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K3 surfaces of genus sixteen

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Abstract.

The generic polarized $K3$ surface (S, h) of genus 16, that is, $(h^2) = 30$, is described in a certain compactified moduli space $\mathcal{T}$ of twisted cubics in $\mathbb{P}^3$, as a complete intersection with respect to an almost homogeneous vector bundle of rank 10. As corollary we prove the unirationality of the moduli space $\mathcal{F}_{16}$ of such K3 surfaces.

§1. Introduction

Let $\mathcal{F}_g$ be the moduli space of quasi-polarized $K3$ surface (S, h) of genus g, *i.e.*, $(h^2) = 2g - 2$. $\mathcal{F}_g$ is an arithmetic quotient of the 19-dimensional bounded symmetric domain of type IV, and a quasi-projective variety. It is shown in [3] that $\mathcal{F}_g$ is of general type for $g \geq 63$ but the birational classification of $\mathcal{F}_g$ is still far from being complete. For $g \leq 10$ and $g = 12, 13, 18, 20$, the generic (S, h) is a complete intersection in a suitable homogeneous space with respect to a suitable homogeneous vector bundle. As corollary the unirationality of $\mathcal{F}_g$ is proved for those values of g in [5, 7, 8]. In this article we shall describe the generic member of $\mathcal{F}_{16}$ using the EPS moduli space $\mathcal{T} := G(2, 3; \mathbb{C}^4)$ of twisted cubics in $\mathbb{P}^3$.

The EPS moduli space $\mathcal{T}$ is constructed by Ellingsrud-Piene-Strømme [2] as the GIT quotient of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes V$, V being a 4-dimensional vector space, by the obvious action of $GL(2) \times GL(3)$. $\mathcal{T}$ is a smooth equivariant compactification of the 12-dimensional homogeneous space $PGL(V)/PGL(2)$. A point $t \in \mathcal{T}$ represents an equivalence class of 2×3 matrices whose entries belong to V. Its three minors define a subscheme R_t of the projective space $\mathbb{P}(V)$. R_t is a cubic curve mostly and a plane with an embedded point in exceptional cases. By

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construction there exists two natural vector bundles $\mathcal{E}, \mathcal{F}$ of rank 3, 2, respectively, with $\det \mathcal{E} \simeq \det \mathcal{F}$, and the tautological homomorphism

$$\mathcal{E} \otimes V^\vee \longrightarrow \mathcal{F}$$

on $\mathcal{T}$. (Be cautioned that the same letters $\mathcal{E}$ and $\mathcal{F}$ are used for the dual vector bundles in [2].) The tautological homomorphism induces linear maps

$$(S^2V)^\vee \longrightarrow H^0(\mathcal{E}) \quad \text{and} \quad (S^{2,1}V)^\vee \longrightarrow H^0(\mathcal{F}). \tag{1}$$

(See §2.) Here S^2V is the second symmetric tensor product, and

$$S^{2,1}V = \ker[V \otimes S^2V \to S^3V] \tag{2}$$

is the space of linear syzygies among second symmetric tensors. $S^{2,1}V$ is of dimension 20.

For two subspaces $M \subset (S^2V)^\vee$ and $N \subset (S^{2,1}V)^\vee$, we consider the common zero locus

$$\bigcap_{s \in \bar{M}} (s)_0 \cap \bigcap_{t \in \bar{N}} (t)_0 \subset \mathcal{T}. \tag{3}$$

of global sections $s \in \bar{M}$ and $t \in \bar{N}$, where $\bar{M}$ and $\bar{N}$ are the images of M and N in $H^0(\mathcal{E})$ and $H^0(\mathcal{F})$, respectively. The case $\dim M = \dim N = 2$ is most interesting. We denote the common zero locus (3) by $S_{M,N}$ in this case.

Theorem 1.1. *If M and N are general, then $S_{M,N}$ is a (smooth) $K3$ surface, and the restriction of $H := c_1(\mathcal{E})$ is a polarization of genus* 16.

For general M and N, $S_{M,N}$ is a complete intersection in $\mathcal{T}$ with respect to the vector bundle $\mathcal{E}^{\oplus 2} \oplus \mathcal{F}^{\oplus 2}$ of rank 10. Furthermore the following converse also holds:

Theorem 1.2. *Generic $K3$ surface of genus* 16 *is isomorphic to the complete intersection $S_{M,N}$.*

A twisted cubic

$$R : \text{rank} \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{pmatrix} \leq 1, \quad f_{ij} \in V \tag{4}$$

in $\mathbb{P}(V) = \mathbb{P}^3$ is *apolar to M* if all minors of the matrix are perpendicular to M. Similarly R is *apolar to N* if all linear syzygies among the three minors are perpendicular to N. The K3 surface $S_{M,N}$ in Theorem 1.1 parametrizes all R which are apolar to both M and N.

If a 2-dimensional subspace $M \subset (S^2V)^\vee$ is general, then the kernel Syz_M of the multiplication map $V \otimes M^\perp \to S^3V$ is of dimension 12, where $M^\perp \subset S^2V$ is the space of quadratic forms apolar to M. Hence the totality of $S_{M,N}$ are parametrized by an open subset of a generic $G(2,12)$-bundle $\mathcal{P}$ over the 16-dimensional Grassmannian $G(2,(S^2V)^\vee)$ which parametrizes M. By Theorem 1.1, we have the rational map

$$\Psi_{16} : \mathcal{P} \cdots \to \mathcal{F}_{16}, \quad (M, \bar{N}) \mapsto (S_{M,N}, H|_S), \tag{5}$$

whose dominance is Theorem 1.2, where $\bar{N} \subset (Syz_M)^\vee$ is the image of N by the linear map $S^{2,1}V^\vee \to (Syz_M)^\vee$. Therefore, as bi-product, we have

Corollary *The moduli space $\mathcal{F}_{16}$ of polarized $K3$ surface of genus 16 is unirational.*

In order to prove the theorems, we study a certain special case in detail. More explicitly, we consider the space $M_0 \subset (S^2V)^\vee \simeq S^2(V^\vee)$ spanned by two *reducible* quadratic forms $q_1 = XY, q_2 = ZT$, and study the common zero locus $\mathcal{T}_{M_0} := (q_1)_0 \cap (q_2)_0$ of M_0 in $\mathcal{T}$. $\mathcal{T}_{M_0}$ parametrizes all twisted cubics whose defining quadratic forms do not contain the term xy or zt, where $(x : y : z : t)$ is a homogeneous coordinate $\mathbb{P}^3$ and $(X : Y : Z : T)$ is the dual coordinate of $\mathbb{P}^{3,*}$.

