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Local structure of principally polarized stable Lagrangian fibrations

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Dedicated to Professor Shigefumi Mori

Abstract.

A holomorphic Lagrangian fibration is stable if the characteristic cycles of the singular fibers are of type $I_m, 1 \leq m < \infty$, or A_∞. We will give a complete description of the local structure of a stable Lagrangian fibration when it is principally polarized. In particular, we give an explicit form of the period map of such a fibration and conversely, for a period map of the described type, we construct a principally polarized stable Lagrangian fibration with the given period map. This enables us to give a number of examples exhibiting interesting behavior of the characteristic cycles.

§1. Introduction

For a holomorphic symplectic manifold (M, ω), i.e., a $2n$-dimensional complex manifold with a holomorphic symplectic form $\omega \in H^0(M, \Omega^2_M)$, a proper flat morphism $f : M \to B$ over an n-dimensional complex manifold B is called a (holomorphic) Lagrangian fibration if all smooth fibers are Lagrangian submanifolds of M. The discriminant $D \subset B$, i.e., the set of critical values of f, is a hypersurface if it is non-empty. In [7], the structure of the singular fiber of f at a general point $b \in D$ was studied. By introducing the notion of characteristic cycles, [7] shows that the structure of such a singular fiber can be described in a manner completely parallel to Kodaira's classification ([12], see also V. 7 in [1])

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of singular fibers of elliptic fibrations. Furthermore, to study the multiplicity of the singular fibers, [8] generalized the stable reduction theory of elliptic fibrations (cf. V.10 in [1]), explicitly describing how arbitrary singular fiber over a general point of D can be transformed to a stable singular fiber, a singular fiber of particularly simple type. These results exhibit that the theory of general singular fibers of a holomorphic Lagrangian fibration gives a very natural generalization of Kodaira's theory of elliptic fibrations.

The current work is yet another manifestation of this principle. An important part of Kodaira's theory is the study of the asymptotic behavior of the elliptic modular function of a given elliptic fibration near a singular fiber. As a generalization of this we will study the asymptotic behavior of the periods of the abelian fibers near a general singular fiber of a holomorphic Lagrangian fibration. Here we need to make two additional assumptions on the Lagrangian fibration.

First, we will assume that the singular fibers are of stable type, i.e., its characteristic cycles are of type $I_k, 0 \leq k \leq \infty$ (I_∞ meaning A_∞).

The second assumption we will make is that the Lagrangian fibration is *principally polarized*, in the sense explained in Definition 2. This condition is satisfied if there exists an f-ample line bundle on $M \setminus f^{-1}(D)$ whose restriction on smooth fibers gives principal polarizations on the abelian varieties.

We believe that these assumptions are natural and understanding the structure of Lagrangian fibration under these assumptions is essential for the study of general cases.

The main results of the paper are the following two theorems.

Theorem 1. *Let $f : M \to B$ be a principally polarized stable Lagrangian fibration (cf. Definition 1 and Definition 2). Then at a general point $b \in D$ of the discriminant, there exist a coordinate system $(z_1, \ldots, z_n)$ of B where D is defined by $z_n = 0$, a positive integer ℓ and a holomorphic function Ψ with the following properties: for a suitable choice of an integral frame of the local system $R^1 f_* \mathbb{Z}$ on $B \setminus D$, the period matrices $(\theta^i_j) = (\theta^i_j(z_1, \ldots, z_n))$ of the fibers over $B \setminus D$ have the form*

$$\theta^i_j = \frac{\partial^2 \Psi}{\partial z_i \partial z_j} \text{ for } (i,j) \neq (n,n) \text{ and } \theta^n_n = \frac{\partial^2 \Psi}{\partial z_n \partial z_n} + \frac{\ell}{2\pi\sqrt{-1}} \log z_n.$$

Theorem 2. *On the polydisc $\Delta^n = \{(z_1, \ldots, z_n), |z_i| < 1\}$, let ℓ be a positive integer and $\Psi(z_1, \ldots, z_n)$ be a holomorphic function such that* *the matrix function* $(\theta^i_j(z_1, \ldots, z_n))$ *defined by*

$$\theta^i_j = \frac{\partial^2 \Psi}{\partial z_i \partial z_j} \text{ for } (i,j) \neq (n,n) \text{ and } \theta^n_n = \frac{\partial^2 \Psi}{\partial z_n \partial z_n} + \frac{\ell}{2\pi\sqrt{-1}} \log z_n$$

has positive definite imaginary part on the complement of $z_n = 0$. *Then there exists a principally polarized stable Lagrangian fibration* $f^{\mathrm{model}} : M^{\mathrm{model}} \to \Delta^n$ *whose period matrices in a suitable frame of* $R^1 f_* \mathbb{Z}$ *are* $(\theta^i_j(z_1, \ldots, z_n))$. *Moreover,* ℓ *is the number of irreducible components of a singular fiber of* f^{model}.

Theorem 1 and Theorem 2 are a sort of converse to each other. More precisely, we can prove the following.

Proposition 1. *In the setting of Theorem 1, choose a polydisc neighborhood* $U \subset B$ *of* b *such that the coordinates* $(z_1, \ldots, z_n)$ *and the function* Ψ *are defined on* U. *Applying Theorem 2, we get a stable Lagrangian fibration* $f^{\mathrm{model}} : M^{\mathrm{model}} \to U$. *Then the two fibrations,* $f^U : f^{-1}(U) \to U$ *and* $f^{\mathrm{model}} : M^{\mathrm{model}} \to U$ *are biholomorphic as fiber spaces in a neighborhood of a general point of* $D \cap U$. *More precisely, there exist a proper analytic subset* $G \subset D \cap U$ *and a biholomorphic morphism*

$$g : f^{-1}(U \setminus G) \cong (f^{\mathrm{model}})^{-1}(U \setminus G)$$

such that g *preserves the symplectic structures and* $f = f^{\mathrm{model}} \circ g$.

See also Proposition 19 and Corollary 1 for slightly stronger results.

These results say that, in a neighborhood of a general singular fiber, a principally polarized stable Lagrangian fibration determines/is determined by a *single* holomorphic function $\Psi(z_1, \ldots, z_n)$ modulo terms of degree ≤ 1 and a positive integer ℓ with the specified properties.

That the period matrix of a *smooth* Lagrangian fibration is the Hessian of a potential function is a consequence of the action-angle variables, one of the most important properties of Lagrangian fibrations (cf. [2]). The logarithmic behavior of the multi-valued function θ^n_n reflects the stability assumption on the singular fiber. The novelty in Theorem 1 lies in the choice of the variable z_n through which these two aspects are intertwined. In this sense, it can be interpreted as a generalization of action-angle variables to *singular (stable)* Lagrangian fibrations. The existence of z_n follows from the fact proved in Proposition 15 that the characteristic foliation accounts for the degenerate part of the polarization restricted to the fixed part of the monodromy.

Theorem 2 is shown by explicitly constructing a principally polarized stable Lagrangian fibration from a given potential function $\Psi(z)$ with positive definite $\mathrm{Im}\,(\theta_i^j)$. This part is a generalization of Nakamura's construction [13] to principally polarized stable Lagrangian fibration in arbitrary dimension.

Using our construction, we shall give a concrete 4-dimensional example of a principally polarized stable Lagrangian fibration in which the types of characteristic cycles of singular fibers change fiber by fiber, too. To our knowledge, such an example has not been noticed previously.

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§2. Stable Lagrangian fibrations

Definition 1. *A* Lagrangian fibration *is a proper flat morphism $f : M \to B$ from a holomorphic symplectic manifold (M, ω) of dimension $2n$ to a complex manifold B of dimension n such that the smooth locus of each fiber is a Lagrangian submanifold of M. The* discriminant $D \subset B$ *is the set of the critical values of f, which is a hypersurface in B if it is non-empty.* Throughout this paper, we assume that D is non-empty. *We say that f is a* stable Lagrangian fibration *if $D \subset B$ is a submanifold and each singular fiber $f^{-1}(b), b \in D$, is stable, i.e., it is reduced and its normalization is a disjoint union of a finite number of compact complex manifolds $Y^1, \ldots, Y^\ell$ for some positive integer $\ell = \ell(b)$ (which might depend on $b \in D$) such that*

(i) *each Y^i is a $\mathbb{P}^1$-bundle over an $(n-1)$-dimensional complex torus A^i whose fibers are sent to characteristic leaves of $f^{-1}(b)$ in the sense of [7], i.e., for a defining function $h \in \mathcal{O}(B)$ of the divisor D, the Hamiltonian vector field $\iota_\omega(f^*dh)$, where $\iota_\omega : \Omega^1_M \to T(M)$ is the vector bundle isomorphism induced by ω, is tangent to the image of the fibers in M;*

(ii) *there exist submanifolds $S^i_1, S^i_2 \subset Y^i$, with $S^i_1 \neq S^i_2$ except possibly when $\ell = 1, 2$, such that $S^i_1 \cup S^i_2$ is a 2-to-1 unramified cover of A^i under the $\mathbb{P}^1$-bundle projection;*

(iii) *the normalization $\nu : \bigcup Y^i \to f^{-1}(b)$ is obtained by the identification via a collection of biholomorphic morphisms $g_i : S^i_2 \to S^{i+1}_1$ for $1 \le i \le \ell - 1$ and $g_\ell : S^\ell_2 \to S^1_1$ with the additional requirement $g_1 = g_2^{-1}$ if $S^1_1 = S^1_2$ and $S^2_1 = S^2_2$ for $\ell = 2$.*

A maximal connected union of the $\mathbb{P}^1$-fibers in (i) under the identification in (iii) is called a characteristic cycle*. A characteristic cycle can be either of finite type (I_m-type, $1 \le m < \infty$) or of infinite type A_∞, which we also denote by I_∞.*

Remark 1. *There is a notion of characteristic cycle, as explained in p.983 of* [7]*, for a general singular fiber of a Lagrangian fibration whose fibers are of Fujiki class. As proved in Theorem 1.4 of* [7]*, a characteristic cycle is isomorphic to one of the Kodaira's list of singular fibers of elliptic fibrations of surfaces or of I_∞ type. Furthermore, it was proved in* [7] *that the fibration is stable in the sense of Definition 1 if and only if the characteristic cycle is of type I_m, $1 \leq m \leq \infty$. Thus it is possible to define a stable Lagrangian fibration in a simpler way, in terms of its characteristic cycle. However, Definition 1 is more convenient for the proof of Theorem 1.1.*

Recall (cf. [8] Section 4) that in a neighborhood of a general singular fiber, any Lagrangian fibration whose fibers are of Fujiki class can be transformed to a stable Lagrangian fibration by certain explicitly given bimeromorphic modifications and branched covering. We will be interested in the local property of the fibration at a point of D. Thus we will make the following

(*Assumption*) the pair $(D \subset B)$ is a germ of the pair satisfying the condition that D is a a smooth hypersurface in an n-dimensional complex manifold such that $B \setminus D$ has a cyclic fundamental group.

