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More lectures on Hilbert schemes of points on surfaces

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Dedicated to Professor Shigeru Mukai on the occasion of his 60th birthday

§ Introduction

This paper is based on author's lectures at Kyoto University in 2010 Summer, and in the 6th MSJ-SI 'Development of Moduli Theory' at RIMS in June 2013.

The purpose of lectures was to review several results on Hilbert schemes of points which were obtained after author's lecture note [24] was written. Among many results, we choose those which are about equivariant homology groups $H_*^T(X^{[n]})$ of Hilbert schemes of points on the affine plane $X = \mathbb{C}^2$ with respect to the torus action. Study of equivariant homology groups increases its importance recently. In particular, it is a basis of the AGT correspondence between instanton moduli spaces on $\mathbb{C}^2$ and the representation theory of W-algebras, which is a very hot topic now (see e.g., [20]).

We omit proofs if they are present in [24], but give self-contained proofs otherwise. In this sense, this should be read after [24].

The paper is organized as follows. In §1, we review basics on equivariant (co)homology groups. It will be basis of subsequent sections. In §2 we construct the Fock representation of the Heisenberg algebra on $\bigoplus H_*^T(X^{[n]})$, following [24, Ch. 8] as well as an idea of Vasserot [27]. In §3 we explain a geometric realization of Jack symmetric functions as fixed point classes in $H_*^T(X^{[n]})$ by Li-Qin-Wang [17]. We also give author's unpublished result, which was used in [17]. As applications, we give geometric proofs of the norm formula and Pieri formula of Jack

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symmetric functions. In §4 we construct a representation of the Virasoro algebra on $\bigoplus H_*^T(X^{[n]})$. It is a special case of Lehn's result [14] for $X = \mathbb{C}^2$, but the proof is different.

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§1. Equivariant (co)homology groups

In this section, we review basics on equivariant cohomology and homology groups, which will be used in later sections. Our definition of equivariant cohomology groups is slightly different from the usual one (e.g., as in [1]). We replace the classifying space BT by its finite dimensional approximations. This approach is suitable for equivariant homology groups, and was taken by Lusztig [18]. The same approach was used for equivariant Chow groups [8].

We assume coefficients of cohomology groups are $\mathbb{C}$.

1(i). Equivariant cohomology groups

Let $T = (\mathbb{C}^*)^r$ be an algebraic torus. Let $V = (\mathbb{C}^{N+1} \setminus \{0\})^r$ and let T act on V so that each $\mathbb{C}^*$ acts on the corresponding $\mathbb{C}^{N+1}$ by multiplication. Then $V \to (\mathbb{P}^N)^r$ is a T-bundle, and the universal T-bundle $ET \to BT$ is the inductive limit when $N \to \infty$. Let M be a T-variety, i.e., an algebraic variety with an algebraic T-action. We further assume that M admits a locally closed T-embedding into a smooth projective T-variety. This restriction can be weakened, but it is enough for our purpose.

We define the equivariant cohomology of M by

$$H_T^i(M) \overset{\text{def.}}{=} H^i(M_V) \quad \text{where } M_V = V \times_T M.$$

Here for a given i, we take V with sufficiently large N. Then it is well-defined thanks to the following lemma:

Lemma 1.1. *$H_T^i(M)$ is independent of the choice of V.*

Proof. A key point is that $H^i(V) \cong H^i((S^{2N+1})^r) = 0$ if $0 < i < 2N + 1$.

Take two large integers N_1, N_2 and consider the diagram

$$M_{V_1} \leftarrow (V_1 \times V_2) \times_T M \to M_{V_2},$$

where $V_1 = (\mathbb{C}^{N_1+1} \setminus \{0\})^r$, $V_2 = (\mathbb{C}^{N_2+1} \setminus \{0\})^r$. The left and right arrows are fiber bundles with fibers V_2, V_1 respectively. Since their cohomology groups vanishes in degree between 1 and sufficiently large number (in particular, larger than i), $H^i(M_{V_1})$ and $H^i(M_{V_2})$ are isomorphic to H^i of the middle under the pull-back homomorphisms. Q.E.D.

Let us briefly explain several important properties of equivariant cohomology groups.

When $M = \mathrm{pt}$, we have $H^i_T(\mathrm{pt}) = H^i((\mathbb{P}^N)^r)$. Note that $H^*(\mathbb{P}^N) = \mathbb{C}[a]/(a^{N+1} = 0)$. Taking $N \to \infty$, we have

$$H^*_T(\mathrm{pt}) \cong \mathbb{C}[a_1, \dots, a_r], \tag{1.2}$$

where a_i is the first Chern class of the hyperplane bundle $\mathcal{O}(1)$ of the i^{th} factor of $(\mathbb{P}^N)^r$. As this example shows, $H^i_T(M)$ may be nonzero in arbitrary large degree i unlike ordinary cohomology groups.

When T acts trivially on M, we have

$$H^*_T(M) \cong H^*(M) \otimes H^*_T(\mathrm{pt}) \tag{1.3}$$

from the definition.

We have a cup product

$$H^i_T(M) \otimes H^j_T(M) \to H^{i+j}_T(M).$$

The isomorphism (1.2) is a ring isomorphism.