If N is general, then $S_{M_0,N}$ is a quartic surface in $\mathbb{P}^3$ which contains two quintic elliptic curves E_1 and E_2 with $(E_1.E_2) = 3$. In particular we have Theorem 1.1. Moreover, the restriction of $\mathcal{E}$ to $S_{M_0,N}$ is an extension of three line bundles $\mathcal{O}_S(E_1), \mathcal{O}_S(E_2)$ and $\mathcal{O}_S(H - E_1 - E_2)$ ((21) in §6), and the restriction of $\mathcal{F}$ contains $\mathcal{O}_S(E_1)^{\oplus 2}$ as a subsheaf (Proposition 3.1). These give us the following vanishing of higher cohomology groups which is the key of the proof of Theorem 1.2.

Proposition 1.3. *If both M and N are general, then the restrictions of $\mathcal{E}, \mathcal{F}$ to $S := S_{M,N}$ are simple and satisfy*

$$\mathrm{Ext}^i(\mathcal{E}|_S, \mathcal{F}|_S) = H^i(S, \mathcal{E}|_S) = H^i(S, \mathcal{F}|_S) = 0, \text{ for all } \ i > 0.$$

After preparing some basic facts on the EPS moduli space $\mathcal{T} = G(2,3;\mathbb{C}^4)$ in §2, we first study the locus $\mathcal{T}_Q$ of twisted cubics apolar to one reducible quadric in §3. We next study the locus $\mathcal{T}_{B_1,B_2}$ of twisted cubics which have two skew lines as their bisecants in §4 and the above $\mathcal{T}_{M_0}$ in §5. We prove Theorem 1.1 at the end of §6 and Theorem 1.2 in §7 using *doubly octagonal* K3 surfaces $S_{M_0,N}$. The final §8 is logically unnecessary but explains how Theorem 1.2 originates from the description of Fano 3-fold of genus 12 in [6].

Notations and convention All varieties are considered over the complex number field $\mathbb{C}$. The projective space $\mathbb{P}(V)$ associated to a vector space V is that in Grothendieck's sense. The Grassmann variety of s-dimensional subspaces of V is denoted by $G(s, V)$. The isomorphism class of $G(s, V)$ is denoted by $G(s, n)$ when $\dim V = n$. The dual vector space (and more generally the dual vector bundle) is denoted by $V^\vee$. *Twisted cubic* is used in the generalized sense of [2]. But the locus where twisted cubics are not curves is of sufficiently large codimension, and hence is never crucial in our argument.

§2. Pair of vector bundles whose ranks differ by one

Let (E, F) be a pair of vector bundles on a scheme S such that

$$\det E \simeq \det F, \quad \operatorname{rank} E = \operatorname{rank} F + 1. \tag{6}$$

Let r be the rank of F. r homomorphisms $f_1, \dots, f_r \in \operatorname{Hom}(E, F)$ give rise the homomorphism

$$f_1 \wedge \cdots \wedge f_r : \wedge^r E \to \wedge^r F \simeq \det F$$

which can be regarded as a global section of E by our assumption (6). Since $f_1 \wedge \cdots \wedge f_r$ is symmetric with respect to $f_1, \dots, f_r$, we have a linear map

$$S^r \operatorname{Hom}(E, F) \to \operatorname{Hom}(\wedge^r E, \wedge^r F) \simeq H^0(E). \tag{7}$$

If $g : E \to F$ is a homomorphism, then $g(f_1 \wedge \cdots \wedge f_r)$ is a global section of F. Hence we have another linear map

$$S^r \operatorname{Hom}(E, F) \otimes \operatorname{Hom}(E, F) \to H^0(F), \quad ((f_1, \dots, f_r), g) \mapsto g(f_1 \wedge \ldots \wedge f_r).$$

Since $S^{r+1}\operatorname{Hom}(E, F)$ lies in the kernel of this linear map, we have

$$S^{r,1}\operatorname{Hom}(E, F) \to H^0(F). \tag{8}$$

Let V be a vector space and let $G(r, r+1; V)$ be the GIT quotient of the tensor product $\mathbb{C}^r \otimes \mathbb{C}^{r+1} \otimes V$ by $GL(r) \times GL(r+1)$. There are two natural vector bundles $\mathcal{E}, \mathcal{F}$ of rank $r+1$, r, respectively, with $\det \mathcal{E} \simeq \det \mathcal{F}$, and the tautological homomorphism

$$\mathcal{E} \otimes V^\vee \to \mathcal{F} \tag{9}$$

on $G(r, r+1; V)$. This has the following universal property.

(*) If a homomorphism $E \otimes V^\vee \to F$ satisfies (6) and if the induced homomorphism $S^r V^\vee \otimes \mathcal{O}_S \to E$ is surjective, then there exists a unique morphism $\Phi : S \to G(r, r+1; V)$ such that $E \otimes V^\vee \to F$ coincides with the pull-back of (9). This Φ will be denoted by $\Phi_{E,F,V^\vee} : S \to G(r, r+1; V)$, or $\Phi_{E,F}$ if $V^\vee = \mathrm{Hom}(E, F)$.

Remark 2.1. If E, F are vector bundles of rank $r+1, r$, respectively. Then, putting $L = (\det E)^{-1} \otimes \det F$, we have

$$\mathrm{Hom}(E, F) \simeq \mathrm{Hom}(E \otimes L, F \otimes L) \quad \text{and} \quad \det(E \otimes L) \simeq \det(F \otimes L).$$

Hence, the assumption (6) is not restrictive.

In the sequel we apply the case $r = 2, \dim V = 4$ to $K3$ surfaces of genus 16. $G(2, 3; V)$ is regarded as a subvariety of the Grassmannian $G(3, S^2 V)$ by $R \mapsto H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2 - R))$, where $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2 - R))$ is the 3-dimensional space of quadratic forms vanishing on R. $G(2, 3; V)$ is also a subvariety of another Grassmannian $G(2, S^{2,1} V)$ by $R \mapsto Syz_R$, where Syz_R is the 2-dimensional space of linear syzygies among $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2 - R))$.

Let $S_{M,N} \subset \mathcal{T} = G(2, 3; V)$ be as in the introduction for general 2-dimensional subspaces M and N.

Proposition 2.2. 1) *$S_{M,N}$ is the disjoint union of $K3$ surfaces and abelian surfaces.*

2) *The degree of $S_{M,N}$ with respect to $H := c_1(\mathcal{E})$ is equal to* 30.

3) *The second Chern number of the restrictions of $\mathcal{E}$ and $\mathcal{F}$ to $S_{M,N}$ are equal to* 13 *and* 9, *respectively.*

Proof. 1) The vector bundles $\mathcal{E}$ and $\mathcal{F}$ are generated by the global sections coming from $S^2 V^\vee$ and $S^{2,1} V^\vee$, respectively. Hence by the Bertini type theorem (see Remark 2.4 below), the general complete intersection $S_{M,N}$ is smooth of expected dimension, which is equal to $\dim \mathcal{T} - 2 \cdot \mathrm{rank}\, \mathcal{E} - 2 \cdot \mathrm{rank}\, \mathcal{F} = 2$. The canonical bundle of $S_{M,N}$ is trivial by the adjunction formula [7, (1.5)] since $c_1(\mathcal{T}) = 4H$.