The following is immediate from Proposition 2.2 of [7].

Proposition 2. *Given a stable Lagrangian fibration, we can assume that there exists an action of the complex Lie group $\mathbb{C}^{n-1}$ on M preserving the fibers and the symplectic form such that S_1^i, S_2^i are orbits of this action for all $1 \leq i \leq \ell$. This action of $\mathbb{C}^{n-1}$ on Y^i descends to the translation action on A^i. The patching biholomorphisms g_i in Definition 1 (iii) as well as the $\mathbb{P}^1$-bundle structure in (i) are equivariant under this action. In particular, if $S_1^1 = S_2^1$ (resp. $S_1^2 = S_2^2$), the Galois action of the double cover $S_1^1 \to A^1$ (resp. $S_1^2 \to A^2$) is given by a translation on the torus S_1^1 (resp. S_1^2).*

Proposition 3. *In the notation of Definition 1, let $f^{-1}(b)$ be a singular fiber of a stable Lagrangian fibration. Then all the complex tori A^i's are biholomorphic to each other by biholomorphic maps induced by g_i's. Thus we may denote them by a single letter A. When $S_1^1 = S_2^1$ and $S_1^2 = S_2^2$ with $\ell = 2$, there exists a morphism $\mu : f^{-1}(b) \to A$ whose fibers are isomorphic to a characteristic cycle of I_2-type.*

Proof. If $S_1^i \neq S_2^i$ for some (and hence all) i, then by the $\mathbb{P}^1$-bundle projection, $S_1^i \cong S_2^i \cong A^i$. Thus the biholomorphisms g_i give biholomorphisms between A^i's.

It remains to handle the case $S_1^1 = S_2^1$ and $S_1^2 = S_2^2$ with $\ell = 2$. By Proposition 2, the Galois action on S_1^1 (resp. S_1^2) of the double cover

over A^1 (resp. A^2) is given by a translation, say, by $\gamma_1 \in \mathbb{C}^{n-1}$ (resp. $\gamma_2 \in \mathbb{C}^{n-1}$). By the $\mathbb{C}^{n-1}$-equivariance of $g_1 = g_2^{-1}$, for each $\alpha \in S_1^1$, we have $g_1(\gamma_1 \cdot \alpha) = \gamma_2 \cdot g_1(\alpha)$. Thus by the normalization morphism $\nu : Y^1 \cup Y^2 \to f^{-1}(b)$, a point $\alpha \in S_1^1$ is identified with $g_1(\alpha) \in S_1^2$, and the point $\gamma_1 \cdot \alpha \in S_1^1$, which lies in the $\mathbb{P}^1$-fiber through α, is identified with $\gamma_2 \cdot g_1(\alpha)$, which lies in the $\mathbb{P}^1$-fiber through $g_1(\alpha)$. Thus $A^1 \cong A^2$ and we get a morphism $\mu : f^{-1}(b) \to A^1$ whose fiber is a union of two $\mathbb{P}^1$'s identified at two points. Q.E.D.

In general, the $\mathbb{P}^1$-bundles $Y^i \to A$ do not patch together to define a morphism $\mu : f^{-1}(b) \to A$, except the case described in Proposition 3. However, we have the following topological substitute.

Proposition 4. *For the singular fiber $f^{-1}(b)$ in Definition 1, denote by Y^i and $\mu^i : Y^i \to A$ the $\mathbb{P}^1$-bundle structure on the components of the normalization of $f^{-1}(b)$ and set $Y_o := Y^1 \setminus (S_1^1 \cup S_2^1)$. There exists an analytic variety $f^{-1}(b)^1$ with the following properties.*

(1) *The normalization of $f^{-1}(b)^1$ is biholomorphic to the normalization of $f^{-1}(b)$ and there exists a morphism $\mu : f^{-1}(b)^1 \to A$ such that the induced morphism on each component Y^i of the normalization agrees with μ^i.*

(2) *There exists a homeomorphism $h : f^{-1}(b) \to f^{-1}(b)^1$ such that, denoting by $j : Y_0 \to f^{-1}(b)$ and $j^1 : Y_0 \to f^{-1}(b)^1$ the natural inclusions, the continuous maps $h \circ j$ and j^1 are homotopic.*

Proof. Let us use the notation introduced in Definition 1 (iii) for the description of the normalization morphism $\nu : \bigcup Y^i \to f^{-1}(b)$.

First, we consider the case $S_1^i \neq S_2^i$ for all $1 \leq i \leq \ell$. Define $\widetilde{f^{-1}(b)}$ as the variety obtained from $\bigcup Y^i$ with all the patching identification $g_1, \ldots, g_{\ell-1}$ such that the normalization factors through

$$\nu : \bigcup Y^i \to \widetilde{f^{-1}(b)} \to f^{-1}(b)$$

with the second arrow given by the identification via g_ℓ. When $\ell = 1$, $\widetilde{f^{-1}(b)} = Y^1$. It is immediate that the morphisms μ^i patch together to define a morphism $\widetilde{f^{-1}(b)} \to A$, which determines a biholomorphism $\zeta : S_2^\ell \to S_1^1$. Fix a point $\alpha \in S_2^\ell$ and let $\beta = \zeta(\alpha) \in S_1^1$. For $t \in [0,1] \subset \mathbb{R}$, let $\tau_t : S_1^1 \to S_1^1$ be the translation by $t(\beta - g_\ell(\alpha))$.

Define a new family of biholomorphic morphisms $g_\ell^t : S_2^\ell \to S_1^1$ by $g_\ell^t = \tau_t \circ g_\ell$. Clearly, $g_\ell^0 = g_\ell$. We claim that $g_\ell^1 = \zeta$. In fact, $\zeta^{-1} \circ g_\ell^1$ is an automorphism of S_2^ℓ which fixes the point α. But both g_ℓ^1 and ζ must be equivariant under the $\mathbb{C}^{n-1}$-action of Proposition 2. Thus $\zeta^{-1} \circ g_\ell^1$ must be the identity map of S_2^ℓ, proving the claim.

Let $f^{-1}(b)^t$ be the variety obtained from $\widetilde{f^{-1}(b)}$ by identifying S_1^1 and S_2^ℓ via g_ℓ^t. Then

$$f^{-1}(b)^0 = f^{-1}(b)$$

and $f^{-1}(b)^1$ is homeomorphic to $f^{-1}(b)$. The $\mathbb{C}^{n-1}$-action descends to $f^{-1}(b)^t$ for each t as τ_t commutes with the $\mathbb{C}^{n-1}$-action. By our choice of α and β, the morphisms μ^i patch up to give a morphism $\mu : f^{-1}(b)^1 \to A$.

For each t, we have a homeomorphism $h_t : f^{-1}(b) \to f^{-1}(b)^t$ such that h_0 is the identity morphism. The homeomorphism $h := h_1$ certainly satisfies the required properties.

Now consider the case when $\ell = 1$ and $S_1^1 = S_2^1$. Set $\widetilde{f^{-1}(b)} = Y^1$ and define $\zeta : S_2^1 \to S_1^1$ as the Galois action of the double covering $S_1^1 \to A^1$ in Definition 1 (ii). Then the same argument as in the previous case applies.

Finally, when $\ell = 2$, $S_1^1 = S_2^1$ and $S_1^2 = S_2^2$, we may put $f^{-1}(b)^1 = f^{-1}(b)$ and use Proposition 3. Q.E.D.

We have a generalization of the classical action-angle correspondence as follows.

Proposition 5. *Let $f : M \to B$ be a stable Lagrangian fibration that admits a Lagrangian section $\Sigma \subset M$ of f. Then we have a natural surjective unramified morphism $\Phi : T^*B \to M \setminus (E \cup \operatorname{Sing}(f))$ where E is the union of the irreducible components of the fibers of f disjoint from Σ such that*

(1) *$f \circ \Phi$ agrees with the natural projection $g : T^*B \to B$,*
(2) *Φ sends the zero section of T^*B to Σ and*
(3) *$\Phi^*\omega$ coincides with the standard symplectic form on T^*B.*

In particular, $\Gamma := \Phi^{-1}(\Sigma)$ is a Lagrangian submanifold (with infinitely many connected components) in T^*B. For each $b \in B$, $\Phi_b := \Phi|_{T_b^*(B)} : T_b^*(B) \to f^{-1}(b) \setminus E$ is the universal covering and $\Gamma_b := \Gamma \cap T_b^*(B)$ is naturally isomorphic to $H_1(f^{-1}(b) \setminus E, \mathbb{Z})$.

Proof. Over $B \setminus D$, this is just a holomorphic version (cf. Proposition 3.5 in [6]) of the classical action-angle correspondence as described in Section 44 of [5]. The statement over D follows by the same argument as for the smooth fibers. In fact, for each $b \in B$, the vector group $T_b^*(B)$ acts on the fiber $f^{-1}(b)$ with n-dimensional orbits on the smooth locus of $f^{-1}(b)$ (cf. Proposition 3.3 in [6]). The morphism Φ_b is defined by taking the orbit map of the point $\Sigma \cap f^{-1}(b)$ under this action, which is a universal covering map for the smooth locus of the component of

$f^{-1}(b)$ containing $\Sigma \cap f^{-1}(b)$. This shows (1) and (2). The proof of (3) is the same as that of Theorem 44.2 of [5]. Q.E.D.

