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Quantum Lakshmibai-Seshadri paths and root operators

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Abstract.

We give an explicit description of the image of a quantum LS path, regarded as a rational path, under the action of root operators, and show that the set of quantum LS paths is stable under the action of the root operators. As a by-product, we obtain a new proof of the fact that a projected level-zero LS path is just a quantum LS path.

§1. Introduction.

In our previous papers [NS1], [NS3], [NS2], we gave a combinatorial realization of the crystal bases of level-zero fundamental representations $W(\varpi_i)$, $i \in I_0$, and their tensor products $\bigotimes_{i \in I_0} W(\varpi_i)^{\otimes m_i}$, $m_i \in \mathbb{Z}_{\geq 0}$, over a quantum affine algebra $U_q'(\mathfrak{g})$, by using projected level-zero Lakshmibai-Seshadri (LS for short) paths. Here, for a level-zero dominant integral weight $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with ϖ_i the i-th level-zero fundamental weight, the set of projected level-zero LS paths of shape λ, which is a "simple" crystal denoted by $\mathbb{B}(\lambda)_{\mathrm{cl}}$, is obtained from the set $\mathbb{B}(\lambda)$ of LS paths of shape λ (in the sense of [L2]) by factoring out the null root δ of an affine Lie algebra $\mathfrak{g}$. However, from the nature of the above definition of projected level-zero LS paths, our description of these objects in [NS1], [NS3], [NS2] was not as explicit as the one (given in [L1]) of usual LS paths, the shape of which is a dominant integral weight.

Recently, in [LNSSS1], [LNSSS2], we proved that a projected level-zero LS path is identical to a certain "rational path", which we call a quantum LS path. A quantum LS path is described in terms of the (parabolic) quantum Bruhat graph (QBG for short), which was introduced by [BFP] (and by [LS] in the parabolic case) in the study of the quantum

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cohomology ring of the (partial) flag variety; see §3.1 for the definition of the (parabolic) QBG. It is noteworthy that the description of a quantum LS path as a rational path is very similar to the one of a usual LS path given in [L1], in which we replace the Hasse diagram of the (parabolic) Bruhat graph by the (parabolic) QBG. Also, remark that the vertices of the (parabolic) QBG are the minimal-length representatives for the cosets of a parabolic subgroup W_J of the finite Weyl group W_0, though we consider finite-dimensional representations $W(\varpi_i)$, $i \in I_0$, of the quantum affine algebra $U_q'(\mathfrak{g})$.

The purpose of this paper is to give an explicit description, in terms of rational paths, of the image of a quantum LS path (= projected level-zero LS path) under root operators in a way similar to the one given in [L1]; see Theorem 4.1.1 for details. This explicit description, together with the Diamond Lemmas [LNSSS1, Lemma 5.14], for the parabolic QBG, provides us with a proof of the fact that the set of quantum LS paths (the shape of which is a level-zero dominant integral weight λ) is stable under the action of the root operators.

As a by-product of the stability property above, we obtain another (but somewhat roundabout) proof of the fact that a projected level-zero LS path is just a quantum LS path; see [LNSSS1], [LNSSS2] for a more direct proof. This new proof is accomplished by making use of a characterization (Theorem 2.4.1) of the set $\mathbb{B}(\lambda)_{\mathrm{cl}}$ of projected level-zero LS paths of shape λ in terms of root operators, which is based upon the connectedness of the (crystal graph for the) tensor product crystal $\bigotimes_{i\in I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m_i} \simeq \mathbb{B}(\lambda)_{\mathrm{cl}}$; recall from [NS1], [NS3], [NS2] that for a level-zero dominant integral weight $\lambda = \sum_{i\in I_0} m_i\varpi_i$, the crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ decomposes into the tensor product $\bigotimes_{i\in I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m_i}$ of crystals, and that $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ for each $i \in I_0$ is isomorphic to the crystal basis of the level-zero fundamental representation $W(\varpi_i)$.

This paper is organized as follows. In §2, we fix our basic notation, and recall some fundamental facts about (level-zero) LS path crystals. Also, we give a characterization (Theorem 2.4.1) of projected level-zero LS paths, which is needed to obtain our main result (Theorem 4.1.1). In §3, we recall the notion of the (parabolic) quantum Bruhat graph, and then give the definition of quantum LS paths. In §4, we first state our main result. Then, after preparing several technical lemmas, we finally obtain an explicit description (Proposition 4.2.1) of the image of a quantum LS path as a rational path under the action of root operators. Our main result follows immediately from this description, together with the characterization above of projected level-zero LS paths.

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§2. Lakshmibai-Seshadri paths.

2.1. Basic notation.

Let $\mathfrak{g}$ be an untwisted affine Lie algebra over $\mathbb{C}$ with Cartan matrix $A = (a_{ij})_{i,\,j\in I}$; throughout this paper, the elements of the index set I are numbered as in [Kac, §4.8, Table Aff 1]. Take a distinguished vertex $0 \in I$ as in [Kac], and set $I_0 := I\setminus\{0\}$. Let $\mathfrak{h} = \left(\bigoplus_{j\in I}\mathbb{C}\alpha_j^\vee\right)\oplus\mathbb{C}d$ denote the Cartan subalgebra of $\mathfrak{g}$, where $\Pi^\vee := \left\{\alpha_j^\vee\right\}_{j\in I} \subset \mathfrak{h}$ is the set of simple coroots, and $d \in \mathfrak{h}$ is the scaling element (or degree operator). Also, we denote by $\Pi := \left\{\alpha_j\right\}_{j\in I} \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h},\mathbb{C})$ the set of simple roots, and by $\Lambda_j \in \mathfrak{h}^*$, $j \in I$, the fundamental weights; note that $\alpha_j(d) = \delta_{j,0}$ and $\Lambda_j(d) = 0$ for $j \in I$. Let $\delta = \sum_{j\in I} a_j\alpha_j \in \mathfrak{h}^*$ and $c = \sum_{j\in I} a_j^\vee\alpha_j^\vee \in \mathfrak{h}$ denote the null root and the canonical central element of $\mathfrak{g}$, respectively. The Weyl group W of $\mathfrak{g}$ is defined by $W := \langle r_j \mid j \in I\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$, where $r_j \in \mathrm{GL}(\mathfrak{h}^*)$ denotes the simple reflection associated to α_j for $j \in I$, with $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function on W. Denote by Δ_{re} the set of real roots, i.e., $\Delta_{\mathrm{re}} := W\Pi$, and by $\Delta_{\mathrm{re}}^+ \subset \Delta_{\mathrm{re}}$ the set of positive real roots; for $\beta \in \Delta_{\mathrm{re}}$, we denote by $\beta^\vee$ the dual root of β, and by $r_\beta \in W$ the reflection with respect to β. We take a dual weight lattice $P^\vee$ and a weight lattice P as follows:

(2.1.1)

$$P^\vee = \left(\bigoplus_{j\in I}\mathbb{Z}\alpha_j^\vee\right)\oplus\mathbb{Z}d \subset \mathfrak{h} \quad \text{and} \quad P = \left(\bigoplus_{j\in I}\mathbb{Z}\Lambda_j\right)\oplus\mathbb{Z}\delta \subset \mathfrak{h}^*.$$

It is clear that P contains $Q := \bigoplus_{j\in I}\mathbb{Z}\alpha_j$, and that $P \cong \mathrm{Hom}_{\mathbb{Z}}(P^\vee,\mathbb{Z})$.

Let W_0 be the subgroup of W generated by r_j, $j \in I_0$, and set $\Delta_0 := \Delta_{\mathrm{re}} \cap \bigoplus_{j\in I_0}\mathbb{Z}\alpha_j$, $\Delta_0^+ := \Delta_{\mathrm{re}} \cap \bigoplus_{j\in I_0}\mathbb{Z}_{\geq 0}\alpha_j$, and $\Delta_0^- := -\Delta_0^+$. Note that W_0 (resp., Δ_0, Δ_0^+, Δ_0^-) can be thought of as the (finite) Weyl group (resp., the set of roots, the set of positive roots, the set of negative roots) of the finite-dimensional simple Lie subalgebra corresponding to I_0. Denote by $\theta \in \Delta_0^+$ the highest root for the (finite) root system Δ_0; note that $\alpha_0 = -\theta + \delta$ and $\alpha_0^\vee = -\theta^\vee + c$.

Definition 2.1.1.

(1) An integral weight $\lambda \in P$ is said to be of level zero if $\langle \lambda, c \rangle = 0$.

(2) An integral weight $\lambda \in P$ is said to be level-zero dominant if $\langle \lambda, c \rangle = 0$, and $\langle \lambda, \alpha_j^\vee \rangle \geq 0$ for all $j \in I_0 = I \setminus \{0\}$.

Remark 2.1.2. If $\lambda \in P$ is of level zero, then $\langle \lambda, \alpha_0^\vee \rangle = -\langle \lambda, \theta^\vee \rangle$.

For each $i \in I_0$, we define a level-zero fundamental weight $\varpi_i \in P$ by

$$\varpi_i := \Lambda_i - a_i^\vee \Lambda_0. \tag{2.1.2}$$

The ϖ_i for $i \in I_0$ is actually a level-zero dominant integral weight; indeed, $\langle \varpi_i, c \rangle = 0$ and $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for $j \in I_0$.

Let $\mathrm{cl} : \mathfrak{h}^* \twoheadrightarrow \mathfrak{h}^*/\mathbb{C}\delta$ be the canonical projection from $\mathfrak{h}^*$ onto $\mathfrak{h}^*/\mathbb{C}\delta$, and define P_{cl} and $P_{\mathrm{cl}}^\vee$ by

$$P_{\mathrm{cl}} := \mathrm{cl}(P) = \bigoplus_{j \in I} \mathbb{Z}\,\mathrm{cl}(\Lambda_j) \quad \text{and} \quad P_{\mathrm{cl}}^\vee := \bigoplus_{j \in I} \mathbb{Z}\alpha_j^\vee \subset P^\vee. \tag{2.1.3}$$

We see that $P_{\mathrm{cl}} \cong P/\mathbb{Z}\delta$, and that P_{cl} can be identified with $\mathrm{Hom}_{\mathbb{Z}}(P_{\mathrm{cl}}^\vee, \mathbb{Z})$ as a $\mathbb{Z}$-module by

$$\langle \mathrm{cl}(\lambda), h \rangle = \langle \lambda, h \rangle \quad \text{for } \lambda \in P \text{ and } h \in P_{\mathrm{cl}}^\vee. \tag{2.1.4}$$

Also, there exists a natural action of the Weyl group W on $\mathfrak{h}^*/\mathbb{C}\delta$ induced by the one on $\mathfrak{h}^*$, since $W\delta = \delta$; it is obvious that $w \circ \mathrm{cl} = \mathrm{cl} \circ w$ for all $w \in W$.

Remark 2.1.3. Let $\lambda \in P$ be a level-zero integral weight. It is easy to check that $\mathrm{cl}(W\lambda) = W_0\,\mathrm{cl}(\lambda)$ (see the proof of [NS4, Lemma 2.3.3]). In particular, we have $\mathrm{cl}(r_0\lambda) = r_\theta\,\mathrm{cl}(\lambda)$ since $\alpha_0 = -\theta + \delta$ and $\alpha_0^\vee = -\theta^\vee + c$.

For simplicity of notation, we often write β instead of $\mathrm{cl}(\beta) \in P_{\mathrm{cl}}$ for $\beta \in Q = \bigoplus_{j \in I} \mathbb{Z}\alpha_j$; note that $\alpha_0 = -\theta$ in P_{cl} since $\alpha_0 = -\theta + \delta$ in P.

2.2. Paths and root operators.

A path with weight in $P_{\mathrm{cl}} = \mathrm{cl}(P)$ is, by definition, a piecewise-linear, continuous map $\pi : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{cl}}$ such that $\pi(0) = 0$ and $\pi(1) \in P_{\mathrm{cl}}$. We denote by $\mathbb{P}_{\mathrm{cl}}$ the set of all paths with weight in P_{cl}, and define $\mathrm{wt} : \mathbb{P}_{\mathrm{cl}} \to P_{\mathrm{cl}}$ by

$$\mathrm{wt}(\eta) := \eta(1) \quad \text{for } \eta \in \mathbb{P}_{\mathrm{cl}}. \tag{2.2.1}$$

For $\eta \in \mathbb{P}_{\mathrm{cl}}$ and $j \in I$, we set

$$\begin{aligned} H_j^\eta(t) &:= \langle \eta(t), \alpha_j^\vee \rangle \quad \text{for } t \in [0,1], \\ m_j^\eta &:= \min\bigl\{ H_j^\eta(t) \mid t \in [0,1] \bigr\}. \end{aligned} \tag{2.2.2}$$

For each $j \in I$, let $\mathbb{P}_{\mathrm{cl,\,int}}^{(j)}$ denote the subset of $\mathbb{P}_{\mathrm{cl}}$ consisting of all paths η for which all local minima of the function $H_j^\eta(t)$ are integers; note that if $\eta \in \mathbb{P}_{\mathrm{cl,\,int}}^{(j)}$, then $m_j^\eta \in \mathbb{Z}_{\le 0}$ and $H_j^\eta(1) - m_j^\eta \in \mathbb{Z}_{\ge 0}$. We set

$$\mathbb{P}_{\mathrm{cl,\,int}} := \bigcap_{j \in I} \mathbb{P}_{\mathrm{cl,\,int}}^{(j)};$$

see also [NS2, §2.3]. Here we should warn the reader that the set $\mathbb{P}_{\mathrm{cl,\,int}}$ itself is not necessarily stable under the action of the root operators e_j and f_j for $j \in I$, defined below.

