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Order 40 automorphisms of K3 surfaces

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Abstract.

In each characteristic $p \neq 2, 5$, it is shown that order 40 automorphisms of K3 surfaces are purely non-symplectic. Moreover, a K3 surface in characteristic $p \neq 2, 5$, with a cyclic action of order 40 is isomorphic to the Kondō's example.

Let X be a K3 surface over an algebraically closed field k of characteristic $p \geq 0$. An automorphism g of X is called *symplectic* if it preserves a non-zero regular 2-form ω_X, and *purely non-symplectic* if no power of g is symplectic except the identity.

Over $k = \mathbb{C}$ Kondō [7] gave an example of a complex K3 surface with a purely non-symplectic automorphism of order 40 as the minimal resolution X_{40} of $X'_{40} \subset \mathbf{P}(1,1,1,3)$:

$$X'_{40} : w^2 = x(x^4 z + y^5 - z^5), \tag{0.1}$$

$$g_{40}(x,\, y,\, z,\, w) = (x,\, \zeta_{40}^2 y,\, \zeta_{40}^{10} z,\, \zeta_{40}^5 w) \tag{0.2}$$

where $\zeta_{40} \in k$ is a primitive 40th root of unity. The surface X'_{40} is a double plane branched along the union of a line and a smooth quintic curve. The surface X'_{40} is defined over the integers and both the surface and the automorphism have a good reduction mod p unless $p = 2, 5$.

The main result of this short note is the following.

Theorem 0.1. *Let k be an algebraically closed field of characteristic $p \neq 2, 5$. Let X be a K3 surface defined over k with an automorphism g of order 40. Then*

(1) *g is purely non-symplectic;*
(2) *the pair $(X, \langle g \rangle)$ is isomorphic to the pair $(X_{40}, \langle g_{40} \rangle)$.*

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Over $k = \mathbb{C}$ the second statement of Theorem 0.1 was proved by Machida and Oguiso [8], under the assumption that g is purely non-symplectic. Our proof is characteristic free, does not use lattice theory and the holomorphic Lefschetz formula.

A similar characterization of K3 surfaces with a tame cyclic action of order 60 was given in [6], where it was proven that for such a pair $(X, \langle g\rangle)$ the K3 surface X admits a g-invariant elliptic fibration, thus can be given by a g-invariant Weierstrass equation. In the case of order 40, we show that the K3 surface admits a g-invariant double plane presentation.

Using the algorithm for determining the Artin invariant of a weighted Delsarte surface whose minimal resolution is a K3 surface ([11], [3]), one can show that in characteristic $p \equiv -1 \pmod{40}$ the surface X_{40} is a supersingular K3 surface with Artin invariant 1.

Remark 0.2. In characteristic 5, cyclic actions of order 40 are wild and have been classified in [5] (Section 9 and Example 9.5). More precisely, a K3 surface in characteristic 5 with a cyclic action of order 40 must be isomorphic to the minimal resolution of the pair $(Y, \langle g\rangle)$:

$$Y : w^2 = z(y^5 - yx^4 + xz^4 + B_5x^5), \;\; B_5 \in k \tag{0.3}$$

$$g(x, y, z, w) = (x, 2x + y, \zeta_8^2 z, \zeta_8 w) \tag{0.4}$$

where $\zeta_8 \in k$ is a primitive 8th root of unity. The surface Y is a double plane branched along the union of a line and a smooth quintic curve.

Notation

- $\mathrm{NS}(X)$: the Néron-Severi group of X.
- $X^g = \mathrm{Fix}(g)$: the fixed locus of an automorphism g of X.
- $e(g) := e(\mathrm{Fix}(g))$, the Euler characteristic of $\mathrm{Fix}(g)$ for g tame.
- $\mathrm{Tr}(g^*|H^*(X)) := \sum_{j=0}^{2\dim X}(-1)^j\mathrm{Tr}(g^*|H^j_{\mathrm{et}}(X, \mathbb{Q}_l))$.

  For an automorphism g of a K3 surface X,
- $[g^*] = [\lambda_1, \ldots, \lambda_{22}]$: the eigenvalues of $g^*|H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$.
- ζ_a : a primitive a-th root of unity in $\overline{\mathbb{Q}_l}$.
- $\zeta_a : \phi(a)$: all primitive a-th roots of unity where ϕ is the Euler function and $\phi(a)$ the number of conjugates of ζ_a.
- $[\lambda.r] \subset [g^*]$: λ repeats r times in $[g^*]$.
- $[(\zeta_a : \phi(a)).r] \subset [g^*]$: the list $\zeta_a : \phi(a)$ repeats r times in $[g^*]$.