Proposition 6. *Given a stable Lagrangian fibration $f : M \to B$ and a fixed point $b \in D$, since we are assuming that $(D \subset B)$ is a germ as explained in (Assumption), we have a Lagrangian section $\Sigma \subset M$ from Darboux theorem. From Proposition 5, we know that the connected component of Γ containing a point of $\Gamma \cap T_b^*(B)$ is a Lagrangian section of $T^*(B) \to B$, i.e., a closed 1-form on B. Let $\Gamma' \subset \Gamma$ be the union of all connected components of Γ that intersect the fiber $T_b^*(B)$. Then for each $s \in B \setminus D$, $\Gamma'_s := \Gamma' \cap T_s^*(B)$ is a sublattice of Γ_s satisfying $\Gamma_s/\Gamma'_s \cong \mathbb{Z}$.*

Proof. Since $f^{-1}(b) \setminus E$ is a $\mathbb{C}^*$-bundle over an $(n-1)$-dimensional torus, we see that $\Gamma \cap T_b^*(B)$ has rank $2n-1$. Thus Γ'_s has rank $2n-1$. It remains to show that Γ_s/Γ'_s is torsion-free. Suppose it has k-torsion, $0 < k \in \mathbb{Z}$, i.e., there exists a point $\alpha \in \Gamma_s \setminus \Gamma'_s$ such that $k\alpha \in \Gamma'_s$. Let $\widetilde{k\alpha}$ be a closed 1-form given by the component of Γ' containing $k\alpha$. Then the closed 1-form $\tilde{\alpha} = \frac{1}{k}\widetilde{k\alpha}$ is also a component of Γ' containing α, which implies $\alpha \in \Gamma'_s$, a contradiction. Q.E.D.

Proposition 7. *In the notation of Proposition 5, let Y_o be the fiber of $M \setminus E$ at a point $b \in D$. Let $\Phi_b : T_b^*(B) \to Y_o$ be the universal covering map and $\varrho : Y_o \to A$ be the $\mathbb{C}^*$-bundle over an $(n-1)$-dimensional torus. Let $\Upsilon \subset \Gamma_b = \Gamma'_b$ be the rank-1 sublattice corresponding to the kernel of*

$$\varrho_* : H_1(Y_o, \mathbb{Z}) \to H_1(A, \mathbb{Z}).$$

Then for any $v \in \Gamma'_b \setminus \Upsilon$, there exists $\varpi \in H^1(f^{-1}(b), \mathbb{Z})$ such that $\langle \varpi, j_* v \rangle \neq 0$ where $j_* : H_1(Y_o, \mathbb{Z}) \to H_1(f^{-1}(b), \mathbb{Z})$ is induced by the inclusion $j : Y_o \subset f^{-1}(b)$.

Proof. Since $\varrho_*(v) \in H_1(A, \mathbb{Z})$ is non-zero, there exists an element $\varphi \in H^1(A, \mathbb{Z})$ such that $\langle \varphi, \varrho_*(v) \rangle \neq 0$. Let $\varpi = (\mu \circ h)^* \varphi$ where the continuous map $\mu \circ h : f^{-1}(b) \to A$ is as defined in Proposition 4 such that $\mu \circ h \circ j$ is homotopic to ϱ. Then

$$\langle \varpi, j_*(v) \rangle = \langle (\mu \circ h)^* \varphi, j_*(v) \rangle =$$

$$\langle (\mu \circ h \circ j)^* \varphi, v \rangle = \langle \varrho^* \varphi, v \rangle = \langle \varphi, \varrho_*(v) \rangle \neq 0.$$

Q.E.D.

Proposition 8. *For a stable Lagrangian fibration $f : M \to B$ and $s \in B \setminus D$, set $\Lambda_s := H_1(f^{-1}(s), \mathbb{Z})$ for $s \in B \setminus D$. Fix a generator of the cyclic fundamental group of $\pi_1(B \setminus D, s)$ and denote by $\tau_s : \Lambda_s \to \Lambda_s$ the monodromy operator of the generator. Then the fixed part $\Lambda'_s \subset \Lambda_s$ of τ_s at $s \in B \setminus D$ has corank 1.*

Proof. For any $s \in B\backslash D$, we can identify each fiber $\Lambda_s = H_1(M_s, \mathbb{Z})$ with the fiber Γ_s of Proposition 5. Thus the result follows from Proposition 6. Q.E.D.

§3. Principally polarized stable Lagrangian fibration

Definition 2. *For a stable Lagrangian fibration $f : M \to B$, denote by Λ the local system on $B \setminus D$ defined by the lattice Λ_s in Proposition 8. A* principal polarization *on $M \setminus f^{-1}(D)$ is a unimodular anti-symmetric form $Q : \wedge^2\Lambda \to \mathbb{Z}_{B\setminus D}$ where $\mathbb{Z}_{B\setminus D}$ denotes the constant sheaf of integers on $B \setminus D$, which induces a principal polarization on each smooth fiber of f. A stable Lagrangian fibration with a choice of principal polarization is called a* principally polarized stable Lagrangian fibration.

Remark 2. *In Definition 2, the polarization on $M \setminus f^{-1}(D)$ may not extend to an f-ample class of the whole M. In fact, f need not be projective. This definition is useful because there are many situations where the polarization exists* a priori *only on the smooth fibers, e.g., in Kodaira's study of elliptic fibrations and also in our construction in Section 5.*

Proposition 9. *Let $f : M \to B$ be a principally polarized stable Lagrangian fibration. Then the monodromy operator in Proposition 8 satisfies $\tau_s \neq \mathrm{Id}$ and $\tau_s \circ \tau_s \neq \mathrm{Id}$.*

Proof. If $\tau_s = \mathrm{Id}$, then we see that f is a smooth fibration, as in the proof of Proposition 3.2 in [6]. In fact, since there is no monodromy and f is polarized over $B \setminus D$, we can extend the period map of the abelian family on $B \setminus D$ to the whole B ([3], Theorem 9.5). Thus, we obtain a smooth abelian fibration $f' : M' \to B$ such that f and f' are bimeromorphic outside D. Since M' contains no rational curves and both M and M' have trivial canonical bundles, this implies M and M' are biholomorphic, a contradiction to the non-emptiness of the discriminant D of f.

If $\tau_s \circ \tau_s = \mathrm{Id}$, take a double cover $g : B' \to B$ branched along D and let $D' = g^{-1}(D)$. Denote by $\hat{f} : \hat{M} \to B'$ the fiber product of f and g, which has no monodromy on $B' \setminus D'$. By the $\mathbb{C}^{n-1}$-action of Proposition 2 which lifts to $\hat{M}$, the following property of $\hat{M}$ can be seen from the corresponding properties in the case of $n = 1$ (cf. Proof of Proposition 9.2 in [1]): $\hat{M}$ is normal, Gorenstein with singularities of type $A_1 \times$ (germ of $2(n-1)$-dimensional manifold) and has trivial canonical bundle. Thus we have a crepant resolution $f' : M' \to B'$, which is a family with trivial canonical bundle and no monodromy. Then we get a contradiction as in the previous case. Q.E.D.

Lemma 1. *Let $\tau : \Lambda \to \Lambda$ be an automorphism of a lattice such that $\Lambda' := \{v \in \Lambda, \tau(v) = v\}$ is a sublattice of corank 1, i.e., $\Lambda/\Lambda' \cong \mathbb{Z}$. If $\tau \circ \tau \neq \mathrm{Id}$, then $\eta := \tau - \mathrm{Id}$ satisfies $\eta \circ \eta = 0$.*

Proof. Note that $\Lambda' \subset \mathrm{Ker}(\eta)$. The induced automorphism $\bar{\tau} : \Lambda/\Lambda' \to \Lambda/\Lambda'$ is either Id or $-$Id. If $\bar{\tau} = \mathrm{Id}$, then for a non-zero $v \in \Lambda \backslash \Lambda'$, we have $\tau(v) = v + \lambda$ for some $\lambda \in \Lambda'$. Then $\eta(v) = \lambda \in \Lambda' \subset \mathrm{Ker}(\eta)$. This proves that $\eta \circ \eta = 0$. If $\bar{\tau} = -\mathrm{Id}$, then for a non-zero $v \in \Lambda \setminus \Lambda'$, we have $\tau(v) = -v + \lambda$ for some $\lambda \in \Lambda'$. Then

$$\tau \circ \tau(v) = -\tau(v) + \tau(\lambda) = -(-v + \lambda) + \lambda = v.$$

Thus $\tau \circ \tau = \mathrm{Id}$, a contradiction. Q.E.D.

Proposition 10. *In the setting of Proposition 9, let $\eta := \tau_s - \mathrm{Id}$. Then for any $\beta \in \mathrm{Im}(\eta)$ and an element $\varphi \in H^1(M, \mathbb{Z})$,*

$$\langle i^* \varphi, \beta \rangle = \langle \varphi, i_* \beta \rangle = 0$$

where $i^* : H^1(M, \mathbb{Z}) \to H^1(M_s, \mathbb{Z})$ and $i_* : H_1(M_s, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ are the homomorphisms induced by the inclusion $i : M_s := f^{-1}(s) \subset M$.

Proof. Let $\mathcal{H}$ be the local system on $B \setminus D$ given by $H^1(M_s, \mathbb{Z}), s \in B \setminus D$. Denote by $\tau^* : \mathcal{H}_s \to \mathcal{H}_s$ the transformation dual to τ, i.e., for any $\varpi \in H^1(M_s, \mathbb{Z})$ and $u \in H_1(M_s, \mathbb{Z})$,

$$\langle \tau^*(\varpi), u \rangle = \langle \varpi, \tau(u) \rangle.$$

By Proposition 8, Proposition 9 and Lemma 1, we have $\eta \neq 0$ and $\eta \circ \eta = 0$, i.e.,

$$0 \neq \mathrm{Im}(\eta) \subset \mathrm{Ker}(\eta) = \Lambda'_s.$$

Similarly, $\eta^* := \tau^* - \mathrm{Id}$ is an endomorphism of $\mathcal{H}_s$ with $\eta^* \neq 0$ and $\eta^* \circ \eta^* = 0$. Since $i^* \varphi \in \mathrm{Ker}(\eta^*)$ by (the easy half of) the global invariant cycles theorem (cf. Theorem 4.24 of [14]), for any $\psi \in \mathrm{Ker}(\eta^*)$ and $u \in \Lambda_s$,

$$\langle \psi, \eta(u) \rangle = \langle \eta^*(\psi), u \rangle = 0.$$

It follows that $\langle i^* \varphi, \mathrm{Im}(\eta) \rangle = 0$. Q.E.D.