Now, for $j \in I$ and $\eta \in \mathbb{P}_{\mathrm{cl,\,int}}^{(j)}$, we define $e_j \eta$ as follows. If $m_j^\eta = 0$, then $e_j \eta := \mathbf{0}$, where $\mathbf{0}$ is an additional element not contained in $\mathbb{P}_{\mathrm{cl}}$. If $m_j^\eta \le -1$, then we define $e_j \eta \in \mathbb{P}_{\mathrm{cl}}$ by

$$(e_j \eta)(t) := \begin{cases} \eta(t) & \text{if } 0 \le t \le t_0, \\ \eta(t_0) + r_j(\eta(t) - \eta(t_0)) & \text{if } t_0 \le t \le t_1, \\ \eta(t) + \alpha_j & \text{if } t_1 \le t \le 1, \end{cases} \tag{2.2.3}$$

where we set

$$\begin{aligned} t_1 &:= \min\bigl\{ t \in [0,1] \mid H_j^\eta(t) = m_j^\eta \bigr\}, \\ t_0 &:= \max\bigl\{ t \in [0,t_1] \mid H_j^\eta(t) = m_j^\eta + 1 \bigr\}; \end{aligned} \tag{2.2.4}$$

note that the function $H_j^\eta(t)$ is strictly decreasing on $[t_0, t_1]$ since $\eta \in \mathbb{P}_{\mathrm{cl,\,int}}^{(j)}$. Because

$$H_j^{e_j \eta}(t) = \begin{cases} H_j^\eta(t) & \text{if } 0 \le t \le t_0, \\ 2(m_j^\eta + 1) - H_j^\eta(t) & \text{if } t_0 \le t \le t_1, \\ H_j^\eta(t) + 2 & \text{if } t_1 \le t \le 1, \end{cases}$$

it is easily seen that $e_j \eta \in \mathbb{P}_{\mathrm{cl,\,int}}^{(j)}$, and $m_j^{e_j \eta} = m_j^\eta + 1$. Therefore, if we set

$$\varepsilon_j(\eta) := \max\bigl\{ n \ge 0 \mid e_j^n \eta \ne \mathbf{0} \bigr\} \tag{2.2.5}$$

for $j \in I$ and $\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$, then $\varepsilon_j(\eta) = -m^\eta_j$ (see also [L2, Lemma 2.1 c)]). By convention, we set $e_j\mathbf{0} := \mathbf{0}$ for all $j \in I$.

Remark 2.2.1. Assume that $\eta \in \mathbb{P}^{(0)}_{\mathrm{cl,\,int}}$ satisfies the condition that $m^\eta_0 \le -1$ and $\langle \eta(t),\, c\rangle = 0$ for all $t \in [0,1]$. Then we have

$$(e_0\eta)(t) = \begin{cases} \eta(t) & \text{if } 0 \le t \le t_0, \\ \eta(t_0) + r_\theta(\eta(t) - \eta(t_0)) & \text{if } t_0 \le t \le t_1, \\ \eta(t) - \theta & \text{if } t_1 \le t \le 1, \end{cases} \tag{2.2.6}$$

where t_0 and t_1 are defined by (2.2.4) for $j = 0$.

Similarly, for $j \in I$ and $\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$, we define $f_j\eta$ as follows. If $H^\eta_j(1) - m^\eta_j = 0$, then $f_j\eta := \mathbf{0}$. If $H^\eta_j(1) - m^\eta_j \ge 1$, then we define $f_j\eta \in \mathbb{P}_{\mathrm{cl}}$ by

$$(f_j\eta)(t) := \begin{cases} \eta(t) & \text{if } 0 \le t \le t_0, \\ \eta(t_0) + r_j(\eta(t) - \eta(t_0)) & \text{if } t_0 \le t \le t_1, \\ \eta(t) - \alpha_j & \text{if } t_1 \le t \le 1, \end{cases} \tag{2.2.7}$$

where we set

$$\begin{aligned} t_0 &:= \max\bigl\{t \in [0,1] \mid H^\eta_j(t) = m^\eta_j\bigr\}, \\ t_1 &:= \min\bigl\{t \in [t_0,1] \mid H^\eta_j(t) = m^\eta_j + 1\bigr\}; \end{aligned} \tag{2.2.8}$$

note that the function $H^\eta_j(t)$ is strictly increasing on $[t_0, t_1]$ since $\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$. Because

$$H^{f_j\eta}_j(t) = \begin{cases} H^\eta_j(t) & \text{if } 0 \le t \le t_0, \\ 2m^\eta_j - H^\eta_j(t) & \text{if } t_0 \le t \le t_1, \\ H^\eta_j(t) - 2 & \text{if } t_1 \le t \le 1, \end{cases}$$

it is easily seen that $f_j\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$, and $m^{f_j\eta}_j = m^\eta_j - 1$. Therefore, if we set

$$\varphi_j(\eta) := \max\bigl\{n \ge 0 \mid f^n_j\eta \ne \mathbf{0}\bigr\} \tag{2.2.9}$$

for $j \in I$ and $\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$, then $\varphi_j(\eta) = H^\eta_j(1) - m^\eta_j$ (see also [L2, Lemma 2.1 c)]). By convention, we set $f_j\mathbf{0} := \mathbf{0}$ for all $j \in I$.

Remark 2.2.2. Assume that $\eta \in \mathbb{P}^{(0)}_{\mathrm{cl,\,int}}$ satisfies the condition that $H^{\eta}_0(1) - m^{\eta}_0 \geq 1$ and $\langle \eta(t), c \rangle = 0$ for all $t \in [0,1]$. Then we have

$$(f_0\eta)(t) = \begin{cases} \eta(t) & \text{if } 0 \leq t \leq t_0, \\ \eta(t_0) + r_\theta(\eta(t) - \eta(t_0)) & \text{if } t_0 \leq t \leq t_1, \\ \eta(t) + \theta & \text{if } t_1 \leq t \leq 1, \end{cases} \tag{2.2.10}$$

where t_0 and t_1 are defined by (2.2.8) for $j = 0$.

We know the following theorem from [L2, §2] (see also [NS2, Theorem 2.4]); for the definition of crystals, see [Kas1, §7.2] or [HK, §4.5] for example.

Theorem 2.2.3.

(1) *Let $j \in I$, and $\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$. If $e_j\eta \neq \mathbf{0}$, then $f_je_j\eta = \eta$. Also, if $f_j\eta \neq \mathbf{0}$, then $e_jf_j\eta = \eta$.*

(2) *Let $\mathbb{B}$ be a subset of $\mathbb{P}_{\mathrm{cl,\,int}}$ such that the set $\mathbb{B} \cup \{\mathbf{0}\}$ is stable under the action of the root operators e_j and f_j for all $j \in I$. The set $\mathbb{B}$, equipped with the root operators e_j, f_j for $j \in I$ and the maps* (2.2.1), (2.2.5), (2.2.9), *is a crystal with weights in P_{cl}.*

Remark 2.2.4. In §2.3, we will give a typical example of a subset $\mathbb{B}$ of $\mathbb{P}_{\mathrm{cl,\,int}}$ such that $\mathbb{B} \cup \{\mathbf{0}\}$ is stable under the action of the root operators.

For each path $\eta \in \mathbb{P}_{\mathrm{cl}}$ and $N \in \mathbb{Z}_{\geq 1}$, we define a path $N\eta \in \mathbb{P}_{\mathrm{cl}}$ by: $(N\eta)(t) = N\eta(t)$ for $t \in [0,1]$; by convention, we set $N\mathbf{0} := \mathbf{0}$ for all $N \in \mathbb{Z}_{\geq 1}$. It is easily verified that if $\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$ for some $j \in I$, then $N\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$ for all $N \in \mathbb{Z}_{\geq 1}$.

Lemma 2.2.5 (see [L2, Lemma 2.4] and also [NS2, Lemma 2.5])**.** *Let $j \in I$. For every $\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$ and $N \in \mathbb{Z}_{\geq 1}$, we have*

$$\varepsilon_j(N\eta) = N\varepsilon_j(\eta) \quad \text{and} \quad \varphi_j(N\eta) = N\varphi_j(\eta),$$
$$N(e_j\eta) = e_j^N(N\eta) \quad \text{and} \quad N(f_j\eta) = f_j^N(N\eta).$$

For $j \in I$ and $\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$, we define $e_j^{\max}\eta := e_j^{\varepsilon_j(\eta)}\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$ and $f_j^{\max}\eta := f_j^{\varphi_j(\eta)}\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$. The next lemma follows immediately from Lemma 2.2.5.

Lemma 2.2.6. *Let $j \in I$. For every $\eta \in \mathbb{P}^{(j)}_{\mathrm{cl,\,int}}$ and $N \in \mathbb{Z}_{\geq 1}$, we have $e_j^{\max}(N\eta) = N(e_j^{\max}\eta)$ and $f_j^{\max}(N\eta) = N(f_j^{\max}\eta)$.*

Now, for $\eta_1, \eta_2, \ldots, \eta_n \in \mathbb{P}_{\mathrm{cl}}$, define the concatenation $\eta_1 * \eta_2 * \cdots * \eta_n \in \mathbb{P}_{\mathrm{cl}}$ by

$$(\eta_1 * \eta_2 * \cdots * \eta_n)(t) := \sum_{l=1}^{k-1} \eta_l(1) + \eta_k(nt - k + 1) \quad \text{for } \frac{k-1}{n} \le t \le \frac{k}{n} \text{ and } 1 \le k \le n. \tag{2.2.11}$$

For a subset $\mathbb{B}$ of $\mathbb{P}_{\mathrm{cl}}$ and $n \in \mathbb{Z}_{\ge 1}$, we set $\mathbb{B}^{*n} := \bigl\{\eta_1 * \eta_2 * \cdots * \eta_n \mid \eta_k \in \mathbb{B} \text{ for } 1 \le k \le n\bigr\}$.

Proposition 2.2.7 (see [L2, Lemma 2.7], [NS2, Proposition 1.3.3])**.** *Let $\mathbb{B}$ be a subset of $\mathbb{P}_{\mathrm{cl,\,int}}$ such that the set $\mathbb{B} \cup \{\mathbf{0}\}$ is stable under the action of the root operators e_j and f_j for all $j \in I$; note that $\mathbb{B}$ is a crystal with weights in P_{cl} by Theorem 2.2.3.*

(1) *For every $n \in \mathbb{Z}_{\ge 1}$, the set $\mathbb{B}^{*n} \cup \{\mathbf{0}\}$ is stable under the root operators e_j and f_j for all $j \in I$. Therefore, $\mathbb{B}^{*n}$ is a crystal with weights in P_{cl} by Theorem 2.2.3.*

(2) *For every $n \in \mathbb{Z}_{\ge 1}$, the crystal $\mathbb{B}^{*n}$ is isomorphic as a crystal to the tensor product $\mathbb{B}^{\otimes n} := \mathbb{B} \otimes \cdots \otimes \mathbb{B}$ (n times), where the isomorphism is given by: $\eta_1 * \eta_2 * \cdots * \eta_n \mapsto \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n$ for $\eta_1 * \eta_2 * \cdots * \eta_n \in \mathbb{B}^{*n}$.*

2.3. Lakshmibai-Seshadri paths.

Let us recall the definition of Lakshmibai-Seshadri (LS for short) paths from [L2, §4]. In this subsection, we fix an integral weight $\lambda \in P$, which is not necessarily dominant.

Definition 2.3.1. For $\mu, \nu \in W\lambda$, let us write $\mu \ge \nu$ if there exists a sequence $\mu = \mu_0, \mu_1, \ldots, \mu_n = \nu$ of elements in $W\lambda$ and a sequence $\beta_1, \ldots, \beta_n \in \Delta_{\mathrm{re}}^{+}$ of positive real roots such that $\mu_k = r_{\beta_k}\mu_{k-1}$ and $\langle \mu_{k-1}, \beta_k^{\vee} \rangle < 0$ for $k = 1, 2, \ldots, n$. If $\mu \ge \nu$, then we define $\mathrm{dist}(\mu, \nu)$ to be the maximal length n of all possible such sequences $\mu_0, \mu_1, \ldots, \mu_n$ for (μ, ν).

Definition 2.3.2. For $\mu, \nu \in W\lambda$ with $\mu > \nu$ and a rational number $0 < \sigma < 1$, a σ-chain for (μ, ν) is, by definition, a sequence $\mu = \mu_0 > \mu_1 > \cdots > \mu_n = \nu$ of elements in $W\lambda$ such that $\mathrm{dist}(\mu_{k-1}, \mu_k) = 1$ and $\sigma\langle \mu_{k-1}, \beta_k^{\vee} \rangle \in \mathbb{Z}_{<0}$ for all $k = 1, 2, \ldots, n$, where β_k is the positive real root such that $r_{\beta_k}\mu_{k-1} = \mu_k$.