2) The degree of $S_{M,N}$ is equal to

$$(H^2 . c_{\mathrm{top}}(\mathcal{E}^{\oplus 2} \oplus \mathcal{F}^{\oplus 2})) = (H^2 . c_3(\mathcal{E})^2 . c_2(\mathcal{F})^2),$$

which is equal to $(c_1^2 c_3^2 d_2^2) = 30$ by [1, Table 1].

3) The Chern numbers are equal to

$$c_2(\mathcal{E}|_S) = (c_2 c_3^2 d_2^2) = 13 \quad \text{and} \quad c_2(\mathcal{F}|_S) = (c_3^2 d_2^3) = 9,$$

respectively, again by [1, Table 1]. Q.E.D.

Remark 2.3. A computation using the description of the tangent bundle of $\mathcal{T}$ in [1, (4.4)] shows that the Euler number of $S_{M,N}$ is equal to 24. This shows that a K3 surface appears in $S_{M,N}$ and it is unique.

Remark 2.4. The Bertini type theorem proved in [7, Theorem 1.10] holds in the following more general form: if a subspace $W \subset H^0(\mathcal{G})$ generates a vector bundle $\mathcal{G}$ of rank r and if a global section $s \in W$ is general, then the scheme of zeroes of s is smooth of codimension r.

§3. Twisted cubics apolar to a reducible quadric

We fix a line l in $\mathbb{P}(V) = \mathbb{P}^3$ and consider the subvariety

$$\mathcal{T}_B := \{R \,|\, \mathrm{length}(R \cap l) \geq 2\} \subset \mathcal{T}$$

consisting of twisted cubics which have l as a bisecant line. Here "B" stands for bisecant. $\mathcal{T}_B$ is a 10-dimensional variety. Assigning the intersection $l \cap R$ to l, we obtain the rational map

$$f_B : \mathcal{T}_B \cdots \to \mathbb{P}^2 = \mathrm{Sym}^2 l. \tag{10}$$

Let $\mathcal{D}$ be the subvariety of $\mathcal{T}$ consisting of reducible twisted cubics. $\mathcal{D}$ is a divisor. Let $\mathcal{D}_B$ be the intersection $\mathcal{D} \cap \mathcal{T}_B$. $\mathcal{D}_B$ decomposes into the union of two irreducible components $\mathcal{D}_{B,1}$ and $\mathcal{D}_{B,2}$ according as the intersection of the conical part of R and l. Every general member R of $\mathcal{D}_{B,2}$ is the union of a line and a conic which meets l at two points. Oppositely every general member R of $\mathcal{D}_{B,1}$ is the union of a line and a conic both of which meet l.

The restriction of the syzygy bundle $\mathcal{F}$ to $\mathcal{T}_B$ is described using this former divisor $\mathcal{D}_{B,2}$.

Proposition 3.1. *The restriction $\mathcal{F}|_{\mathcal{T}_B}$ contains the rank 2 vector bundles $f_B^*\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$ as a subsheaf, and the quotient $(\mathcal{F}|_{\mathcal{T}_B})/(f_B^*\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2})$ is a line bundle on the divisor $\mathcal{D}_{B,2}$.*

Proof. We take a homogeneous coordinate $(x : y : z : t)$ of $\mathbb{P}^3$ and assume that the line l is defined, say, by $x = y = 0$. We describe the syzygy space Syz_R of a twisted cubic R in $\mathcal{T}_B$ using the two quadrics containing the union $R \cup l$.

First we construct a homomorphism $f_B^*\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2} \to \mathcal{F}|_{\mathcal{T}_B}$. Since l is a bisecant of R, the union $R \cup l$ is contained in two different quadrics, say, $cx - ay = 0$ and $dx - by = 0$ with $a, b, c, d \in V = \langle x, y, z, t\rangle_{\mathbb{C}}$. The third quadric containing R is defined by $ad - bc = 0$. Hence R is defined by the three minors of the matrix $\begin{pmatrix} x & a & b \\ y & c & d \end{pmatrix}$. Therefore, the syzygy space Syz_R of R is spanned by

$$x \otimes (ad - bc) - a \otimes (dx - by) + b \otimes (cx - ay) \tag{11}$$

and

$$y \otimes (ad - bc) - c \otimes (dx - by) + d \otimes (cx - ay). \tag{12}$$

When R runs over $\mathcal{T}_B$, these syzygies generate a subspace $Syz_1 \subset S^{2,1}V$ of codimension 2. More precisely, $S^{2,1}V$ has 20 tensors of the form (monomial) $\otimes$ (monomial) $-$ (monomial) $\otimes$ (monomial) as its basis, and Syz_1 is generated by all except $z \otimes zt - t \otimes z^2, t \otimes zt - z \otimes t^2$. The syzygies

$$a \otimes by - b \otimes ay, \quad c \otimes dx - d \otimes cx, \quad a, b, c, d \in V$$

are contained in the vector space Syz_1, and generate a subspace Syz_2 isomorphic to $(\bigwedge^2 V)^{\oplus 2}$. The quotient Syz_1/Syz_2 is canonically isomorphic to $\langle x, y\rangle_{\mathbb{C}} \otimes S^2(V/\langle x, y\rangle_{\mathbb{C}})$. Since the quadric $ad - bc = 0$ cut the two points $f_B(R)$ from l, we have a homomorphism $f_B^* \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2} \to \mathcal{F}|_{\mathcal{T}_B}$ on $\mathcal{T}_B$.

If R does not belong to the divisor $\mathcal{D}_{B,2}$, then the union $R \cup l$ is the intersection of two quadrics $cx - ay = 0$ and $dx - by = 0$. Moreover, the residual classes of (11) and (12) are $x \otimes \overline{ad - bc}$ and $y \otimes \overline{ad - bc}$, respectively. Hence $f_B^* \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2} \to \mathcal{F}|_{\mathcal{T}_B}$ is an isomorphism outside $\mathcal{D}_{B,2}$.

On the contrary assume that $[R] \in \mathcal{D}_{B,2}$. Then the intersection of two quadrics containing $R \cup l$ is the union of a plane containing l, say $x = 0$, and a line. R is defined by the three minors of the matrix of the form $\begin{pmatrix} x & a & b \\ 0 & c & d \end{pmatrix}$, and Syz_R is spanned by

$$x \otimes (ad - bc) - a \otimes dx + b \otimes cx, \tag{13}$$

which is a specialization of (11), and $-c \otimes dx + d \otimes cx \in Syz_2$, a specialization of (12). Since (13) is not contained in the subspace Syz_2, the cokernel of $f_B^* \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2} \hookrightarrow \mathcal{F}|_{\mathcal{T}_B}$ is a line bundle on $\mathcal{D}_{B,2}$. Q.E.D.