If $f\colon M_1 \to M_2$ is a T-equivariant continuous map, we have a pull-back map $H^*_T(M_2) \to H^*_T(M_1)$. It is a ring homomorphism. In particular, we always have $H^*_T(\mathrm{pt}) \to H^*_T(M)$. Therefore $H^*_T(M)$ is a ring over $H^*_T(\mathrm{pt}) \cong \mathbb{C}[a_1, \dots, a_r]$. We consider $H^*_T(M)$ as a coherent sheaf on $\mathrm{Spec}(\mathbb{C}[a_1, \dots, a_r]) = \mathbb{C}^r$ in this way, and this view point is useful in the localization theorem below.

Suppose T acts freely on M, and $M \to M/T$ is a principal T-bundle. Then M_V is a fiber bundle over M/T with fiber V. Since the fiber has trivial cohomology groups, the spectral sequence for a fiber bundle gives us

$$H^i_T(M) \cong H^i(M/T).$$

Moreover, $H^{>0}_T(\mathrm{pt})$ acts by 0, as it is so on the E^2 term. In this case, $H^i_T(M)$ vanishes if i is sufficiently large for a reasonable M/T.

Note that $M_V \to (\mathbb{P}^N)^r$ is a fiber bundle with fiber M. The restriction to a fiber gives a forgetful homomorphism

$$H^*_T(M) \to H^*(M).$$

More generally, we have a restriction homomorphism

$$H_T^*(M) \to H_{T'}^*(M)$$

for a subtorus $T' \subset T$. Considering the case $M = \mathrm{pt}$, we find that we have an intrinsic description of (1.2):

$$H_T^*(\mathrm{pt}) \cong \mathbb{C}[\operatorname{Lie} T],$$

where $\mathbb{C}[\operatorname{Lie} T]$ is the ring of polynomial functions on $\operatorname{Lie} T$. The homomorphism $H_T^*(\mathrm{pt}) = \mathbb{C}[\operatorname{Lie} T] \to H_{T'}^*(\mathrm{pt}) = \mathbb{C}[\operatorname{Lie} T']$ is induced from the embedding $\operatorname{Lie} T' \to \operatorname{Lie} T$.

If E is a T-equivariant (complex) vector bundle over M, it induces a vector bundle $E_V = V \times_T E$ over M_V. We define an equivariant Chern class $c_i(E)$ by $c_i(E_V)$. If E is rank r, the top Chern class $c_r(E)$ is equal to the equivariant Euler class $e(E)$, defined in the same way.

If $M = \mathrm{pt}$, a T-equivariant vector bundle is nothing but a representation of T. The above a_i in (1.2) is the equivariant first Chern class of the representation $T \to \mathbb{C}^*$, the projection to the i^{th} factor. Then $c_i(E)$ is the i^{th} elementary symmetric function of weights of E, regarded as a representation of T. Here weights are considered as linear functions $\operatorname{Lie} T \to \mathbb{C}$. The equivariant Euler class $e(E)$ is the product of weights.

1(ii). Equivariant homology groups

In [24, Ch.8] we used both ordinary and Borel-Moore (or locally finite) homology groups to deal with Hilbert schemes of points on a noncompact surface X, like $X = \mathbb{C}^2$. Here in the equivariant case, we mainly use Borel-Moore homology groups, since it fits better with convolution products.

Let M be a T-variety as above. We define

$$H_i^{T,lf}(M) \overset{\text{def.}}{=} H_{i+2\dim V - 2\dim T}^{lf}(M_V).$$

It is independent of V by the same argument as above, where we use $H_i^{lf}(V) = 0$ for $2(\dim V - N - 1) < i < 2\dim V$. Then $H_*^{T,lf}(M)$ is a module over $H_T^*(M)$ under the cap product. In particular, it is also a module over $H_T^*(\mathrm{pt})$.

If T acts free on M and $M \to M/T$ is a fiber bundle, we have

$$H_i^{T,lf}(M) \cong H_{i-2\dim T}^{lf}(M/T). \tag{1.4}$$

If M is a smooth manifold of $\dim M = m$ with a smooth T-action, we define the equivariant fundamental class $[M]$ as

$$[M_V] \in H_{m+2\dim V - 2\dim T}^{lf}(M_V).$$

We have the Poincaré duality

$$H_T^i(M) \cong H_{m-i}^{T,lf}(M); \qquad c \mapsto c \cap [M]. \tag{1.5}$$

Even for an irreducible complex algebraic variety M with a T-action, not necessarily smooth, its fundamental class $[M] \in H_{2\dim M}^{T,lf}(M)$ is defined, as $[M_V]$ is defined. However the homomorphism $H_T^i(M) \to H_{2\dim M-i}^{T,lf}(M)$ may not be an isomorphism in general. Note also that

$$H_i^{T,lf}(M) = 0 \quad \text{if } i > 2\dim M. \tag{1.6}$$

On the other hand, $H_i^{T,lf}(M)$ may be nonzero even for $i < 0$.