Remark 3. *If the family $f : M \to B$ is projective, we could have used the Monodromy Theorem (cf. Theorem 3.15 in* [14]*) in place of Proposition 3.3 and Lemma 1 in the above proof. We have used the above approach because we do not want to assume that f is projective.*

Proposition 11. *For a principally polarized stable Lagrangian fibration $f : M \to B$ and $s \in B \setminus D$, let $\tau_s : \Lambda_s \to \Lambda_s$ be the monodromy operator of Proposition 8. Set $\eta := \tau_s - \mathrm{Id}$ as in Proposition 10 and $\Lambda'_s = \mathrm{Ker}(\eta)$. Then τ_s preserves the polarization $Q_s : \wedge^2\Lambda_s \to \mathbb{Z}$ and $\mathrm{Im}(\eta) \subset \Lambda_s$ is contained in*

$$\Xi_s := \{v \in \Lambda'_s \,|\, Q(v,w) = 0 \textit{ for all } w \in \Lambda'_s\}.$$

Proof. It is clear that τ_s preserves the polarization Q_s. Since $\eta \circ \eta = 0$ by Lemma 1,

$$Q_s(\eta(v), u) + Q_s(v, \eta(u)) = 0 \text{ for all } v, u \in \Lambda_s.$$

Thus for any $v \in \Lambda_s$ and $u \in \mathrm{Ker}(\eta) = \Lambda'_s$, we have $Q_s(\eta(v), u) = -Q_s(v, \eta(u)) = 0$, which means $\eta(v) \in \Xi_s$. Q.E.D.

Definition 3. *Let Λ be a free abelian group of rank $2n$. Given a unimodular non-degenerate anti-symmetric form $Q : \wedge^2\Lambda \to \mathbb{Z}$, a basis $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ of Λ is called a* symplectic basis *of Λ with respect to Q if, in terms of the dual basis $\{p^1, \ldots, p^n, q^1, \ldots, q^n\}$ of* $\mathrm{Hom}(\Lambda, \mathbb{Z})$,

$$Q = p^1 \wedge q^1 + p^2 \wedge q^2 + \cdots + p^n \wedge q^n.$$

Lemma 2. *In the setting of Definition 3, let $\tau : \Lambda \to \Lambda$ be a group automorphism preserving Q. Assume that the subgroup $\Lambda' \subset \Lambda$ of elements fixed under τ has corank 1. Then there exists a symplectic basis $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ of Λ such that*

$$\{p_1, \ldots, p_n, q_1, \ldots, q_{n-1}\} \subset \Lambda'.$$

Proof. Fix a symplectic basis $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ such that

$$Q = a^1 \wedge b^1 + \cdots + a^n \wedge b^n.$$

The anti-symmetric form $Q|_{\Lambda'}$ must have a kernel of rank 1, i.e.,

$$\Xi := \{v \in \Lambda',\ Q(v, u) = 0 \text{ for all } u \in \Lambda'\}$$

has rank 1. Pick a generator p_n of Ξ. Since Ξ is primitive, i.e., Λ/Ξ has no torsion, we can write

$$p_n = \alpha_1 a_1 + \cdots + \alpha_n a_n + \beta_1 b_1 + \cdots + \beta_n b_n$$

with some integers α_i, β_i satisfying $gcd(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = 1$. Thus there exists integers $\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_n$ such that

$$\alpha'_1 \cdot \alpha_1 + \cdots + \alpha'_n \cdot \alpha_n + \beta'_1 \cdot \beta_1 + \cdots + \beta'_n \cdot \beta_n = 1.$$

Let

$$q_n := -\beta_1' a_1 - \cdots - \beta_n' a_n + \alpha_1' b_1 + \cdots + \alpha_n' b_n.$$

Then $Q(p_n, q_n) = 1$. Define

$$\Lambda^{''} := \{v \in \Lambda, Q(p_n, v) = 0 = Q(q_n, v)\}.$$

Then $\Lambda^{''} \subset \Lambda'$ is a lattice of rank $2n-2$ such that $Q|_{\Lambda''}$ is unimodular and non-degenerate (cf. [4], the proof of Lemma in p.304). Let $\{p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}\}$ be a symplectic basis of $\Lambda^{''}$. Then

$$\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$$

is a symplectic basis of Λ with the required property. Q.E.D.

Proposition 12. *In the setting of Proposition 6, identify $\Lambda_s = H_1(M_s, \mathbb{Z})$ with Γ_s for $s \in B \setminus D$ as in the proof of Proposition 8. Assume that we have a principal polarization Q. Then we can find a collection of connected components $\{p_1, \ldots, p_n, q_1, \ldots, q_{n-1}\}$ of Γ' such that for each $x \in B \setminus D$, there exists $q_{n,x} \in \Gamma_x$ such that*

$$\{p_{1,x}, \ldots, p_{n,x}, q_{1,x}, \ldots, q_{n-1,x}, q_{n,x}\}$$

is a symplectic basis of $\Lambda_x = \Gamma_x$ with respect to Q_x.

Proof. Fix a point $s \in B \setminus D$. The monodromy $\tau_s : \Lambda_s \to \Lambda_s$ preserves the polarization Q_s on Λ_s and fixes $\Lambda'_s = \Gamma'_s$. Applying Lemma 2, we have a symplectic basis $\{p_{1,s}, \ldots, p_{n,s}, q_{1,s}, \ldots, q_{n,s}\}$ with $p_{1,s}, \ldots, p_{n,s}, q_{1,s}, \ldots, q_{n-1,s} \in \Lambda'_s$. Since Γ' consists of sections of $g : T^*B \to B$, the vectors $p_{1,s}, \ldots, p_{n,s}, q_{1,s}, \ldots, q_{n-1,s}$ uniquely determine components $p_1, \ldots, p_n, q_1, \ldots, q_{n-1}$ of Λ'. To check the existence of $q_{n,x}$ for any $x \in B \setminus D$, just pick as $q_{n,x}$ any vector in Λ_x contained in the component of Λ containing $q_{n,s}$. Q.E.D.

Proposition 13. *In the notation of Proposition 12, when $b \in D$, the vector $p_{n,b} \in \Gamma'_b$ regarded as an element of $H_1(Y_o, \mathbb{Z})$ in the notation of Proposition 7, lies in the lattice Υ of Proposition 7.*

Proof. Suppose not. By our (Assumption) after Definition 1, we may assume that M is topologically retractable to $f^{-1}(b)$ and identify $H^1(f^{-1}(b), \mathbb{Z})$ with $H^1(M, \mathbb{Z})$. Then by Proposition 7, there exists $\varpi \in H^1(f^{-1}(b), \mathbb{Z}) = H^1(M, \mathbb{Z})$ such that $\langle \varpi, j_* p_{n,b} \rangle \neq 0$. For a point $s \in B \setminus D$, the choice in Proposition 12 implies that $p_{n,s} \in \Xi_s$ of Proposition 11. Denote by ϖ_s the element in $H^1(M_s, \mathbb{Z})$ induced by $\varpi \in H^1(M, \mathbb{Z})$ under the identification $H^1(f^{-1}(b), \mathbb{Z}) = H^1(M, \mathbb{Z})$. Since $j_* p_{n,b} \in H_1(f^{-1}(b), \mathbb{Z}) = H_1(M, \mathbb{Z})$ and the image of $p_{n,s} \in H_1(M_s, \mathbb{Z})$

in $H_1(M, \mathbb{Z})$ belongs to the same class, Proposition 10 and Proposition 11 say that

$$\langle \varpi, j_* p_{n,b} \rangle = \langle \varpi_s, p_{n,s} \rangle = 0.$$

This is a contradiction. Q.E.D.

Proposition 14. *In Proposition 13, the $\mathbb{C}$-linear span of Υ in $T_b^*(B)$ is exactly $\mathbb{C} \cdot dh$ where $h \in \mathcal{O}(B)$ is a defining equation of the divisor D.*

Proof. From the definition of Υ in Proposition 7, the linear span of Υ is sent to a fiber of the $\mathbb{C}^*$-bundle. By Definition 1 (i), this fiber is a leaf of the characteristic foliation, which is given by the Hamiltonian vector field $\iota_\omega(f^* dh)$ on M. Under the symplecto-morphism Φ in Proposition 5, this corresponds to $\mathbb{C} \cdot dh$. Q.E.D.

Proposition 15. *Let $\{p_1, \ldots, p_n, q_1, \ldots, q_{n-1}\}$ be as in Proposition 12. Then there exists a holomorphic coordinate system $\{z_1, \ldots, z_n\}$ on B such that, regarded as sections of $T^*(B)$,*

$$p_1 = dz_1, \ \ldots, \ p_n = dz_n$$

and D is given by $z_n = 0$.

Proof. Since $p_1, \ldots, p_n$ are closed 1-forms which are point-wise linearly independent at every point of B, we can find coordinates $z_1, \ldots, z_n$ with $p_i = dz_i$. By Proposition 14, we may choose z_n to be a defining equation of D. Q.E.D.

Let us recall the classical Riemann condition (e.g. [4], p.306).

Proposition 16. *Let V be a complex vector space of dimension n and let $\Lambda \subset V$ be a lattice of rank $2n$ such that V/Λ is an abelian variety with a principal polarization. For a symplectic basis*

$$\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$$

of Λ with respect to the principal polarization $Q : \wedge^2 \Lambda \to \mathbb{Z}$, $\{p_1, \ldots, p_n\}$ becomes a $\mathbb{C}$-basis of V and the period matrix (θ_i^j) defined by

$$q_i = \sum_{j=1}^{n} \theta_i^j p_j \quad \textit{in } V$$

is symmetric in (i, j) and $\mathrm{Im}(\theta_i^j) > 0$.