§1. Preliminaries

We first recall the following basic result used in the paper [5].

Proposition 1.1. (*Proposition 2.1* [5]) *Let g be an automorphism of a projective variety X over an algebraically closed field k of characteristic $p > 0$. Let l be a prime $\neq p$. Then the following hold true.*

(1) (*3.7.3* [4]) *The characteristic polynomial of $g^*|H^j_{\mathrm{et}}(X, \mathbb{Q}_l)$ has integer coefficients for each j. The characteristic polynomial does not depend on the choice of cohomology, l-adic or crystalline. In particular, if a primitive m-th root of unity appears with multiplicity r as an eigenvalue of $g^*|H^j_{\mathrm{et}}(X, \mathbb{Q}_l)$, then so does each of its conjugates.*

(2) *If g is of finite order, then g has an invariant ample divisor, and $g^*|H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ has 1 as an eigenvalue.*

(3) *If X is a K3 surface, g is tame and $g^*|H^0(X, \Omega^2_X)$ has $\zeta_n \in k$ as an eigenvalue, then $g^*|H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ has $\zeta_n \in \overline{\mathbb{Q}_l}$ as an eigenvalue.*

For a smooth projective variety Z in characteristic $p > 0$, the Kummer sequence in étale cohomology [9] induces an exact sequence of $\mathbb{Q}_l$-vector spaces

$$0 \to \mathrm{NS}(Z) \otimes \mathbb{Q}_l \to H^2_{\mathrm{et}}(Z, \mathbb{Q}_l) \to T^2_l(Z) \to 0 \tag{1.1}$$

where $T^2_l(Z) = T_l(\mathrm{Br}(Z))$ in the standard notation in the theory of étale cohomology (cf. [11]). The Brauer group $\mathrm{Br}(Z)$ is known to be a birational invariant, and it is trivial when Z is a rational variety. In fact, one can show that

$$\mathrm{NS}(Z) \otimes \mathbb{Q}_l = \mathrm{Ker}(H^2_{\mathrm{et}}(Z, \mathbb{Q}_l) \to H^2(k(Z), \mathbb{Q}_l));$$

$$T^2_l(Z) = \mathrm{Im}(H^2_{\mathrm{et}}(Z, \mathbb{Q}_l) \to H^2(k(Z), \mathbb{Q}_l)).$$

Here $H^2(k(Z), \mathbb{Q}_l) = \varinjlim_U H^2(U, \mathbb{Q}_l)$, where U runs through the set of open subsets of Z. It is known that the dimension of all $\mathbb{Q}_l$-spaces from above do not depend on l prime to the characteristic p.

Proposition 1.2. (*Proposition 2.2* [5]) *Let Z be a smooth projective variety in characteristic $p > 0$ and g be an automorphism of Z of finite order. Assume $l \neq p$. Then the following assertions are true.*

(1) $\mathrm{Tr}(g^*|H^2_{\mathrm{et}}(X, \mathbb{Q}_l)) = \mathrm{Tr}(g^*|\mathrm{NS}(Z)) + \mathrm{Tr}(g^*|T^2_l(Z))$ *and the three traces are integers.*

(2) $\mathrm{rank}\ \mathrm{NS}(Z)^g = \mathrm{rank}\ \mathrm{NS}(Z/\langle g\rangle)$.

(3) $\dim H^2_{\mathrm{et}}(Z, \mathbb{Q}_l)^g = \mathrm{rank}\ \mathrm{NS}(Z)^g + \dim T^2_l(Z)^g$.

