z}) \simeq \mathbb{Z}$ and characterizes the circle bundle up to isomorphism, see e.g. [57, Section 2]. Note that $|e| = 2g-2$ if and only if ρ is an isomorphism between $\pi_1(C)$ and a Fuchsian group. In this case M_ρ is diffeomorphic to the unitary tangent bundle of C, see [72, Proposition 6.2].

Proposition 2.9. *Let C be a compact oriented surface of genus $g \geq 2$ and let $e \in \mathbb{Z}$ satisfying $|e| \leq 2g-2$. There exists a flat $\mathbb{P}^1(\mathbb{C})$-bundle S over C and a Levi-flat hypersurface $M \subset S$ which is diffeomorphic to a circle bundle over C with Euler class e.*

Proof. If $|e| \leq 2g-2$ then there exists a representation $\rho\colon \pi_1(C) \to \mathrm{PSL}(2,\mathbb{R})$ such that M_ρ has Euler class e, see [38, Theorems A and B]. Q.E.D.

2.3. Levi-flat circle bundles in surfaces of general type

We begin with an upper bound on the Euler class of Levi-flat circle bundles.

Proposition 2.10. *Let S be a surface of general type and M be a Levi-flat hypersurface of class C^2 in S. Assume that M is an oriented circle bundle over a compact oriented surface C of genus $g \geq 2$. Then the Euler class of M satisfies $|e| \leq 2g-2$.*

Sketch of Proof. We can assume $e \neq 0$. We first prove that the Cauchy–Riemann foliation has no compact leaf. As we will see later, see Section 2.4, the general type assumption implies that every leaf is hyperbolic. Assuming by contradiction that there exists a compact leaf L, it would have genus ≥ 2, and the Euler class being different from 0, it is easy to see that L would be compressible, namely the map $\pi_1(L) \to \pi_1(M)$ would not be injective. Novikov's theorem would then provide a Reeb component, which contradicts the fact that the surface is Kähler. Hence, there are no compact leaves, and the result follows from the combination of the next two results. Q.E.D.

Theorem 2.11 (Thurston)**.** *Let M be an oriented circle bundle over a compact oriented surface Σ of genus $g \geq 2$. Assume that $\mathcal{F}$ is an oriented 2-dimensional foliation on M of class C^2, and that $\mathcal{F}$ does not have any compact leaf. Then there exists a diffeomorphism Ψ of M*

of class C^2 isotopic to the identity such that $\Psi_*\mathcal{F}$ is transverse to the circle fibration.

Theorem 2.12 (Milnor–Wood)**.** *Let M be an oriented circle bundle over a compact oriented surface Σ of genus $g \geq 2$. If M supports a transversally oriented 2-dimensional foliation which is transverse to the circle fibration, then its Euler class satisfies $|e| \leq 2g-2$.*

Remark 2.13. The question of the existence of Levi-flats in algebraic surfaces diffeomorphic to circle bundles over hyperbolic compact surface with arbitrarily large Euler class, whose Cauchy–Riemann foliation is obtained by the technique called in french “tourbillonement de Reeb”, remains open.

The following result provides a construction of Levi-flat hypersurfaces in surfaces of general type with a non trivial Euler class.

Theorem 2.14. *For every $\epsilon > 0$ there exist a surface of general type S_ϵ and a Levi-flat hypersurface $M_\epsilon \subset S_\epsilon$ which is diffeomorphic to an oriented circle bundle M_ϵ over a compact oriented surface C_ϵ of genus ≥ 2. We have $|e(M_\epsilon)/\mathrm{Eu}(C_\epsilon)| \in [1/5-\epsilon, 1/5]$, where $e(M_\epsilon)$ denotes the Euler class of M_ϵ and $\mathrm{Eu}(C_\epsilon)$ denotes the Euler characteristic of C_ϵ.*

Sketch of Proof. Here we only prove that there exists a Levi-flat in a surface of general type which is diffeomorphic to a non trivial circle bundle, hence carrying the geometry $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Let C be a compact algebraic curve of genus $g \geq 2$. By the uniformization theorem, see Theorem 1.9, there is a biholomorphism $D\colon \widetilde{C} \to \mathbb{H}$ which is equivariant w.r.t. some representation $\rho\colon \pi_1(C) \to \mathrm{Aut}(\mathbb{H}) \subset \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$. Let $(S_\rho, \mathcal{F}_\rho)$ be the flat $\mathbb{P}^1(\mathbb{C})$-bundle over C of monodromy ρ, defined as in Example 1.1. There is a Levi-flat defined by $M_\rho = C \ltimes \mathbb{P}^1(\mathbb{R})$, which is diffeomorphic to the unitary tangent bundle of the surface C equipped with e.g. its Poincaré metric. The bundle $S_\rho \to C$ has a holomorphic section $s\colon C \to S_\rho$ defined as the level of the universal covers by $\tilde{s}(x) = (x, D(x))$. Of course we are not done since the Kodaira dimension of S_ρ is $-\infty$, hence S_ρ is not of general type.

We construct $(S_\varepsilon, M_\varepsilon)$ as a double ramified covering of (S_ρ, M_ρ). To define such a ramified cover, let $E \to S_\rho$ be a holomorphic line bundle and $h\colon S_\rho \to 2E$ (recall our additive notation for tensor product of line bundles) be a holomorphic section, whose zero divisor $h^{-1}(0)$ is a smooth reduced algebraic curve in S_ρ. The algebraic surface

$$S_\varepsilon = \{(w,\zeta) \in E \mid \zeta^2 = h(w)\}. \tag{17}$$

is a $2:1$ ramified cover (defined by $\pi(x,\zeta)=x$), ramifying over $h^{-1}(0)$. We easily verify that the pull-back of $\mathcal{F}_\rho$ is a singular holomorphic foliation $\mathcal{F}_\varepsilon$ whose singularities are the pull-back in S_ε of the points of tangency between $\mathcal{F}_\rho$ and $h^{-1}(0)$. Hence assuming that $h^{-1}(0)$ intersects M_ρ transversally, the set $M_\varepsilon=\pi^{-1}(M_\rho)$ is a Levi-flat hypersurface of S_ε. To understand its topology, one has to understand the topology of the link $h^{-1}(0)\cap M_\rho$ in M_ρ.

It is well-known that if E is sufficiently ample[1] then the surface S_ε constructed above is of general type (e.g. if F is ample, then the sufficiently large powers of F will work). For such a line bundle, choosing at random the section h of its square would probably lead to a hyperbolic manifold M_ε. Hence we will need to make a very particular choice. Define $F=\mathcal{O}\big(ks+\sum_{j\in J}f_j\big)$ where k is an integer, and f_j are distinct fibers of the fibration $S_\rho\to C$. If we assume furthermore that k and the number $|J|$ of fibers f_j are both even, then it is possible to find a line bundle E such that $2E=F$. By definition of F there exists a holomorphic section $h_0\colon S_\rho\to F$ such that $h_0^{-1}(0)=s\cup\bigcup_j f_j$. Observe that the zero set of h_0 is transverse to M_ρ and that its intersection with M_ρ is a union of $|J|$ fibers of the circle fibration $M_\rho\to C$, hence is a quite simple link. The section h_0 is not convenient for our purpose, since its zero set is not smooth (at the intersection points of f_j and s). Nevertheless, we can show that if k and $|J|$ are large enough, the line bundle E is ample, and one can make a small perturbation h of h_0 with a smooth zero set. For such a choice, the couple (E,h) yields the desired Levi-flat $M_\varepsilon\subset S_\varepsilon$ diffeomorphic to a non trivial circle bundle. See details in [20]. Q.E.D.