Theorem 1 now follows from the next Theorem:

Theorem 3. *Given a principally polarized stable Lagrangian fibration $f : M \to B$ with a Lagrangian section $\Sigma \subset M$, there exists a holomorphic coordinate system $(z_1, \ldots, z_n)$ on B such that*

(i) *$z_n = 0$ is a local defining equation of D;*

(ii) *on $B \setminus D$, $dz_1, \ldots, dz_{n-1}, dz_n$ belong to Γ' in the notation of Proposition 6;*

(iii) *there exists a symplectic basis $\{p_{1,s}, \ldots, p_{n,s}, q_{1,s}, \ldots, q_{1,n}\}$ on each $\Lambda_s = \Gamma_s, s \in B \setminus D$ satisfying*

$$p_{1,s} = (dz_1)_s, \ldots, p_{n,s} = (dz_n)_s$$

and the associated period matrix in the sense of Proposition 16 is given by

$$\theta_i^j = \frac{\partial^2 \Psi}{\partial z_i \partial z_j} + \delta_{in}\delta_{jn}\frac{\ell}{2\pi\sqrt{-1}}\log z_n$$

for some holomorphic function Ψ on B, which we call a potential function *of the Lagrangian fibration, and some integer ℓ.*

Proof. Let $\{p_1, \ldots, p_n, q_1, \ldots, q_{n-1}\}$ be as in Proposition 12 and Proposition 15. At a point $s \in B \setminus D$, we add $q_{n,s}$ to get a symplectic basis of Λ_s. By analytic continuation, we get a multi-valued 1-form q_n over $B \setminus D$ such that any choice of a value $q_{n,t}$ of q_n at a point $t \in B \setminus D$, together with $p_{1,t}, \ldots, p_{n,t}, q_{1,t}, \ldots, q_{n-1,t}$, gives a symplectic basis of Λ_t. Using the coordinate system in Proposition 15, we can write

$$q_i = \sum_{j=1}^{n} \theta_i^j dz_j,$$

where θ_i^j is a (univalent) holomorphic function on B for each $1 \le i \le n-1$ and $1 \le j \le n$, while θ_n^j is a multi-valued holomorphic function on $B \setminus D$ for each $1 \le j \le n$. By Proposition 16, $\theta_j^i = \theta_i^j$ for each $1 \le i, j \le n$. It follows that θ_n^j is univalent holomorphic function on B for each $1 \le j \le n-1$. By the choice of $p_{n,s} \in \Xi_s$ and Proposition 11, the monodromy operator $\tau_s : \Lambda_s \to \Lambda_s$ is of the form

$$\tau_s(q_{n,s}) = q_{n,s} + \ell p_{n,s}$$

for some integer ℓ. Thus

$$\tilde{\theta}_n^n := \theta_n^n - \frac{\ell}{2\pi\sqrt{-1}}\log z_n$$

is univalent. $\tilde{\theta}_n^n$ is also holomorphic on B by the positive definiteness of $\mathrm{Im}\,(\theta_i^j)$. Set $\tilde{\theta}_i^j := \theta_i^j$ if $(i,j) \neq (n,n)$. Then $\tilde{\theta}_i^j$ is a univalent holomorphic function on B for all values of $1 \leq i, j \leq n$ and

$$q_i = \sum_{j=1}^{n} \tilde{\theta}_i^j dz_j \text{ for } 1 \leq i \leq n-1$$

$$q_n = \sum_{j=1}^{n} \tilde{\theta}_n^j dz_j + \frac{\ell}{2\pi\sqrt{-1}} \log z_n \; dz_n.$$

Since q_i's are closed 1-forms on B, we have

$$\frac{\partial \tilde{\theta}_i^j}{\partial z_k} = \frac{\partial \tilde{\theta}_i^k}{\partial z_j} = \frac{\partial \tilde{\theta}_j^i}{\partial z_k}$$

for any $1 \leq i, j, k \leq n$. By Poincaré's lemma, there exists a holomorphic function Ψ such that

$$\tilde{\theta}_i^j = \frac{\partial^2 \Psi}{\partial z_i \partial z_j}.$$

Q.E.D.

§4. Construction of principally polarized stable Lagrangian fibrations with given potential functions

In this section, we shall prove Theorem 2 and Proposition 1. For an n-dimensional polydisc B with a given holomorphic function $\Psi(z)$ with the property specified in Theorem 2, we shall construct a principally polarized stable Lagrangian fibration $f : (M, \omega_M) \to B$ having $\Psi(z)$ as its potential function. This will turn out to be $f^{\mathrm{model}} : M^{\mathrm{model}} \to \Delta^n$ in Theorem 2. Our construction closely follows Nakamura's toroidal construction [Na]. However, main differences are the following:

(i) the base space B is of dimension n (rather than 1).

(ii) the total space should be not only smooth but also symplectic.

(I) Construction of a non-proper Lagrangian fibration $\tilde{M} \to B$.

For each integer $k \in \mathbb{Z}$, let E_k be a copy of $\mathbb{C} \times \mathbb{C}$ equipped with linear coordinates (x_k, y_k). We define a complex manifold E by identifying points in $\cup_{k \in \mathbb{Z}} E_k$ by the following rule: a point (x_k, y_k) of E_k with $x_k \neq 0$ is identified with a point (x_{k+1}, y_{k+1}) of E_{k+1} with $y_{k+1} \neq 0$, if and only if

$$x_{k+1} = x_k^2 y_k, \quad \text{and} \quad y_{k+1} = \frac{1}{x_k}.$$

On E, $z_n := x_k y_k$ is a well-defined holomorphic function independent of k and

$$w_n := x_k^{-k+1} y_k^{-k}$$

is a meromorphic function independent of k, with zeros and poles supported on

$$\cup_{k \in \mathbb{Z}} (x_k y_k = 0).$$

Moreover, the 2-forms $dy_k \wedge dx_k$ glue together yielding a holomorphic symplectic form ω_E on E, satisfying

$$\omega_E = dz_n \wedge \frac{dw_n}{w_n}.$$

Fix coordinates

$$(z_1, \ldots, z_{n-1}, w_1, \ldots, w_{n-1})$$

on $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$ and regard them as functions on the open subset $\mathbb{C}^{n-1} \times (\mathbb{C}^\times)^{n-1}$ defined by

$$w_1 \neq 0, \ \ldots, \ w_{n-1} \neq 0.$$

Define

$$\tilde{X} := \mathbb{C}^{n-1} \times (\mathbb{C}^\times)^{n-1} \times E.$$

On $\tilde{X}$, we have the holomorphic functions $z_1, \ldots, z_n, w_1, \ldots, w_{n-1}$ and the meromorphic function w_n.

Define a morphism $\tilde{p} : \tilde{X} \to \mathbb{C}^n$ by $(z^1, \ldots, z^n)$. The fiber of $\tilde{p}$ over b with $z_n(b) \neq 0$ is isomorphic to

$$(\mathbb{C}^\times)^{n-1} \times \mathbb{C}^\times$$

with coordinates $(w_1, \ldots, w_{n-1}, w_n)$ and the fiber over b with $z_n(b) = 0$ is isomorphic to

$$(\mathbb{C}^\times)^{n-1} \times \cup_{k \in \mathbb{Z}} \mathbb{P}^1_k$$

where $\mathbb{P}^1_k$ is a copy of the projective line $\mathbb{P}^1$ with affine coordinate x_k.

We have a holomorphic symplectic 2-form

$$\omega_{\tilde{X}} := \sum_{i=1}^{n-1} dz_i \wedge \frac{dw_i}{w_i} + \omega_E = \sum_{i=1}^{n} dz_i \wedge \frac{dw_i}{w_i}$$

on $\tilde{X}$. From now, we regard $\tilde{X}$ as a symplectic manifold by this symplectic form. From the coordinate expression of $\omega_{\tilde{X}}$ and $\tilde{p}$, it is immediate that $\tilde{p}$ is a *non-proper* Lagrangian fibration.

We set

$$\tilde{M} := \tilde{X} \times_{\mathbb{C}^n} B$$

where

$$B = \{(z_1, \ldots, z_{n-1}, z_n) \mid |z_i| < 1 (\forall i)\}.$$

We denote the natural projection $\tilde{M} \to B$ induced from $\tilde{p}$ by

$$\tilde{f} : \tilde{M} \to B \ .$$

Note that the restriction $\omega_{\tilde{M}}$ of $\omega_{\tilde{X}}$ is a symplectic 2-form on $\tilde{M}$ and $\tilde{f}$ is a non-proper Lagrangian fibration.

(II) Group action of $\Gamma = \mathbb{Z}^n$ *on* $\tilde{M}$*.*

Let $\Psi(z_1, z_2, \cdots, z_n)$ be a holomorphic function on B and ℓ be a positive integer such that the imaginary part $\operatorname{Im}\theta(z)$ of the period matrix

$$\theta(z) = \tilde{\theta}(z) + \frac{\log z_n}{2\pi\sqrt{-1}} \begin{pmatrix} O_{n-1} & 0 \\ 0 & \ell \end{pmatrix}$$

is positive definite. Here

$$\tilde{\theta}(z) = (\frac{\partial^2 \Psi}{\partial z_i \partial z_j})$$

is the Hessian matrix of $\Psi(z_1, z_2, \cdots, z_n)$. We will write

$$\tilde{\theta}(z) = \begin{pmatrix} \tilde{\Theta}_1(z) & \tilde{\Theta}_2(z) \\ \tilde{\Theta}_2^t(z) & \tilde{\theta}_n^n(z) \end{pmatrix} \ ,$$

where $\tilde{\Theta}_1(z)$ is an $(n-1) \times (n-1)$ matrix, $\tilde{\Theta}_2(z)$ is an $(n-1) \times 1$ matrix, $\tilde{\Theta}_2^t(z)$ is the transpose of $\tilde{\Theta}_2(z)$ and $\tilde{\theta}_n^n(z)$ is a 1×1 matrix.

Set $\Gamma = \mathbb{Z}^{n-1} \oplus \mathbb{Z}$. We define a group action of Γ on $\tilde{M}$ as follows. Let $\gamma = (j, m) \in \Gamma$. Then the action $T_\gamma : \tilde{M} \to \tilde{M}$ is defined in terms of the coordinate functions on $\mathbb{C}^{n-1} \times (\mathbb{C}^\times)^{n-1} \times E_k \subset \tilde{M}$ by

$$T_\gamma^* z_i = z_i \text{ for } i = 1, \ldots, n-1$$

$$T_\gamma^*(\Pi_{i=1}^{n-1} w_i^{b_i}) = \exp(2\pi\sqrt{-1}(j\tilde{\Theta}_1(z)b + m\tilde{\Theta}_2^t(z)b)\Pi_{i=1}^{n-1} w_i^{b_i}$$

where $b = (b_i)_{i=1}^{n-1}$ is an $(n-1) \times 1$ matrix, and

$$T_\gamma^* x_k = \exp(2\pi\sqrt{-1}(j\tilde{\Theta}_2(z) + m\tilde{\theta}_n^n(z))x_{k+m\ell}$$

$$T_\gamma^* y_k = (\exp(2\pi\sqrt{-1}(j\tilde{\Theta}_2(z) + m\tilde{\theta}_n^n(z)))^{-1} y_{k+m\ell} \ .$$

It is immediate that $T_\gamma^* T_{\gamma'}^* = T_{\gamma+\gamma'}^*$. Then $T_\gamma \in \mathrm{Aut}\,(\tilde{M}/B)$ and $\gamma \mapsto T_\gamma$ defines an injective group homomorphism from Γ to $\mathrm{Aut}\,(\tilde{M}/B)$. Here $\mathrm{Aut}\,(\tilde{M}/B)$ is the group of automorphisms of $\tilde{M}$ over B, i.e., the group of automorphisms g of $\tilde{M}$ such that $\tilde{f} \circ g = \tilde{f}$.