Definition 2.3.3. An LS path of shape $\lambda \in P$ is, by definition, a pair $(\underline{\nu}\,; \underline{\sigma})$ of a sequence $\underline{\nu} : \nu_1 > \nu_2 > \cdots > \nu_s$ of elements in $W\lambda$ and a sequence $\underline{\sigma} : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers satisfying the condition that there exists a σ_k-chain for $(\nu_k,\, \nu_{k+1})$ for each $k = 1, 2, \dots, s-1$. We denote by $\mathbb{B}(\lambda)$ the set of all LS paths of shape λ.

Let $\pi = (\nu_1, \nu_2, \dots, \nu_s\,;\, \sigma_0, \sigma_1, \dots, \sigma_s)$ be a pair of a sequence $\nu_1, \nu_2, \dots, \nu_s$ of integral weights with $\nu_k \neq \nu_{k+1}$ for $1 \le k \le s-1$ and a sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers. We identify π with the following piecewise-linear, continuous map $\pi : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P$:
(2.3.1)

$$\pi(t) = \sum_{l=1}^{k-1} (\sigma_l - \sigma_{l-1})\nu_l + (t - \sigma_{k-1})\nu_k \quad \text{for } \sigma_{k-1} \le t \le \sigma_k,\ 1 \le k \le s.$$

Remark 2.3.4. It is obvious from the definition that for each $\nu \in W\lambda$, $\pi_\nu := (\nu\,;\, 0, 1)$ is an LS path of shape λ, which corresponds (under (2.3.1)) to the straight line $\pi_\nu(t) = t\nu$, $t \in [0,1]$, connecting 0 to ν.

For each $\pi \in \mathbb{B}(\lambda)$, we define $\mathrm{cl}(\pi) : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{cl}}$ by: $(\mathrm{cl}(\pi))(t) = \mathrm{cl}(\pi(t))$ for $t \in [0,1]$. We set

$$\mathbb{B}(\lambda)_{\mathrm{cl}} := \bigl\{ \mathrm{cl}(\pi) \mid \pi \in \mathbb{B}(\lambda) \bigr\}.$$

We know from [NS2, §3.1] that $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is a subset of $\mathbb{P}_{\mathrm{cl,\,int}}$ such that $\mathbb{B}(\lambda)_{\mathrm{cl}} \cup \{\mathbf{0}\}$ is stable under the action of the root operators e_j and f_j for all $j \in I$. In particular, $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is a crystal with weights in P_{cl} by Theorem 2.2.3.

Here we recall the notion of simple crystals. A crystal B with weights in P_{cl} is said to be regular if for every proper subset $J \subsetneq I$, B is isomorphic, as a crystal for $U_q(\mathfrak{g}_J)$, to the crystal basis of a finite-dimensional $U_q(\mathfrak{g}_J)$-module, where $\mathfrak{g}_J$ is the (finite-dimensional) Levi subalgebra of $\mathfrak{g}$ corresponding to J (see [Kas2, §2.2]). A regular crystal B with weights in P_{cl} is said to be simple if the set of extremal elements in B coincides with a W-orbit in B through an (extremal) element in B (cf. [Kas2, Definition 4.9]).

Remark 2.3.5.

(1) The crystal graph of a simple crystal is connected (see [Kas2, Lemma 4.10]).

(2) A tensor product of simple crystals is also a simple crystal (see [Kas2, Lemma 4.11]).

We know the following theorem from [NS1, Proposition 5.8], [NS3, Theorem 2.1.1 and Proposition 3.4.2], and [NS2, Theorem 3.2].

Theorem 2.3.6.

(1) *For each $i \in I_0$, the crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is isomorphic, as a crystal with weights in P_{cl}, to the crystal basis of the level-zero fundamental representation $W(\varpi_i)$, introduced in* [Kas2, Theorem 5.17]*, of the quantum affine algebra $U'_q(\mathfrak{g})$. In particular, $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is a simple crystal.*

(2) *Let $i_1, i_2, \dots, i_p$ be an arbitrary sequence of elements of I_0 (with repetitions allowed), and set $\lambda := \varpi_{i_1} + \varpi_{i_2} + \cdots + \varpi_{i_p}$. The crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is isomorphic, as a crystal with weights in P_{cl}, to the tensor product $\mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_p})_{\mathrm{cl}}$. In particular, $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is also a simple crystal by Remark 2.3.5 (2).*

Remark 2.3.7. Let $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \varpi_i$ be a level-zero dominant integral weight.

(1) It is easily seen from Remark 2.3.4 that $\eta_\mu(t) := t\mu$ is contained in $\mathbb{B}(\lambda)_{\mathrm{cl}}$ for all $\mu \in \mathrm{cl}(W\lambda) = W_0\,\mathrm{cl}(\lambda)$.

(2) We know from [NS2, Lemma 3.19] that $\eta_{\mathrm{cl}(\lambda)} \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ is an extremal element in the sense of [Kas2, §3.1]. Therefore, it follows from [AK, Lemma 1.5] and the definition of simple crystals that for each $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, there exist $j_1, j_2, \dots, j_p \in I$ such that

$$e_{j_p}^{\max} \cdots e_{j_2}^{\max} e_{j_1}^{\max} \eta = \eta_{\mathrm{cl}(\lambda)}.$$

Also, by the same argument as for [AK, Lemma 1.5], we can show that for each $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, there exist $k_1, k_2, \dots, k_q \in I$ such that

$$f_{k_q}^{\max} \cdots f_{k_2}^{\max} f_{k_1}^{\max} \eta = \eta_{\mathrm{cl}(\lambda)}.$$

Lemma 2.3.8. *Let $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \varpi_i$ be a level-zero dominant integral weight, and let $n \in \mathbb{Z}_{\geq 1}$. Then, the set $\mathbb{B}(\lambda)_{\mathrm{cl}}^{*n}$ is identical to $\mathbb{B}(n\lambda)_{\mathrm{cl}}$.*

Proof. First, let us show the inclusion $\mathbb{B}(\lambda)_{\mathrm{cl}}^{*n} \supset \mathbb{B}(n\lambda)_{\mathrm{cl}}$. It is easily seen that the element $\eta_{\mathrm{cl}(\lambda)} * \cdots * \eta_{\mathrm{cl}(\lambda)} \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{*n}$ is identical to $\eta_{\mathrm{cl}(n\lambda)}$. Hence it follows that the crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}^{*n}$ contains the connected component containing $\eta_{\mathrm{cl}(n\lambda)} \in \mathbb{B}(n\lambda)_{\mathrm{cl}}$. Here we recall that the crystal $\mathbb{B}(n\lambda)_{\mathrm{cl}}$ is simple (see Theorem 2.3.6), and hence connected (see Remark 2.3.5 (1)). Therefore, the connected component above is identical to $\mathbb{B}(n\lambda)_{\mathrm{cl}}$. Thus, we have shown the inclusion $\mathbb{B}(\lambda)_{\mathrm{cl}}^{*n} \supset \mathbb{B}(n\lambda)_{\mathrm{cl}}$.

Now, it follows from Proposition 2.2.7 that $\mathbb{B}(\lambda)_{\mathrm{cl}}^{*n}$ is isomorphic as a crystal to the tensor product $\mathbb{B}(\lambda)_{\mathrm{cl}}^{\otimes n}$. Therefore, $\mathbb{B}(\lambda)_{\mathrm{cl}}^{*n} \cong \mathbb{B}(\lambda)_{\mathrm{cl}}^{\otimes n}$ is a simple crystal by Theorem 2.3.6 (2) and Remark 2.3.5 (2), and hence connected by Remark 2.3.5 (1). From this, we conclude that $\mathbb{B}(\lambda)_{\mathrm{cl}}^{*n} = \mathbb{B}(n\lambda)_{\mathrm{cl}}$, as desired. Q.E.D.

2.4. Characterization of the set $\mathbb{B}(\lambda)_{\mathrm{cl}}$ of paths.

Theorem 2.4.1. *Let* $\lambda \in \sum_{i\in I_0} \mathbb{Z}_{\geq 0}\varpi_i$ *be a level-zero dominant integral weight. If a subset* $\mathbb{B}$ *of* $\mathbb{P}_{\mathrm{cl,\,int}}$ *satisfies the following two conditions, then the set* $\mathbb{B}$ *is identical to* $\mathbb{B}(\lambda)_{\mathrm{cl}}$.

(a) *The set* $\mathbb{B} \cup \{\mathbf{0}\}$ *is stable under the action of the root operators* f_j *for all* $j \in I$.

(b) *For each* $\eta \in \mathbb{B}$, *there exist a sequence* $\mu_1, \mu_2, \ldots, \mu_s$ *of elements in* $\mathrm{cl}(W\lambda) = W_0\,\mathrm{cl}(\lambda)$ *and a sequence* $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ *of rational numbers such that*

(2.4.1)
$$\eta(t) = \sum_{l=1}^{k-1}(\sigma_l - \sigma_{l-1})\mu_l + (t - \sigma_{k-1})\mu_k \quad \text{for } \sigma_{k-1} \leq t \leq \sigma_k,\ 1 \leq k \leq s.$$

Remark 2.4.2. The equality $\mathbb{B} = \mathbb{B}(\lambda)_{\mathrm{cl}}$ also holds when we replace the root operators f_j for $j \in I$ by e_j for $j \in I$ in the theorem above; for its proof, simply replace f_j's by e_j's in the proof below.

Proof of Theorem 2.4.1. First, let us show the inclusion $\mathbb{B} \subset \mathbb{B}(\lambda)_{\mathrm{cl}}$. Fix an element $\eta \in \mathbb{B}$ arbitrarily, and assume that η is of the form (2.4.1). Take $N \in \mathbb{Z}_{\geq 1}$ such that $N\sigma_u \in \mathbb{Z}$ for all $0 \leq u \leq s$. Then, the element $N\eta \in \mathbb{P}_{\mathrm{cl,\,int}}$ is of the form:

$$N\eta = \underbrace{\eta_{\mu_1} * \cdots * \eta_{\mu_1}}_{N(\sigma_1 - \sigma_0)\text{-times}} * \underbrace{\eta_{\mu_2} * \cdots * \eta_{\mu_2}}_{N(\sigma_2 - \sigma_1)\text{-times}} * \cdots * \underbrace{\eta_{\mu_s} * \cdots * \eta_{\mu_s}}_{N(\sigma_s - \sigma_{s-1})\text{-times}}.$$

Since $\eta_\mu \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ for every $\mu \in \mathrm{cl}(W\lambda)$ (see Remark 2.3.7 (1)), we have $N\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{*N}$, and hence $N\eta \in \mathbb{B}(N\lambda)_{\mathrm{cl}}$ by Lemma 2.3.8. Hence, by Remark 2.3.7, there exists $k_1, k_2, \ldots, k_q \in I$ such that

$$f_{k_q}^{\max} \cdots f_{k_2}^{\max} f_{k_1}^{\max}(N\eta) = \eta_{\mathrm{cl}(N\lambda)}.$$

Also, by using Lemma 2.2.6 and condition (a) repeatedly, we deduce that

$$f_{k_q}^{\max} \cdots f_{k_2}^{\max} f_{k_1}^{\max}(N\eta) = N(f_{k_q}^{\max} \cdots f_{k_2}^{\max} f_{k_1}^{\max}\eta).$$

Combining these equalities, we obtain $N(f_{k_q}^{\max} \cdots f_{k_2}^{\max} f_{k_1}^{\max}\eta) = \eta_{\mathrm{cl}(N\lambda)}$. Since $\eta_{\mathrm{cl}(N\lambda)} = N\eta_{\mathrm{cl}(\lambda)}$, we get

(2.4.2)
$$f_{k_q}^{\max} \cdots f_{k_2}^{\max} f_{k_1}^{\max}\eta = \eta_{\mathrm{cl}(\lambda)} \in \mathbb{B}(\lambda)_{\mathrm{cl}}.$$

Therefore, by Theorem 2.2.3 (1), $\eta = e_{k_1}^{c_1} e_{k_2}^{c_2} \cdots e_{k_q}^{c_q} \eta_{\mathrm{cl}(\lambda)} \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ for some $c_1, c_2, \ldots, c_q \in \mathbb{Z}_{\geq 0}$. Thus we have shown the inclusion $\mathbb{B} \subset \mathbb{B}(\lambda)_{\mathrm{cl}}$. In addition, we should remark that $\eta_{\mathrm{cl}(\lambda)} \in \mathbb{B}$ by (2.4.2) and condition (a).

Next, let us show the opposite inclusion $\mathbb{B} \supset \mathbb{B}(\lambda)_{\mathrm{cl}}$. Fix an element $\eta' \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ arbitrarily. By Remark 2.3.7, there exists $j_1, j_2, \ldots, j_p \in I$ such that

$$e_{j_p}^{\max} \cdots e_{j_2}^{\max} e_{j_1}^{\max} \eta' = \eta_{\mathrm{cl}(\lambda)}.$$

Therefore, by Theorem 2.2.3 (1), $\eta' = f_{j_1}^{d_1} f_{j_2}^{d_2} \cdots f_{j_p}^{d_p} \eta_{\mathrm{cl}(\lambda)}$ for some $d_1, d_2, \ldots, d_p \in \mathbb{Z}_{\geq 0}$. Since $\eta_{\mathrm{cl}(\lambda)} \in \mathbb{B}$ as shown above, it follows from condition (a) that $\eta' \in \mathbb{B}$. Thus we have shown the inclusion $\mathbb{B} \supset \mathbb{B}(\lambda)_{\mathrm{cl}}$, thereby completing the proof of the theorem. Q.E.D.