The sup of the ratios $|e(M)/\mathrm{Eu}(C)|$, where M is a Levi-flat in a surface of general type diffeomorphic to a circle bundle of Euler class $e(M)$ over a hyperbolic compact surface C, is unknown. The following result shows that the value $|e(M)/\,\mathrm{Eu}(C)|=1$ (the maximal permitted by Proposition 2.10) is not reached:

Theorem 2.15. *A Levi-flat hypersurface of class C^2 in a surface of general type is not diffeomorphic to the unitary tangent bundle of a hyperbolic compact two dimensional orbifold.*

The proof of this result uses a foliated Lyapunov exponent associated to the Cauchy–Riemann foliation and its sketch is postponed to Corollary 2.22. See [20] for details.

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[1]Ample means that it carries a metric of positive curvature.

2.4. Hyperbolicity and topological consequences

The following result will be crucial for studying the topology of Levi-flats in surfaces of general type.

Proposition 2.16. *Let M be a Levi-flat of class C^2 in a surface of general type. Then the Cauchy–Riemann foliation of $\mathcal{F}$ has hyperbolic leaves.*

Sketch of Proof. We prove Proposition 2.16 under the assumption that K_S is ample, namely that it has a metric of positive curvature. Assume that $\mathcal{F}$ has a compact leaf L. Adjunction formula then gives

$$\mathrm{Eu}(L) = -L \cdot K_S - L \cdot N_L.$$

The first term of the right hand side is < 0 because K_S has a metric of positive curvature, and the second one is zero because the normal bundle of L has a flat connexion (the Bott connexion induced by the foliation), hence L is hyperbolic.

Assume now that there exists a parabolic leaf L. A theorem of Candel shows that there exists an Ahlfors current T such that $T \cdot K_{\mathcal{F}} = 0$ (see [11]). Using the leafwise adjunction formula we obtain

$$T \cdot K_{\mathcal{F}} = T \cdot K_S + T \cdot N_{\mathcal{F}}.$$

The right hand side is > 0 for the same reason as before (take the Bott connexion on $N_{\mathcal{F}}$ in equation (7)). This yields a contradiction. Q.E.D.

We deduce the following application:

Theorem 2.17. *Let S be a surface of general type and let M be an immersed Levi-flat hypersurface of class C^2 in S. Then the fundamental group of M has exponential growth. In particular M does not carry the geometries $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{R}^3$ nor* Nil.

Sketch of Proof. Since there is no Reeb component, Novikov's theory shows that the leaves of the pull-back of the Cauchy–Riemann foliation in the universal cover $\widetilde{M}$ of M are simply connected, and that their growth is bounded by the one of $\widetilde{M}$ (if it were not the case, we could find a leaf in $\widetilde{M}$ that intersects a flow box in two different plaques, those finding a closed transverse loop to the foliation in $\widetilde{M}$, which is impossible by the last part of Novikov's theorem (Theorem 2.4)). Hence, $\widetilde{M}$ has exponential growth, by Proposition 2.16 and by Verjovsky–Candel result on the continuity of the Poincaré metric, see Theorem 1.11. Q.E.D.

Remark 2.18. The hyperbolicity of the Cauchy–Riemann foliation is related to the following open conjecture.

Conjecture 2.19 (Green–Griffiths)**.** *Let S be a surface of general type. There exists a proper subvariety $Y \subset S$ such that every entire curve $f\colon \mathbb{C} \to S$ satisfies $f(\mathbb{C}) \subset Y$.*

This problem was solved by McQuillan [59] for surfaces of general type satisfying $c_1^2(S) > c_2(S)$. He proved that every non-degenerate entire curve $f\colon \mathbb{C} \to S$ is tangent to a singular holomorphic foliation on (a finite cover of) S. A contradiction is deduced from positivity properties of the tangent bundle of the foliation. Brunella provided an alternative proof in [7] by using the normal bundle of the foliation. An important difficulty in these works is that $f(\mathbb{C})$ can contain a singular point of the foliation. In our non-singular context the proof is simpler because we directly use adjunction formula. We refer to the survey [23] for recent results concerning Green–Griffiths conjecture.

2.5. The Anosov property and application to the topology of Levi-flats

A Levi-flat $M \subset S$ in a complex surface is called *Anosov* if its Cauchy–Riemann foliation is topologically conjugate to the weak unstable foliation of a 3-dimensional Anosov flow on some compact 3-manifold N. Classical examples of Anosov flows are the geodesic flow on the unitary tangent bundle of compact orientable surfaces of genus ≥ 2 and the horizontal flow on hyperbolic torus bundles. There are many other examples, for instance on hyperbolic 3-manifolds and graph 3-manifolds, see [30, 39, 41]. One can verify that Anosov Levi-flat hypersurfaces do not have any transverse invariant measure, their foliation $\mathcal{F}$ is therefore hyperbolic. We have the following upper bound for the Lyapunov exponent.

Theorem 2.20. *Let S be a complex surface and M be an immersed Anosov Levi-flat hypersurface in S. We endow the leaves of the Cauchy–Riemann foliation $\mathcal{F}$ with the Poincaré metric g_P. Let T be an ergodic foliated harmonic current of $\mathcal{F}$. Then the Lyapunov exponent of T satisfies $\lambda(T) \leq -1$.*

Sketch of Proof. We use that the trajectories of the Anosov flow in the hyperbolic uniformizations of the leaves are quasigeodesics for the Poincaré metric, to produce a new flow by stretching these trajectories to geodesics. We obtain a continuous flow on M whose orbits are leafwise geodesics for the Poincaré metric. Let v_P be the leafwise Poincaré volume form. Since the result does not depend on the projective class of T, we can assume that the foliated harmonic measure $T \wedge v_P$ has mass one. This latter is shown to be a SRB measure for the stretched flow. Moreover, the Lyapunov exponents of this measure are 1, 0, λ. (The

Lyapunov exponents are not a priori defined since the stretched flow is only continuous. However, it is smooth along the leaves, which gives the exponents 1 and 0, and using the C^1 transverse structure of the foliation we can define another exponent, which we identify with λ). The ingredients for this computation involve the shadowing property of geodesics by Brownian paths due to Ancona, see [2, Théorème 7.3, p. 103]. The bound $\lambda(T) + 1 \leq 0$ to be proved then relies on volume estimates in the spirit of Margulis–Ruelle's inequality. Q.E.D.

Corollary 2.21. *Let S be a surface of general type and let M be an immersed Levi-flat hypersurface in S. Then M is not Anosov.*

Sketch of Proof. We indicate the proof when K_S has a metric of positive curvature. The proof then relies on the leafwise adjunction formula, which gives $T \cdot K_{\mathcal{F}} = T \cdot N_{\mathcal{F}} + T \cdot K_S > T \cdot N_{\mathcal{F}}$. We deduce that the Lyapunov exponent verifies the following pinching estimates

$$-1 < \lambda(T) \leq 0 \tag{18}$$

which is contradictory with being Anosov by Theorem 2.20. Q.E.D.

Corollary 2.22. *A Levi-flat in a surface of general type is not diffeomorphic to a quotient of the Lie groups* Sol *or* $\mathrm{PSL}(2, \mathbb{R})$ *by a cocompact lattice.*

Sketch of Proof. The proof is by contradiction. Assuming that a Levi-flat is diffeomorphic to one of those manifolds, we use deep results of resp. Ghys/Sergiescu, see [37], and Matsumoto, see [57], which enable to prove that the Levi-flat is Anosov. Hence the contradiction comes from Corollary 2.21. In order to apply the mentioned theorems, one needs to verify that the Cauchy–Riemann foliation has no compact leaf, which is done by using the hyperbolicity of the leaves together with Novikov's theory. Q.E.D.