Proposition 17. *The action Γ on $\tilde{M}$ is properly discontinuous, free and symplectic, in the sense that $T_\gamma^* \omega_{\tilde{M}} = \omega_{\tilde{M}}$ for each $\gamma \in \Gamma$.*

Proof. Freeness of the action is clear from the description of the action. The proof of proper discontinuity is essentially the same as the proof of [13], Theorem 2.6. This can be also seen from the concrete description of fibers below in (III).

Let us show that the action is symplectic, i.e., $\omega_{\tilde{M}} = T_\gamma^* \omega_{\tilde{M}}$ for each $\gamma = (j, m)$. This is a new part not considered by [13]. We have $T_\gamma^* z_i = z_i$, $T_\gamma^* w_i = \exp\,(2\pi\sqrt{-1} f_i(z)) w_i$ for all i, where, in terms of the standard basis $\langle e_i \rangle_{i=1}^{n-1}$ of $\mathbb{C}^{n-1}$,

$$f_i(z) = j\tilde{\Theta}_1(z)e_i + m\tilde{\Theta}_2^t(z)e_i$$

for $1 \le i \le n-1$ and

$$f_n(z) = j\tilde{\Theta}_2(z) + m(\tilde{\theta}_n^n(z) + \frac{\ell \log z_n}{2\pi\sqrt{-1}}) \ .$$

Thus, for i with $1 \le i \le n$, we have

$$T_\gamma^* dz_i = dz_i \ ,$$

$$T_\gamma^* \frac{dw_i}{w_i} = T_\gamma^*(d \log w_i)$$

$$= d\log(T_\gamma^* w_i) = d(2\pi\sqrt{-1} f_i(z)) + d(\log w_i)$$

$$= \frac{dw_i}{w_i} + 2\pi\sqrt{-1} \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k} dz_k \ .$$

Using these identities, we can compute

$$T_\gamma^* \omega_{\tilde{M}} = \sum_{i=1}^{n} T_\gamma^*(dz_i) \wedge T_\gamma^*(\frac{dw_i}{w_i})$$

$$= \omega_{\tilde{M}} - 2\pi\sqrt{-1} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k} dz_k \wedge dz_i$$

$$= \omega_{\tilde{M}} - 2\pi\sqrt{-1} \sum_{1 \le i < k \le n} (\frac{\partial f_i}{\partial z_k} - \frac{\partial f_k}{\partial z_i}) dz_k \wedge dz_i \ .$$

On the other hand, by definition of $f_i(z)$ $(1 \le i \le n)$ and definition of $\tilde{\theta}(z)$ from the potential function $\Psi(z)$, we know that

$$f_i(z) = \sum_{\alpha=1}^{n-1} j_\alpha \tilde{\theta}^i_\alpha(z) + m\tilde{\theta}^i_n$$

$$= \sum_{\alpha=1}^{n-1} j_\alpha \frac{\partial^2 \Psi}{\partial z_\alpha \partial z_i} + m\frac{\partial^2 \Psi}{\partial z_n \partial z_i} + \delta_{i,n}\frac{m\ell \log z_n}{2\pi\sqrt{-1}} \ .$$

Since $\Psi(z)$ is holomorphic, it follows that

$$\frac{\partial}{\partial z_k}(\frac{\partial^2 \Psi}{\partial z_\alpha \partial z_i}) = \frac{\partial^3 \Psi}{\partial z_k \partial z_\alpha \partial z_i} = \frac{\partial}{\partial z_i}(\frac{\partial^2 \Psi}{\partial z_\alpha \partial z_k}) \ .$$

Substituting this into the formula above, we obtain that $T^*_\gamma \omega_{\tilde{M}} = \omega_{\tilde{M}}$. Q.E.D.

(III) Group quotient of $\tilde{M}$ by $\Gamma = \mathbb{Z}^n$.

Let $M = \tilde{M}/\Gamma$. By Proposition 17, M is a smooth symplectic manifold with symplectic form ω_M induced by $\omega_{\tilde{M}}$ and M admits a fibration $f : M \to B$ induced by $\tilde{f}$. We denote the (scheme theoretic) fiber $f^{-1}(b)$ over $b \in B$ by M_b. Let us describe the fibers M_b.

(III-1) Smooth fibers M_b

First consider the case where $z_n(b) \neq 0$, i.e., the case where M_b is smooth. We have

$$\tilde{M}_b = (\mathbb{C}^\times)^{n-1} \times \{(x_k, y_k) | x_k y_k = b_n\} \simeq (\mathbb{C}^\times)^n_{(w_1,\dots,w_{n-1},w_n)} \ .$$

and $w_n = z_n(b)^{-(k+1)} x_k$. Let $\langle e_i \rangle_{i=1}^n$ be the ordered standard basis of Γ. From the description in (II), the action of Γ is given by:

$$T^*_{e_i} w_j = \exp(2\pi\sqrt{-1}\theta^j_i(b)) w_j$$

$$T^*_{e_i} w_n = \exp(2\pi\sqrt{-1}\theta^n_i(b)) w_n$$

for $1 \le i \le n-1$ with $\theta^j_i = \tilde{\theta}^j_i$ and

$$T^*_{e_n} w_j = \exp(2\pi\sqrt{-1}\theta^j_n(b)) w_j$$

$$T^*_{e_n} w_n = \exp(2\pi\sqrt{-1}\tilde{\theta}^n_n(b)) z_n(b)^\ell w_n = \exp(2\pi\sqrt{-1}\theta^n_n(b)) w_n \ .$$

Let us consider the universal covering map of fibers over $B \setminus D$:

$$\pi_b : \mathbb{C}^n \to (\mathbb{C}^\times)^n = \tilde{M}_b \ ; \ (t_i)_{i=1}^n \mapsto (\exp(2\pi\sqrt{-1}t_i))_{i=1}^n \ .$$

Then

$$M_b = \tilde{M}_b/\Gamma = \mathbb{C}^n/\Lambda_b$$

where $\Lambda_b \subset \mathbb{C}^n$ is the sublattice generated by the following $2n$ vectors:

$$e_i := (\delta_{ij})_{j=1}^n \ , \ \theta_i(b) := (\theta_{ij}(b))_{j=1}^n \ 1 \le i \le 2n \ .$$

Hence M_b is an n-dimensional principally polarized abelian variety of period $\theta(b)$, as desired. By the description of ω_M, the fibers M_b, $z_n(b) \neq 0$, are also Lagrangian submanifolds.

Proposition 18. *For $b \in B, z_n(b) \neq 0$, choose the basis*

$$p_{1,b}, \dots, p_{n,b},\ q_{1,b}, \dots, q_{n,b}$$

of $H_1(M_b, \mathbb{Z})$ such that $\tilde{M}_b = \mathbb{C}^n/\langle p_{j,b}\rangle_{j=1}^n$ and

$$q_{i,b} = \sum_{j=1}^n \theta_i^j(b) p_{j,b}$$

for each i $(1 \le i \le n)$. Let

$$p_b^1, \dots, p_b^n,\ q_b^1, \dots, q_b^n$$

be the dual basis of $H^1(M_b, \mathbb{Z})$. Then the integral 2-form

$$L_b := \sum_{i=1}^n p_b^i \wedge q_b^i$$

gives a monodromy invariant principal polarization of M over $B \setminus D$ where $D = (z_n = 0)$.

Proof. When $z_n(b) \neq 0$, the fiber $\tilde{M}_b$ of $\tilde{p} : \tilde{M} \to B$ is $(\mathbb{C}^\times)^n$ and this family has no monodromy over $B \setminus (z_n = 0)$. Thus we can fix a basis $p_{1,b}, \dots, p_{n,b}$ of $H_1(\tilde{M}_b, \mathbb{Z})$ uniformly in $b, z_n(b) \neq 0$.

To get a basis of $H_1(M_b, \mathbb{Z})$, we choose additional elements

$$q_{1,b}, \dots, q_{n,b} \in H_1(M_b, \mathbb{Z})$$

determined by the deck-transformation of $\tilde{M}_b$ induced by the action $T_{e_1}, \dots, T_{e_n}$. From the description of $T_{e_i}^*$ on w_j, they satisfy the relation

$$q_{i,b} = \sum_{j=1}^n \theta_i^j(b) p_{j,b}.$$

We see that $q_{1,b},\dots,q_{n-1,b}$ are invariant under the monodromy, while $q_{n,b}\mapsto q_{n,b}+\ell p_{n,b}$ under the monodromy of the generator γ of $\pi_1(B\setminus D)$, i.e., the circle around discriminant divisor $z_n=0$. The 2-form L_b is a principal polarization on M_b. It remains to show that L_b is invariant under the monodromy. By definition of $\theta(b)$, we compute that

$$\gamma^*(L_b)=\gamma^*(\sum_{i=1}^{n-1}p_b^i\wedge q_b^i)+\gamma^*(p_b^n\wedge q_b^n)$$

$$=\sum_{i=1}^{n-1}p_b^i\wedge q_b^i+p_b^n\wedge(q_b^n-\ell p_b^n)=L_b.$$

This implies the invariance. Q.E.D.