(4) *If* $\dim Z = 2$ *and the minimal resolution* Y *of* $Z/\langle g\rangle$ *has* $T_l^2(Y) = 0$ *(e.g., if* $Z/\langle g\rangle$ *is rational or birational to an Enriques surface), then*

$$\dim H^2_{\mathrm{et}}(Z, \mathbb{Q}_l)^g = \mathrm{rank}\ \mathrm{NS}(Z)^g.$$

Proposition 1.3. (*Topological Lefschetz formula, cf.* [1] *Theorem 3.2*) *Let* X *be a smooth projective variety over an algebraically closed field* k *of characteristic* $p > 0$ *and let* g *be a tame automorphism of* X*. Then* $X^g = \mathrm{Fix}(g)$ *is smooth and*

$$e(g) := e(X^g) = \mathrm{Tr}(g^*|H^*(X)).$$

A tame symplectic automorphism h of a K3 surface has finitely many fixed points, the number of fixed points $f(h)$ depends only on the order of h and the list of possible pairs $(\mathrm{ord}(h), f(h))$ is the same as in the complex case (Theorem 3.3 and Proposition 4.1 [2], see also Nikulin's paper [10] for the complex case) :

$$(\mathrm{ord}(h), f(h)) = (2,8),\ (3,6),\ (4,4),\ (5,4),\ (6,2),\ (7,3),\ (8,2).$$

Thus by the topological Lefschetz formula, we obtain the following.

Lemma 1.4. (*Lemma 2.6* [5]) *Let* h *be a tame symplectic automorphism of a K3 surface* X*. Then* $h^*|H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ *has eigenvalues*

$$\begin{array}{lcl}
\mathrm{ord}(h) = 2 & : & [h^*] = [1,\ 1.13,\ -1.8] \\
\mathrm{ord}(h) = 3 & : & [h^*] = [1,\ 1.9,\ (\zeta_3 : 2).6] \\
\mathrm{ord}(h) = 4 & : & [h^*] = [1,\ 1.7,\ (\zeta_4 : 2).4,\ -1.6] \\
\mathrm{ord}(h) = 5 & : & [h^*] = [1,\ 1.5,\ (\zeta_5 : 4).4] \\
\mathrm{ord}(h) = 6 & : & [h^*] = [1,\ 1.5,\ (\zeta_3 : 2).4,\ (\zeta_6 : 2).2,\ -1.4] \\
\mathrm{ord}(h) = 7 & : & [h^*] = [1,\ 1.3,\ (\zeta_7 : 6).3] \\
\mathrm{ord}(h) = 8 & : & [h^*] = [1,\ 1.3,\ (\zeta_8 : 4).2,\ (\zeta_4 : 2).3,\ -1.4]
\end{array}$$

where the first eigenvalue corresponds to an invariant ample divisor.

Lemma 1.5. (*Lemma 1.6* [6]) *Let* X *be a K3 surface in characteristic* $p \neq 2$*, admitting an automorphism* h *of order* 2 *with* $\dim H^2_{\mathrm{et}}(X, \mathbb{Q}_l)^h = 2$*. Then* h *is non-symplectic and has an* h*-invariant elliptic fibration* $\psi : X \to \mathbf{P}^1$*,*

$$X/\langle h\rangle \cong \mathbf{F}_e$$

a rational ruled surface, and X^h *is either a curve of genus* 9 *which is a* 4*-section of* ψ *or the union of a section and a curve of genus* 10 *which is a* 3*-section. In the first case* $e = 0, 1$ *or* 2*, and in the second* $e = 4$*. Each* *singular fibre of ψ is of type I_1 (nodal), I_2, II (cuspidal) or III, and is intersected by X^h at the node and two smooth points if of type I_1, at the two singular points if of type I_2, at the cusp with multiplicity 3 and a smooth point if of type II, at the singular point tangentially to both components if of type III. If X^h contains a section, then each singular fibre is of type I_1 or II.*

Remark 1.6. If $e \neq 0$, the h-invariant elliptic fibration ψ is the pull-back of the unique ruling of $\mathbf{F}_e$. If $e = 0$, either ruling of $\mathbf{F}_0$ lifts to an h-invariant elliptic fibration.

The following easy lemmas also will be used frequently.

Lemma 1.7. (*Lemma 2.10* [5]) *Let S be a set and $\mathrm{Aut}(S)$ be the group of bijections of S. For any $g \in \mathrm{Aut}(S)$ and positive integers a and b,*

(1) $\mathrm{Fix}(g) \subset \mathrm{Fix}(g^a)$;
(2) $\mathrm{Fix}(g^a) \cap \mathrm{Fix}(g^b) = \mathrm{Fix}(g^d)$ *where* $d = \gcd(a, b)$;
(3) $\mathrm{Fix}(g) = \mathrm{Fix}(g^a)$ *if* $\mathrm{ord}(g)$ *is finite and prime to* a.