§3. Lecture 3—Complex projective structures: Lyapunov exponent, degree and harmonic measure

3.1. A rough guide to complex projective structures

Let C be a smooth complex quasi-projective curve of negative Euler characteristic. We denote by g its genus and by n its number of punctures. A complex projective structure on C is a maximal atlas of holomorphic charts $z_j \colon U_j \subset C \to \mathbb{P}^1(\mathbb{C})$ (called projective charts) which overlap as

$$z_j = \frac{az_k + b}{cz_k + d},$$

on the intersection $U_j \cap U_k$, where a, b, c, d are complex numbers such that $ad - bc \neq 0$. We will denote $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$, and will refer to $\mathbb{P}^1$-structures instead of complex projective structures. Two $\mathbb{P}^1$-structures on C are equivalent if they define the same atlas of projective charts.

It is convenient to define a $\mathbb{P}^1$-structure on C in terms of the so-called development-holonomy pair $(\mathsf{dev}, \mathsf{hol})$. Each projective chart can be extended analytically as a locally injective meromorphic map $\mathsf{dev}\colon \widetilde{C} \to \mathbb{P}^1$, satisfying the equivariance property $\mathsf{dev} \circ \gamma = \mathsf{hol}(\gamma) \circ \mathsf{dev}$, where hol is a representation $\pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$. A development-holonomy pair is not unique for a given projective structure. Namely, if $A \in \mathrm{PSL}(2, \mathbb{C})$, $(A \circ \mathsf{dev}, A \circ \mathsf{hol} \circ A^{-1})$ gives another development-holonomy pair. We refer here to the survey paper by Dumas, see [24] for a comprehensive treatment of this notion.

When the surface C is not compact (hence by assumption it is biholomorphic to a compact Riemann surface punctured at a finite set), we restrict ourselves to the subclass of *parabolic* $\mathbb{P}^1$-structures. Such a structure has the following well-defined local model around the punctures: each puncture has a neighborhood which is *projectively* equivalent to the quotient of the upper half plane by the translation $z \mapsto z + 1$.

A $\mathbb{P}^1$-structure on C can be understood by the way of the Schwarzian derivative. Indeed, introduce the following differential operator called the Schwarzian:

$$S(f) := \{f, z\}\, dz^2 \quad \text{where} \quad \{f, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{19}$$

for every holomorphic local diffeomorphism $f\colon U \subset \mathbb{C} \to \mathbb{C}$. We have the following two fundamental properties

(1) $S(g \circ f) = S(f) + f^* S(g)$ for every local diffeomorphisms $f\colon U \subset \mathbb{C} \to V \subset \mathbb{C}$ and $g\colon V \subset \mathbb{C} \to W \subset \mathbb{C}$.
(2) $S(f) = 0$ iff $f(z) = \frac{az+b}{cz+d}$ for some complex numbers a, b, c, d such that $ad - bc \neq 0$.

In particular, let σ_1 and σ_2 be two $\mathbb{P}^1$-structures on C. Pick projective charts z_1 and z_2 defined on some common open set $U \subset C$ of σ_1 and σ_2 respectively, and define the holomorphic quadratic differential $q_{\sigma_1,\sigma_2} = \{z_2, z_1\}\, dz_1^2$. Properties (1) and (2) show that q_{σ_1,σ_2} does not depend on the chosen projective charts z_1 and z_2, and thus defines a holomorphic quadratic differential on the curve C. Reciprocally, given a $\mathbb{P}^1$-structure σ_1 and a holomorphic quadratic differential q on C, there exists a unique $\mathbb{P}^1$-structure σ_2 on C such that $q = q_{\sigma_1,\sigma_2}$. In particular, at least when C is compact, the set of projective structures on C is an affine space directed by the vector space of holomorphic quadratic differentials on C.

This shows that the set $P(C)$ of $\mathbb{P}^1$-structures on a compact algebraic curve of genus $g \geq 2$ is isomorphic to $\mathbb{C}^{3g-3}$. We will not discuss here the analogous computation in the punctured case, which relies on results of Fuchs and Schwarz, but we state the result: the set $P(C)$ of parabolic $\mathbb{P}^1$-structures on C is isomorphic to $\mathbb{C}^{3g-3+n}$.

One of the interest in studying complex projective structures comes from their relations to uniformization problems in two or three dimensions. The main illustration of this is certainly given by the uniformization theorem of Poincaré–Koebe, which in particular defines a canonical projective structure σ_{Fuchs} (by viewing C as a quotient of $\mathbb{H}$ under a Fuchsian group). Other kind of uniformizations have been considered, e.g. Schottky uniformizations, and lead to parabolic $\mathbb{P}^1$-structures as well. More generally, the Ahlfors finiteness theorem provides many examples of parabolic $\mathbb{P}^1$-structures:

Theorem 3.1 (Ahlfors finiteness theorem)**.** *Let* Γ *be a finitely generated discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$. *Then the quotient of the discontinuity set* $\Omega \subset \mathbb{P}^1$ *by* Γ *is a finite type Riemann surface. Moreover*, *if* Γ *is torsion free*, *then the* $\mathbb{P}^1$*-structure that it inherits is parabolic.*

The last (less known) part of the theorem is proved in [1, Lemma 1]. The structures produced by Theorem 3.1 have been known as *covering projective structures*, because they are characterized by the fact that the developing map is a covering onto its image [49, 50]. A particular example is given by quasi-Fuchsian deformations of the canonical structure σ_{Fuchs}. These structures play an important role in Teichmüller theory. Recall that the Teichmüller space $T(C)$ is defined as the set of equivalence classes of couples $(D, [\Psi])$ where D is a Riemann surface and $[\Psi]$ is a homotopy class of diffeomorphism between C and D. Two couples $(D_1, [\Psi_1])$ and $(D_2, [\Psi_2])$ are considered as equivalent if $\Psi_2 \circ \Psi_1^{-1}$ is homotopic to a biholomorphism from D_1 to D_2. Recall the following important result.

Theorem 3.2 (Bers simultaneous uniformization theorem)**.** *For every* $(D, [\Psi]) \in T(C)$, *there exists a unique representation* ρ *from* $\pi_1(C)$ *to* $\mathrm{PSL}(2,\mathbb{C})$ (*up to conjugation*) *preserving a partition* $\mathbb{P}^1 = \mathcal{D}_C \cup \Lambda \cup \mathcal{D}_D$, *where* Λ *is a topological circle*, *and* $\mathcal{D}_C$ (*resp.* $\mathcal{D}_D$) *is the image of a* ρ*-equivariant univalent holomorphic* (*resp. anti-holomorphic*) *map from* $\widetilde{C}$ (*resp.* $\widetilde{D}$, *observe that we have an identification of* $\pi_1(D)$ *with* $\pi_1(C)$ *induced by* Ψ) *to* $\mathbb{P}^1$.

Let $P(C)$ be the set of (parabolic) $\mathbb{P}^1$-structures on C. Observe that for every $(D, [\Psi]) \in T(C)$, the holomorphic univalent ρ-equivariant mapping given by Theorem 3.2 produces a (parabolic) $\mathbb{P}^1$-structure, and that this later determines the element $(D, [\Psi])$. This defines an embedding $B\colon T(C) \to P(C)$, called the Bers embedding. Bers proved that the map B is holomorphic, and that its image $B(C)$ is relatively compact in $P(C)$. This later is called the *Bers slice*.

There are many other examples of parabolic $\mathbb{P}^1$-structures. For instance surgery operations such as grafting (see Hejhal's original construction in [42]) may produce a parabolic $\mathbb{P}^1$-structure with holonomy a Kleinian group that is not of covering type.