If $f\colon M_1 \to M_2$ is a proper T-equivariant map, we have the push-forward homomorphism $f_*\colon H_*^{T,lf}(M_1) \to H_*^{T,lf}(M_2)$.

Considering the spectral sequence for the fiber bundle $M_V \to (\mathbb{P}^N)^r$, we have a forgetful homomorphism

$$H_i^{T,lf}(M) \to H_i^{lf}(M).$$

We define the ordinary equivariant homology group $H_i^T(M)$ as the dual space to $H_T^i(M)$. This is enough for our purpose thanks to the universal coefficient theorem, as we only consider complex coefficients. We have $f_*\colon H_i^T(M_1) \to H_i^T(M_2)$ for a T-equivariant map, not necessarily proper. It is defined as the transpose of f^*. If M is a smooth T-manifold, we have the cap product

$$\cap\colon H_i^{T,lf}(M) \otimes H_j^T(M) \to H_{i+j-\dim M}^T(M)$$

thanks to (1.5). We define the intersection pairing

$$H_i^{T,lf}(M) \otimes H_j^T(M) \to H_{i+j-\dim M}^T(\mathrm{pt}) \cong H_T^{\dim M-i-j}(\mathrm{pt})$$

as $a_{M*}(\bullet \cap \bullet)$, where $a_M\colon M \to \mathrm{pt}$. The second isomorphism is the Poincaré duality for pt.

We do not review further properties of equivariant Borel-Moore homology groups, which we will use implicitly in the next section. They are listed in [24, §8.2] for nonequivariant Borel-Moore homology groups, and equivariant versions are simple consequences of nonequivariant ones applied to M_V.

1(iii). Localization theorem

In this subsection we explain the localization theorem in equivariant Borel-Moore homology groups, which relates the equivariant Borel-Moore homology of M and that of the fixed point set M^T. In many situations, the fixed point set consists of a finite set, so the latter is just a direct sum of the equivariant cohomology of points (cf. (1.3)). Therefore the localization theorem is useful to say something on $H_*^{T,lf}(M)$.

A key point is that we view $H_*^{T,lf}(M)$ as a module over $H_T^*(\mathrm{pt}) \cong \mathbb{C}[\operatorname{Lie} T]$, or a sheaf on $\operatorname{Lie} T$.

Recall that $H_T^{>0}(\mathrm{pt})$ acts trivially on $H_*^{T,lf}(M)$ when T acts freely on M, see (1.4). Therefore the support of $H_*^{T,lf}(M)$ is 0 in $\operatorname{Lie} T$. More generally, we have

Lemma 1.7. *Suppose that the stabilizers of arbitrary points $x \in M$ is a fixed subgroup of $T' \subset T$. Then the support of $H_*^{T,lf}(M)$ is contained in* $\operatorname{Lie}(T')$.

Proof. By the assumption, M_V is a fiber bundle over $M/(T/T')$ with fiber V/T'. Then the action of $H_T^*(\mathrm{pt}) \cong \mathbb{C}[\operatorname{Lie} T]$ factors through $\mathbb{C}[\operatorname{Lie} T']$ on the E^2 term, and hence also on $H_*^{T,lf}(M)$. Q.E.D.

Let M^T be the fixed point set in M, and consider $H_*^{T,lf}(M \setminus M^T)$. Since M can be equivariantly embedded into a projective space, there are only finitely many stabilizers occur. We claim

$$\operatorname{Supp} H_*^{T,lf}(M \setminus M^T) \subset \bigcup_{x \in M \setminus M^T} \operatorname{Lie}(\operatorname{Stab}(x)). \tag{1.8}$$

We decompose $M \setminus M^T$ according to stabilizers into $\bigsqcup M_\alpha$. And we can order the index set $\{\alpha\}$ so that $M_{\le\alpha} = \bigcup_{\beta:\beta\le\alpha} M_\beta$ is closed in $M \setminus M^T$. We set $M_{<\alpha} = \bigcup_{\beta:\beta<\alpha} M_\beta$. From an exact sequence

$$\cdots \to H_i^{T,lf}(M_{<\alpha}) \to H_i^{T,lf}(M_{\le\alpha}) \to H_i^{T,lf}(M_\alpha) \to H_{i-1}^{T,lf}(M_{<\alpha}) \to \cdots$$

and Lemma 1.7 applied to M_α, we deduce that the support of $H_*^{T,lf}(M_{\le\alpha})$ is contained in the right hand side of (1.8) by an induction on α. Therefore using the exact sequence for $H_*^{T,lf}(M)$, $H_*^{T,lf}(M^T)$, $H_*^{T,lf}(M \setminus M^T)$, we get

Theorem 1.9. *Let $i\colon M^T \to M$ be the inclusion of the fixed point set. Then the kernel and cokernel of the homomorphism*

$$i_*\colon H_*^{T,lf}(M^T) \to H_*^{T,lf}(M)$$

are supported in $\bigcup_{x \in M\setminus M^T} \operatorname{Lie}(\operatorname{Stab}(x))$.