§3. Quantum Lakshmibai-Seshadri paths.

3.1. Quantum Bruhat graph.

In this subsection, we fix a subset J of I_0. Set

$$W_J := \langle r_j \mid j \in J \rangle \subset W_0.$$

It is well-known that each coset in W_0/W_J has a unique element of minimal length, called the minimal coset representative for the coset; we denote by $W_0^J \subset W_0$ the set of minimal coset representatives for the cosets in W_0/W_J, and by $\lfloor \cdot \rfloor = \lfloor \cdot \rfloor_J : W_0 \twoheadrightarrow W_0^J \cong W_0/W_J$ the canonical projection. Also, we set $\Delta_J := \Delta_0 \cap \left(\bigoplus_{j \in J} \mathbb{Z}\alpha_j\right)$, $\Delta_J^{\pm} := \Delta_0^{\pm} \cap \left(\bigoplus_{j \in J} \mathbb{Z}\alpha_j\right)$, and $\rho := (1/2) \sum_{\alpha \in \Delta_0^+} \alpha$, $\rho_J := (1/2) \sum_{\alpha \in \Delta_J^+} \alpha$.

Definition 3.1.1. The (parabolic) quantum Bruhat graph is the $(\Delta_0^+ \setminus \Delta_J^+)$-labeled, directed graph with vertex set W_0^J and $(\Delta_0^+ \setminus \Delta_J^+)$-labeled, directed edges of the following form: $\lfloor wr_\beta \rfloor \xleftarrow{\beta} w$ for $w \in W_0^J$ and $\beta \in \Delta_0^+ \setminus \Delta_J^+$ such that either

(i) $\ell(\lfloor wr_\beta \rfloor) = \ell(w) + 1$, or

(ii) $\ell(\lfloor wr_\beta \rfloor) = \ell(w) - 2\langle \rho - \rho_J, \beta^\vee \rangle + 1$;

if (i) holds (resp., (ii) holds), then the edge is called a Bruhat edge (resp., a quantum edge).

Remark 3.1.2. If $w \in W_0^J$ and $\beta \in \Delta_0^+ \setminus \Delta_J^+$ satisfy the condition that $\ell(\lfloor wr_\beta \rfloor) = \ell(w) + 1$, then $wr_\beta \in W_0^J$. Indeed, since $\ell(wr_\beta) \geq \ell(\lfloor wr_\beta \rfloor) = \ell(w) + 1$, it follows that wr_β is greater than w in the ordinary Bruhat order. Therefore, by [BB, Proposition 2.5.1], $\lfloor wr_\beta \rfloor$ is greater than or equal to $\lfloor w \rfloor = w$ in the ordinary Bruhat order. Since $\ell(\lfloor wr_\beta \rfloor) =$

$\ell(w)+1$ by the assumption, there exists $\gamma \in \Delta_0^+$ such that $\lfloor wr_\beta \rfloor = wr_\gamma$. Now, we take a dominant integral weight $\Lambda \in P_{\mathrm{cl}}$ with respect to the finite root system Δ_0 such that $\bigl\{ j \in I_0 \mid \langle \Lambda,\, \alpha_j^\vee \rangle = 0 \bigr\} = J$; note that $\langle \Lambda,\, \beta^\vee \rangle > 0$ since $\beta \in \Delta_0^+ \setminus \Delta_J^+$. Then we have $wr_\beta \Lambda = \lfloor wr_\beta \rfloor \Lambda = wr_\gamma \Lambda$, and hence $r_\beta \Lambda = r_\gamma \Lambda$. It follows that $\langle \Lambda,\, \beta^\vee \rangle \beta = \langle \Lambda,\, \gamma^\vee \rangle \gamma$. Since β and γ are both contained in Δ_0^+, and since $\langle \Lambda,\, \beta^\vee \rangle > 0$, we deduce that $\beta = \gamma$. Thus, we obtain $\lfloor wr_\beta \rfloor = wr_\gamma = wr_\beta$, which implies that $wr_\beta \in W_0^J$.

Remark 3.1.3. We know from [LS, Lemma 10.18] that the condition (ii) above is equivalent to the following condition:

(iii) $\ell(\lfloor wr_\beta \rfloor) = \ell(w) - 2\langle \rho - \rho_J,\, \beta^\vee \rangle + 1$ and $\ell(wr_\beta) = \ell(w) - 2\langle \rho,\, \beta^\vee \rangle + 1$.

Let $x,\, y \in W_0^J$. A directed path $\mathbf{d}$ from y to x in the parabolic quantum Bruhat graph is, by definition, a pair of a sequence $w_0,\, w_1,\, \dots,\, w_n$ of elements in W_0^J and a sequence $\beta_1,\, \beta_2,\, \dots,\, \beta_n$ of elements in $\Delta_0^+ \setminus \Delta_J^+$ such that

$$\mathbf{d} : x = w_0 \overset{\beta_1}{\leftarrow} w_1 \overset{\beta_2}{\leftarrow} \cdots \overset{\beta_n}{\leftarrow} w_n = y. \tag{3.1.1}$$

A directed path $\mathbf{d}$ from y to x is said to be shortest if its length n is minimal among all possible directed paths from y to x. Denote by $\ell(y,\, x)$ the length of a shortest directed path from y to x in the parabolic quantum Bruhat graph.

3.2. Definition of quantum Lakshmibai-Seshadri paths.

In this subsection, we fix a level-zero dominant integral weight $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \varpi_i$, and set $\Lambda := \mathrm{cl}(\lambda)$ for simplicity of notation. Also, we set

$$J := \bigl\{ j \in I_0 \mid \langle \Lambda,\, \alpha_j^\vee \rangle = 0 \bigr\} \subset I_0.$$

Definition 3.2.1. Let $x,\, y \in W_0^J$, and let $\sigma \in \mathbb{Q}$ be such that $0 < \sigma < 1$. A directed σ-path from y to x is, by definition, a directed path

$$x = w_0 \overset{\beta_1}{\leftarrow} w_1 \overset{\beta_2}{\leftarrow} w_2 \overset{\beta_3}{\leftarrow} \cdots \overset{\beta_n}{\leftarrow} w_n = y$$

from y to x in the parabolic quantum Bruhat graph satisfying the condition that

$$\sigma \langle \Lambda,\, \beta_k^\vee \rangle \in \mathbb{Z} \quad \text{for all } 1 \leq k \leq n.$$

Definition 3.2.2. Denote by $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ (resp., $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$) the set of all pairs $\eta = (\underline{x}\,;\, \underline{\sigma})$ of a sequence $\underline{x} \,:\, x_1,\, x_2,\, \dots,\, x_s$ of elements in W_0^J, with $x_k \neq x_{k+1}$ for $1 \leq k \leq s-1$, and a sequence $\underline{\sigma} \,:\, 0 = \sigma_0 < \sigma_1 <$

$\cdots < \sigma_s = 1$ of rational numbers satisfying the condition that there exists a directed σ_k-path (resp., a directed σ_k-path of length $\ell(x_{k+1}, x_k)$) from x_{k+1} to x_k for each $1 \le k \le s-1$; observe that $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}} \subset \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$. We call an element of $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ a quantum Lakshmibai-Seshadri path of shape λ.

Let $\eta = (x_1, x_2, \ldots, x_s\,; \sigma_0, \sigma_1, \ldots, \sigma_s)$ be a rational path, that is, a pair of a sequence $x_1, x_2, \ldots, x_s$ of elements in W_0^J, with $x_k \neq x_{k+1}$ for $1 \le k \le s-1$, and a sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers. We identify η with the following piecewise-linear, continuous map $\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{cl}}$ (cf. (2.3.1)):
(3.2.1)

$$\eta(t) = \sum_{l=1}^{k-1}(\sigma_l - \sigma_{l-1})x_l\Lambda + (t - \sigma_{k-1})x_k\Lambda \quad \text{for } \sigma_{k-1} \le t \le \sigma_k,\ 1 \le k \le s;$$

note that the map $W_0^J \to W_0\Lambda$, $w \mapsto w\Lambda$, is bijective. We will prove that under this identification, both $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ and $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ can be regarded as a subset of $\mathbb{P}_{\mathrm{cl,\,int}}$ (see Proposition 4.1.12). Furthermore, we will prove that both of the sets $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}} \cup \{\mathbf{0}\}$ and $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}} \cup \{\mathbf{0}\}$ are stable under the action of the root operators (see Proposition 4.2.1).

§4. Main result.

4.1. Statement of the main result and some technical lemmas.

In this subsection and the next subsection, we fix a level-zero dominant integral weight $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{\ge 0}\varpi_i$. Set $\Lambda := \mathrm{cl}(\lambda)$, and

$$J := \bigl\{ j \in I_0 \mid \langle \Lambda, \alpha_j^{\vee} \rangle = 0 \bigr\} \subset I_0.$$

The following theorem is the main result of this paper; it is obtained as a by-product of an explicit description, given in §4.2, of the image of a quantum LS path as a rational path under the action of root operators on quantum LS paths.

Theorem 4.1.1. *With the notation and setting above, we have*

$$\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}} = \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}} = \mathbb{B}(\lambda)_{\mathrm{cl}}.$$

In view of Theorem 2.4.1, in order to prove Theorem 4.1.1, it suffices to prove that both $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ and $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ are contained in $\mathbb{P}_{\mathrm{cl,\,int}}$ (see Proposition 4.1.12 below), and that both of the sets $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}} \cup \{\mathbf{0}\}$ and $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}} \cup \{\mathbf{0}\}$ are stable under the action of the root operators f_j for all $j \in I$ (see Proposition 4.2.1 below). To prove these, we need some lemmas.

Lemma 4.1.2 ([LNSSS1, Proposition 5.11])**.** *Let* $w \in W_0^J$. *If* $w^{-1}\theta \in \Delta_0^-$, *then there exists a quantum edge* $\lfloor r_\theta w \rfloor \xleftarrow{-w^{-1}\theta} w$ *from* w *to* $\lfloor r_\theta w \rfloor$ *in the parabolic quantum Bruhat graph.*

Lemma 4.1.3 ([LNSSS1, Proposition 5.10 (1) and (3)])**.** *Let* $w \in W_0^J$ *and* $j \in I_0$. *If* $w^{-1}\alpha_j \in \Delta_0 \setminus \Delta_J$, *then* $r_j w \in W_0^J$.

Lemma 4.1.4. *Let* $w \in W_0^J$ *and* $\beta \in \Delta_0^+ \setminus \Delta_J^+$ *be such that* $\lfloor w r_\beta \rfloor \xleftarrow{\beta} w$. *Let* $j \in I_0$.

(1) *If* $\langle w\Lambda, \alpha_j^\vee \rangle > 0$ *and* $w\beta \neq \pm\alpha_j$, *then* $\langle w r_\beta \Lambda, \alpha_j^\vee \rangle > 0$. *Also, both* $r_j \lfloor w r_\beta \rfloor$ *and* $r_j w$ *are contained in* W_0^J, *and* $r_j \lfloor w r_\beta \rfloor \xleftarrow{\beta} r_j w$.

(2) *If* $\langle w r_\beta \Lambda, \alpha_j^\vee \rangle < 0$ *and* $w\beta \neq \pm\alpha_j$, *then* $\langle w\Lambda, \alpha_j^\vee \rangle < 0$. *Also, both* $r_j \lfloor w r_\beta \rfloor$ *and* $r_j w$ *are contained in* W_0^J, *and* $r_j \lfloor w r_\beta \rfloor \xleftarrow{\beta} r_j w$.

(3) *If* $\langle w r_\beta \Lambda, \alpha_j^\vee \rangle < 0$ *and* $\langle w\Lambda, \alpha_j^\vee \rangle \geq 0$, *then* $w\beta = \pm\alpha_j$.

(4) *If* $\langle w r_\beta \Lambda, \alpha_j^\vee \rangle \leq 0$ *and* $\langle w\Lambda, \alpha_j^\vee \rangle > 0$, *then* $w\beta = \pm\alpha_j$.

Proof. (1) Since $\langle w\Lambda, \alpha_j^\vee \rangle > 0$, we see that $w^{-1}\alpha_j \in \Delta_0^+ \setminus \Delta_J^+$. By [LNSSS1, Proposition 5.10 (3)], there exists a Bruhat edge $r_j w \xleftarrow{w^{-1}\alpha_j} w$ in the parabolic quantum Bruhat graph, with $r_j w \in W_0^J$. If the edge $\lfloor w r_\beta \rfloor \xleftarrow{\beta} w$ is a Bruhat (resp., quantum) edge, then it follows from the left diagram of (5.3) (resp., (5.4)) in part (1) (resp., part (2)) of [LNSSS1, Lemma 5.14] that $r_j \lfloor w r_\beta \rfloor = \lfloor r_j w r_\beta \rfloor \in W_0^J$, and that there exists a Bruhat (resp., quantum) edge $r_j \lfloor w r_\beta \rfloor \xleftarrow{\beta} r_j w$ and a Bruhat edge $r_j \lfloor w r_\beta \rfloor \xleftarrow{\lfloor w r_\beta \rfloor^{-1}\alpha_j} \lfloor w r_\beta \rfloor$ in the parabolic quantum Bruhat graph. In particular, we have $\lfloor w r_\beta \rfloor^{-1}\alpha_j \in \Delta_0^+ \setminus \Delta_J^+$, which implies that $\langle w r_\beta \Lambda, \alpha_j^\vee \rangle > 0$. This proves part (1).