Lemma 1.8. (*Lemma 2.11* [5]) *Let $R(n)$ be the sum of all primitive n-th root of unity in $\overline{\mathbb{Q}}$ or in $\overline{\mathbb{Q}_l}$, where $\gcd(l, n) = 1$. Then*

$$R(n) = \begin{cases} 0 & \text{if} \quad n \, \text{has a square factor,} \\ (-1)^t & \text{if} \quad n \, \text{is a product of } t \text{ distinct primes.} \end{cases}$$

For an automorphism g of finite order of a K3 surface X, tame or wild, we write

$$\mathrm{ord}(g) = m.n$$

if g is of order mn and the natural homomorphism $\langle g \rangle \to \mathrm{GL}(H^0(X, \Omega^2_X))$ has kernel of order m and image of order n.

§2. Proof: the Tame Case

Throughout this section, we assume that the characteristic $p > 0$, $p \neq 2, 5$. Let g be an automorphism of order 40 of a K3 surface X.

Lemma 2.1. $[g^*] \neq [1,\ \zeta_8 : 4,\ \pm 1,\ \zeta_{40} : 16]$.

Proof. Suppose that $[g^*] = [1,\ \zeta_8 : 4,\ \pm 1,\ \zeta_{40} : 16]$. Then

$$[g^{20*}] = [1,\ -1.4,\ 1,\ -1.16].$$

One can apply Lemma 1.5 to $h = g^{20}$. The quotient surface $X/\langle g^{20} \rangle$ is isomorphic to a rational ruled surface

$$X/\langle g^{20} \rangle \cong \mathbf{F}_e,$$

X has a g^{20}-invariant elliptic fibration

$$\psi : X \to \mathbf{P}^1$$

and $\mathrm{Fix}(g^{20})$ is either a curve C_9 of genus 9 which is a 4-section of ψ or the union of a section R and a curve C_{10} of genus 10 which is a 3-section. The automorphism $\bar{g}$ of $X/\langle g^{20}\rangle \cong \mathbf{F}_e$ induced by g preserves the ruling of $\mathbf{F}_e$ if $e \neq 0$, and either preserves or interchanges the two rulings of $\mathbf{F}_e$ if $e = 0$.

Case 1: $\bar{g}$ preserves the ruling(s) of $\mathbf{F}_e$.
In this case g preserves the fibration $\psi : X \to \mathbf{P}^1$. The fibre class is g-invariant and linearly independent from any g-invariant ample divisor class. Thus the eigenvalue 1 appears twice in $[g^*]$ and $[g^*] = [1, \zeta_8 : 4, 1, \zeta_{40} : 16]$. Then we compute $e(g) = e(g^2) = e(g^4) = e(g^5) = e(g^{10}) = 4$. Consider the order 20 action of g on C_9 or $C_{10} \subset \mathrm{Fix}(g^{20})$. The action of g on C_9 (resp. C_{10}) has 4 (resp. 2) points of ramification index 20 and no other ramifications. Neither is compatible with the Hurwitz formula.

Case 2: $e = 0$ and $\bar{g}$ interchanges the rulings of $\mathbf{F}_0$.
In this case $\mathrm{Fix}(g^{20}) = C_9$ and g interchanges the two elliptic fibrations coming from the rulings. Since g interchanges the two elliptic fibrations, -1 should appear as an eigenvalue and $[g^*] = [1, \zeta_8 : 4, -1, \zeta_{40} : 16]$. We compute $e(g) = e(g^5) = 2, e(g^2) = e(g^4) = e(g^{10}) = 4$. The order 20 action of g on C_9 has 2 points of ramification index 20, 2 points of ramification index 10 and no other ramifications. Unfortunately the Hurwitz formula cannot rule out this case.

The automorphism g^2 preserves the elliptic fibration $\psi : X \to \mathbf{P}^1$. It preserves two fibres. By Lemma 1.5 a fibre of ψ is of type I_0 (smooth), I_1, I_2, II or III. We claim that g^2 does not preserve a reducible fibre of type I_2 or III. Suppose it does. Then g^4 preserves both components which, with an invariant ample class, give 3 linearly independent g^4-invariant classes, hence $[g^{4*}] \supset [1, 1, 1]$, absurd. If g^2 preserves a fibre F_0 of type I_1 or II, then g^4 fixes all points in the set $F_0 \cap \mathrm{Fix}(g^{20})$, which consists of 3 points if F_0 is of type I_1 and 2 points if F_0 is of type II. If g^2 preserves a smooth fibre F_0, then the involution $g^{20}|F_0$ of the elliptic curve F_0 has non-empty fixed locus, hence must have 4 fixed points, thus the set $F_0 \cap \mathrm{Fix}(g^{20})$ consists of 4 distinct points and g^4 fixes all of them. Since $e(g^4) = 4$ and $\mathrm{Fix}(g^4) \subset \mathrm{Fix}(g^{20}) = C_9$, we see that $\mathrm{Fix}(g^4)$ consists of 4 points. From these we infer that g^2 preserves two fibres of type II, say F_1 and F_2. Let