Remark 4. *As in* [13], *one can also describe* $\tilde f:\tilde M\to B$ *in terms of toric geometry. Following an argument similar to* [13], *Section 4, it seems possible to give a relatively principally polarized divisor (the relative theta divisor) which is defined globally over* $B\setminus D$*. However, its closure is not necessarily* f*-ample even if total space is of dimension* 4 *(cases of stable principally polarized Lagrangian* 4*-folds). In fact, a failure of* f*-ampleness of the closure already happens when the fiber dimension is* 2 *and the base dimension is* 1 *as explicitly described in* [13] *Section 4, Page 219. See also Remark 3.6.*

(III-2) Singular fibers M_b

Next consider the singular fibers of f. They are M_b with $z_n(b)=0$. Recall from (I) that $\tilde M_b$ is the product of $(\mathbb{C}^\times)^{n-1}$ with coordinate $(w_i)_{i=1}^{n-1}$ and the infinite tree $\cup_{k\in\mathbb{Z}}\mathbb{P}^1_k$ of projective lines $\mathbb{P}^1_k$ with affine coordinate x_k, and $M_b=\tilde M_b/\Gamma$. Let us denote by $(0)_k,(\infty)_k\in\mathbb{P}^1_k$ the two points on the projective line $\mathbb{P}^1_k$ such that $(0)_k$ is identified with $(\infty)_{k-1}$ in the tree.

From (II), the action of Γ is given by:

$$T^*_{e_i}w_j=\exp(2\pi\sqrt{-1}\tilde\theta_i^j(b))w_j$$

$$T^*_{e_i}x_k=\exp(2\pi\sqrt{-1}\tilde\theta_i^n(b))x_k$$

for $1\le i\le n-1$ and

$$T^*_{e_n}w_j=\exp(2\pi\sqrt{-1}\tilde\theta_n^j(b))w_j$$

$$T^*_{e_n}x_k=\exp(2\pi\sqrt{-1}\tilde\theta_n^n(b))x_{k+\ell}\ ,$$

where $\langle e_i \rangle_{i=1}^n$ is the ordered standard basis of Γ. Here we note that the last equality shows that the monodromy operation corresponds to the shift of the components of the infinite tree $\cup_{k\in\mathbb{Z}}\mathbb{P}^1_k$. Thus $\tilde{M}/\langle e_n\rangle$ can be described as the variety obtained from

$$(\mathbb{C}^\times)^{n-1} \times \cup_{k=0}^{\ell-1}\mathbb{P}^1_k$$

by identifying the point

$$(w^1, \dots, w^{n-1}) \times (0)_0 \ \in \ (\mathbb{C}^\times)^{n-1} \times (0)_0$$

with the point

$$(\exp(2\pi\sqrt{-1}\tilde{\theta}_n^1)w^1, \dots, \exp(2\pi\sqrt{-1}\tilde{\theta}_n^{n-1})w^{n-1}) \ \in \ (\mathbb{C}^\times)^{n-1} \times (\infty)_{\ell-1}.$$

From this description, M_b consists of ℓ irreducible components, each of whose normalization is isomorphic to a $\mathbb{P}^1$-bundle over a complex torus of dimension $(n-1)$ isogenous to $(\mathbb{C}^\times)^{n-1}/\langle e_i\rangle_{i=1}^{n-1}$, where the action of $\langle e_i\rangle_{i=1}^{n-1}$ is given by the coordinate action $T_{e_i}^*$ $(1 \le i \le n-1)$ on w_j $(1 \le j \le n-1)$ described above. Note that the quotient $(\mathbb{C}^\times)^{n-1}/\langle e_i\rangle_{i=1}^{n-1}$ is compact because the imaginary part of $\tilde{\Theta}_1(z)$ is positive definite from the assumption that the imaginary part of $\theta(z)$ is positive definite. We also note that the characteristic cycles are of type I_m for some $1 \le m \le \infty$.

Let us denote the fibration $f : M \to B$ constructed above by $f^{\text{model}} : M^{\text{model}} \to B$. From the construction and the description of singlar fibers in (III-2), we obtain Theorem 2.

To prove Proposition 1, we need the following proposition, which is essentially proved in [8] Proposition 5.1:

Proposition 19. *Let $B = \{(z_1, \dots, z_n) | |z_i| < 1\}$ be the polydisc. Let $f : M \to B$ and $f' : M' \to B$ be two Lagrangian fibrations with the same discriminant $D \subset B$, having Lagrangian sections $\Sigma \subset M$ and $\Sigma' \subset M'$. Suppose there exists a biholomorphic morphism $\Phi : M \setminus f^{-1}(D) \to M' \setminus f'^{-1}(D)$ over B such that $\Phi(\Sigma) = \Sigma'$ and Φ is symplectomorphic, i.e., $\Phi^*\omega_{M'} = \omega_M$. Then*

(1) Φ extends to a bimeromorphic map $M \cdots \to M'$ over B and the extended bimeromorphic map, denoted by the same latter Φ, is isomorphic in codimension 1.

(2) Let $b \in D$ and $L \subset B$ be the intersection of B with a general line passing through b, in terms of the coordinates $(z_1, \dots, z_n)$. Let $M_L = f^{-1}(L)$, $M'_L = (f')^{-1}(L)$ and $\Phi_L : M_L \cdots \to M'_L$, the bimeromprphic map over L induced from Φ. Then Φ_L extends to a biholomorphic map over L. In particular, $M_b \simeq M'_b$ for all $b \in D$.

Proof. The statement (2) is exactly the same as [8] Proposition 5.1. Since both K_M and $K_{M'}$ are trivial, the same argument of [8] Proposition 5.1 concludes that Φ and Φ^{-1} extend to bimeromorphic maps such that the indeterminacies are of codimension at least 2. (However, the argument there does not automatically exclude the possible indeterminacies of Φ and Φ^{-1}.) Q.E.D.

Proof of Proposition 1. We can apply Proposition 19 (1) for $f : M \to B$ and $f^{\text{model}} : M^{\text{model}} \to B$ with the same potential function $\Psi(z)$ and ℓ to obtain Proposition 1. Q.E.D.

Applying Proposition 19 (2) for $f : M \to B$ and $f^{\text{model}} : M^{\text{model}} \to B$, we also obtain

Corollary 1. *For a general principally polarized stable Lagrangian fibration $f : M \to B$, the singular fibers are isomorphic to the singular fibers of its model $f^{\text{model}} : M^{\text{model}} \to B$. In particular, the positive integer $|\ell|$ in Theorem 3 is the number of components of the singular fiber and $S_1^i \neq S_2^i$ for each i in the notation of Definition 1.*

It is natural to ask whether Φ in Proposition 19 can be extended to a biholomorphic map from M to M'. We do not know the answer in the general case. But this is indeed the case if f and f' are projective morphisms, as explained by the next proposition which we learned from Kawamata.

Proposition 20. *Let $f : M \to B$ a principally polarized stable Lagrangian fibration and $f^{\text{model}} : M^{\text{model}} \to B$ be the model with the same potential function $\Psi(z)$ and ℓ as f. Assume that the morphisms f and f^{model} are projective. Then M and M^{model} are isomorphic over B.*

Proof. By assumption, M and M^{model} are minimal, birational and projective over B. Let $\Phi : M \cdots \to M^{\text{model}}$ be a birational map over B. If Φ is not an isomorphism, then Φ is decomposed into a finite sequence of flops over B by Theorem 1 of [10]. In particular, M^{model} admits at least one flopping contraction. The exceptional locus of a flopping contraction is covered by rational curves by Theorem 1 of [9]. Moreover, all rational curves numerically equivalent to any of these rational curves are contracted under this flopping contraction by Contraction Theorem (Theorem 3.2.1 (i) of [11]). On the other hand, by the construction of M^{model}, the total space of deformation of any rational curve C in M^{model} is a codimension 1 subset of M^{model}. This contradicts the fact that the exceptional locus of a flopping contraction has codimension ≥ 2 (cf. p. 420 Proof in [10]). It follows that M^{model} has no flopping contraction and Φ is an isomorphism. Q.E.D.

§5. Periods and the characteristic cycles

In this section, we will examine the relation between types of the characteristic cycles and the periods. For simplicity, we will restrict our discussion to the case of $\ell = 1$. The generalization to arbitrary ℓ is straightforward. Explicit constructions (Constructions I -III) will be given when $n = 2$, i.e., constructions of 4-dimensional principally polarized stable Lagrangian fibrations. Construction I gives an explicit example in which the types of characteristic cycles change fiber by fiber. Construction II gives an explicit example in which the types of characteristic cycles are constant type I_m ($m < \infty$) and Construction III gives an explicit example in which the types of characteristic cycles are constant type A_∞.

Proposition 21. *Let $f : M \to B$ be a $2n$-dimensional principally polarized stable Lagrangian fibration with potential function $\Psi(z)$ and $\ell = 1$. We denote the (univalent) period matrix by $\tilde{\theta}(z) = (\tilde{\theta}_i^j(z))_{i,j=1}^n$ and the multi-valued period matrix $\theta(z)$ of f as*

$$\theta(z) = \tilde{\theta}(z) + \frac{\log z_n}{2\pi\sqrt{-1}} \begin{pmatrix} O_{n-1} & 0 \\ 0 & 1 \end{pmatrix} .$$

For $b \in D$ for which M_b is singular, define $n(b)$ ($1 \le n(b) \le \infty$) to be the order of

$$(\exp(2\pi\sqrt{-1}\tilde{\theta}_n^j(b)))_{j=1}^{n-1} \bmod \langle(\exp(2\pi\sqrt{-1}\tilde{\theta}_i^j(b)))_{j=1}^{n-1} \,|\, 1 \le i \le n-1\,\rangle$$

in the multiplicative group

$$(\mathbb{C}^\times)^{n-1}/\langle(\exp(2\pi\sqrt{-1}\tilde{\theta}_i^j(b)))_{j=1}^{n-1} \,|\, 1 \le i \le n-1\,\rangle \ .$$

Then the characteristic cycle of M_b is of type $I_{n(b)}$.

Remark 5. *When $n = 2$, we can state the above result more explicitly as follows. We shall use this description in Constructions (I)-(III) below. We write the (univalent) period matrix $\tilde{\theta}(z)$ and the multi-valued period matrix $\theta(z)$ of f as*

$$\tilde{\theta}(z) = \begin{pmatrix} \tilde{\theta}_1(z) & \tilde{\theta}_2(z) \\ \tilde{\theta}_2(z) & \tilde{\theta}_3(z) \end{pmatrix} , \theta(z) = \tilde{\theta}(z) + \frac{\log z_2}{2\pi\sqrt{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

For $b = (b_1, 0) \in B$ for which M_b is singular, the characteristic cycle of M_b is then of type $I_{n(b)}$, where $n(b)$ ($1 \le n(b) \le \infty$) is exactly the order of

$$\exp(2\pi\sqrt{-1}\tilde{\theta}_2(b)) \bmod \langle\exp(2\pi\sqrt{-1}\tilde{\theta}_1(b))\rangle$$

in the multiplicative group $\mathbb{C}^\times/\langle\exp(2\pi\sqrt{-1}\tilde{\theta}_1(b))\rangle$.