(2) Since $\langle w r_\beta \Lambda, \alpha_j^\vee \rangle < 0$, we see that $\lfloor w r_\beta \rfloor^{-1}\alpha_j \in \Delta_0^- \setminus \Delta_J^-$. By [LNSSS1, Proposition 5.10 (1)], there exists a Bruhat edge $\lfloor w r_\beta \rfloor \xleftarrow{-\lfloor w r_\beta \rfloor^{-1}\alpha_j} r_j \lfloor w r_\beta \rfloor$ in the parabolic quantum Bruhat graph, with $r_j \lfloor w r_\beta \rfloor \in W_0^J$. If the edge $\lfloor w r_\beta \rfloor \xleftarrow{\beta} w$ is a Bruhat (resp., quantum) edge, then it follows from the right diagram of (5.3) (resp., (5.4)) in part (1) (resp., part (2)) of [LNSSS1, Lemma 5.14] that $r_j w \in W_0^J$, and that there exists a Bruhat (resp., quantum) edge $r_j \lfloor w r_\beta \rfloor \xleftarrow{\beta} r_j w$ and a Bruhat edge $w \xleftarrow{-w^{-1}\alpha_j} r_j w$ in the parabolic quantum Bruhat graph. In

particular, we have $w^{-1}\alpha_j \in \Delta_0^- \setminus \Delta_J^-$, which implies that $\langle w\Lambda, \alpha_j^\vee \rangle < 0$. This proves part (2).

(3) (resp., (4)) Assume that $\langle wr_\beta\Lambda, \alpha_j^\vee \rangle < 0$ and $\langle w\Lambda, \alpha_j^\vee \rangle \geq 0$ (resp., $\langle wr_\beta\Lambda, \alpha_j^\vee \rangle \leq 0$ and $\langle w\Lambda, \alpha_j^\vee \rangle > 0$). Suppose that $w\beta \neq \pm\alpha_j$. Then it follows from part (2) (resp., (1)) that $\langle w\Lambda, \alpha_j^\vee \rangle < 0$ (resp., $\langle wr_\beta\Lambda, \alpha_j^\vee \rangle > 0$), which is a contradiction. Thus we obtain $w\beta = \pm\alpha_j$. This completes the proof of Lemma 4.1.4. Q.E.D.

Lemma 4.1.5. *Let $w \in W_0^J$ and $\beta \in \Delta_0^+ \setminus \Delta_J^+$ be such that $\lfloor wr_\beta \rfloor \xleftarrow{\beta} w$. Let $z \in W_J$ be such that $r_\theta w = \lfloor r_\theta w \rfloor z$; note that $z\beta \in \Delta_0^+ \setminus \Delta_J^+$.*

(1) *If $\langle w\Lambda, \alpha_0^\vee \rangle > 0$ and $w\beta \neq \pm\theta$, then $\langle wr_\beta\Lambda, \alpha_0^\vee \rangle > 0$ and $\lfloor r_\theta wr_\beta \rfloor \xleftarrow{z\beta} \lfloor r_\theta w \rfloor$.*

(2) *If $\langle wr_\beta\Lambda, \alpha_0^\vee \rangle < 0$ and $w\beta \neq \pm\theta$, then $\langle w\Lambda, \alpha_0^\vee \rangle < 0$ and $\lfloor r_\theta wr_\beta \rfloor \xleftarrow{z\beta} \lfloor r_\theta w \rfloor$.*

(3) *If $\langle wr_\beta\Lambda, \alpha_0^\vee \rangle < 0$ and $\langle w\Lambda, \alpha_0^\vee \rangle \geq 0$, then $w\beta = \pm\theta$.*

(4) *If $\langle wr_\beta\Lambda, \alpha_0^\vee \rangle \leq 0$ and $\langle w\Lambda, \alpha_0^\vee \rangle > 0$, then $w\beta = \pm\theta$.*

Proof. (1) Since $\langle w\Lambda, \alpha_0^\vee \rangle > 0$, we see that $w^{-1}\theta \in \Delta_0^- \setminus \Delta_J^-$. By [LNSSS1, Proposition 5.11 (1)], there exists a quantum edge $\lfloor r_\theta w \rfloor \xleftarrow{-w^{-1}\theta} w$ in the parabolic quantum Bruhat graph. If the edge $\lfloor wr_\beta \rfloor \xleftarrow{\beta} w$ is a Bruhat (resp., quantum) edge, then it follows from the left diagram of (5.5) or (5.6) (resp., (5.7) or (5.8)) in part (3) (resp., part (4)) of [LNSSS1, Lemma 5.14] that there exists an edge $\lfloor r_\theta wr_\beta \rfloor \xleftarrow{z\beta} \lfloor r_\theta w \rfloor$ and a quantum edge $\lfloor r_\theta wr_\beta \rfloor \xleftarrow{-\lfloor wr_\beta \rfloor^{-1}\theta} \lfloor wr_\beta \rfloor$ in the parabolic quantum Bruhat graph. In particular, we have $\lfloor wr_\beta \rfloor^{-1}\theta \in \Delta_0^- \setminus \Delta_J^-$, which implies that $\langle wr_\beta\Lambda, \alpha_0^\vee \rangle > 0$. This proves part (1).

(2) Since $\langle wr_\beta\Lambda, \alpha_0^\vee \rangle < 0$, we see that $\lfloor wr_\beta \rfloor^{-1}\theta \in \Delta_0^+ \setminus \Delta_J^+$. By [LNSSS1, Proposition 5.11 (3)], there exists a quantum edge $\lfloor wr_\beta \rfloor \xleftarrow{z'\lfloor wr_\beta \rfloor^{-1}\theta} \lfloor r_\theta wr_\beta \rfloor$ in the parabolic quantum Bruhat graph, where $z' \in W_J$ is defined by: $r_\theta \lfloor wr_\beta \rfloor = \lfloor r_\theta wr_\beta \rfloor z'$. If the edge $\lfloor wr_\beta \rfloor \xleftarrow{\beta} w$ is a Bruhat (resp., quantum) edge, then it follows from the right diagram of (5.5) or (5.6) (resp., (5.7) or (5.8)) in part (3) (resp., part (4)) of [LNSSS1, Lemma 5.14] that there exists an edge $\lfloor r_\theta wr_\beta \rfloor \xleftarrow{z\beta} \lfloor r_\theta w \rfloor$ and a quantum edge $w \xleftarrow{zw^{-1}\theta} \lfloor r_\theta w \rfloor$ in the parabolic quantum Bruhat graph. In particular, we have $w^{-1}\theta \in \Delta_0^+ \setminus \Delta_J^+$, which implies that $\langle w\Lambda, \alpha_0^\vee \rangle < 0$. This proves part (2).

Parts (3) and (4) can be shown by using parts (2) and (1) in the same way as parts (3) and (4) of Lemma 4.1.4, respectively. This completes the proof of Lemma 4.1.5. Q.E.D.

Lemma 4.1.6. *Let λ, Λ, and J be as above. Let $x, y \in W_0^J$, and let $\sigma \in \mathbb{Q}$ be such that $0 < \sigma < 1$. Assume that there exists a directed σ-path from y to x as follows:*

$$x = w_0 \xleftarrow{\beta_1} w_1 \xleftarrow{\beta_2} w_2 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} w_n = y.$$

Then, $\sigma(x\Lambda - y\Lambda)$ is contained in $Q_0 := \bigoplus_{j \in I_0} \mathbb{Z}\alpha_j$.

Proof. We have

$$\sigma(x\Lambda - y\Lambda) = \sum_{k=1}^{n} \sigma(w_{k-1}\Lambda - w_k\Lambda) = \sum_{k=1}^{n} \sigma(w_k r_{\beta_k}\Lambda - w_k\Lambda)$$
$$= -\sum_{k=1}^{n} \sigma\langle \Lambda, \beta_k^\vee \rangle\, w_k\beta_k.$$

It follows from the definition of a directed σ-path that $\sigma\langle \Lambda, \beta_k^\vee \rangle \in \mathbb{Z}$ for all $1 \le k \le n$. Also, it is obvious that $w_k\beta_k \in Q_0$ for all $1 \le k \le n$. Therefore, we conclude that $\sigma(x\Lambda - y\Lambda) \in Q_0$. This proves the lemma. Q.E.D.

Lemma 4.1.7. *Let λ, Λ, and J be as above. If $\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, then $\eta(1)$ is contained in $\Lambda + Q_0$, and hence in P_{cl}.*

Proof. Let $\eta = (x_1, x_2, \ldots, x_s\,; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$. Then we have (see (3.2.1))

$$\eta(1) = x_s\Lambda + \sum_{k=1}^{s-1} \sigma_k(x_k\Lambda - x_{k+1}\Lambda).$$

It is obvious that $x_s\Lambda \in \Lambda + Q_0$. Also, it follows from Lemma 4.1.6 that $\sigma_k(x_k\Lambda - x_{k+1}\Lambda) \in Q_0$ for each $1 \le k \le s-1$. Therefore, we conclude that $\eta(1) \in \Lambda + Q_0$. This proves the lemma. Q.E.D.

In what follows, we set $s_j := r_j$ for $j \in I_0$, and $s_0 := r_\theta \in W_0$, in order to state our results and write their proofs in a way independent of whether $j = 0$ or not.

Lemma 4.1.8. *Let λ, Λ, and J be as above. Let $x, y \in W_0^J$, and assume that there exists a directed path*

$$x = w_0 \xleftarrow{\beta_1} w_1 \xleftarrow{\beta_2} w_2 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} w_n = y. \qquad (4.1.1)$$

from y *to* x*. Let* $j \in I$*.*

(1) *If there exists* $1 \le p \le n$ *such that* $\langle w_k\Lambda, \alpha_j^\vee\rangle < 0$ *for all* $0 \le k \le p-1$ *and* $\langle w_p\Lambda, \alpha_j^\vee\rangle \ge 0$*, then* $\lfloor s_j w_{p-1}\rfloor = w_p$*, and there exists a directed path from* y *to* $\lfloor s_j x\rfloor$ *of the form:*

(4.1.2)

$$\lfloor s_j x\rfloor = \lfloor s_j w_0\rfloor \xleftarrow{z_1\beta_1} \cdots \xleftarrow{z_{p-1}\beta_{p-1}} \lfloor s_j w_{p-1}\rfloor = w_p \xleftarrow{\beta_{p+1}} \cdots \xleftarrow{\beta_n} w_n = y.$$

Here, if $j \in I_0$*, then we define* $z_k \in W_J$ *to be the identity element for all* $1 \le k \le p-1$*; if* $j = 0$*, then we define* $z_k \in W_J$ *by* $r_\theta w_k = \lfloor r_\theta w_k\rfloor z_k$ *for each* $1 \le k \le p-1$*.*

(2) *If the directed path* (4.1.1) *from* y *to* x *is shortest, i.e.,* $\ell(y, x) = n$*, then the directed path* (4.1.2) *from* y *to* $\lfloor s_j x\rfloor$ *is also shortest, i.e.,* $\ell(y, \lfloor s_j x\rfloor) = n-1$*.*

(3) *If the directed path* (4.1.1) *is a directed* σ*-path from* y *to* x *for some rational number* $0 < \sigma < 1$*, then the directed path* (4.1.2) *is a directed* σ*-path from* y *to* $\lfloor s_j x\rfloor$*.*

Proof. (1) We give a proof only for the case $j \in I_0$. The proof for the case $j = 0$ is similar; replace α_j and $\alpha_j^\vee$ by $-\theta$ and $-\theta^\vee$, respectively, and use Lemma 4.1.5 instead of Lemma 4.1.4. First, let us check that $w_k\beta_k \ne \pm\alpha_j$ for any $1 \le k \le p-1$. Suppose, contrary to our claim, that $w_k\beta_k = \pm\alpha_j$ for some $1 \le k \le p-1$. Then,

$$w_{k-1}\Lambda = w_k r_{\beta_k}\Lambda = r_{w_k\beta_k} w_k\Lambda = s_j w_k\Lambda,$$

and hence $\langle w_{k-1}\Lambda, \alpha_j^\vee\rangle = \langle s_j w_k\Lambda, \alpha_j^\vee\rangle = -\langle w_k\Lambda, \alpha_j^\vee\rangle > 0$, which contradicts our assumption. Thus, $w_k\beta_k \ne \pm\alpha_j$ for any $1 \le k \le p-1$. It follows from Lemma 4.1.4 (2) and our assumption that $\lfloor s_j w_{k-1}\rfloor \xleftarrow{\beta_k} \lfloor s_j w_k\rfloor$ for all $1 \le k \le p-1$. Also, since $\langle w_{p-1}\Lambda, \alpha_j^\vee\rangle < 0$ and $\langle w_p\Lambda, \alpha_j^\vee\rangle \ge 0$, it follows from Lemma 4.1.4 (3) that $w_p\beta_p = \pm\alpha_j$, and hence

$$s_j w_{p-1}\Lambda = s_j w_p r_{\beta_p}\Lambda = s_j r_{w_p\beta_p} w_p\Lambda = s_j s_j w_p\Lambda = w_p\Lambda.$$

Thus, we obtain a directed path of the form (4.1.2) from y to $\lfloor s_j x\rfloor$. This proves part (1).