$$F_i' \subset X/\langle g^{20}\rangle \cong \mathbf{F}_0$$

be the line, the image of F_i. On $\mathbf{F}_0$ consider the rectangle with 4 sides F_1', F_2', $\bar{g}(F_1')$ and $\bar{g}(F_2')$. Let $C_9' \subset \mathbf{F}_0$ be the image of C_9. At each vertex of the rectangle, the genus 9 curve C_9' intersects one side with multiplicity 3 and the other with multiplicity 1. This configuration admits no symmetry except rotations, and $\bar{g}$ must be a rotation by 90 degrees. Then $\bar{g}^2$ interchanges F_1' and F_2', so g^2 interchanges F_1 and F_2, a contradiction. Q.E.D.

By [5] Lemma 4.5 and 4.7, g cannot be of order 2.20, 4.10 or 8.5. It remains to exclude the possibility 5.8.

Lemma 2.2. $\mathrm{ord}(g) \neq 5.8$.

Proof. Suppose that $\mathrm{ord}(g) = 5.8$. Then by Proposition 1.1 the action of g^* on $H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ has $\zeta_8 \in \overline{\mathbb{Q}_l}$ as an eigenvalue. Hence $[\zeta_8 : 4] \subset [g^*]$. By Lemma 1.4,

$$[g^{8*}] = [1,\, 1.5,\, (\zeta_5 : 4).4].$$

From this we infer that

$$[g^*] = [1,\, \zeta_8 : 4,\, \pm 1,\, \eta_1, \ldots, \eta_{16}]$$

where $[\eta_1, \ldots, \eta_{16}]$ is a combination of $\zeta_5 : 4$, $\zeta_{10} : 4$, $\zeta_{20} : 8$, $\zeta_{40} : 16$, and the first eigenvalue corresponds to a g-invariant ample divisor.
By Lemma 2.1, $\zeta_{40} : 16$ cannot appear. Then

$$[g^{4*}] = [1,\, -1.4,\, 1,\, (\zeta_5 : 4).4],$$

hence $e(g^4) = \mathrm{Tr}(g^{4*}|H^*(X)) = -4$. But $\mathrm{Fix}(g^4) \subset \mathrm{Fix}(g^8)$ and the latter consists of finitely many points, hence $e(g^4) \geq 0$. Q.E.D.

We have proved that g is purely non-symplectic, the first statement of Theorem 0.1.

Lemma 2.3. *If* $\mathrm{ord}(g) = 1.40$, *then*

(1) $[g^*] = [1,\, \zeta_{40} : 16,\, \zeta_5 : 4,\, 1]$
where the first eigenvalue corresponds to a g-invariant ample class;

(2) $\mathrm{Fix}(g^{20}) = R \cup C_6$
where R is a smooth rational curve and C_6 a curve of genus 6*;*

(3) $\mathrm{Fix}(g^8) = D_2 \cup \{\text{ one point on } R\,\}$
where D_2 is a curve of genus 2 *with $D_2C_6 = 5$, $D_2R = 1$.*

Proof. Suppose that $\mathrm{ord}(g) = 1.40$. Then by Proposition 1.1 the action of g^* on $H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ has $\zeta_{40} \in \overline{\mathbb{Q}_l}$ as an eigenvalue. Thus

$$[g^*] = [1,\, \zeta_{40} : 16,\, \eta_1, \ldots, \eta_5]$$

where the first eigenvalue corresponds to a g-invariant ample divisor and $[\eta_1, \ldots, \eta_5]$ is a combination of $\zeta_8 : 4$, $\zeta_{10} : 4$, $\zeta_5 : 4$, $\zeta_4 : 2$, ± 1.
By Lemma 2.1, $[\eta_1, \ldots, \eta_5] \neq [\zeta_8 : 4,\, \pm 1]$.