Now we study the locus $\mathcal{T}_Q$ of twisted cubics which are apolar to a quadric $Q : q = 0 \subset \mathbb{P}^{3,*}$ when $q \in (S^2V)^\vee \simeq S^2(V^\vee)$ is of rank 2. The quadric Q is the union of two distinct planes P_1 and P_2. Let l be the line joining the two points $[P_1]$ and $[P_2] \in \mathbb{P}^3 = (\mathbb{P}^{3,*})^*$. q is the pull-back of a quadratic form $\bar{q}$ on $l^* \simeq \mathbb{P}^1$ by the projection $\mathbb{P}^{3,*} \cdots \to l^*$. A twisted cubic R is apolar to q if and only if the restriction of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2 - R))$ to l is apolar to $\bar{q}$.

Proposition 3.2. *The following are equivalent to each other.*

1) *A twisted cubic* $R \subset \mathbb{P}^3$ *is apolar to* q.

2) l *is a bisecant line of* R *and the intersection* $R \cap l$ *is apolar to* $\bar{q}$.

Proof. 2) $\Longrightarrow$ 1) If l is a bisecant of R, then the union $R \cup l$ is contained in two distinct quadrics. Hence the restriction map

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2-R)) \to H^0(l, \mathcal{O}_l(2)) \tag{14}$$

is of rank ≤ 1. Hence, if furthermore $R \cap l$ is apolar to $q|_l$, then R is apolar to q.

1) $\Longrightarrow$ 2) Let $W \subset H^0(l, \mathcal{O}_l(2))$ be the space of quadratic forms apolar to $\bar{q}$. If R is apolar to q, then the image of the restriction map (14) is contained in W. Since $\dim W = 2$, the linear map (14) is not injective, that is, the union $R \cup l$ is contained in a quadric. Hence $R \cap l$ is non-empty. Since the quadratic forms in W has no common zero, the rank of (14) is at most one, which shows (2). Q.E.D.

By the proposition, $\mathcal{T}_Q$ is contained in $\mathcal{T}_B$. More precisely, it coincides with the pull-back of a line by the rational map (10). In particular, we have the rational map

$$f_Q : \mathcal{T}_Q \cdots \to \mathbb{P}^1 \subset \mathbb{P}^2 = \mathrm{Sym}^2 l, \quad R \mapsto R \cap l. \tag{15}$$

§4. Twisted cubics with two fixed bisecant lines

We fix a pair of skew lines l_1 and l_2 in $\mathbb{P}(V) = \mathbb{P}^3$ and consider the (8-dimensional) subvariety

$$\mathcal{T}_{B_1,B_2} := \{R \,|\, \mathrm{length}(R \cap l_1) \geq 2,\ \mathrm{length}(R \cap l_2) \geq 2\} \subset \mathcal{T}$$

consisting of twisted cubics which have both l_1 and l_2 as bisecant lines. Restricting (10) we have two rational maps

$$f_{B_i} : \mathcal{T}_{B_1,B_2} \cdots \to \mathbb{P}^2 = \mathrm{Sym}^2 l_i, \quad i = 1, 2. \tag{16}$$

Now we consider the correspondence

$$Y = \{(R, Q) \,|\, R \subset Q\} \subset \mathcal{T}_{B_1,B_2} \times \Lambda \tag{17}$$

between $\mathcal{T}_{B_1,B_2}$ and the linear web $\Lambda := |\mathcal{O}_{\mathbb{P}}(2 - l_1 - l_2)|$ of quadrics Q containing l_1 and l_2. Assume that a twisted cubic R belongs to $\mathcal{T}_{B_1,B_2}$. As we saw in the proof of Poposition 3.2, the restriction maps

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2-R)) \to H^0(l_i, \mathcal{O}_l(2)), \quad i = 1, 2$$

are of rank at most one. Hence there exists a quadric which contains $R \cup l_1 \cup l_2$. Therefore, the first projection $\pi : Y \to \mathcal{T}_{B_1,B_2}$ is surjective. π is not an isomorphism at $[R]$ if and only if $\dim |\mathcal{O}_{\mathbb{P}}(2 - l_1 - l_2 - R)| > 0$.

Proposition 4.1. *The following are equivalent for a twisted cubic* $[R]$ *in* $\mathcal{T}_{B_1,B_2}$.

1) $\dim|\mathcal{O}_{\mathbb{P}}(2 - l_1 - l_2 - R)| > 0$.

2) $R \supset l_1$ *or* $R \supset l_2$.

Proof. 1) $\Rightarrow$ 2) There exist two distinct quadrics Q_1 and Q_2 which contains $C = l_1 \cup l_2 \cup R$. If $\deg C \leq 4$ then 2) follows. Otherwise, we have $\deg C > \deg Q_1 \cdot \deg Q_2$, and Q_1 and Q_2 have a common component. Therefore, the intersection $Q_1 \cap Q_2$ is the union of plane and a line. Hence 2) holds.

2) $\Rightarrow$ 1) If R contains both l_1 and l_2, then 1) is obvious. If $R \supset l_1$ and $R \not\supset l_2$, then $R \cup l_2$ is contained in two distinct quadrics. Hence 1) holds true. Similarly 1) holds in the case where $R \supset l_2$ and $R \not\supset l_1$. Q.E.D.

More explicitly we have the following whose proof is straightforward.

Proposition 4.2. *If a twisted cubic* $[R] \in \mathcal{T}_{B_1,B_2}$ *satisfies the equivalent conditions of the preceding proposition, then it satisfies one of the following:*

(*a*) *R is the union of l_1 and a conic which have l_2 as a bisecant line, or vice versa, or*

(*b*) *R is the union $m_1 \cup m_2 \cup l_i$ of three lines, with $i = 1$ or 2, such that both m_1 and m_2 intersect l_1 and l_2, or*

(*c*) *R is the union $l_1 \cup l_2 \cup m$ of three lines such that m intersects both l_1 and l_2.*

The twisted cubics satisfying (a) are parametrized by open subsets of two $\mathbb{P}^4$-bundles A_1 and A_2 over $\mathbb{P}^1$. More precisely, A_1 is a $\mathbb{P}^4$-bundle over $|\mathcal{O}_{\mathbb{P}}(1 - l_1)| \simeq \mathbb{P}^1$, the pencil of planes P containing l_1, and its fiber over $[P]$ parametrizes the conics in P passing through the intersection point $P \cap l_2$. In particular, both A_1 and A_2 are of dimension 5. The twisted cubics satisfying (c) are parametrized by the intersection $A_1 \cap A_2$, which is isomorphic to $l_1 \times l_2$. The twisted cubics satisfying (b) are parametrized by two copies of $\mathrm{Sym}^2(\mathbb{P}^1 \times \mathbb{P}^1)$. In particular they are 4-dimensional families. Therefore, the first projection $\pi : Y \to \mathcal{T}_{B_1,B_2}$ of (17) is birational and we have the rational map

$$\mathcal{T}_{B_1,B_2} \cdots \to \Lambda \simeq \mathbb{P}^3, \quad R \mapsto Q$$

assigning the unique quadric $Q \in |\mathcal{O}_{\mathbb{P}}(2 - R - l_1 - l_2)|$ to R. The correspondence Y in (17) is nothing but the graph of this rational map.