(2) Assume that $\ell(y, x) = n$. By the argument above, we have $\ell(y, \lfloor s_j x\rfloor) \le n-1$. Suppose, for a contradiction, that $\ell(y, \lfloor s_j x\rfloor) < n-1$, and take a directed path

$$\lfloor s_j x\rfloor = z_0 \xleftarrow{\gamma_1} z_1 \xleftarrow{\gamma_2} z_2 \xleftarrow{\gamma_3} \cdots \xleftarrow{\gamma_l} z_l = y$$

from y to $\lfloor s_j x\rfloor$ whose length l is less than $n-1$. Let us show that $x \xleftarrow{\gamma} \lfloor s_j x\rfloor$ for some $\gamma \in \Delta_0^+ \setminus \Delta_J^+$. Assume first that $j \in I_0$. Since

$\langle x\Lambda,\, \alpha_j^\vee\rangle < 0$ by the assumption, we have $x^{-1}\alpha_j \in \Delta_0^- \setminus \Delta_J^-$, and hence $\ell(x) = \ell(s_j x) + 1$. Also, since $x \in W_0^J$, it follows from Lemma 4.1.3 that $s_j x \in W_0^J$. Therefore, if we set $\gamma := x^{-1}s_j\alpha_j = -x^{-1}\alpha_j \in \Delta_0^+ \setminus \Delta_J^+$, then we obtain $x \overset{\gamma}{\leftarrow} s_j x = \lfloor s_j x\rfloor$. Assume next that $j = 0$. Since $\langle x\Lambda,\, -\theta^\vee\rangle = \langle x\Lambda,\, \alpha_0^\vee\rangle < 0$ by the assumption, we have $x^{-1}\theta \in \Delta_0^+ \setminus \Delta_J^+$. Define an element $v \in W_J$ by $r_\theta x = \lfloor r_\theta x\rfloor v$. Then we see that $\gamma := vx^{-1}\theta$ is contained in $\Delta_0^+ \setminus \Delta_J^+$, and that

$$\begin{aligned}\lfloor \lfloor s_0 x\rfloor r_\gamma\rfloor &= \lfloor \lfloor r_\theta x\rfloor r_\gamma\rfloor = \lfloor r_\theta x v^{-1} r_{vx^{-1}\theta}\rfloor \\ &= \lfloor r_\theta x v^{-1} v x^{-1} r_\theta x v^{-1}\rfloor = \lfloor x v^{-1}\rfloor = x\end{aligned}$$

since $x \in W_0^J$ and $v \in W_J$. Also, note that $\lfloor s_0 x\rfloor^{-1}\theta = \lfloor r_\theta x\rfloor^{-1}\theta = vx^{-1}r_\theta\theta = -\gamma \in \Delta_0^- \setminus \Delta_J^-$. Therefore, we deduce from Lemma 4.1.2 that

$$x = \lfloor \lfloor s_0 x\rfloor r_\gamma\rfloor \overset{\gamma}{\leftarrow} \lfloor r_\theta x\rfloor = \lfloor s_0 x\rfloor.$$

Thus, we obtain a directed path

$$x \overset{\gamma}{\leftarrow} \lfloor s_j x\rfloor = z_0 \overset{\gamma_1}{\leftarrow} z_1 \overset{\gamma_2}{\leftarrow} z_2 \overset{\gamma_3}{\leftarrow} \cdots \overset{\gamma_l}{\leftarrow} z_l = y$$

from y to x whose length is $l + 1 < n = \ell(y,\, x)$. This contradicts the definition of $\ell(y,\, x)$. This proves part (2).

(3) We should remark that $\langle \Lambda,\, z_k\beta_k^\vee\rangle = \langle \Lambda,\, \beta_k^\vee\rangle$ for each $1 \le k \le p-1$, since $z_k \in W_J$. Hence the assertion of part (3) follows immediately from the definition of a directed σ-path. This completes the proof of Lemma 4.1.8. Q.E.D.

The following lemma can be shown in the same way as Lemma 4.1.8. If $j \in I_0$, then use Lemma 4.1.4 (1) and (4) instead of Lemma 4.1.4 (2) and (3), respectively; if $j = 0$, then use Lemma 4.1.5 (1) and (4) instead of Lemma 4.1.5 (2) and (3), respectively.

Lemma 4.1.9. *Keep the notation and setting of Lemma 4.1.8.*

(1) *If there exists $1 \le p \le n$ such that $\langle w_k\Lambda,\, \alpha_j^\vee\rangle > 0$ for all $p \le k \le n$ and $\langle w_{p-1}\Lambda,\, \alpha_j^\vee\rangle \le 0$, then $w_{p-1} = \lfloor s_j w_p\rfloor$, and there exists a directed path from $\lfloor s_j y\rfloor$ to x of the form:*

(4.1.3)

$$x = w_0 \overset{\beta_1}{\leftarrow} \cdots \overset{\beta_{p-1}}{\leftarrow} w_{p-1} = \lfloor s_j w_p\rfloor \overset{z_{p+1}\beta_{p+1}}{\longleftarrow} \cdots \overset{z_n\beta_n}{\longleftarrow} \lfloor s_j w_n\rfloor = \lfloor s_j y\rfloor.$$

Here, if $j \in I_0$, then we define $z_k \in W_J$ to be the identity element for all $p+1 \le k \le n$; if $j = 0$, then we define $z_k \in W_J$ by $r_\theta w_k = \lfloor r_\theta w_k\rfloor z_k$ for each $p+1 \le k \le n$.

(2) *If the directed path* (4.1.1) *from* y *to* x *is shortest, i.e.,* $\ell(y, x) = n$*, then the directed path* (4.1.3) *from* $\lfloor s_j y \rfloor$ *to* x *is also shortest, i.e.,* $\ell(\lfloor s_j y \rfloor, x) = n - 1$*.*

(3) *If the directed path* (4.1.1) *is a directed* σ*-path from* y *to* x *for some rational number* $0 < \sigma < 1$*, then the directed path* (4.1.3) *is a directed* σ*-path from* $\lfloor s_j y \rfloor$ *to* x*.*

Lemma 4.1.10. *Let* $\eta = (x_1, x_2, \ldots, x_s\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s) \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$*. Let* $j \in I$ *and* $1 \le u \le s-1$ *be such that* $\langle x_{u+1}\Lambda, \alpha_j^\vee \rangle > 0$*. Let*

$$x_u = w_0 \overset{\beta_1}{\leftarrow} w_1 \overset{\beta_2}{\leftarrow} w_2 \overset{\beta_3}{\leftarrow} \cdots \overset{\beta_n}{\leftarrow} w_n = x_{u+1}$$

be a directed σ_u*-path from* x_{u+1} *to* x_u*. If there exists* $0 \le k < n$ *such that* $\langle w_k\Lambda, \alpha_j^\vee \rangle \le 0$*, then* $H_j^\eta(\sigma_u) \in \mathbb{Z}$*. In particular, if* $\langle x_u\Lambda, \alpha_j^\vee \rangle \le 0$*, then* $H_j^\eta(\sigma_u) \in \mathbb{Z}$*.*

Proof. We see from the definition that

$$\eta' := (x_1, x_2, \ldots, x_u, x_{u+1}\,;\, \sigma_0, \sigma_1, \ldots, \sigma_u, \sigma_s)$$

is an element of $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$. Also, observe that $\eta'(t) = \eta(t)$ for $0 \le t \le \sigma_{u+1}$, and hence $H_j^{\eta'}(t) = H_j^\eta(t)$ for $0 \le t \le \sigma_{u+1}$. It follows that

$$H_j^\eta(\sigma_u) = H_j^{\eta'}(\sigma_u) = H_j^{\eta'}(1) - (1 - \sigma_u)\langle x_{u+1}\Lambda, \alpha_j^\vee \rangle.$$

Since $\eta'(1) \in P_{\mathrm{cl}}$ (and hence $H_j^{\eta'}(1) \in \mathbb{Z}$) by Lemma 4.1.7, it suffices to show that $(1 - \sigma_u)\langle x_{u+1}\Lambda, \alpha_j^\vee \rangle \in \mathbb{Z}$.

We deduce from Lemma 4.1.9 that there exists a directed σ_u-path from $\lfloor s_j x_{u+1} \rfloor$ to x_u. Therefore,

$$\eta'' = (x_1, x_2, \ldots, x_u, \lfloor s_j x_{u+1} \rfloor\,;\, \sigma_0, \sigma_1, \ldots, \sigma_u, \sigma_s)$$

is also an element of $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$. Since both $\eta'(1)$ and $\eta''(1)$ are contained in $\Lambda + Q_0$ by Lemma 4.1.7, we have $\eta'(1) - \eta''(1) \in Q_0$. Also, we have

$$\begin{aligned}(Q_0 \ni)\ \eta'(1) - \eta''(1) &= (1 - \sigma_u)x_{u+1}\Lambda - (1 - \sigma_u)s_j x_{u+1}\Lambda \\ &= \begin{cases} (1 - \sigma_u)\langle x_{u+1}\Lambda, \alpha_j^\vee \rangle \alpha_j & \text{if } j \in I_0, \\ (1 - \sigma_u)\langle x_{u+1}\Lambda, \alpha_j^\vee \rangle (-\theta) & \text{if } j = 0. \end{cases}\end{aligned}$$

Here we remark that $\theta = \delta - \alpha_0 = \sum_{j \in I_0} a_j \alpha_j$, and the greatest common divisor of the a_j, $j \in I_0$, is equal to 1. From these, we conclude that $(1 - \sigma_u)\langle x_{u+1}\Lambda, \alpha_j^\vee \rangle \in \mathbb{Z}$, thereby completing the proof of the lemma. Q.E.D.

The following lemma can be shown in the same way as Lemma 4.1.10; noting that $\pi' := (x_u,\, x_{u+1}\, \ldots,\, x_s;\, \sigma_0,\, \sigma_u,\, \sigma_{u+1},\, \ldots,\, \sigma_s)$ is an element of $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, use π' instead of η' and the fact that $H_j^{\pi'}(1) - H_j^{\pi'}(1-t) = H_j^{\eta}(1) - H_j^{\eta}(1-t)$ for $0 \le t \le 1 - \sigma_{u-1}$.

Lemma 4.1.11. *Let* $\eta = (x_1,\, x_2,\, \ldots,\, x_s\,;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s) \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$. *Let* $j \in I$ *and* $1 \le u \le s-1$ *be such that* $\langle x_u\Lambda,\, \alpha_j^{\vee}\rangle < 0$. *Let*

$$x_u = w_0 \xleftarrow{\beta_1} w_1 \xleftarrow{\beta_2} w_2 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} w_n = x_{u+1}$$

be a directed σ_u*-path from* x_{u+1} *to* x_u. *If there exists* $0 < k \le n$ *such that* $\langle w_k\Lambda,\, \alpha_j^{\vee}\rangle \ge 0$, *then* $H_j^{\eta}(\sigma_u) \in \mathbb{Z}$. *In particular, if* $\langle x_{u+1}\Lambda,\, \alpha_j^{\vee}\rangle \ge 0$, *then* $H_j^{\eta}(\sigma_u) \in \mathbb{Z}$.

Proposition 4.1.12. *Let* $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{\ge 0}\varpi_i$ *be as above. Both* $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ *and* $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ *are contained in* $\mathbb{P}_{\mathrm{cl,\,int}}$ *under the identification* (3.2.1) *of a rational path with a piecewise-linear, continuous map.*

Proof. Since $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}} \subset \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ by the definitions, it suffices to show that $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}} \subset \mathbb{P}_{\mathrm{cl,\,int}}$. Let $\eta = (x_1,\, x_2,\, \ldots,\, x_s\,;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s) \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$. We have shown that $\eta(1) \in P_{\mathrm{cl}}$ for every $\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ (see Lemma 4.1.7). It remains to show that for every $j \in I$, all local minima of the function $H_j^{\eta}(t)$ are integers. Fix $j \in I$, and assume that the function $H_j^{\eta}(t)$ attains a local minimum at $t' \in [0,1]$; we may assume that $t' = \sigma_u$ for some $0 \le u \le s$. If $u = 0$ (resp., $u = s$), then $H_j^{\eta}(t') = H_j^{\eta}(0) = 0 \in \mathbb{Z}$ (resp., $H_j^{\eta}(t') = H_j^{\eta}(1) \in \mathbb{Z}$) since $\eta(0) = 0$ (resp., $\eta(1) \in P_{\mathrm{cl}}$). If $0 < u < s$, then we have either of the following: $\langle x_u\Lambda,\, \alpha_j^{\vee}\rangle \le 0$ and $\langle x_{u+1}\Lambda,\, \alpha_j^{\vee}\rangle > 0$, or $\langle x_u\Lambda,\, \alpha_j^{\vee}\rangle < 0$ and $\langle x_{u+1}\Lambda,\, \alpha_j^{\vee}\rangle \ge 0$. Therefore, it follows from Lemma 4.1.10 or 4.1.11 that $H_j^{\eta}(\sigma_u) \in \mathbb{Z}$. This proves the proposition. Q.E.D.