Claim: $[\eta_1, \ldots, \eta_5]$ is not a combination of $\zeta_4 : 2$, ± 1.

Suppose that it is. Then

$$[g^{20*}] = [1,\, -1.16,\, 1.5], \quad e(g^{20}) = -8$$

$$[g^{8*}] = [1,\, (\zeta_5 : 4).4,\, 1.5], \quad e(g^8) = 4$$

$$[g^{4*}] = [1,\, (\zeta_{10} : 4).4,\, 1.5], \quad e(g^4) = 12.$$

Thus

$$\mathrm{Fix}(g^{20}) = R_1 \cup \ldots \cup R_d \cup C_{d+5}, \quad d \leq 5$$

where R_j's are smooth rational curves and C_{d+5} a curve of genus $d+5$. Note that a non-symplectic automorphism of order 2 like g^{20} cannot have an isolated fixed point. The locus $\mathrm{Fix}(g^4)$, being a subset of $\mathrm{Fix}(g^{20})$, consists of isolated points and possibly some $R_j's$. The action of g^4 on $\mathrm{Fix}(g^{20})$ has order 5. If $d = 5$ and g^4 permutes R_j's, then $g^4|C_{d+5}$ would have 12 fixed points, too many for an order 5 automorphism of a genus 10 curve. Thus

$(*)\qquad g^4(R_j) = R_j$ for each j and $g^4|C_{d+5}$ has $12 - 2d$ fixed points.

If $d \leq 2$, then the order 5 automorphism $g^4|C_{d+5}$ would have too many fixed points. Thus $d \geq 3$.

Now we consider $\mathrm{Fix}(g^8)$. Suppose that $\mathrm{Fix}(g^8)$ does not contain a curve of genus > 1. Then it consists of $2k$ points, d' smooth rational curves and possibly some elliptic curves. Since $e(g^8) = 4$, $2k + 2d' = 4$. The action of g^4 on $\mathrm{Fix}(g^8)$ has order 2. We infer that there are 2 elliptic curves $E_1, E_2 \subset \mathrm{Fix}(g^8)$ on each of which g^4 has 4 fixed points. Then g^4 fixes more than two fibres of the elliptic fibration $|E_1| = |E_2|$, hence fixes all fibres. Note that $\mathrm{Fix}(g^8) \cap \mathrm{Fix}(g^{20}) = \mathrm{Fix}(g^4)$. By $(*)$, $g^4|C_{d+5}$ has at most 6 fixed points, so we see that there is an R_j, say R_1, that meets E_1. If $R_1E_1 = R_1E_2 > 1$, then $g^4|R_1$ has more than two fixed points, hence is the identity, then $R_1 \subset \mathrm{Fix}(g^4) \subset \mathrm{Fix}(g^8)$, contradicting the smoothness of a fixed locus. If $R_1E_1 = R_1E_2 = 1$, then R_1 is a section of $|E_1|$, hence $R_1 \subset \mathrm{Fix}(g^4)$, again contradicting the smoothness of $\mathrm{Fix}(g^8)$.

We have proved that $\mathrm{Fix}(g^8)$ contains a curve D_a of genus $a > 1$. Note that $D_a \cap C_{d+5} \subset \mathrm{Fix}(g^4)$, hence the intersection number

$$D_a C_{d+5} \leq 12 - 2d.$$

By the Hodge Index Theorem

$$(D_a^2)(C_{d+5}^2) \leq (12 - 2d)^2.$$

This is possible only if

$$d = 3, \ \ a = 2 \ \text{ and } \ D_2 C_8 = 6.$$

Therefore

$$\mathrm{Fix}(g^{20}) = R_1 \cup R_2 \cup R_3 \cup C_8,$$
$$\mathrm{Fix}(g^8) = D_2 \cup \{2k \text{ points}\} \cup R'_1 \cup \ldots \cup R'_c$$

where $R'_1, \ldots, R'_c$ are smooth rational curves. We know that $e(g^4) = 12$. By $(*)$ g^4 acts on R_j for $j = 1, 2, 3$, hence $g^4|R_j$ fixes 2 points or the whole R_j. Thus the action of $g^4|C_8$ fixes 6 points. Since $\mathrm{Fix}(g^4) = \mathrm{Fix}(g^{20}) \cap \mathrm{Fix}(g^8)$, we see that $C_8 \cap \mathrm{Fix}(g^8)$ consists of 6 points. Since $C_8 D_2 = 6$, we infer that

$$(**) \qquad C_8 \cap \mathrm{Fix}(g^8) = C_8 \cap D_2 = \{6 \text{ points}\}.$$