Proposition 4.3. *Y is an 8-dimensional irreducible variety, and a generic $\mathbb{P}^5$-bundle over $\Lambda = |\mathcal{O}_{\mathbb{P}}(2 - l_1 - l_2)|$.*

Proof. We denote the second projection $Y \to |\mathcal{O}_{\mathbb{P}}(2 - l_1 - l_2)|$ by g, and the locus of singular members of $|\mathcal{O}_{\mathbb{P}}(2 - l_1 - l_2)|$ by Λ_0. Every member of Λ_0 is the union of two distinct planes. If $Q \notin \Lambda_0$, the fiber of g over Q is $|\mathcal{O}_{\mathbb{P}\times\mathbb{P}}(1, 2)| \simeq \mathbb{P}^5$. The fiber over $Q \in \Lambda_0$ is reducible. But it is easily checked that it is also of dimension 5. Q.E.D.

Assume that a smooth member $Q \in |\mathcal{O}_{\mathbb{P}}(2 - l_1 - l_2)|$ is defined by $xt - yz = 0$ for a homogeneous coordinate $(x; y; z; t)$ of $\mathbb{P}^3$. Then Q contains two 5-dimensional families of twisted cubics. They correspond to the matrices of the form

$$\begin{pmatrix} x & z & f \\ -y & -t & g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & y & f \\ -z & -t & g \end{pmatrix},$$

where f and g are linear forms. The former family is characterized by the property that the $x = y = 0$ is a bisecant line, and the latter family has $x = z = 0$ as a bisecant line.

§5. Twisted cubics apolar to two reducible quadrics

In this section we study the locus $\mathcal{T}_{M_0}$ of twisted cubics apolar to $M_0 \subset (S^2V)^\vee$ when M_0 is spanned by two quadratic forms q_1 and q_2 of rank 2. q_i is the pull-back of a quadratic form $\bar{q}_1$ on a line l_i for $i = 1, 2$. We assume that two lines l_1 and l_2 are skew. By Proposition 3.2, $\mathcal{T}_{M_0}$ is the pull-back of $\mathbb{P}^1 \times \mathbb{P}^1$ by the rational map $\mathcal{T}_{M_0} \cdots \to \mathbb{P}^2 \times \mathbb{P}^2$ defined by (16). We denote the restriction of (16) by

$$f_i : \mathcal{T}_{M_0} \cdots \to \mathbb{P}^1 \subset \mathrm{Sym}^2 l_i, \quad i = 1, 2. \tag{18}$$

Similar to the previous section, we consider the correspondence

$$X = \{(R, Q) \mid R \subset Q\} \subset \mathcal{T}_{M_0} \times \Lambda \tag{19}$$

between T_{M_0} and Λ. We denote the second projection $X \to \Lambda$ by g. When a quadric Q in Λ is smooth, the fiber of g over $[Q]$ is a 3-dimensional projective subspace of $|\mathcal{O}_{\mathbb{P}\times\mathbb{P}}(1, 2)| \simeq \mathbb{P}^5$. Similar to Proposition 4.3, X is irreducible of dimension 6, and a generic $\mathbb{P}^3$-bundle over Λ.

Proposition 4.1 holds for $\mathcal{T}_{M_0}$ too, and we have the following by Proposition 4.2.

Proposition 5.1. *If the first projection $\pi : X \to \mathcal{T}_{M_0}$ is not an isomorphism at $[R]$, then one of the following holds:*

(a) R is the union of l_1 and a conic which have l_2 as a bisecant line, or vice versa, or

(*b*) *R is the union $m_1 \cup m_2 \cup l_i$ of three lines, with $i = 1$ or 2, such that both m_1 and m_2 intersect l_1 and l_2, or*

(*c*) *R is the union $l_1 \cup l_2 \cup m$ of three lines such that m intersects both l_1 and l_2.*

The twisted cubics satisfying (a) are parametrized by open subsets of A'_1 and A'_2 which are $\mathbb{P}^3$-bundles over $\mathbb{P}^1$. In particular, both A'_1 and A'_2 are of dimension 4. The twisted cubics satisfying (c) are parametrized by the intersection $A'_1 \cap A'_2 \simeq l_1 \times l_2$. Since the twisted cubics satisfying (b) forms a 3-dimensional family, the first projection π is birational, and we obtain the rational map

$$\mathcal{T}_{M_0} \cdots \to \Lambda \simeq \mathbb{P}^3$$

which assigns the unique quadric $Q \in |\mathcal{O}_{\mathbb{P}}(2 - R - l_1 - l_2)|$ to R. The correspondence X in (19) is nothing but the graph of this rational map. $\pi^{-1}(A'_1)$ is of dimension 5 and its image by g is $\Lambda_0 \simeq l_2 \times l_1$.

We need also the following information on the restriction of the syzygy bundle $\mathcal{F}$ to a general fiber of the second projection $g : X \to \Lambda$.

Lemma 5.2. *If Q in Λ is smooth, then the restriction of $\mathcal{F}$ to $g^{-1}[Q] \simeq \mathbb{P}^3$ is isomorphic to $\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$.*

Proof. We take a homogeneous coordinate $(x : y : z : t)$ of $\mathbb{P}^3$ such that

$$Q_1 : XY = 0, \quad Q_2 : ZT = 0, \quad Q : xt - yz = 0,$$

where $(X : Y : Z : T)$ is the dual coordinate of $\mathbb{P}^{3,*}$. A twisted cubic in the fiber $g^{-1}[Q]$ is defined by the three minors of the matrix $\begin{pmatrix} x & z & by + b't \\ -y & -t & ax + a'z \end{pmatrix}$, where a, a', b, b' are constants. (See the argument at the end of §4.) The syzygy space Syz_R of R is generated by

$$x \otimes \{(ax + a'z)z + (by + b't)t\} - z \otimes \{(ax + a'z)x + (by + b't)y\} + (by + b't) \otimes q$$

and

$$-y \otimes \{(ax + a'z)z + (by + b't)t\} + t \otimes \{(ax + a'z)x + (by + b't)y\} + (ax + a'z) \otimes q,$$

where we put $q = xt - yz$. Hence when R runs over the fiber $g^{-1}[Q]$, Syz_R generates the vector space of dimension 8 with the following basis:

$$x \otimes xz - z \otimes x^2, x \otimes z^2 - z \otimes xz, x \otimes yt - z \otimes y^2 - y \otimes q, x \otimes t^2 - z \otimes yt - t \otimes q,$$

$$-y \otimes xz + t \otimes x^2 - x \otimes q, -y \otimes z^2 - t \otimes xz - z \otimes q, y \otimes yt - t \otimes y^2, -y \otimes t^2 - t \otimes yt.$$

Syz_R has a 1-dimensional intersection with the vector space spanned by the first four syzygies, and so does with that spanned by the last four. Hence the fiber $g^{-1}[Q]$ is the projective space with $(a : a' : b : b')$ as its homogeneous coordinate, and $\mathcal{F}|_{g^{-1}[Q]}$ is isomorphic to $\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$. Q.E.D.