Proof. In the description (III-2), M_b is the quotient of

$$\tilde{M}_b = \cup_{k\in\mathbb{Z}}(\mathbb{C}^\times)^{n-1} \times \mathbb{P}^1_k$$

with coordinates $((w_j)_{j=1}^{n-1}, x_k)$ $(k \in \mathbb{Z})$ by the action of $\Gamma = \mathbb{Z}^n$ with ordered standard basis

$$\langle e_1, \dots, e_{n-1}, e_n\rangle \ .$$

In terms of the standard basis, the action is given by:

$$T^*_{e_i} : (w_j)_{j=1}^{n-1} \mapsto (\exp(2\pi\sqrt{-1}\tilde{\theta}^j_i(b))w_j)_{j=1}^{n-1}$$

$$T^*_{e_i} : x_k \mapsto \exp(2\pi\sqrt{-1}\tilde{\theta}^n_i(b))x_k$$

for e_i $(1 \le i \le n-1)$, and for e_n

$$T^*_{e_n} : (w_j)_{j=1}^{n-1} \mapsto (\exp(2\pi\sqrt{-1}\tilde{\theta}^j_n(b))w_j)_{j=1}^{n-1}$$

$$T^*_{e_n} : x_k \mapsto \exp(2\pi\sqrt{-1}\tilde{\theta}^n_n(b))x_{k+1} \ .$$

Thus $\tilde{M}_b/\langle e_n\rangle$ is $(\mathbb{C}^\times)^{n-1} \times \mathbb{P}^1_0$ in which $((w_j)_{j=1}^{n-1}, 0)$ and $((w'_j)_{j=1}^{n-1}, \infty)$ are identified exactly when the two points

$$(w_j)_{j=1}^{n-1} \text{ and } (w'_j)_{j=1}^{n-1} \text{ of } (\mathbb{C}^\times)^{n-1}$$

are in the same orbit under the action of the cyclic subgroup

$$G(b) := \langle(\exp(2\pi\sqrt{-1}\tilde{\theta}^j_n(b)))_{j=1}^{n-1}\rangle$$

of $(\mathbf{C}^\times)^{n-1}$. On the other hand, $((\mathbb{C}^\times)^{n-1} \times \mathbb{P}^1_0)/\langle e_i\rangle_{i=1}^{n-1}$ is the normalization of M_b. Thus, M_b is obtained from $((\mathbb{C}^\times)^{n-1} \times \mathbb{P}^1_0)/\langle e_i\rangle_{i=1}^{n-1}$ by identifying the two $(n-1)-$dimensional complex tori $((\mathbb{C}^\times)^{n-1} \times \{\infty\})/\langle e_i\rangle_{i=1}^{n-1}$ and $((\mathbb{C}^\times)^{n-1} \times \{0\})/\langle e_i\rangle_{i=1}^{n-1}$,by the action of $G(b)$ above. Here, as a subgroup of $(\mathbb{C}^\times)^{n-1}$, the group $\langle e_i\rangle_{i=1}^{n-1}$ is the multiplicative subgroup generated by the $n-1$ elements

$$(\exp(2\pi\sqrt{-1}\tilde{\theta}^j_i(b)))_{j=1}^{n-1} \ , \ 1 \le i \le n-1 \ .$$

This implies the result. Q.E.D.

Construction I.

Under the notation of Proposition 21, we set $n = 2, \ell = 1$ and

$$\Psi(z_1, z_2) := \frac{(z_1 + 5\sqrt{-1})^3 + (z_2 + 5\sqrt{-1})^3 + 3z_1^2z_2 + 3z_1z_2^2}{6} \ ,$$

Then

$$\tilde{\theta}(z) = \begin{pmatrix} z_1 + z_2 + 5\sqrt{-1} & z_1 + z_2 \\ z_1 + z_2 & z_1 + z_2 + 5\sqrt{-1} \end{pmatrix},$$

$$\theta(z) = \tilde{\theta}(z) + \frac{\log z_2}{2\pi\sqrt{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathrm{Im}\,\theta(z) = \begin{pmatrix} y_1 + y_2 + 5 & y_1 + y_2 \\ y_1 + y_2 & y_1 + y_2 + 5 - \log|z_2|/2\pi \end{pmatrix}.$$

Here and hereafter x_i and y_i are the real and imaginary part of z_i respectively. Since $t + 5 > 0$ and $(t+5)^2 - t^2 > 0$ when $-2 < t < 2$ and $|z_2| < 1$, it follows that $\mathrm{Im}\,\theta(z)$ is positive definite on the polydisc

$$\{(z_1, z_2) \,|\, |z_i| < 1\}.$$

Taking a smaller 2-dimensional polydisc B with multi-radius ϵ, we then obtain a 4-dimensional Lagrangian fibration $f : M \to B$, associated with the potential function $\Psi(z)$ and $\ell = 1$. The discriminant set is $z_2 = 0$. Define $N = N(z)$ to be the order of

$$e^{2\pi\sqrt{-1}z_1} \bmod \langle e^{2\pi\sqrt{-1}(z_1+5\sqrt{-1})}\rangle$$

in the multiplicative group $\mathbb{C}^\times/\langle e^{2\pi\sqrt{-1}(z_1+5\sqrt{-1})}\rangle$. By abuse of language, we include $N = \infty$ when the order is not finite. Then, the characteristic cycle of $M_{(z_1,0)}$ is of type I_N.

Proposition 22. *In Construction 1, the characteristic cycle on $M_{(z_1,0)}$ is of Type I_k with $k < \infty$ if and only if*

$$z_1 \in \mathbb{Q}(\sqrt{-1}).$$

So, the singular fibers of finite characteristic cycle I_k ($k < \infty$) and the singular fibers of infinite characteristic cycle I_∞ are both dense over the discriminant set. Moreover, the characteristic cycle of $M_{(z_1,0)}$ is precisely of type I_k ($k < \infty$) for $z_1 = 1/k$. So, the singular fibers with characteristic cycles of type I_k with any sufficiently large k appear in this family.

Proof. By the definition of $N = N(z)$, it follows that $N < \infty$ for $M_{(z_1,0)}$ if and only if there are integers $k > 0$ and m such that

$$(e^{2\pi\sqrt{-1}z_1})^k = (e^{2\pi\sqrt{-1}(z_1+5\sqrt{-1})})^m.$$

The last condition is equivalent to

$$kz_1 - m(z_1 + 5\sqrt{-1}) \in \mathbb{Z}$$

which is also equivalent to

$$(k-m)x_1 \in \mathbb{Z} \text{ and } (k-m)y_1 - 5m = 0 \ .$$

Note that $k - m \neq 0$ in the last equivalent condition, as otherwise $k = m = 0$. It is immediate to see that two integers $k > 0$ and m satisfying last equivalent condition exist if and only if $z_1 \in \mathbb{Q}(\sqrt{-1})$. Since $|e^{2\pi\sqrt{-1}(z_1+5\sqrt{-1})}| > 1$ for $|z_1| < 1$, whereas $|e^{2\pi\sqrt{-1}/k}| = 1$ for $k \in \mathbb{Z}$, it follows that the order $N(z)$ for $z_1 = 1/k$ is precisely the order of $e^{2\pi\sqrt{-1}/k}$ in the multiplicative group $\mathbb{C}^\times$. This implies the last statement. Q.E.D.

Construction II.

Under the notation of Proposition 21, we set $n = 2, \ell = 1$ and

$$\Psi(z_1, z_2) := \frac{\sqrt{-1}(z_1^2 + z_2^2)}{2} + \frac{z_1 z_2}{k} \,,$$

where n is a positive integer. Then

$$\tilde{\theta}(z) = \begin{pmatrix} \sqrt{-1} & 1/k \\ 1/k & \sqrt{-1} \end{pmatrix} , \theta(z) = \tilde{\theta}(z) + \frac{\log z_2}{2\pi\sqrt{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\operatorname{Im}\theta(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \log|z_2|/2\pi \end{pmatrix} .$$

The matrix $\operatorname{Im}\theta(z)$ is positive definite. So, taking a smaller polydisc B of dimension 2, we obtain a 4-dimensional Lagrangian fibration $f : M \to B$, associated with the potential function $\Psi(z)$ and $\ell = 1$ above. The discriminant set is $z_2 = 0$. The order of

$$e^{2\pi\sqrt{-1}/k} \bmod \langle e^{-2\pi} \rangle$$

is exactly k in the multiplicative group $\mathbb{C}^\times / \langle e^{-2\pi} \rangle$, where

$$-2\pi = 2\pi\sqrt{-1} \cdot \sqrt{-1}.$$

Then, the characteristic cycle of $M_{(z_1,0)}$ is of type I_k, and in particular, the type is constant.

Construction III.

Under the notation of Proposition 21, we set $n = 2, \ell = 1$ and

$$\Psi(z_1, z_2) := \frac{\sqrt{-1}(z_1^2 + z_2^2)}{2} + \alpha z_1 z_2 \, ,$$

where α is any irrational, real number, say $\sqrt{2}$. Then

$$\tilde{\theta}(z) = \begin{pmatrix} \sqrt{-1} & \alpha \\ \alpha & \sqrt{-1} \end{pmatrix}, \theta(z) = \tilde{\theta}(z) + \frac{\log z_2}{2\pi\sqrt{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathrm{Im}\, \theta(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \log |z_2|/2\pi \end{pmatrix} .$$

The matrix $\mathrm{Im}\, \theta(z)$ is positive definite. So, taking a smaller polydisc B of dimension 2, we obtain a 4-dimensional Lagrangian fibration $f : M \to B$, associated with the potential function $\Psi(z)$ and $\ell = 1$ above. The discriminant set is $z_2 = 0$. Since α is an irrational real number, the element

$$e^{2\pi\sqrt{-1}\cdot\alpha} \bmod \langle e^{-2\pi} \rangle$$

is of infinite order in the multiplicative group $\mathbb{C}^\times / \langle e^{-2\pi} \rangle$, where $-2\pi = 2\pi\sqrt{-1} \cdot \sqrt{-1}$. Then, the characteristic cycle of $M_{(z_1, 0)}$ is of type A_∞, and in particular the type is constant.

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