Corollary 1.10. *Let* $\operatorname{Frac}(H_T^*(\mathrm{pt}))$ *be the fractional field of* $H_T^*(\mathrm{pt})$ *and let* $H_*^{T,lf}(M)_{\mathbb{F}} = H_*^{T,lf}(M) \otimes_{H_T^*(\mathrm{pt})} \operatorname{Frac}(H_T^*(\mathrm{pt}))$*, and similarly for*

$H_*^{T,lf}(M^T)_{\mathbb{F}}$. *Then*

$$i_* \colon H_*^{T,lf}(M^T)_{\mathbb{F}} \to H_*^{T,lf}(M)_{\mathbb{F}}$$

is an isomorphism.

1(iv). Fixed point formula

When M is nonsingular, we have the Poincaré duality (1.5). Therefore we have two isomorphisms $i_* \colon H_*^{T,lf}(M^T)_{\mathbb{F}} \to H_*^{T,lf}(M)_{\mathbb{F}}$ and $i^* \colon H_*^{T,lf}(M)_{\mathbb{F}} \to H_*^{T,lf}(M^T)_{\mathbb{F}}$. Having both are very useful, as $i^* i_*$ can be explicitly written down. This leads us to the *fixed point formula.*

Let $M^T = \bigsqcup M_\alpha$ be the decomposition to connected components. Each M_α is a nonsingular subvariety of M. Let N_α denote its normal bundle. We have

$$H_*^{T,lf}(M^T) = \bigoplus_\alpha H_*^{T,lf}(M_\alpha).$$

Lemma 1.11. *Let i_α be the inclusion of M_α into M. Then $i_\alpha^* i_{\alpha *}$ is given by the cap product $e(N_\alpha) \cap \bullet$ of the equivariant Euler class of N_α.*

Proof. In the neighborhood of M_α, M is isomorphic to a neighborhood of the 0-section of N_α. Therefore we may replace $i_\alpha \colon M_\alpha \to M$ by the inclusion $M_\alpha \to N_\alpha$ of the 0-section. We then obtain the assertion by the Thom isomorphism. Q.E.D.

Lemma 1.12. $e(N_\alpha) \cap \bullet$ *is invertible in $H_*^{T,lf}(M_\alpha)_{\mathbb{F}}$.*

Proof. Since T acts trivially on M_α, we have $H_T^*(M_\alpha) = H^*(M_\alpha) \otimes H_T^*(\mathrm{pt})$. Note that $H^{>0}(M_\alpha) \otimes H_T^*(\mathrm{pt})$ is nilpotent as $H^{>2\dim M_\alpha}(M_\alpha) = 0$. Therefore it is enough to check that $H^0(M_\alpha) \otimes H_T^*(\mathrm{pt})$ part of $e(N_\alpha)$ is nonzero. It is enough to study it after restricting N_α to a point $x \in M_\alpha$.

For $x \in M_\alpha$, $T_x M$ is a T-module so that $T_x M_\alpha$ is its weight 0 subspace. Therefore $e(N_\alpha)|_x$ is the product of nonzero weights of $T_x M$. Therefore it is nonzero. Q.E.D.

Let us denote the inverse of $e(N_\alpha) \cap \bullet$ by $\frac{1}{e(N_\alpha)}$.

Theorem 1.13. *Suppose M is nonsingular. Then the inverse of i_* in Corollary 1.10 is given by*

$$\sum_\alpha \frac{1}{e(N_\alpha)} i_\alpha^*.$$

Proof. Note that we have decomposition

$$H_*^{T,lf}(M^T) = \bigoplus H_*^{T,lf}(M_\alpha),$$

and hence we have $i_* = \sum i_{\alpha*}$. By Lemmas 1.11, 1.12, $e(N_\alpha)^{-1} i_\alpha^* i_{\alpha*}$ is the identity operator on $H_*^{T,lf}(M_\alpha)_{\mathbb{F}}$. Therefore we have the assertion. Q.E.D.

We now arrive at Atiyah-Bott-Berline-Vergne fixed point formula.

Theorem 1.14. *Assume M is proper and nonsingular. Let $a\colon M \to \mathrm{pt}$, $a_\alpha\colon M_\alpha \to \mathrm{pt}$. Then we have an equality in $H_*^T(\mathrm{pt})_{\mathbb{F}} \cong \mathbb{C}(a_1, \dots, a_r)$ for $\omega \in H_*^T(M)$:*

$$a_*(\omega) = \sum_\alpha a_{\alpha*}\left(\frac{1}{e(N_\alpha)} i_\alpha^* \omega\right).$$

Proof. We have

$$a_*(\omega) = a_* i_* i_*^{-1}(\omega) = \sum_\alpha a_* i_{\alpha*} \frac{1}{e(N_\alpha)} i_\alpha^*(\omega).$$

Since $a_\alpha = a \circ i_\alpha$, we get the assertion. Q.E.D.

§2. Equivariant homology groups of Hilbert schemes of points

We explain the construction of the Fock space representation of the Heisenberg algebra in [24, Ch. 8] in equivariant homology groups in this section. It was first noticed by Vasserot [27] that the arguments in [24, Ch. 8] work in the equivariant setting, and such a generalization is quite useful.