Theorem 3.3 (Hejhal)**.** *There exist* $\mathbb{P}^1$*-structures on compact curves such that the developing map is not a covering onto its image, but whose holonomy has image a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$*.*

Such projective structures are usually called *exotic*. The prototype of such an exotic projective structure is obtained by inserting a Hopf annulus after cutting a given $\mathbb{P}^1$-structure along a simple closed curve. More precisely start with the quotient C_u of $\mathbb{H}$ by a lattice $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ containing as a primitive element the hyperbolic transformation $\gamma(z) = \alpha z$ for $\alpha > 1$, and consider

$$C = (\overline{C_u \setminus \gamma_u} \cup \overline{H \setminus \gamma_H})/\{\gamma_u^{\pm} \simeq \gamma_H^{\mp}\},$$

where $\gamma_u = \alpha\backslash i\mathbb{R}^{+*} \subset C_u$, $H = \alpha\backslash\mathbb{C}^*$ is the Hopf torus, and $\gamma_H = \alpha\backslash i\mathbb{R}^{+*} \subset H$. The set of exotic $\mathbb{P}^1$-structures in $P(C)$ is organized as a countable union of non empty connected open subsets called *exotic components*.

Using the point of view of the Schwarzian derivative, one can construct yet other examples of $\mathbb{P}^1$-structures on C. For instance, one can prove that there exists a non empty open subset of $P(C)$ formed by $\mathbb{P}^1$-structures on C whose holonomy has image a dense subgroup of $\mathrm{PSL}(2,\mathbb{C})$. We refer to [9] for a proof of this fact in the case of the fourth punctured sphere, which readily extends to all algebraic curves.

There are nice pictures of the decomposition of $P(C)$ into the various subsets described above: Bers slice, exotic components, etc. We refer e.g. to [48].

3.2. The degree of a $\mathbb{P}^1$-structure

Let g_P be the unique complete conformal metric of curvature -1 on C. It is well known that when C is of finite type, the hyperbolic metric has finite volume. Recall that a representation $\pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ is non elementary if it does not preserve any probability measure on the Riemann sphere. The holonomy of a parabolic projective structure is always non elementary: see [33, Theorem 11.6.1, p. 695] for the compact case, and [9, Lemma 10] for the punctured case.

If σ is a parabolic projective structure, we want to define $\delta(\sigma)$ as a nonnegative number counting the average asymptotic covering degree of $\mathsf{dev}_\sigma\colon \widetilde{C} \to \mathbb{P}^1$. For any $x \in \widetilde{C}$ we denote by $B(x, R)$ the ball centered at x of radius R in the Poincaré metric, and by vol the hyperbolic volume.

Definition-Proposition 3.4. *Let C be a Riemann surface of finite type and σ be a parabolic $\mathbb{P}^1$-structure on X. Choose a universal covering $c\colon \widetilde{C} \to C$, and a developing map $\mathsf{dev}\colon \widetilde{C} \to \mathbb{P}^1$. Let (x_n) be a sequence of points in $\widetilde{C}$ whose projections $c(x_n)$ stay in a compact subset of C, R_n be a sequence of radii tending to infinity, and (z_n) be an arbitrary sequence in $\mathbb{P}^1$. Then the limit*

$$\delta = \lim_{n\to\infty} \frac{\#B(x_n, R_n) \cap \mathsf{dev}^{-1}(z_n)}{\mathrm{vol}(B(x_n, R_n))} \tag{20}$$

exists, and does not depend on the chosen sequences (x_n), (R_n), (z_n) nor on the developing map dev. *The number δ is invariant by taking finite covering, so does not behave like a degree. We define* $\deg(\sigma) = \mathrm{vol}(C)\delta$, *and call this number the* degree *of the $\mathbb{P}^1$-structure.*

The very reason for the normalization $\deg(\sigma) = \mathrm{vol}(C)\delta$ is clearer when dealing with branched projective structures. Such structures are defined by non constant equivariant meromorphic maps defined on the universal cover w.r.t. a representation of the covering group to $\mathrm{PSL}(2, \mathbb{C})$. The most basic example of a branched projective structure is a non constant meromorphic function $f\colon C \to \mathbb{P}^1$. For such a structure, one verifies that the limit (20) exists, and that the average degree in the sense of Definition-Proposition 3.4 coincides with the topological degree of the map f.

The existence of the limit in (20) is not obvious, in particular due to the possibility of boundary effects. The proof ultimately relies on the equidistribution theorem of Bonatti and Gomez-Mont [5] mentioned in the first lecture, Theorem 1.19.

It also makes use of the following dictionary between projective structures on curves and transverse sections of flat $\mathbb{P}^1$-bundles over curves, which was developed in depth in the survey [53].

Suppose that σ is a $\mathbb{P}^1$-structure. Introduce the flat $\mathbb{P}^1$-bundle $(S_{\mathsf{hol}}, \mathcal{F}_{\mathsf{hol}})$, see Example 1.1, where $(\mathsf{dev}, \mathsf{hol})$ is a development-holonomy pair for the structure σ. Observe that the bundle map $S_{\mathsf{hol}} \to C$ has a section $s\colon C \to S_{\mathsf{hol}}$ defined at the level of the universal covers by $x \mapsto (x, \mathsf{dev}(x))$. This section—we identify the section and its image here—is transverse to the foliation $\mathcal{F}_{\mathsf{hol}}$.

Reciprocally, if $\rho\colon \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$ is any representation, a section of S_ρ transverse to the foliation $\mathcal{F}_\rho$ gives rise to a projective structure on C, by restricting the transverse projective structure of the foliation $\mathcal{F}_\rho$ to the section. This operation is the inverse of the one described in the last paragraph.

Sketch of Proof of Definition-Proposition 3.4. After these preliminaries, let us sketch the proof of the convergence (20). We will give the proof only in the case C is compact. The punctured case necessitates a separate technical analysis. We refer to [19] for the details. Let σ be a $\mathbb{P}^1$-structure and s its corresponding section of S_{hol}. We denote by T a foliated harmonic current on $(S_{\mathsf{hol}}, \mathcal{F}_{\mathsf{hol}})$ normalized so that its product with the Poincaré volume form is 1. The number $\# B(x_n, R_n) \cap \mathsf{dev}^{-1}(z_n)$ is easily seen to be the number of intersection of points of the leafwise ball $B_{\mathcal{F}}(w_n, R_n)$ with s, where w_n is the projection in S_ρ of the point (x_n, z_n). Hence since the leafwise balls normalized by their volume (considered as currents) tend to T (Theorem 1.19), one shows (with a little additional technical work) that $\frac{\# B(x_n,R_n)\cap \mathsf{dev}^{-1}(z_n)}{\mathrm{vol}(B(x_n,R_n))}$ tends to the *geometric* intersection product of T with s. This product is defined in the following way: T can be thought of as a family of transverse measures for the foliation $\mathcal{F}_\rho$, and it induces a Radon measure on any curve of S_ρ. The mass of this measure is by definition the intersection product of T with s and is denoted $T \dot{\wedge} s$. Q.E.D.

A corollary from Proof of Definition-Proposition 3.4 yields the following.

Corollary 3.5. *The degree vanishes iff σ is a covering projective structure.*

3.3. Lyapunov exponent of $\mathbb{P}^1$-structures

Fix a basepoint $\star \in C$, in particular an identification between the covering group $\pi_1(C)$ and the usual fundamental group $\pi_1(C, \star)$. As C is endowed with its Poincaré metric, Brownian motion on C is well-defined. Let $W_\star$ be the Wiener measure on the set of continuous paths $\omega\colon [0,\infty) \to X$ starting at $\omega(0) = \star$.