The order 5 automorphism g^8 is non-symplectic and tame. The quotient

$$Y := X/\langle g^8 \rangle$$

is a normal surface. The image $\bar{C}_8 \subset Y$ of C_8 has self intersection number

$$\bar{C}_8^2 = \frac{1}{5} C_8^2 = \frac{14}{5}$$

which is not an integer, so $\bar{C}_8$ must pass through some singular points of Y, then C_8 must pass through some isolated fixed points of g^8, contradicting $(**)$. The claim is proved.

Now we may assume that $[\eta_1, \ldots, \eta_5] = [\zeta_{10} : 4, \pm 1]$ or $[\zeta_5 : 4, \pm 1]$. In these cases we have

$$[g^{20*}] = [1, \, -1.16, \, 1.5], \quad e(g^{20}) = -8$$

$$[g^{10*}] = [1, \, (\zeta_4 : 2).8, \, 1.5], \quad e(g^{10}) = 8$$

$$[g^{8*}] = [1, \, (\zeta_5 : 4).4, \, \zeta_5 : 4, \, 1], \quad e(g^8) = -1$$

$$[g^{4*}] = [1, \, (\zeta_{10} : 4).4, \, \zeta_5 : 4, \, 1], \quad e(g^4) = 7$$

$$[g^{2*}] = [1,\, (\zeta_{20} : 4).4,\, \zeta_5 : 4,\, 1], \quad e(g^2) = 3.$$

Thus

$$\mathrm{Fix}(g^{20}) = R_1 \cup \ldots \cup R_d \cup C_{d+5}, \quad d \leq 5$$

where R_j's are smooth rational curves and C_{d+5} a curve of genus $d+5$. For a divisor r of 20 with $r < 20$, the locus $\mathrm{Fix}(g^r)$, being a subset of $\mathrm{Fix}(g^{20})$, consists of isolated points and possibly some $R_j's$. Consider the action of g on the set $\{R_1, \ldots, R_d\}$ of d elements. In its cycle decomposition each cycle has length 1, 2, 4 or 5. If $d = 5$, then g^4 or g^{10} preserves each R_j, so has a negative number of fixed points on C_{10}, absurd. If $d = 4$, then g^4 preserves each R_j, so has a negative number of fixed points on C_9. If $d = 2$ or 3, then g^2 preserves each R_j, so has a negative number of fixed points on C_8. If $d = 0$, then g^4 has 7 fixed points on C_5, too many for an order 5 automorphism. We have shown that $d = 1$, $\mathrm{Fix}(g^{20}) = R \cup C_6$, giving (2).

Since $e(g^8) < 0$, $\mathrm{Fix}(g^8)$ must contain a curve D_a of genus $a > 1$ (hence no elliptic curves). Since $g^4|C_6$ has 5 fixed points and $D_a \cap C_6 \subset \mathrm{Fix}(g^4)$, the intersection number $D_a C_6 \leq 5$. By the Hodge Index Theorem

$$(D_a^2)(C_6^2) \leq 5^2.$$

This is possible only if

$$a = 2 \ \text{and} \ D_2 C_6 = 5.$$

Since $e(g^8) = -1$, $\mathrm{Fix}(g^8)$ consists of D_2 and a point p. Since g^4 acts on $\mathrm{Fix}(g^8)$, it is easy to see that $p \in \mathrm{Fix}(g^4)$. Since $\mathrm{Fix}(g^4) = \mathrm{Fix}(g^8) \cap \mathrm{Fix}(g^{20})$ and $e(g^4) = 7$, we infer that $p \in R$ and $D_2 R = 1$. This proves (3).

Since g^5 acts on $\mathrm{Fix}(g^{20}) = R \cup C_6$ and $\mathrm{Fix}(g^5) \subset \mathrm{Fix}(g^{20})$, we see that $e(g^5) > 0$. This rules out the possibility $[\eta_1, \ldots, \eta_5] = [\zeta_{10} : 4, \pm 1]$. Thus we have

$$[\eta_1, \ldots, \eta_5] = [\zeta_5 : 4, \pm 1].$$

Finally, g^{10} fixes 6 points on C_6 (and g^5 fixes 6 or 4 points on C_6). Considering the action of g on these 6 points, we see that g fixes at least one of them, hence $e(g) \geq 3$. Thus the last eigenvalue must be 1, proving (1). Q.E.D.