§6. Doubly octagonal K3 surface and proof of Theorem 1.1

Let $S_{M_0,N} \subset T_{M_0}$ be the zero locus of the global section of $\mathcal{F}^{\oplus 2}$ corresponding to a 2-dimensional subspace $N \subset (S^{2,1}V)^\vee$.

Lemma 6.1. *If N is general, then $S_{M_0,N}$ is disjoint from A'_1 and A'_2, that is, a twisted cubic in $S_{M_0,N}$ does not contain the line l_1 or l_2 as a component.*

Proof. We may assume that $q_1 = XY$ and $q_2 = ZT$ for a homogeneous coordinate $(x : y : z : t)$ of $\mathbb{P}^3$, where $(X : Y : Z : T)$ is the dual coordinate of $\mathbb{P}^{3,*}$.

Since $\mathcal{F}^{\oplus 2}$ is of rank 4 and generated by its global sections, it suffices to show that a twisted cubic satisfying (a) does not belong to $S_{M_0,N}$. Assume that such a cubic R satisfies the first half of the statement (a) of Proposition5.1. Then R is defined by three minors of a matrix of the form $\begin{pmatrix} f & * & * \\ 0 & x & y \end{pmatrix}$ and has $x \otimes yf - y \otimes xf$ as its syzygy, where f is a linear commination of x and y. When R runs over A'_1 these syzygies span the 2-dimensional vector space $\langle x \otimes yz - y \otimes xz, x \otimes yt - y \otimes xt \rangle_{\mathbb{C}}$ in $S^{2,1}V$. Since $N \subset (S^{2,1}V)^\vee$ is a general 2-dimensional space, its intersection with $N^\perp$ is zero. Hence A'_1 is disjoint from $S_{M_0,N}$. The same holds for A'_2. Q.E.D.

By the lemma, the morphism $\pi : X \to \mathcal{T}_{M_0}$ is an isomorphism over $S_{M_0,N}$. Hence we denote its pull-back in X by the same symbol $S_{M_0,N} \subset X$. The restriction of the rational map f_i $(i = 1, 2)$ to $S_{M_0,N}$ is a morphism, which we also denote by the same symbol $f_i : S_{M_0,N} \to \mathbb{P}^1 \subset \mathrm{Sym}^2 l_i$.

Now we study the intersection of divisor $\mathcal{D}_{B,2}$ (§3) with $S_{M_0,N}$. Let $\mathcal{D}_1$ be the locus of reducible twisted cubics R whose conical component has l_1 as a bisecant line.

Lemma 6.2. *If N is general, then the intersection $Z := \mathcal{D}_1 \cap S$ is isomorphic to $\mathbb{P}^1$.*

Proof. More precisely, we show that the restriction of $f_2|_Z : Z \to \mathbb{P}^1 \subset \mathrm{Sym}^2 l_2$ is the double cover induced from $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathrm{Sym}^2 l_2$.

Let (p_1, p_2) be an ordered pair of points of l_2 which is apolar to (or orthogonal with respect to) $\bar{q}_2$. It suffice to show that there exist a unique reducible twisted cubic $R = C \cup l$ in $S_{M_0,N} \cap \mathcal{D}_1$ whose linear part l passes through p_1 and conical part C through p_2. Such a twisted cubic is the common zero locus of the matrix of the form $\begin{pmatrix} f & f_1 & f_2 \\ 0 & g_1 & g_2 \end{pmatrix}$, where f is the equation of the plane spanned by l and p_2, and g_1, g_2 are linear forms vanishing at p_1. One syzygy of R is $s(R) := g_1 \otimes fg_2 - g_2 \otimes fg_1$ which belongs to the space of syzygies

$$\langle x \otimes fy - y \otimes fx, y \otimes fz - z \otimes fy, z \otimes fx - x \otimes z \rangle_{\mathbb{C}}, \tag{20}$$

where $\{x, y, z\}$ is a basis of linear forms vanishing at p_2. Since N is of dimension 2, $s(R)$ belongs to $N^\perp$ for suitable choice of g_1 and g_2. Similarly another syzygy of R independent from $s(R)$ belongs to $N^\perp$ for suitable choice of f_1 and f_2. This shows the existence of the required $R = C \cup l$.

When an unordered pair $\{p_1, p_2\}$ runs over $\mathbb{P}^1 \subset \mathrm{Sym}^2 l_2$, the image of f_2, (20) is a 1-dimensional family of 3-dimensional subspaces. Hence the usual dimension count argument shows that the linear part l is unique for a given (p_1, p_2) if we choose N general enough. Similarly the conical part C is unique also if N is general. Q.E.D.

We now compute the intersection numbers of several divisor classes on S. We denote the restriction of $H = c_1(\mathcal{E})$ to S by h, and the divisor class of a general fiber of $f_i : S \to \mathbb{P}^1$ by a_i for $i = 1, 2$.

For every R in S, $H^0(\mathcal{O}_{\mathbb{P}}(2-R))$ has 1-dimensional intersection with $H^0(\mathcal{O}_{\mathbb{P}}(2 - l_1 - l_2))$ and 2-dimensional intersection with $H^0(\mathcal{O}_{\mathbb{P}}(2 - l_i))$, $i = 1, 2$, by Proposition 4.1 and Lemma 6.1. Hence we have an exact sequence

$$0 \to \mathcal{O}_S(a_1) \oplus \mathcal{O}_S(a_2) \to \mathcal{E}|_S \to \mathcal{O}_S(b) \to 0 \tag{21}$$

on S, where we put $b = h - a_1 - a_2$.

Lemma 6.3. 1) $(h.a_1) = (h.a_2) = 8$.
2) $(a_1.a_2) = 3$.

Proof. 1) A general fiber of the morphism (15) consists of all twisted cubics passing through two points $p_1, p_2 \in l$. Hence its fundamental cohomology class is $(c_2 - d_2)^2$ by [1, Section 7]. Hence $(h.a_1)$ and $(h.a_2)$ are equal to the intersection number $(c_1(c_2 - d_2)^2 d_2^2 c_3)$, which is equal to $82 - 2 \cdot 57 + 40 = 8$ by [1, Table 1].