Definition-Proposition 3.6. *Let C and σ be as above. Define a family of loops as follows: for $t > 0$, consider a Brownian path ω issued from $\star$, and concatenate $\omega|_{[0,t]}$ with a shortest geodesic joining $\omega(t)$ and $\star$, thus obtaining a closed loop $\widetilde{\omega}_t$. Then for $W_\star$ a.e. Brownian path ω the limit*

$$\chi(\sigma) = \lim_{t\to\infty} \frac{1}{t} \log \|\mathsf{hol}(\widetilde{\omega}_t)\| \tag{21}$$

exists and does not depend on ω. This number is by definition the Lyapunov exponent of σ.

Here $\|\cdot\|$ is any matrix norm on $\mathrm{PSL}(2,\mathbb{C})$. The existence of the limit in (21) was established in [18, Definition-Proposition 2.1]. As expected it is a consequence of the subadditive ergodic theorem. In the notation of [18], $\chi(\sigma) = \chi_{\mathrm{Brown}}(\mathsf{hol})$. Another way to define $\chi(\sigma)$ goes as follows (see [18, Remark 3.7]: identify $\pi_1(C)$ with a Fuchsian group Γ and choose independently random elements $\gamma_n \in \Gamma \cap B_{\mathbb{H}}(0, R_n)$, relative to the counting measure. Here (R_n) is a sequence tending to infinity as fast as, say n^α for $\alpha > 0$. Then almost surely

$$\frac{1}{d_{\mathbb{H}}(0, \gamma_n(0))} \log\|\mathsf{hol}(\gamma_n)\| \xrightarrow[n\to\infty]{} \chi(\sigma).$$

The following formula relates the Lyapunov exponent $\chi(\sigma)$ to the degree defined in the last subsection.

Theorem 3.7. *Let σ be a parabolic holomorphic $\mathbb{P}^1$ structure on C. Let as above $\chi(\sigma)$, $\delta(\sigma)$, and $\deg(\sigma)$ respectively denote the Lyapunov exponent, the unnormalized degree and the degree of σ. Then the following formula holds*:

$$\chi(\sigma) = \frac{1}{2} + 2\pi\delta(\sigma) = \frac{1}{2} + \frac{\deg(\sigma)}{|\mathrm{Eu}(C)|}. \tag{22}$$

Theorem 3.7 could be understood as the analogue of the familiar Manning–Przytycki formula [55, 64] for the Lyapunov exponent of the maximal entropy measure of a polynomial. Recall that this formula states that for a polynomial P of degree d in one variable

$$\chi = \log d + \sum_{P'(c)=0} G(c),$$

where G is the Green function. See [55, 64]. The term $\log d$ is constant on parameter space (equal to the entropy of the polynomial P), as the term $\frac{1}{2}$ in formula (22), and the term $\sum_c G(c)$ is non negative, as well as the degree.

This reinforces an analogy between Mandelbrot sets and Bers slices that was brought to light by McMullen [58]. Namely, the Lyapunov exponent is minimal on these sets (equal to $\log d$ for the Mandelbrot set and to $1/2$ for the Bers slice). We will develop more on this analogy later on.

Sketch of proof. Surprisingly enough, the proof is based on the ergodic theory of holomorphic foliations. Again we will indicate the proof only when C is compact, and refer to [19] for the punctured case. Recall that there is a dictionary between $\mathbb{P}^1$-structures and transverse sections of flat $\mathbb{P}^1$-bundles. In this dictionary, there is a simple relation between the Lyapunov exponent χ defined in Definition-Proposition 3.6 and the foliated Lyapunov exponent defined in Section 1.6.

Lemma 3.8. *Let σ be a $\mathbb{P}^1$-structure, $(\mathsf{dev}, \mathsf{hol})$ a development-holonomy pair, and $\lambda(\sigma)$ be the Lyapunov exponent of the foliated complex surface $(S_{\mathsf{hol}}, \mathcal{F}_{\mathsf{hol}})$ computed w.r.t. the leafwise Poincaré metric. Then $\chi(\sigma) = -\frac{1}{2}\lambda(\sigma)$.*

The proof of this lemma essentially follows from the formula of the derivative of a Moebius map in the spherical metric, namely if $h(z) = \frac{az+b}{cz+d}$, then $\|Dh(z)\| = \frac{|ad-bc|}{|az+b|^2+|cz+d|^2}$. We refer to [19] for the detailed proof of Lemma 3.8.

Next, the proof of Theorem 3.7 relies on cohomological computations in $H^{1,1}(S_{\mathsf{hol}}, \mathbb{C})$. Recall that a $\mathbb{P}^1$-bundle is an algebraic surface, by the GAGA principle, and in particular is Kähler. Also recall that by the $\partial\overline{\partial}$-lemma, in a Kähler compact surface, a closed $(1,1)$-form is exact iff it is $\partial\overline{\partial}$-exact. This means that $T \cdot E = T \cdot F$ if E and F have the same Chern classes, see Section 1.4.

The cohomology of S_{hol} is easy to compute. Indeed, a $\mathbb{P}^1$-bundle over a curve is diffeomorphic to a product as soon as there exists a section of even self-intersection. In our situation, we have such a section at hand: the section s being (at the level of the universal covers) the graph of dev. We claim: $s^2 = \mathrm{Eu}(C)$. This is due to the fact that there is an isomorphism between the tangent bundle of C and the normal bundle of s, since s is both transverse to the foliation $\mathcal{F}_{\mathsf{hol}}$ and to the fibration $S_{\mathsf{hol}} \to C$. In particular, we infer $H^{1,1}(S_{\mathsf{hol}}, \mathbb{C}) = \mathbb{C}[s] \oplus \mathbb{C}[f]$, where f is any fiber of the fibration. The intersection product on $H^{1,1}(S_{\mathsf{hol}}, \mathbb{C})$ is given by $s^2 = \mathrm{Eu}(C)$, $f^2 = 0$, and $f \cdot s = 1$.

After these preliminaries, let us use the combination of Lemma 3.8 and Proposition 1.15, to get

$$\chi = \frac{1}{2}\frac{T \cdot N_{\mathcal{F}}}{T \cdot K_{\mathcal{F}}}.$$

We have $N_{\mathcal{F}} \cdot f = 2$ and $N_{\mathcal{F}} \cdot s = \mathrm{Eu}(C)$. So we infer $[N_{\mathcal{F}}] = 2[s] - \mathrm{Eu}(C)[f]$. Let T be the unique harmonic current whose product with the Poincaré volume form is equal to 1. We then have $T \cdot f = \frac{1}{\mathrm{vol}(C)}$ and $T \cdot K_{\mathcal{F}} = \frac{|\mathrm{Eu}(C)|}{\mathrm{vol}(C)}$. This gives

$$\chi = \frac{\mathrm{vol}(C)}{2\,\mathrm{Eu}(C)}(2T \cdot s + |\mathrm{Eu}(C)| T \cdot f) = 2\pi T \cdot s + \frac{1}{2}.$$

The proof is completed by showing that the cohomological intersection $T \cdot s$ coincides with the geometric intersection $\delta = T \dot{\wedge} s$. This last fact is not immediate since one cannot regularize the current of integration on s (recall $s^2 < 0$) but this is done by hand. We refer to [19] for more details. Q.E.D.