2(i). Heisenberg algebra

Let $X = \mathbb{C}^2$ with the linear coordinate system (z, ξ). Two dimensional torus T acts on X by $(t_1, t_2) \cdot (z, \xi) = (t_1 z, t_2 \xi)$. We denote the equivariant variables corresponding to t_1, t_2 by ε_1, ε_2 respectively. Therefore $H_T^*(\mathrm{pt}) = \mathbb{C}[\varepsilon_1, \varepsilon_2]$.

Let $X^{[n]}$ denote the Hilbert scheme of n points in X. Let $\pi\colon X^{[n]} \to S^n X$ be the Hilbert-Chow morphism. We have induced T-actions on $X^{[n]}$ and $S^n X$ so that π is equivariant.

Recall ([24, (8.9)]) that we considered a subvariety $P[i] \subset \bigsqcup_n X^{[n]} \times X^{[n-i]} \times X$ defined by

$$P[i] \overset{\text{def.}}{=} \left\{(I_1, I_2, x) \mid I_1 \subset I_2, \operatorname{Supp}(I_2/I_1) = \{x\}\right\} \tag{2.1}$$

for $i > 0$. Let us omit $\bigsqcup_n$ for brevity hereafter.

The projections $q_1 : P[i] \to X^{[n]}$, $q_2 : P[i] \to X^{[n-i]} \times X$ are proper. Therefore convolution operators

$$\begin{aligned} &H_*^{T,lf}(X^{[n-i]} \times X) \to H_*^{T,lf}(X^{[n]}); \bullet \mapsto q_{1*}(q_2^*(\bullet) \cap [P[i]]), \\ &H_*^{T,lf}(X^{[n]}) \to H_*^{T,lf}(X^{[n-i]} \times X); \bullet \mapsto (-1)^i q_{2*}(q_1^*(\bullet) \cap [P[i]]) \end{aligned} \tag{2.2}$$

are well-defined. The sign $(-1)^i$ is introduced so that the commutation relation below is simplified. We take the direct sum of homology groups over n later, keeping the same notation.

Take an equivariant class $\beta \in H_*^{T,lf}(X)$. We replace $\bullet$ by $\bullet \otimes \beta \in H_*^{T,lf}(X^{[n-i]} \times X) \cong H_*^{T,lf}(X^{[n-i]}) \otimes H_*^{T,lf}(X)$ in the first operator in (2.2). It is considered as an operator $H_*^{T,lf}(X^{[n-i]}) \to H_*^{T,lf}(X^{[n]})$. Let us denote it by $P_{-i}(\beta)$. It was denoted by $P_\beta[-i]$ in [24].

On the other hand, for the second operator in (2.2), we take the intersection pairing with $\alpha \in H_*^T(X)$ via $H_*^{T,lf}(X^{[n-i]} \times X) \cong H_*^{T,lf}(X^{[n-i]}) \otimes H_*^{T,lf}(X)$. We obtain an operator $H_*^{T,lf}(X^{[n]}) \to H_*^{T,lf}(X^{[n-i]})$, which is denoted by $P_i(\alpha)$.

These operators $P_i(\alpha)$, $P_{-i}(\beta)$ are the same as ones in [24, Ch.8] though the current explanation is slightly different.

We can replace H_*^T by the equivariant Borel-Moore homology group $H_*^{T,lf}$ in (2.2). Then

$$\begin{aligned} P_i(\alpha) &\colon H_*^{T,lf}(X^{[n]}) \to H_*^{T,lf}(X^{[n-i]}), \\ P_{-i}(\beta) &\colon H_*^{T,lf}(X^{[n-i]}) \to H_*^{T,lf}(X^{[n]}) \end{aligned} \tag{2.3}$$

are well-defined for $\alpha \in H_*^T(X)$, $\beta \in H_*^{T,lf}(X)$. Note that homology groups containing α, β are swapped from the above case.

We have a perfect pairing

$$\langle\ ,\ \rangle \colon H_*^{T,lf}(X^{[n]}) \otimes H_*^T(X^{[n]}) \to H_*^T(\mathrm{pt}); c \otimes c' \mapsto (-1)^n a_*(c \cap c') \tag{2.4}$$

by the Poincaré duality. Here $a \colon X^{[n]} \to \mathrm{pt}$. The transpose of $P_i(\alpha)$ is equal to $P_{-i}(\alpha)$.

The operators are linear over $H_T^*(\mathrm{pt})$ from the construction: $P_i(\alpha)f = fP_i(\alpha)$, etc. for $f \in H_T^*(\mathrm{pt})$. It is $H_T^*(\mathrm{pt})$-linear on α: $P_i(f\alpha) = fP_i(\alpha)$, etc.

Note that $H_*^T(X) \cong H_T^*(\mathrm{pt})[0]$, $H_*^{T,lf}(X) \cong H_T^*(\mathrm{pt})[X]$. Therefore it is enough to consider the cases $\alpha, \beta = [0]$ and $[X]$.