2) By Proposition 2.2 and the exact sequence (21), we have $c_2(\mathcal{E}|_S) = (a_1.a_2) + (b.a_1 + a_2) = 13$. Hence 2) follows from 1). Q.E.D.

By the lemma, the a_1, a_2 and b spans an integral sublattice of rank 3 in the Picard lattice of S with inner product $\begin{pmatrix} 0 & 3 & 5 \\ 3 & 0 & 5 \\ 5 & 5 & 4 \end{pmatrix}$. Since the discriminant is equal to 14 and square free, $\langle a_1, a_2, b\rangle_{\mathbb{Z}}$ is a primitive sublattice. Theorem 1.1 follows from Proposition 2.2 and the following

Lemma 6.4. *$S = S_{M_0,N}$, for general N, is mapped to a quartic surface by the morphism $g : \mathcal{T}_{M_0} \to \mathbb{P}^3$.*

Proof. The pull-back of the tautological line bundle of $\mathbb{P}^3$ by g is $\mathcal{O}_S(b)$. By Lemma 6.3, we have $(b^2) = (h - a_1 - a_2)^2 = 4$. Hence the restricted morphism $g|_S : S \to \mathbb{P}^3$ is of degree 4. By Lemma 5.2, every general fiber of $g|_S$ is a linear subspace of $\mathbb{P}^3$. Hence $g|_S$ cannot be a double cover of a quadric or a quartic cover of a plane. Hence $g|_S$ is birational onto a quartic surface. Q.E.D.

Since $(a_1.a_2)$ and $(a_1.b)$ are coprime, the divisor class a_1 is primitive. Hence the fiber of f_1 is connected. Therefore, f_1 is an elliptic fibration of degree 8 of the polarized K3 surface $(S_{M_0,N}, h)$. The same holds for f_2. We call $S_{M_0,N}$ *doubly octagonal* for this reason. The Mukai vectors of $\mathcal{E}|_S$ and $\mathcal{F}|_S$ are $(3, h, 5)$ and $(2, h, 8)$, respectively, by Proposition 2.2. Hence, we have $\chi(\mathcal{E}|_S, \mathcal{F}|_S) = 24 - 30 + 10 = 4$, $(v(\mathcal{E}|_S)^2) = 30 - 30 = 0$ and $(v(\mathcal{F}|_S)^2) = 30 - 32 = -2$ ([4, §2]).

§7. Proof of Proposition 1.3 and Theorem 1.2

We prove Proposition 1.3 step by step. Let S be $S_{M_0,N}$ for general N as in the previous section.

claim 1. *$H^i(S, \mathcal{E}|_S) = 0$ for all $i > 0$.*

Proof. Since $\mathcal{O}_S(b)$ is the pull-back of $\mathcal{O}_{\mathbb{P}}(1)$ by g, $H^i(S, \mathcal{O}_S(b)) = 0$ for all $i > 0$. Since $|a_j|$ contains a smooth elliptic curve, $H^i(S, \mathcal{O}_S(a_j)) = 0$, for all $i > 0$ and $j = 1, 2$. Hence the claim follows from the exact sequence (21). Q.E.D.

We need to investigate the restriction of the syzygy bundle $\mathcal{F}$ to S. By Proposition 3.1, we have an exact sequence

$$0 \to \mathcal{O}_S(a_1) \oplus \mathcal{O}_S(a_1) \to \mathcal{F}|_S \to j_*\gamma \to 0, \tag{22}$$

where $j : Z = \mathcal{D}_1 \cap S \hookrightarrow S$ is a natural inclusion and γ is a line bundle on Z. We have $\deg \gamma = 5$ by Lemma 6.2 and Proposition 2.2.

claim 2. *$H^i(S, \mathcal{F}|_S) = 0$ for all $i > 0$.*

Proof. Obvious from (22) and the vanishing $H^1(Z,\gamma) = 0$ and $H^i(S, \mathcal{O}_S(a_1)) = 0$ for $i > 0$. Q.E.D.

claim 3. $\mathrm{Ext}^i(\mathcal{E}|_S, \mathcal{F}|_S) = 0$ *for all* $i > 0$.

Proof. We denote $\mathcal{E}|_S, \mathcal{F}|_S$ by E and F, respectively. Since $\chi(E,F) = 4$, it suffice to show $\dim \mathrm{Hom}(E,F) = 4$ and $\mathrm{Hom}(F,E) = 0$. Since E is extension of three line bundles $\mathcal{O}_S(a_1), \mathcal{O}_S(a_2), \mathcal{O}_S(b)$, it suffice to show

$$h^0(F(-a_1)) + h^0(F(-a_2)) + h^0(F(-b)) \leq 4.$$

Taking dual and twisting by $\mathcal{O}_S(a_1)$, the exact sequence (22) induces an exact sequence

$$0 \to F(-a_2 - b) \to \mathcal{O}_S \oplus \mathcal{O}_S \to j_*\alpha \to 0, \tag{23}$$

where α is a line bundle of degree 1 on Z. The induced linear map $H^0(\mathcal{O}_S \oplus \mathcal{O}_S) \to H^0(\alpha)$ is an isomorphism. Tensoring with $\mathcal{O}_S(a_2)$, we have the exact sequence

$$0 \to F(-b) \to \mathcal{O}_S(a_2) \oplus \mathcal{O}_S(a_2) \to (j_*\alpha) \otimes \mathcal{O}_S(a_2) \to 0$$

The restriction of the linear system $|a_2|$ to Z is of degree 2 and free. Hence

$$H^0(\mathcal{O}_S(a_2) \oplus \mathcal{O}_S(a_2)) \to H^0(j_*\alpha \otimes \mathcal{O}_S(a_2))$$

is injective. Therefore, we have $H^0(F(-b)) = 0$.

The exact sequence (22) twisted by $\mathcal{O}_S(-a_1)$ is

$$0 \to \mathcal{O}_S \oplus \mathcal{O}_S \to F(-a_1) \to j_*\beta \to 0.$$

for a line bundle β of degree -3. Hence we have $h^0(F(-a_1)) = 2$, and similarly $h^0(F(-a_2)) = 2$. This shows $\dim \mathrm{Hom}(E,F) = 4$.

$\mathrm{Hom}(F,E) = 0$ follows from $H^0(F(-a_1 - b)) = H^0(F(-a_2 - b)) = H^0(F(-a_1 - a_2)) = 0$. Q.E.D.

claim 4. *The natural linear map* $V = \mathbb{C}^4 \to \mathrm{Hom}(\mathcal{E}|_S, \mathcal{F}|_S)$ *(via* $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$*) is an isomorphism.*

Proof. It suffice to show the linear map is injective. Assume the contrary. Then there exists a point $p \in \mathbb{P}^3$ such that every R belonging to S is the union of three lines passing through p. This is obviously impossible. Q.E.D.

claim 5. $\mathcal{F}|_S$ *is simple.*

Proof. By the exact sequence (23), $\mathcal{F}|_S$ is the reflection of $j_*\alpha$ by the rigid bundle $\mathcal{O}_S$ Since $j_*\alpha$ is simple so is $\mathcal{F}|_S$ by [4, Proposition 2.25]. (See also the remark below.) Q.E.D.