3.4. Harmonic measures of $\mathbb{P}^1$-structures

Let C be a smooth quasi-projective curve of negative Euler characteristic and σ a parabolic type projective structure on C. As before, we endow C and its universal covering with the Poincaré metric. We associate to σ a family of *harmonic measures* $\{\nu_x\}_{x\in\widetilde{C}}$ on the Riemann sphere, indexed by $\widetilde{C}$. It can be defined in several ways. The following appealing presentation was introduced by Hussenot in his PhD thesis [45]:

Definition-Proposition 3.9 (Hussenot)**.** *Let C be a Riemann surface of finite type and σ be a parabolic projective structure on C. Choose a representing pair* $(\mathsf{dev}, \mathsf{hol})$*. Then for every $x \in \widetilde{C}$, and W_x a.e. Brownian path starting at $\omega(0) = x$, there exists a point* $\mathrm{e}(\omega)$ *on $\mathbb{P}^1$ defined by the property that*

$$\frac{1}{t}\int_0^t \mathsf{dev}_*(\delta_{\omega(s)})\,ds \xrightarrow[t\to+\infty]{} \delta_{\mathrm{e}(\omega)}.$$

The distribution of the point $\mathrm{e}(\omega)$ subject to the condition that $\omega(0) = x$ is the measure ν_x. In particular, due to the conformal invariance of Brownian motion, for a covering $\mathbb{P}^1$-structure, we recognize the classical harmonic measures on the limit set of a Kleinian group.

Another definition of the harmonic measures is based on the so-called Furstenberg boundary map, which was designed in [32], based on the discretization of Brownian motion in the hyperbolic plane $\mathbb{H}$ (see also Margulis [56, Theorem 3] for a different approach). Furstenberg showed that if Γ is a cofinite Fuchsian group and $\rho\colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ is a non-elementary representation, there exists a unique measurable equivariant mapping $\theta\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1$ defined a.e. with respect to Lebesgue measure. Choose a biholomorphism $\widetilde{C} \simeq \mathbb{H}$, thereby identifying $\pi_1(C)$ with a cofinite Fuchsian group. For every $x \in \mathbb{H}$, let m_x be the classical harmonic measure (i.e. the exit distribution of Brownian paths issued from x), which is a probability measure with smooth density on $\mathbb{P}^1(\mathbb{R})$. The harmonic measure ν_x is then defined by $\nu_x = \theta_* m_x$. From this perspective it is clear that, the measures ν_x are mutually absolutely continuous and depend harmonically on x.

Theorem 3.10. *Let C be compact algebraic curve and σ be a parabolic projective structure on C. Let as above χ be its Lyapunov exponent* *and* $(\nu_x)_{x\in\widetilde{C}}$ *be the associated family of harmonic measures. Then for every* x,

$$\dim_H(\nu_x) \leq \frac{1}{2\chi} \leq 1.$$

Furthermore $\dim_H(\nu_x) = 1$ *if and only if the developing maps are injective.*

So, as in the polynomial case, formula (22) provides an alternate approach to the classical bound $\dim_H(\nu) \leq 1$ for the harmonic measure on boundary of discontinuity components of finitely generated Kleinian groups, which follows from the famous results of Makarov [54] and Jones–Wolff [47]. In addition, with this method we are also able to show that $\dim_H(\nu) < 1$ when the component is not simply connected. Indeed we have the more precise bound $\dim_H(\nu) \leq \frac{A}{2\chi}$, where $0 \leq A \leq 1$ is an invariant of the flat foliation, and $A < 1$ when hol is not injective. This A has been defined by Frankel and is called the *action*, see [29].

We also see that the value of the dimension of the harmonic measures detects exotic quasi-Fuchsian structures, that is, projective structures with quasi-Fuchsian holonomy which do not belong to the Bers slice.

Sketch of Proof. The curve C will be assumed to be compact, we refer to [19] for the punctured case. The main observation is to see the family of harmonic measures of a $\mathbb{P}^1$-structure as a foliated harmonic current. This is summarized in the following statement.

Proposition 3.11. *Let* σ *be a* $\mathbb{P}^1$*-structure on a compact* C, *and let* $(\mathsf{dev}, \mathsf{hol})$ *be a development-holonomy pair. Let* $(S_{\mathsf{hol}}, \mathcal{F}_{\mathsf{hol}})$ *be the flat* $\mathbb{P}^1$*-bundle constructed in* Example 1.1. *Let* T' *be the unique foliated harmonic current whose intersection with the fibers of* S_{hol} *is* 1. *The family of harmonic measures of* σ *is the family of disintegration of a (lift) of* T' *to* $\widetilde{C} \times \mathbb{P}^1$ *on the fibers* $x \times \mathbb{P}^1$.

Observe that the current T' in this proposition is equal to $T' = \mathrm{vol}(C)T$, where T is the current such that the foliated harmonic measure $\mu = T \wedge v_P$ has mass one. The proof of proposition relies on the fact that the map $x \mapsto \nu_x$ is harmonic, which is clear from the Furstenberg/Margulis point of view.

We now review an invariant of the harmonic current T that was introduced by Frankel, under the name of *action*. See [29]. It is defined as the non negative number

$$A = A(T) = \int_{S_{\mathsf{hol}}} \|\nabla_{\mathcal{F}} \log \varphi\|^2 \, d\mu, \tag{23}$$

where the functions φ are the densities of the disintegration of T along the leaves. The function φ are positive harmonic functions, so that the integral (23) is convergent. More precisely, by observing that the functions φ can be extended analytically on the universal cover of the leaves, and applying the Schwarz–Pick lemma, one shows that $A(T) \leq 1$. See [16] for more details.

Using the fact that φ is harmonic, one finds the formula $\|\nabla \log \varphi\|^2 = -\Delta \log \varphi$, so that

$$\int_{S_{\mathsf{hol}}} \Delta(\log \varphi)\, d\mu = -A.$$

Using exactly the same argument as in the proof of Lemma 1.16, we infer the following result:

Lemma 3.12. *For μ-a.e. $w \in S_{\mathsf{hol}}$, and W^w-a.e. leafwise Brownian path ω starting at w, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log D_T(h_{\omega|_{[0,t]}})(w) = -A,$$

where $D_T h := \frac{h^{-1}\nu_{\omega(t)}}{\nu_{\omega(0)}}$ is the Radon–Nikodym derivative with respect to the measure induced by T on $\mathbb{P}^1$-fibers, namely the family of harmonic measures.

Hence, for every $\varepsilon > 0$, the maps $h_{\omega|_{[0,t]}}$ contract conformally the spherical distances by the factor $\exp((\lambda \pm \varepsilon)t)$, whereas they contract the harmonic measures by the factor $\exp((-A(T) \pm \varepsilon)t)$. We deduce the heuristic

$$\dim(\nu_x) \leq \frac{A}{|\lambda|} = \frac{A}{2\chi} \leq \frac{1}{2\chi}.$$

Using a weak notion of dimension, the so-called *essential dimension* (denoted by $\dim_{ess}$), one can prove part of this heuristic, namely the inequality

$$\dim_{ess}(\nu_x) \leq \frac{A}{2\chi}. \tag{24}$$

This uses an argument of Ledrappier [52, Theorem 1] in the context of random product of matrices that we adapt to our setting. The proof of Theorem 3.10 then follows from (24) and the fact that the Hausdorff dimension is bounded by the essential dimension. Q.E.D.

3.5. Geometry of Bers slices

As another application of formula (22), we recover a result due to Shiga [66].

Theorem 3.13 (Shiga). *Let C be a hyperbolic Riemann surface of finite type (of genus g with n punctures). The closure of the Bers embedding $B(C)$ is a polynomially convex compact subset of the space $P(C) \simeq \mathbb{C}^{3g-3+n}$ of holomorphic projective structures on C. As a consequence, $B(C)$ is a polynomially convex (or Runge) domain.*

Recall that a compact set K in $\mathbb{C}^N$ is polynomially convex if $\widehat{K} = K$, where

$$\widehat{K} = \left\{ z \in \mathbb{C}^N,\ |P(z)| \leq \sup_K |P| \text{ for every polynomial } P \right\}.$$

An open set $U \subset \mathbb{C}^N$ is said to be polynomially convex (or Runge) if for every $K \Subset U$, $\widehat{K} \subset U$. The theorem may be reformulated by saying that $\overline{B(C)}$ is defined by countably many polynomial inequalities of the form $|P| \leq 1$. This is not an intrinsic property of Teichmüller space, but rather a property of its embedding into the space $P(C)$ of holomorphic projective structures on C (as opposed to the Bers–Ehrenpreis theorem that Teichmüller is holomorphically convex).