Then the following commutation relation of the Heisenberg algebra holds:

$$[P_i(\alpha), P_j(\beta)] = i\delta_{i+j,0}\langle \alpha, \beta\rangle \,\mathrm{id}\,. \tag{2.5}$$

When the right hand side is nonzero, i.e., $i+j=0$, one of α or β is in $H_*^T(X)$ and the other is in $H_*^{T,lf}(X)$. Therefore

$$\langle \alpha, \beta \rangle = -a_{X*}(\alpha \cap \beta) \in H_T^*(\mathrm{pt}) \tag{2.6}$$

is well-defined, where $a_X \colon X \to \mathrm{pt}$. We take the same sign convention as above, understanding $X^{[1]} = X$, i.e., it is (-1) times the intersection pairing.

The proof of (2.5) for nonequivariant homology groups in [24, Ch.8] works also for the equivariant case. Let us briefly explain checkpoints. When $i+j \neq 0$, the right hand side of (2.5) vanishes. In these cases, we study set-theoretical intersections of cycles, check that they are transversal intersections generically and non-generic parts do not contribute to the computation by dimension reason. The last reasoning works in the equivariant case thanks to (1.6).

When $i+j=0$, the argument in [24] works before the sentence "If $\deg \alpha + \deg \beta < 4$, then $\alpha \cap \beta = 0$" in the middle of p.102. This is not true for the equivariant case, as $H^T_{<0}(X)$ could be nonzero. So let us go back a little and see what is actually proved there. We consider

$$L \overset{\text{def.}}{=} \left\{ (I_1, I_3, x) \in X^{[n]} \times X^{[n]} \times X \;\middle|\; I_1 = I_3 \text{ outside } x \right\}. \tag{2.7}$$

Then L has n irreducible components $L_1, \dots, L_n$ with $\dim_{\mathbb{C}} L_i = 2n$, and $(2n+2)$-dimensional irreducible component $\Delta_{X^{[n]}} \times X$, and other irreducible components (if exists) have lower dimension (see [24, Lemma 8.32]). We have the projection $\Pi'' \colon L \to X$ to the third factor. Then by the same argument as the nonequivariant case, we show that there is a class $\mathscr{R}$ in $H^{T,lf}_{4n+4}(L)$ (which is $p_{134*}\iota''_* R'' - p_{134*}\iota'''_* R'''$ in the notation [24, p.101]) such that the left hand side of (2.5) is given by the correspondence

$$p_{13*}(\Pi''^*(\alpha \cap \beta) \cap \mathscr{R}) \in H^{T,lf}_{4n+\deg\alpha+\deg\beta-4}(X^{[n]} \times_{S^n X} X^{[n]}), \tag{2.8}$$

where $p_{13} \colon L \to X^{[n]} \times_{S^n X} X^{[n]}$ is the projection $(I_1, I_3, x) \mapsto (I_1, I_3)$. By dimension reason, we see that $\mathscr{R}$ comes from a class in $H^{T,lf}_{4n+4}(\Delta_{X^{[n]}} \times X)$. Since it is of top degree, it must be $c_{i,n}[\Delta_{X^{[n]}} \times X]$, where the multiple constant $c_{i,n}$ is independent of equivariant variables, i.e., a complex number. Therefore

$$p_{13*}(\Pi''^*(\alpha \cap \beta) \cap \mathscr{R}) = c_{i,n} \langle \alpha, \beta \rangle [\Delta_{X^{[n]}}]. \tag{2.9}$$

Now the only task is to determine $c_{i,n}$. Since $c_{i,n}$ is a complex number, we can take nonequivariant limit: it is the same for equivariant and nonequivariant cases. Therefore we conclude $c_{i,n} = i$.

2(ii). Torus fixed points

For a later purpose, let us recall fixed points with respect to the two dimensional torus T. They are parametrized by partitions λ of n, or equivalently Young diagrams D with n boxes. (See [24, Ch. 5 and 6].) We keep our convention in [24]: for $\lambda = (\lambda_1, \lambda_2, \dots)$, the corresponding torus fixed point is the monomial ideal

$$I_\lambda = (\xi^{\lambda_1}, z\xi^{\lambda_2}, \dots, z^{i-1}\xi^{\lambda_i}, \dots). \tag{2.10}$$

If we put the monomial $z^{i-1}\xi^{j-1}$ in the box at the intersection of the i^{th} column and the j^{th} row, I_λ is generated by monomials which sit outside of D. See Figure 1.

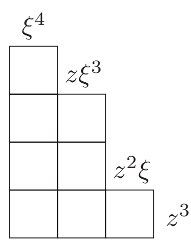

Fig. 1. Young diagram and an ideal

2(iii). Symmetric products of the z-axis

The constant $c_{i,n}$ (in fact, it is easy to see that it is independent of n by a similar argument as above) was determined by studying symmetric products of an embedded curve C in X in the nonequivariant case [24, Ch.9]. We shall briefly review the study here, emphasizing that the argument goes through even when C and X are noncompact. We shall use the construction later.

Let C denote the z-axis, i.e., $C = \{\xi = 0\}$. Let L^*C denote the subvariety consisting of $I \in \bigsqcup_n X^{[n]}$ such that $\mathbb{C}[z, \xi]/I$ is contained in C. In other words, it is the union $\bigsqcup_n \pi^{-1}(S^nC)$ of inverse images of the symmetric products S^nC under the Hilbert-Chow morphism.