Remark 7.1. In the terminology of [9], $\mathcal{F}|_S$ is the spherical twist $T_{\mathcal{O}_S}(j_*\alpha)$ of $j_*\alpha$ by the spherical object $\mathcal{O}_S$. Since $T_{\mathcal{O}_S}$ is an autoequivalence of the derived category of coherent sheaves on S by [9, Theorem 1.2], $\mathcal{F}|_S$ is simple.

Proof of Proposition 1.3. We already proved it mostly in the above claims 1–5 taking $S_{M_0,N}$ as S, except for the simpleness of $\mathcal{E}|_S$. We need an extra argument, since the restriction of $\mathcal{E}$ on $S_{M_0,N}$ is not simple. In fact, the 6-fold $\mathcal{T}_{M_0}$ has an action of the 3-dimensional torus, and the restriction of $\mathcal{E}$ to there is not simple.

By the exact sequence (21), $\mathcal{E}|_S$ is an extension of the direct sum of two line bundles by the line bundle $\mathcal{O}_S(b)$. Now we replace the direct sum by nontrivial extension G of $\mathcal{O}_S(a_1)$ by $\mathcal{O}_S(a_2)$. This is possible since $(a_1 - a_2)^2 = -6$. Furthermore, we take a nontrivial extension E' of G by $\mathcal{O}_S(b)$. This is possible since $(a_1 - b)^2 = (a_2 - b)^2 = -6$. Since $|b - a_i| = |a_i - b| = \emptyset$ for $i = 1, 2$ and since $|a_1 - a_2| = |a_2 - a_1| = \emptyset$, E' is simple. (The emptyness of linear systems follows easily since a_1, a_2 and b are nef.) There is a family of vector bundles $\{E_t\}$ on S parametrized by the affine line $\mathbb{A}^1$ such that $E_0 \simeq \mathcal{E}|_S$ and $E_t \simeq E'$ for every $t \neq 0$.

By the upper semi-continuity of cohomology, E' satisfies the same vanishing as $\mathcal{E}|_S$. In particular, we have $\dim \mathrm{Hom}(E', \mathcal{F}|_S) = \chi(E', \mathcal{F}|_S) = 4$. The universal property (*) in §2 applies to the pair $(E', \mathcal{F}|_S)$ and $V' := \mathrm{Hom}(E', \mathcal{F}|_S)$, and we have a morphism from S to the EPS compactification $\mathcal{T}'(\simeq \mathcal{T})$ of twisted cubics. The morphism is an embedding since it is so for the pair $(\mathcal{E}|_S, \mathcal{F}|_S)$. The image is a complete intersection with respect to $\mathcal{E}^{\oplus 2} \oplus \mathcal{F}^{\oplus 2}$ by virtue of the cohomology vanishing since it is so for the pair $(\mathcal{E}|_S, \mathcal{F}|_S)$. S is isomorphic to $S_{M',N'}$ for a pair (M', N'), which is a deformation of the pair (M_0, N), by the claims 1–4. This K3 surface $S_{M',N'}$ satisfies all the requirement of the proposition. Q.E.D.

Proof of Theorem 1.2. We denote the non-empty open subset of $\mathcal{P}$ (see Introduction) consisting of (M, N) such that the restriction of $\mathcal{E}$ and $\mathcal{F}$ to $S_{M,N}$ satisfies the requirement of Proposition 1.3 by $\mathcal{P}_0$. Let (S, h) be a small deformation of $(S_{M,N}, H|_{S_{M,N}})$ as polarized $K3$ surface. Then by Proposition 1.3 and the proposition below $\mathcal{E}|_{S_{M,N}}$ and $\mathcal{F}|_{S_{M,N}}$ deforms to vector bundles E and F, with $\det E \simeq \det F \simeq \mathcal{O}_S(h)$, on S. Since (E, F) is a small deformation of $(\mathcal{E}|_{S_{M,N}}, \mathcal{F}|_{S_{M,N}})$, it embeds S into $\mathcal{T}$ and the image of S is a complete intersection with respect to $\mathcal{E}^{\oplus 2} \oplus \mathcal{F}^{\oplus 2}$, again by Proposition 1.3 and a similar argument in its proof. Therefore, the image of the classification morphism

$$\mathcal{P}_0 \to \mathcal{F}_{16}, \quad (M, \bar{N}) \mapsto (S_{M,N}, \mathcal{O}_S(1)),$$

is open. Q.E.D.

Proposition 7.2. ([7, Proposition 4.1]) *Let E be a simple vector bundle on a K3 surface S and (S', L') be a small deformation of $(S, \det E)$. Then there is a deformation (S', E') of the pair (S, E) such that* $\det E' \simeq L'$.

Remark 7.3. The rational map (5) factors through the birational quotient $\mathcal{P}/PGL(4)$, which is of dimension 21(=20+16-15). Every general fiber of $\mathcal{P}/PGL(4) \cdots \to \mathcal{F}_{16}$ is birational to a (K3) surface. In fact, it is the moduli space $M_S(3, h, 5)$ of semi-rigid vector bundles E on S with Mukai vector $(3, h, 5)$. See [8, Corollary to Theorem 2] for a similar result in the case of genus 13.

§8. Comparison with the case of genus 12

It is worth recalling here the description of a general K3 surface (S, h) of genus 12, that is, $(h^2) = 22$, which is the origin of our description in this article.

There exist two rigid vector bundles E and F on S with Mukai vector $(3, h, 4)$ and $(2, h, 6)$, respectively. The vector space $V^\vee = \mathrm{Hom}(E, F)$ is of dimension 4 and the universal property (*) applies. Hence we have a morphism from S to the EPS compactification $\mathcal{T}$. The induced linear map $S^2V^\vee \to H^0(E)$ is surjective and its kernel N is of dimension 3. Hence the image of S is contained in the moduli $\mathcal{T}_N$ of twisted cubics apolar to the net of quadrics $|N|$. $\mathcal{T}_N$ is a Fano 3-fold of genus 12 ([6, §3]). The image of S is an anticanonical member of $\mathcal{T}_N$.

Theorem 1.2 concerning on K3 surfaces of genus 16 was found replacing two vector bundles E and F above by a semi-rigid vector bundle ([4, §3]) with Mukai vector $(3, h, 5)$ and a rigid vector bundle with Mukai vector $(2, h, 8)$, respectively.

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