Shiga's proof is based on the Grunsky inequality on univalent functions. Only the polynomial convexity of $B(C)$ is asserted in [66], but the proof covers the case of $\overline{B(C)}$ as well. Our approach is based on the elementary fact that the locus of minima of a global psh function on $\mathbb{C}^N$ is polynomially convex.

Sketch of Proof of Theorem 3.13. We just prove here the polynomial convexity of the Bers slice $B(C)$. The polynomial convexity of $\overline{B(C)}$ is more involved, we refer to [19]. It was shown in [18] that $\sigma \mapsto \chi(\sigma)$ is a continuous (Hölder) plurisubharmonic (psh for short) function on $P(X)$, hence it follows from formula (22) that deg is continuous and psh, too. In addition we see that $\chi(\sigma)$ reaches its minimal value $\frac{1}{2}$ exactly when $\deg(\sigma) = 0$, see Cotollary 3.5. This already proves that the interior of $\{\delta = 0\}$, namely the set of covering $\mathbb{P}^1$-structures, is polynomially convex. But this set is exactly the Bers slice, so we are done. Q.E.D.

We finish this lecture by reviewing yet another application of formula (22) to equidistribution properties in $P(C)$. In [18] we showed that $T_{\text{bif}} := dd^c\chi$ is a *bifurcation current*, in the sense that its support is precisely the set of projective structures whose holonomy representation is not locally structurally stable in $P(X)$. The support of this current is the exterior of the Bers slice $B(C)$.

Analogous bifurcation currents have been defined for families of rational mappings on $\mathbb{P}^1$. It turns out that the exterior powers T_{bif}^k are interesting and rather well understood objects in that context (see [26] for an account). In particular, in the space of polynomials of degree d, the maximal exterior power T_{bif}^{d-1} is a positive measure supported on the boundary of the connectedness locus, which is the right analogue in higher degree of the harmonic measure of the Mandelbrot set [25].

For bifurcation currents associated to spaces of representations, nothing is known in general about the exterior powers T_{bif}^{k}. In our situation, we are able to obtain some information.

Theorem 3.14. *Let C be a compact Riemann surface of genus $g \geq 2$. Let $T_{\mathrm{bif}} = dd^c\chi$ be the natural bifurcation current on $P(C)$. Then $\partial B(C)$ is contained in* $\mathrm{Supp}(T_{\mathrm{bif}}^{3g-3})$.

Notice that $3g-3$ is the maximum possible exponent. It is likely that the support of T_{bif}^{3g-3} is much larger than $\partial B(C)$. The reason for the compactness assumption here is that the proof requires some results of Otal [61] and Hejhal [43] that are known to hold only when X is compact.

If γ is a geodesic on C w.r.t. to the Poincaré metric, we let $Z(\gamma)$ be the subvariety of $P(C)$ defined by the property that $tr^2(\mathsf{hol}(\gamma)) = 4$ (i.e. $\mathsf{hol}(\gamma)$ is parabolic or the identity). As a consequence of Theorem 3.14 and of the equidistribution theorems of [18] we obtain the following result, which contrasts with the description of $\partial B(C)$ "from the inside" in terms of maximal cusps and ending laminations ([60, 6], see [51] for a nice account).

Corollary 3.15. *For every $\varepsilon > 0$ there exist $3g-3$ closed geodesics $\gamma_1, \ldots, \gamma_{3g-3}$ on C such that $\partial B(C)$ is contained in the ε-neighborhood of $Z(\gamma_1) \cap \cdots \cap Z(\gamma_{3g-3})$.*

References

[1] L. Ahlfors, Zur Theorie der Überlagerungsflächen, *Acta Math.* **65** (1935), 157–194.

[2] A. Ancona, Théorie du potentiel sur les graphes et les variétés, in *École d'été de Probabilités de Saint-Flour XVIII—1988*, Lecture Notes in Math. **1427**, Springer, Berlin, 1990, 1–112.

[3] W. Barth, C. Peters and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **4**, Springer, Berlin, 1984.

[4] B. Berndtsson and N. Sibony, The $\overline{\partial}$-equation on a positive current, *Invent. Math.* **147** (2002), 371–428.

[5] C. Bonatti and X. Gómez-Mont, Sur le comportement statistique des feuilles de certains feuilletages holomorphes, in *Essays on geometry and related topics, Vol. 1, 2*, Monogr. Enseign. Math. **38**, Enseignement Math., Geneva, 2001, 15–41.

[6] J. F. Brock, R. D. Canary and Y. N. Minsky, The classification of Kleinian surface groups, II: The ending lamination conjecture, *Ann. of Math.* (2) **176** (2012), 1–149.

[7] M. Brunella, Courbes entières et feuilletages holomorphes, *Enseign. Math.* (2) **45** (1999), 195–216.

[8] M. Brunella, *Birational geometry of foliations*, Monografías de Matemática., Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2000.

[9] G. Calsamiglia, B. Deroin, S. Frankel and A. Guillot, Singular sets of holonomy maps for algebraic foliations, *J. Eur. Math. Soc.* (JEMS) **15** (2013), 1067–1099.

[10] C. Camacho and A. Lins Neto, *Geometric theory of foliations*, Birkhäuser Boston, Boston, MA, 1984.

[11] A. Candel, Uniformization of surface laminations, *Ann. Sci. École Norm. Sup.* (4) **26** (1993), 489–516.

[12] A. Candel, The harmonic measures of Lucy Garnett, *Adv. Math.* **176** (2003), 187–247.

[13] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics **115**, Academic Press, Orlando, FL, 1984.

[14] S. Y. Cheng, P. Li and S. T. Yau, On the upper estimate of the heat kernel of a complete Riemannian manifold, *Amer. J. Math.* **103** (1981), 1021–1063.

[15] H. P. de Saint-Gervais, *Uniformisation des surfaces de Riemann*, ENS Éditions, Lyon, 2010.

[16] B. Deroin, Hypersurfaces Levi-plates immergées dans les surfaces complexes de courbure positive, *Ann. Sci. École Norm. Sup.* (4) **38** (2005), 57–75.

[17] B. Deroin and R. Dujardin, Random walks, Kleinian groups, and bifurcation currents, *Invent. Math.* **190** (2012), 57–118.

[18] B. Deroin and R. Dujardin, *Lyapunov exponents for surface group representations,* I*: bifurcation currents*, preprint (2013), math.GT:1305.0049.

[19] B. Deroin and R. Dujardin, *Complex projective structures: Lyapunov exponent, degree and harmonic measure*, preprint.

[20] B. Deroin, Examples of almost-holomorphic and totally real laminations in complex surfaces, *Topology* **45** (2006), 495–512.

[21] B. Deroin and V. Kleptsyn, Random conformal dynamical systems, *Geom. Funct. Anal.* **17** (2007), 1043–1105.

[22] T.-C. Dinh, V.-A. Nguyên and N. Sibony, Heat equation and ergodic theorems for Riemann surface laminations, *Math. Ann.* **354** (2012), 331–376.

[23] S. Diverio and E. Rousseau, *A survey on hyperbolicity of projective hypersurfaces*, Publicações Matemáticas do IMPA., Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2011.

[24] D. Dumas, Complex projective structures, in *Handbook of Teichmüller theory*, II, IRMA Lect. Math. Theor. Phys. **13**, Eur. Math. Soc., Zürich, 2009, 455–508.