Lemma 4.1.13. *Let* $\eta = (x_1,\, x_2,\, \ldots,\, x_s\,;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s) \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$. *Let* $j \in I$ *and* $1 \le u \le s-1$ *be such that* $\langle x_{u+1}\Lambda,\, \alpha_j^{\vee}\rangle > 0$ *and* $H_j^{\eta}(\sigma_u) \notin \mathbb{Z}$. *Let*

$$x_u = w_0 \xleftarrow{\beta_1} w_1 \xleftarrow{\beta_2} w_2 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} w_n = x_{u+1} \tag{4.1.4}$$

be a directed σ_u*-path from* x_{u+1} *to* x_u. *Then,* $\langle w_k\Lambda,\, \alpha_j^{\vee}\rangle > 0$ *for all* $0 \le k \le n$, *and there exists a directed* σ_u*-path from* $\lfloor s_j x_{u+1} \rfloor$ *to* $\lfloor s_j x_u \rfloor$ *of the form:*

$$\lfloor s_j x_u \rfloor = \lfloor s_j w_0 \rfloor \xleftarrow{z_1\beta_1} \lfloor s_j w_1 \rfloor \xleftarrow{z_2\beta_2} \cdots \xleftarrow{z_n\beta_n} \lfloor s_j w_n \rfloor = \lfloor s_j x_{u+1} \rfloor. \tag{4.1.5}$$

Here, if $j \in I_0$*, then we define* $z_k \in W_J$ *to be the identity element for all* $1 \le k \le n$*; if* $j = 0$*, then we define* $z_k \in W_J$ *by* $r_\theta w_k = \lfloor r_\theta w_k \rfloor z_k$ *for each* $1 \le k \le n$*. Moreover, if* (4.1.4) *is a shortest directed path from* x_{u+1} *to* x_u*, i.e.,* $\ell(x_{u+1}, x_u) = n$*, then* (4.1.5) *is a shortest directed path from* $\lfloor s_j x_{u+1} \rfloor$ *to* $\lfloor s_j x_u \rfloor$*, i.e.,* $\ell(\lfloor s_j x_{u+1} \rfloor, \lfloor s_j x_u \rfloor) = n$*.*

Proof. It follows from Lemma 4.1.10 that if $H^\eta_j(\sigma_u) \notin \mathbb{Z}$, then $\langle w_k \Lambda, \alpha_j^\vee \rangle > 0$ for all $0 \le k \le n$ (in particular, $\langle x_u \Lambda, \alpha_j^\vee \rangle > 0$). Assume that $j \in I_0$ (resp., $j = 0$), and suppose, for a contradiction, that $w_k \beta_k = \pm \alpha_j$ (resp., $= \pm\theta$) for some $1 \le k \le n$. Then, $w_{k-1}\Lambda = w_k r_{\beta_k} \Lambda = r_{w_k \beta_k} w_k \Lambda = s_j w_k \Lambda$, and hence $\langle w_{k-1}\Lambda, \alpha_j^\vee \rangle = \langle s_j w_k \Lambda, \alpha_j^\vee \rangle = -\langle w_k \Lambda, \alpha_j^\vee \rangle$, which contradicts the fact that $\langle w_{k-1}\Lambda, \alpha_j^\vee \rangle > 0$ and $\langle w_k \Lambda, \alpha_j^\vee \rangle > 0$. Thus, we conclude that $w_k \beta_k \ne \pm\alpha_j$ (resp., $\ne \pm\theta$) for any $1 \le k \le n$. Therefore, we deduce from Lemma 4.1.4 (1) (resp., Lemma 4.1.5 (1)) that there exists a directed path of the form (4.1.5) from $\lfloor s_j x_{u+1} \rfloor$ to $\lfloor s_j x_u \rfloor$. Because the directed path (4.1.4) is a directed σ_u-path, we have $\sigma_u \langle \Lambda, \beta_k^\vee \rangle \in \mathbb{Z}$. Also, it follows immediately that $\sigma_u \langle \Lambda, z\beta_k^\vee \rangle = \sigma_u \langle \Lambda, \beta_k^\vee \rangle \in \mathbb{Z}$ since $z \in W_J$. Hence the directed path (4.1.5) is a directed σ_u-path from $\lfloor s_j x_{u+1} \rfloor$ to $\lfloor s_j x_u \rfloor$.

Now, we assume that $\ell(x_{u+1}, x_u) = n$, and suppose, for a contradiction, that there exists a directed path

$$\lfloor s_j x_u \rfloor = z_0 \xleftarrow{\gamma_1} z_1 \xleftarrow{\gamma_2} z_2 \xleftarrow{\gamma_3} \cdots \xleftarrow{\gamma_l} z_l = \lfloor s_j x_{u+1} \rfloor \tag{4.1.6}$$

from $\lfloor s_j x_{u+1} \rfloor$ to $\lfloor s_j x_u \rfloor$ whose length l is less than n. Let us show that $\lfloor s_j x_{u+1} \rfloor \xleftarrow{\gamma} x_{u+1}$ for some $\gamma \in \Delta_0^+ \setminus \Delta_J^+$. Assume first that $j \in I_0$. Since $\langle x_{u+1}\Lambda, \alpha_j^\vee \rangle > 0$, we have $\gamma := x_{u+1}^{-1}\alpha_j \in \Delta_0^+ \setminus \Delta_J^+$, and hence $\ell(s_j x_{u+1}) = \ell(x_{u+1}) + 1$. Also, by Lemma 4.1.3, $s_j x_{u+1} \in W_0^J$. Since $s_j x_{u+1} = x_{u+1} r_\gamma$, we obtain $\lfloor s_j x_{u+1} \rfloor = s_j x_{u+1} \xleftarrow{\gamma} x_{u+1}$. Assume next that $j = 0$. Since $\langle x_{u+1}\Lambda, \theta^\vee \rangle = -\langle x_{u+1}\Lambda, \alpha_0^\vee \rangle < 0$ by the assumption, it follows that $x_{u+1}^{-1}\theta \in \Delta_0^- \setminus \Delta_J^-$. Therefore, if we set $\gamma := -x_{u+1}^{-1}\theta \in \Delta_0^+ \setminus \Delta_J^+$, then $s_0 x_{u+1} = r_\theta x_{u+1} = x_{u+1} r_\gamma$, and we obtain $\lfloor s_0 x_{u+1} \rfloor \xleftarrow{\gamma} x_{u+1}$ by Lemma 4.1.2. By concatenating the directed path (4.1.6) and $\lfloor s_j x_{u+1} \rfloor \xleftarrow{\gamma} x_{u+1}$, we obtain a directed path from x_{u+1} to $\lfloor s_j x_u \rfloor$ whose length is $l+1$. Since $\langle x_{u+1}\Lambda, \alpha_j^\vee \rangle > 0$ and $\langle s_j x_u \Lambda, \alpha_j^\vee \rangle = -\langle x_u \Lambda, \alpha_j^\vee \rangle < 0$, we deduce from Lemma 4.1.8 (1) that there exists a directed path from x_{u+1} to $\lfloor s_j \lfloor s_j x_u \rfloor \rfloor = x_u$ whose length is $(l+1) - 1 = l$. However, this contradicts the fact that $n = \ell(x_{u+1}, x_u)$ since $l < n$. This proves the lemma. Q.E.D.

4.2. Explicit description of the image of a quantum LS path under the action of root operators.

In the course of the proof of the following proposition, we obtain an explicit description of the image of a quantum LS path as a rational path under the action of root operators; this description is similar to the one given in [L1].

Proposition 4.2.1. *Both of the sets* $\widetilde{\mathbb{B}}(\lambda) \cup \{\mathbf{0}\}$ *and* $\widehat{\mathbb{B}}(\lambda) \cup \{\mathbf{0}\}$ *are stable under the action of the root operators* f_j *for all* $j \in I$*.*

Proof. Fix $j \in I$. Let $\eta = (x_1, x_2, \dots, x_s\,; \sigma_0, \sigma_1, \dots, \sigma_s) \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, and assume that $f_j\eta \neq \mathbf{0}$. It follows that the point $t_0 = \max\bigl\{t \in [0,1] \mid H_j^\eta(t) = m_j^\eta\bigr\}$ is equal to σ_u for some $0 \le u < s$. Let $u \le m < s$ be such that $\sigma_m < t_1 \le \sigma_{m+1}$; recall that $t_1 = \min\bigl\{t \in [t_0, 1] \mid H_j^\eta(t) = m_j^\eta + 1\bigr\}$. Note that the function $H_j^\eta(t)$ is strictly increasing on $[t_0, t_1]$, which implies that $\langle x_p\Lambda, \alpha_j^\vee\rangle > 0$ for all $u+1 \le p \le m+1$.

Case 1. Assume that $x_u \neq \lfloor s_j x_{u+1} \rfloor$ or $u = 0$, and that $\sigma_m < t_1 < \sigma_{m+1}$. Then we deduce from the definition of the root operator f_j (for the case $j = 0$, see also Remark 2.2.2; cf. [L2, Proposition 4.7 a)]) that

$$\begin{aligned} f_j\eta = (x_1, x_2, \dots, x_u, \lfloor s_j x_{u+1} \rfloor, \dots, \\ \lfloor s_j x_m \rfloor, \lfloor s_j x_{m+1} \rfloor, x_{m+1}, x_{m+2}, \dots, x_s\,; \\ \sigma_0, \sigma_1, \dots, \sigma_u, \dots, \sigma_m, t_1, \sigma_{m+1}, \dots, \sigma_s); \end{aligned}$$

note that $\lfloor s_j x_p \rfloor \neq \lfloor s_j x_{p+1} \rfloor$ for any $u+1 \le p \le m$, and that $\lfloor s_j x_{m+1} \rfloor \neq x_{m+1}$ since $\langle x_{m+1}\Lambda, \alpha_j^\vee\rangle > 0$ as mentioned above. In order to prove that $f_j\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, we need to verify that

(i) there exists a directed σ_u-path from $\lfloor s_j x_{u+1} \rfloor$ to x_u (when $u > 0$);
(ii) there exists a directed σ_p-path from $\lfloor s_j x_{p+1} \rfloor$ to $\lfloor s_j x_p \rfloor$ for each $u+1 \le p \le m$;
(iii) there exists a directed t_1-path from x_{m+1} to $\lfloor s_j x_{m+1} \rfloor$.

Also, we will show that if $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, then the directed paths in (i)–(iii) above can be chosen from the shortest ones, which implies that $f_j\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$.

(i) We deduce from the definition of $t_0 = \sigma_u$ that $\langle x_u\Lambda, \alpha_j^\vee\rangle \le 0$ and $\langle x_{u+1}\Lambda, \alpha_j^\vee\rangle > 0$. Since $\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, there exists a directed σ_u-path from x_{u+1} to x_u. Hence it follows from Lemma 4.1.9 (1), (3) that there exists a directed σ_u-path from $\lfloor s_j x_{u+1} \rfloor$ to x_u. Furthermore, we see from the definition of $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ and Lemma 4.1.9 (2) that if $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, then there exists a directed σ_u-path from $\lfloor s_j x_{u+1} \rfloor$ to x_u whose length is equal to $\ell(\lfloor s_j x_{u+1} \rfloor,\, x_u)$.

(ii) Recall that $H^\eta_j(t)$ is strictly increasing on $[t_0, t_1]$, and that $H^\eta_j(t_0) = m^\eta_j$ and $H^\eta_j(t_1) = m^\eta_j + 1$. Hence it follows that $H^\eta_j(\sigma_p) \notin \mathbb{Z}$ for any $u+1 \le p \le m$. Therefore, we deduce from Lemma 4.1.13 that there exists a directed σ_p-path from $\lfloor s_j x_{p+1} \rfloor$ to $\lfloor s_j x_p \rfloor$ for each $u+1 \le p \le m$. Furthermore, we see from the definition of $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ and Lemma 4.1.13 that if $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, then for each $u+1 \le p \le m$, there exists a directed σ_p-path from $\lfloor s_j x_{p+1} \rfloor$ to $\lfloor s_j x_p \rfloor$ whose length is equal to $\ell(\lfloor s_j x_{p+1} \rfloor,\, \lfloor s_j x_p \rfloor)$.