By [24, Ch.7] it has irreducible components $L^\lambda C$ parametrized by partitions λ. Moreover, it is a lagrangian subvariety in $X^{[n]}$ with $n = |\lambda|$. We recall two descriptions of irreducible component $L^\lambda C$.

The first one is by the $\mathbb{C}^*$-action given by $t \cdot (z, \xi) \mapsto (z, t\xi)$. Then $I \in \bigsqcup_n X^{[n]}$ is fixed by $\mathbb{C}^*$ if and only if it is of a form

$$I = I_{D_1, z_1} \cap \cdots \cap I_{D_k, z_k}, \tag{2.11}$$

where $z_i \in C$, D_i is a Young diagram and I_{D_i, z_i} is the ideal generated by monomials in $(z - z_i)$ and ξ which sit outside of D_i as in §2(ii). The diagram D_i and the point z_i is uniquely determined by I. Let D be the union of all D_i, reordering columns so that it is a Young diagram. Let λ be the corresponding partition. Then $S^\lambda C$ consisting of fixed points I with $D = D_1 \sqcup D_2 \sqcup \cdots$ is a connected component of $(X^{[n]})^{\mathbb{C}^*}$, and any connected component is of this form.

We have

$$L^*C = \bigsqcup_n \left\{ I \in X^{[n]} \;\middle|\; \lim_{t \to \infty} t \cdot I \text{ exists} \right\}, \tag{2.12}$$

which decomposes according to the limit as $L^*C = \bigsqcup W_\lambda^-$ with

$$W_\lambda^- \overset{\text{def.}}{=} \left\{ I \in L^*C \;\middle|\; \lim_{t \to \infty} t \cdot I \in S^\lambda C \right\}. \tag{2.13}$$

Each W_λ^- is a locally closed lagrangian subvariety in $X^{[n]}$ and we define

$$L^\lambda C \overset{\text{def.}}{=} \text{Closure of } W_\lambda^-. \tag{2.14}$$

For example, $\lambda = (1^n)$, $L^\lambda C$ is isomorphic to the symmetric product $S^n C$ of C, embedded into $X^{[n]}$ via the natural morphism $S^n C \cong C^{[n]} \to X^{[n]}$. This is clearly a lagrangian subvariety in $X^{[n]}$ since the symplectic form is an extension of one on the open subset $\pi^{-1}(S^n_{(1^n)} X)$. The opposite extreme is $L^{(n)} C$, which consists of ideals I such that $\mathbb{C}[z, \xi]/I$ is supported at a single point in C. The lagrangian property is less clear, and follows from the Morse theoretic argument in [24, Ch.7].

The second description is by the Hilbert-Chow morphism $\pi \colon X^{[n]} \to S^n X$. Recall we have $L^*C = \pi^{-1}(S^n C)$. Note that $S^n C$ has the natural stratification $S^n C = \bigsqcup S^n_\lambda C$, where

$$S^n_\lambda C = \left\{ \sum_i \lambda_i [x_i] \in S^n C \;\middle|\; x_i \neq x_j \text{ for } i \neq j \right\}. \tag{2.15}$$

We decompose L^*C accordingly

$$L^*C = \bigsqcup \pi^{-1}(S^n_\lambda C). \tag{2.16}$$

Each $\pi^{-1}(S^n_\lambda C)$ is a locally closed n-dimensional irreducible subvariety in $L^*C(\cap X^{[n]})$, where the irreducibility follows from that of punctual Hilbert schemes $\pi^{-1}(m[p])$ ([24, Th. 5.12]). Hence its closure is an irreducible component.

Proposition 2.17. *$L^\lambda C$ is the closure of $\pi^{-1}(S^n_\lambda C)$.*

Proof. It is enough to note that the above component $S^\lambda C$ of $(X^{[n]})^{\mathbb{C}^*}$ has an open locus consisting of $I = I_{D_1,z_1} \cap \cdots \cap I_{D_k,z_k}$ such that all D_i's have only single column. Q.E.D.

There is the third description, which was not given in [24]: Let us consider the *hyperbolic* $\mathbb{C}^*$-action $t * (z,\xi) = (t^{-1}z, t\xi)$. Then we claim that $I \in \bigsqcup_n X^{[n]}$ has limit when $t \to \infty$ if and only if $I \in L^*C$. If we replace $X^{[n]}$ by the symmetric product S^nX, the corresponding assertion is obvious. A point $\sigma \in S^nX$ has limit when $t \to \infty$ if and only if $\sigma \in S^nC$. Since π is proper, the assertion follows. Then L^*C decomposes according to the decomposition of $(X^{[n]})^{\mathbb{C}^*}$ as in (2.13), where the $\mathbb{C}^*$-action is replaced by the hyperbolic one. Let us first note that $(X^{[n]})^{\mathbb{C}^*} = (X^{[n]})^T$. This follows, for example, from the T-character formula of the tangent space at $I_\lambda \in (X^{[n]})^T$ reviewed in Proposition 3.19 below. Put $t_1 = t_2^{-1} = t$. We see that $T_{I_\lambda}X^{[n]}$ has a trivial weight zero space. It means that $(X^{[n]})^{\mathbb{C}^*}$ cannot be larger than $(X^{[n]})^T$.