[25] R. Dujardin and C. Favre, Distribution of rational maps with a preperiodic critical point, *Amer. J. Math.* **130** (2008), 979–1032.

[26] R. Dujardin, Bifurcation currents and equidistribution in parameter space, in *Frontiers in complex dynamics*, Princeton Math. Ser. **51**, Princeton Univ. Press, Princeton, NJ, 2014, 515–566.

[27] J. E. Fornæss and N. Sibony, Harmonic currents of finite energy and laminations, *Geom. Funct. Anal.* **15** (2005), 962–1003.

[28] J. E. Fornæss and N. Sibony, Unique ergodicity of harmonic currents on singular foliations of $\mathbb{P}^2$, *Geom. Funct. Anal.* **19** (2010), 1334–1377.

[29] S. Frankel, *Harmonic analysis of surface group representations to* $\mathrm{Diff}(S^1)$ *and Milnor type inequalities*, Prépublication de l'École Polytechnique **1125** (1996).

[30] J. Franks and B. Williams, Anomalous Anosov flows, in *Global theory of dynamical systems* (*Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979*), Lecture Notes in Math. **819**, Springer, Berlin, 1980, 158–174.

[31] R. Friedman and J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **27**, Springer, Berlin, 1994.

[32] H. Furstenberg, Boundary theory and stochastic processes on homogeneous spaces, in *Harmonic analysis on homogeneous spaces* (*Proc. Sympos. Pure Math., XXVI, Williams Coll., Williamstown, Mass., 1972*), Amer. Math. Soc., Providence, RI, 1973, 193–229.

[33] D. Gallo, M. Kapovich and A. Marden, The monodromy groups of Schwarzian equations on closed Riemann surfaces, *Ann. of Math.* (2) **151** (2000), 625–704.

[34] L. Garnett, Foliations, the ergodic theorem and Brownian motion, *J. Funct. Anal.* **51** (1983), 285–311.

[35] É. Ghys, Flots transversalement affines et tissus feuilletés, *Mém. Soc. Math. France* (*N.S.*) **46** (1991), 123–150.

[36] É. Ghys, Laminations par surfaces de Riemann, in *Dynamique et géométrie complexes* (*Lyon, 1997*), Panor. Synthèses **8**, Soc. Math. France, Paris, 1999, ix, xi, 49–95.

[37] É. Ghys and V. Sergiescu, Stabilité et conjugaison différentiable pour certains feuilletages, *Topology* **19** (1980), 179–197.

[38] W. M. Goldman, Topological components of spaces of representations, *Invent. Math.* **93** (1988), 557–607.

[39] S. Goodman, Dehn surgery on Anosov flows, in *Geometric dynamics* (*Rio de Janeiro, 1981*), Lecture Notes in Math. **1007**, Springer, Berlin, 1983, 300–307.

[40] A. Haefliger, Variétés feuilletées, *Ann. Scuola Norm. Sup. Pisa* (3) **16** (1962), 367–397.

[41] M. Handel and W. P. Thurston, Anosov flows on new three manifolds, *Invent. Math.* **59** (1980), 95–103.
[42] D. A. Hejhal, Monodromy groups and linearly polymorphic functions, *Acta Math.* **135** (1975), 1–55.
[43] D. A. Hejhal, On Schottky and Koebe-like uniformizations, *Duke Math. J.* **55** (1987), 267–286.
[44] J. H. Hubbard and R. W. Oberste-Vorth, Hénon mappings in the complex domain. I. The global topology of dynamical space, *Inst. Hautes Études Sci. Publ. Math.* **79** (1994), 5–46.
[45] N. Hussenot, *Analytic continuation of holonomy germs of Riccati foliations along brownian paths*, preprint (2013).
[46] T. Inaba and M. A. Mishchenko, On real submanifolds of Kähler manifolds foliated by complex submanifolds, *Proc. Japan Acad. Ser. A Math. Sci.* **70** (1994), 1–2.
[47] P. W. Jones and T. H. Wolff, Hausdorff dimension of harmonic measures in the plane, *Acta Math.* **161** (1988), 131–144.
[48] Y. Komori, T. Sugawa, M. Wada and Y. Yamashita, Drawing Bers embeddings of the Teichmüller space of once-punctured tori, *Experiment. Math.* **15** (2006), 51–60.
[49] I. Kra, Deformations of Fuchsian groups, *Duke Math. J.* **36** (1969), 537–546.
[50] I. Kra, Deformations of Fuchsian groups. II, *Duke Math. J.* **38** (1971), 499–508.
[51] C. Lecuire, *Modèles et laminations terminales* (*d'après Minsky et Brock-Canary-Minsky*), Séminaire N. Bourbaki **1068** (2013).
[52] F. Ledrappier, Quelques propriétés des exposants caractéristiques, in *École d'été de probabilités de Saint-Flour, XII—1982*, Lecture Notes in Math. **1097**, Springer, Berlin, 1984, 305–396.
[53] F. Loray and D. Marín Pérez, Projective structures and projective bundles over compact Riemann surfaces, Astérisque **323** (2009), 223–252.
[54] N. G. Makarov, On the distortion of boundary sets under conformal mappings, *Proc. London Math. Soc.* (3) **51** (1985), 369–384.
[55] A. Manning, The dimension of the maximal measure for a polynomial map, *Ann. of Math.* (2) **119** (1984), 425–430.
[56] G. A. Margulis, Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than 1, *Invent. Math.* **76** (1984), 93–120.
[57] S. Matsumoto, Some remarks on foliated S^1 bundles, *Invent. Math.* **90** (1987), 343–358.
[58] C. McMullen, Rational maps and Kleinian groups, in *Proceedings of the International Congress of Mathematicians, I, II* (*Kyoto, 1990*), Math. Soc. Japan, Tokyo, 1991, 889–899.
[59] M. McQuillan, Diophantine approximations and foliations, *Inst. Hautes Études Sci. Publ. Math.* **87** (1998), 121–174.
[60] Y. N. Minsky, The classification of punctured-torus groups, *Ann. of Math.* (2) **149** (1999), 559–626.

[61] J.-P. Otal, Sur le bord du prolongement de Bers de l'espace de Teichmüller, *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), 157–160.

[62] J. F. Plante, Foliations with measure preserving holonomy, *Ann. of Math.* (2) **102** (1975), 327–361.

[63] H. Poincaré, Sur les cycles des surfaces algébriques, *J. Math. Pures et Appl.* **8** (1902), 169–214.

[64] F. Przytycki, Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map, *Invent. Math.* **80** (1985), 161–179.

[65] P. Scott, The geometries of 3-manifolds, *Bull. London Math. Soc.* **15** (1983), 401–487.

[66] H. Shiga, On analytic and geometric properties of Teichmüller spaces, *J. Math. Kyoto Univ.* **24** (1984), 441–452.

[67] H. Shiga, On monodromies of holomorphic families of Riemann surfaces and modular transformations, *Math. Proc. Cambridge Philos. Soc.* **122** (1997), 541–549.

[68] D. Sullivan, Cycles for the dynamical study of foliated manifolds and complex manifolds, *Invent. Math.* **36** (1976), 225–255.

[69] W. P. Thurston, *Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle*, (1998), arXiv:math/9801045.

[70] W. P. Thurston, *Three-manifolds, foliations, and circles II*, unfinished manuscript (1998).

[71] A. Verjovsky, A uniformization theorem for holomorphic foliations, in *The Lefschetz centennial conference, Part III* (*Mexico City, 1984*), Contemp. Math. **58**, III, Amer. Math. Soc., Providence, RI, 1987, 233–253.

[72] J. W. Wood, Bundles with totally disconnected structure group, *Comment. Math. Helv.* **46** (1971), 257–273.

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