Proof of the second statement of Theorem 0.1.

Lemma 2.3 plays a key role in the proof. We modify the proof of [8] Section 4. The quotient

$$Y := X/\langle g^{20} \rangle$$

is a smooth rational surface with

$$K_Y = -\frac{1}{2}(\bar{R} + \bar{C}_6)$$

where $\bar{R}$, $\bar{C}_6 \subset Y$ are the images of R and C_6. Note that

$$\bar{R}^2 = -4, \quad \bar{C}_6^2 = 20.$$

Let

$$\pi : X \to Y$$

be the projection map. By Proposition 1.2 Y has Picard number $\rho(Y) = 6$.

Claim: If $E \subset Y$ is a smooth rational curve with $E^2 < -1$, then $E = \bar{R}$.

If $E^2 < -2$. Then $K_Y E > 0$, thus $(\bar{R} + \bar{C}_6)E < 0$. This is possible only if $E = \bar{R}$. Suppose that E is a (-2)-curve on Y. Then $K_Y E = 0$, thus E is disjoint from the branch divisor $\bar{R} + \bar{C}_6$, and $\pi^*(E) \subset X$ is a disjoint union of two (-2)-curves, say E_1 and E_2. Note that $E_1.E_2 = 0$, $E_1.g^{10}(E_1) = g^{10}(E_1).E_2 \geq 0$. It is easy to check that the four (-2)-curves E_1, $g^{10}(E_1)$, $g^{20}(E_1) = E_2$, $g^{30}(E_1) = g^{10}(E_2)$ are linearly independent in the Néron-Severi group of X, hence their Chern classes are linearly independent in the second cohomology group. They are rotated by g^{10*}, hence $[1, \zeta_4, \zeta_4^2, \zeta_4^3] \subset [g^{10*}]$, impossible. This proves the claim.

The linear system $|D_2|$ gives a degree 2 morphism $\phi : X \to \mathbf{P}^2$. Take the Stein factorization

$$X \xrightarrow{\mu} X' \xrightarrow{\phi'} \mathbf{P}^2.$$

Let $\tilde{R}$, $\tilde{C}_6 \subset \mathbf{P}^2$ are the images of R and C_6. Since $D_2 C_6 = 5$, $D_2 R = 1$, we infer that $\tilde{R}$ is a line, $\tilde{C}_6$ is a quintic and their union is the branch of the double cover $\phi' : X' \to \mathbf{P}^2$. Thus the map $\phi : X \to \mathbf{P}^2$ factors through

$$X \xrightarrow{\pi} Y \xrightarrow{\nu} \mathbf{P}^2$$

where ν is the contraction of five mutually disjoint (-1)-curves, $E_1, \ldots, E_5 \subset Y$ by Claim. Each E_i satisfies $E_i \bar{R} = E_i \bar{C}_6 = 1$. Their pre-images $E_i^* \subset X$ are (-2)-curves satisfying $E_i^* R = E_i^* C_6 = 1$, and contracted by μ. Our automorphism g induces an automorphism $\tilde{g}$ of

$\mathbf{P}^2$. Thus the automorphism $\bar{g}$ of Y permutes $E_1, \ldots, E_5$. This permutation must be a cycle of length 5, because otherwise g^8 would fix R pointwise. It implies that g^5 fixes R pointwise, hence

$$\mathrm{Fix}(\tilde{g}^5)) = \tilde{R}.$$

We may assume that the equation of $\tilde{R}$ is given by $x = 0$ and $(0, 1, \zeta_5^i)$ are the five intersection points of $\tilde{R}$ and $\tilde{C}_6$. Now a standard computation of invariant polynomials such as in ([8] p. 293) yields the result.

§3. Proof: the Complex Case

We may assume that X is projective, since a non-projective complex K3 surface cannot admit a non-symplectic automorphism of finite order (see [12], [10]) and its automorphisms of finite order are symplectic, hence of order ≤ 8. Now the same proof goes, once $H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ is replaced by $H^2(X, \mathbb{Z})$ and Proposition 1.3 by the usual topological Lefschetz formula.

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