(iii) Since $\langle x_{m+1}\Lambda,\, \alpha_j^\vee \rangle > 0$, by the same argument as in the second paragraph of the proof of Lemma 4.1.13, we obtain $\lfloor s_j x_{m+1} \rfloor \xleftarrow{\gamma} x_{m+1}$, with

$$\gamma := \begin{cases} x_{m+1}^{-1}\alpha_j & \text{if } j \in I_0, \\ x_{m+1}^{-1}(-\theta) & \text{if } j = 0; \end{cases}$$

note that the directed path $\lfloor s_j x_{m+1} \rfloor \xleftarrow{\gamma} x_{m+1}$ is obviously shortest since its length is equal to 1. Let us show that $t_1 \langle \Lambda,\, \gamma^\vee \rangle \in \mathbb{Z}$. It is easily checked that $\langle \Lambda,\, \gamma^\vee \rangle = \langle x_{m+1}\Lambda,\, \alpha_j^\vee \rangle$. Also, we have $\eta(t_1) = t_1 x_{m+1}\Lambda + \sum_{k=1}^{m} \sigma_k (x_k\Lambda - x_{k+1}\Lambda)$, and hence

$$\mathbb{Z} \ni m^\eta_j + 1 = H^\eta_j(t_1) = t_1 \langle x_{m+1}\Lambda,\, \alpha_j^\vee \rangle + \sum_{k=1}^{m} \langle \sigma_k (x_k\Lambda - x_{k+1}\Lambda),\, \alpha_j^\vee \rangle.$$

Since $\sigma_k(x_k\Lambda - x_{k+1}\Lambda) \in Q_0$ for each $1 \le k \le m$ by Lemma 4.1.6, it follows from the equation above that $t_1 \langle x_{m+1}\Lambda,\, \alpha_j^\vee \rangle \in \mathbb{Z}$, and hence $t_1 \langle \Lambda,\, \gamma^\vee \rangle \in \mathbb{Z}$. Thus, we have verified that there exists a directed t_1-path from x_{m+1} to $\lfloor s_j x_{m+1} \rfloor$ whose length is equal to $\ell(x_{m+1},\, \lfloor s_j x_{m+1} \rfloor) = 1$.

Combining these, we conclude that $f_j\eta$ is an element of $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, and that if $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, then $f_j\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$.

Case 2. Assume that $x_u \ne \lfloor s_j x_{u+1} \rfloor$ or $u = 0$, and that $t_1 = \sigma_{m+1}$. Then we deduce from the definition of the root operator f_j (for the case $j = 0$, see also Remark 2.2.2; cf. [L2, Proposition 4.7 a) and Remark 4.8]) that

$$\begin{aligned} f_j\eta = (x_1,\, x_2,\, \dots,\, x_u,\, &\lfloor s_j x_{u+1} \rfloor,\, \dots, \\ &\lfloor s_j x_m \rfloor,\, \lfloor s_j x_{m+1} \rfloor,\, x_{m+2},\, \dots,\, x_s\,; \\ &\sigma_0,\, \sigma_1,\, \dots,\, \sigma_u,\, \dots,\, \sigma_m,\, t_1 = \sigma_{m+1},\, \dots,\, \sigma_s). \end{aligned}$$

First, we observe that $\langle x_{m+2}\Lambda,\, \alpha_j^\vee \rangle \ge 0$. Indeed, suppose, contrary to our claim, that $\langle x_{m+2}\Lambda,\, \alpha_j^\vee \rangle < 0$. Since $H^\eta_j(\sigma_{m+1}) = H^\eta_j(t_1) = m^\eta_j + 1$, it follows immediately that $H^\eta_j(\sigma_{m+1} + \epsilon) < m^\eta_j + 1$ for sufficiently small

$\epsilon > 0$, and hence that the minimum M of the function $H_j^\eta(t)$ on $[t_1,\, 1]$ is (strictly) less than $m_j^\eta + 1$. Here we recall from Proposition 4.1.12 that all local minima of the function $H_j^\eta(t)$ are integers. Hence we deduce that $M = m_j^\eta$, which contradicts the definition of t_0. Thus, we obtain $\langle x_{m+2}\Lambda,\, \alpha_j^\vee \rangle \geq 0$. Since $\langle x_{m+1}\Lambda,\, \alpha_j^\vee \rangle > 0$, and hence $\langle s_j x_{m+1}\Lambda,\, \alpha_j^\vee \rangle < 0$, it follows that $\lfloor s_j x_{m+1} \rfloor \neq x_{m+2}$.

Now, in order to prove that $f_j\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, we need to verify that

(i) there exists a directed σ_u-path from $\lfloor s_j x_{u+1} \rfloor$ to x_u (when $u > 0$);

(ii) there exists a directed σ_p-path from $\lfloor s_j x_{p+1} \rfloor$ to $\lfloor s_j x_p \rfloor$ for each $u + 1 \leq p \leq m$;

(iv) there exists a directed σ_{m+1}-path from x_{m+2} to $\lfloor s_j x_{m+1} \rfloor$ (when $m + 1 < s$).

We can verify (i) and (ii) by the same argument as for (i) and (ii) in Case 1, respectively. Hence it remains to show (iv). Also, in order to prove that $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ implies $f_j\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, it suffices to check that the directed paths in (i), (ii), and (iv) above can be chosen from the shortest ones. We can show this claim for (i) and (ii) in the same way as for (i) and (ii) in Case 1, respectively. So, it remains to show it for (iv).

(iv) As in the proof of (iii) in Case 1, it can be shown that there exists a directed t_1-path (and hence directed σ_{m+1}-path since $t_1 = \sigma_{m+1}$ by the assumption) from x_{m+1} to $\lfloor s_j x_{m+1} \rfloor$ whose length is equal to 1. Also, it follows from the definition that there exists a directed σ_{m+1}-path from x_{m+2} to x_{m+1}. Concatenating these directed σ_{m+1}-paths, we obtain a directed σ_{m+1}-path from x_{m+2} to $\lfloor s_j x_{m+1} \rfloor$. Thus, we have proved that $f_j\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$.

Assume now that $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, and set $n := \ell(x_{m+2},\, x_{m+1})$. We see from the argument above that there exists a directed σ_{m+1}-path from x_{m+2} to $\lfloor s_j x_{m+1} \rfloor$ whose length is equal to $n + 1$. Suppose, for a contradiction, that there exists a directed path from x_{m+2} to $\lfloor s_j x_{m+1} \rfloor$ whose length l is less than $n+1$. Since $\langle s_j x_{m+1}\Lambda,\, \alpha_j^\vee \rangle < 0$ and $\langle x_{m+2}\Lambda,\, \alpha_j^\vee \rangle \geq 0$ as seen above, we deduce from Lemma 4.1.8 that there exists a directed path from x_{m+2} to $\lfloor s_j \lfloor s_j x_{m+1} \rfloor \rfloor = \lfloor x_{m+1} \rfloor = x_{m+1}$ whose length is equal to $l-1 < n$, which contradicts $n = \ell(x_{m+2},\, x_{m+1})$. Thus, we have proved that if $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, then $f_j\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$.

Case 3. Assume that $x_u = \lfloor s_j x_{u+1} \rfloor$ and $\sigma_m < t_1 < \sigma_{m+1}$. Then we deduce from the definition of the root operator f_j (for the case $j = 0$, see also Remark 2.2.2; cf. [L2, Proposition 4.7 a) and Remark 4.8]) that

$$f_j\eta = (x_1,\, x_2,\, \ldots,\, x_u = \lfloor s_j x_{u+1} \rfloor,\, \lfloor s_j x_{u+2} \rfloor,\, \ldots,$$

$$\lfloor s_j x_m \rfloor,\ \lfloor s_j x_{m+1} \rfloor,\ x_{m+1},\ x_{m+2},\ \ldots,\ x_s\,;$$
$$\sigma_0,\ \sigma_1,\ \ldots,\ \sigma_{u-1},\ \sigma_{u+1},\ \ldots,\ \sigma_m,\ t_1,\ \sigma_{m+1},\ \ldots,\ \sigma_s);$$

note that $\lfloor s_j x_{m+1} \rfloor \neq x_{m+1}$ since $\langle x_{m+1}\Lambda,\, \alpha_j^{\vee} \rangle > 0$. In order to prove that $f_j\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, we need to verify that

(ii) there exists a directed σ_p-path from $\lfloor s_j x_{p+1} \rfloor$ to $\lfloor s_j x_p \rfloor$ for each $u+1 \leq p \leq m$;

(iii) there exists a directed t_1-path from x_{m+1} to $\lfloor s_j x_{m+1} \rfloor$.

We can verify (ii) and (iii) by the same argument as for (ii) and (iii) in Case 1, respectively. Also, in the same way as in the proofs of (ii) and (iii) in Case 1, respectively, we can check that if $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, then the directed paths in (ii) and (iii) above can be chosen from the shortest ones. Thus we have proved that $f_j\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, and that $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ implies $f_j\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$.

Case 4. Assume that $x_u = \lfloor s_j x_{u+1} \rfloor$ and $t_1 = \sigma_{m+1}$. Then we deduce from the definition of the root operator f_j (for the case $j = 0$, see also Remark 2.2.2; cf. [L2, Proposition 4.7 a) and Remark 4.8]) that

$$f_j\eta = (x_1,\ x_2,\ \ldots,\ x_u = \lfloor s_j x_{u+1} \rfloor,\ \lfloor s_j x_{u+2} \rfloor,\ \ldots,$$
$$\lfloor s_j x_m \rfloor,\ \lfloor s_j x_{m+1} \rfloor,\ x_{m+2},\ \ldots,\ x_s\,;$$
$$\sigma_0,\ \sigma_1,\ \ldots,\ \sigma_{u-1},\ \sigma_{u+1},\ \ldots,\ \sigma_m,\ t_1 = \sigma_{m+1},\ \ldots,\ \sigma_s);$$

note that $\lfloor s_j x_{m+1} \rfloor \neq x_{m+2}$ since $\langle s_j x_{m+1}\Lambda,\, \alpha_j^{\vee} \rangle < 0$ and $\langle x_{m+2}\Lambda,\, \alpha_j^{\vee} \rangle \geq 0$ (see Case 2 above). In order to prove that $f_j\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, we need to verify that

(ii) there exists a directed σ_p-path from $\lfloor s_j x_{p+1} \rfloor$ to $\lfloor s_j x_p \rfloor$ for each $u+1 \leq p \leq m$;

(iv) there exists a directed σ_{m+1}-path from x_{m+2} to $\lfloor s_j x_{m+1} \rfloor$ (when $m+1 < s$).

We can verify (ii) and (iv) by the same argument as for (ii) in Case 1 and (iv) in Case 2, respectively. Also, as in the proofs of (ii) in Case 1 and (iv) in Case 2, respectively, we can check that if $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, then the directed paths in (ii) and (iv) above can be chosen from the shortest ones. Thus we have proved that $f_j\eta \in \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$, and that $\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ implies $f_j\eta \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$.

This completes the proof of Proposition 4.2.1. Q.E.D.

Combining Theorem 2.4.1 with Propositions 4.1.12 and 4.2.1, we obtain Theorem 4.1.1.

References

[AK] T. Akasaka and M. Kashiwara, Finite-dimensional representations of quantum affine algebras, *Publ. Res. Inst. Math. Sci.* **33** (1997), 839–867.

[BB] A. Björner and F. Brenti, "Combinatorics of Coxeter Groups", Graduate Texts in Mathematics Vol. 231, Springer, New York, 2005.

[BFP] F. Brenti, S. Fomin, and A. Postnikov, Mixed Bruhat operators and Yang-Baxter equations for Weyl groups, *Int. Math. Res. Not.* **1999** (1999), no. 8, 419–441.

[HK] J. Hong and S.-J. Kang, "Introduction to Quantum Groups and Crystal Bases", Graduate Studies in Mathematics Vol. 42, Amer. Math. Soc., Providence, RI, 2002.

[Kac] V. G. Kac, "Infinite Dimensional Lie Algebras", 3rd Edition, Cambridge University Press, Cambridge, UK, 1990.

[Kas1] M. Kashiwara, On crystal bases, *in* "Representations of Groups" (B.N. Allison and G.H. Cliff, Eds.), CMS Conf. Proc. Vol. 16, pp. 155–197, Amer. Math. Soc., Providence, RI, 1995.

[Kas2] M. Kashiwara, On level-zero representations of quantized affine algebras, *Duke Math. J.* **112** (2002), 117–175.

[LS] T. Lam and M. Shimozono, Quantum cohomology of G/P and homology of affine Grassmannian, *Acta Math.* **204** (2010), 49–90.

[LNSSS1] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono, A uniform model for Kirillov-Reshetikhin crystals I: Lifting the parabolic quantum Bruhat graph, *Int. Math. Res. Not.* **2015** (2015), no. 7, 1848–1901.

[LNSSS2] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono, A uniform model for Kirillov-Reshetikhin crystals Ⅱ: Alcove model, path model, and $P = X$, to appear in *Int. Math. Res. Not.*, doi:10.1093/imrn/rnw 129, arXiv:1402.2203.

[L1] P. Littelmann, A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras, *Invent. Math.* **116** (1994), 329–346.

[L2] P. Littelmann, Paths and root operators in representation theory, *Ann. of Math.* (2) **142** (1995), 499–525.

[NS1] S. Naito and D. Sagaki, Path model for a level-zero extremal weight module over a quantum affine algebra, *Int. Math. Res. Not.* **2003** (2003), no. 32, 1731–1754.

[NS2] S. Naito and D. Sagaki, Crystal of Lakshmibai-Seshadri paths associated to an integral weight of level zero for an affine Lie algebra, *Int. Math. Res. Not.* **2005** (2005), no. 14, 815–840.

[NS3] S. Naito and D. Sagaki, Path model for a level-zero extremal weight module over a quantum affine algebra. Ⅱ, *Adv. Math.* **200** (2006), 102–124.

[NS4] S. Naito and D. Sagaki, Crystal structure on the set of Lakshmibai-Seshadri paths of an arbitrary level-zero shape, *Proc. Lond. Math. Soc.* (3) **96** (2008), 582–622.

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