Proposition 2.18.

$$W_\lambda^- = \left\{ I \in L^*C \;\middle|\; \lim_{t\to\infty} t * I = I_\lambda \right\}.$$

Proof. Since both left and right hand sides are T-invariant locally closed submanifolds, it is enough to prove the equality in a neighborhood of the fixed point I_λ.

The tangent space of W_λ^- at I_λ is equal to the sum of nonpositive weight spaces in $T_{I_\lambda}X^{[n]}$ with respect to the $\mathbb{C}^*$-action given by $t\cdot(z,\xi) = (z, t\xi)$. Similarly, the tangent space of the right hand side is the sum of negative weight spaces with respect to the hyperbolic $\mathbb{C}^*$-action $t * (z,\xi) = (t^{-1}z, t\xi)$. Looking at the formula in Proposition 3.19 below, one finds that both spaces are the same space. In fact, it corresponds to $\sum t_1^{l(s)+1} t_2^{-a(s)}$. (One can also check that it is a lagrangian subspace in $T_{I_\lambda}X^{[n]}$, as the symplectic form is of weight t_1t_2.)

There is no nontrivial T-homomorphism from any symmetric power of $T_{I_\lambda}W_\lambda^-$ to the sum of complementary weight spaces, and hence a T-invariant submanifold with the tangent space $T_{I_\lambda}W_\lambda^-$ is unique, as in the proof of the existence of Białynicki-Birula decomposition (see [5, Th. 2.2]). Therefore two submanifolds are equal. Q.E.D.

Let us consider the top degree Borel-Moore homology group of L^*C:

$$H^{lf}_{\mathrm{top}}(L^*C) = \bigoplus_n H^{lf}_{2n}(L^*C \cap X^{[n]}). \tag{2.19}$$

This is a vector space with a base $\{[L^\lambda C]\}$. The creation operator $P_{-i}(\beta)$ acts on $H^{lf}_{\mathrm{top}}(L^*C)$ for $\beta = [C] \in H^{lf}_2(C)$. On the other hand, for the annihilation operator $P_i(\alpha)$, we take $\alpha = [\xi\text{-axis}]$. Then the intersection pairing of α and β is well-defined (in fact, it is 1), as $\{z\text{-axis}\}\cap\{\xi\text{-axis}\}$ is a single point in X, and $P_i(\alpha)$ is a well-defined operator on $H^{lf}_{\mathrm{top}}(L^*C)$. The top degree is preserved by the convolution product, as $P[i]$ in (2.1) is middle dimensional in $X^{[n]} \times X^{[n-i]} \times X$, and α, β are so in X.

We have a linear map from the ring Λ of symmetric functions to $H^{lf}_{\mathrm{top}}(L^*C)$ by

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \mapsto P_{-\lambda_1}(\beta) P_{-\lambda_2}(\beta) \cdots 1, \tag{2.20}$$

for $\lambda = (\lambda_1, \lambda_2, \dots)$. Here p_k is the k^{th} power sum. From the Heisenberg algebra relation (2.5), it is injective. (For this we only need that the constant $c_{i,n}$ is nonzero, which is a consequence of the Poincaré duality.) Since $\dim H^{lf}_{\mathrm{top}}(L^*C \cap X^{[n]})$ is equal to the number of partitions of n, it is an isomorphism.

In order to determine the coefficient i in (2.5), it is enough to compute it in the current situation. A key result is

Proposition 2.21. *Under the isomorphism $\Lambda \cong H^{lf}_{\mathrm{top}}(L^*C)$, the class $[L^\lambda C]$ corresponds to the monomial symmetric function m_λ.*

This result together with the formula

$$P_i(\alpha)[L^{(1^n)}C] = [L^{(1^{n-i})}C] \tag{2.22}$$

(see [24, Lemma 9.21]) determines the coefficient.

§3. Jack symmetric functions and torus fixed points

The goal of this section is to give a geometric realization of Jack symmetric functions in the T-equivariant homology groups of Hilbert schemes of points in $X = \mathbb{C}^2$.

The result presented here was stated and proved by Li-Qin-Wang [17]. Both the framework and the proof were based on an earlier work by Vasserot [27], who considered Schur functions and the $\mathbb{C}^*$-equivariant homology groups, instead of T-equivariant ones. Vasserot's work, in turn, was motivated by author's unpublished paper [22], where Jack symmetric functions were considered in a different setting, i.e., the case when X is the total space of a line bundle over a Riemann surface.

Here we present the proof in [17], as well as materials in [22] used there.

The result here could be viewed as a homological version of Haiman's result [12], which relates Macdonald polynomials to T-equivariant K-theory of Hilbert schemes of points in X. However there is a big difference: we relate $H_*^T(X^{[n]})$ with the ring of symmetric functions via the Heisenberg representation in the previous section. On the other hand, Haiman relates them via the *Procesi bundle*, a rank $n!$ vector bundle over $X^{[n]}$. The definition of the Procesi bundle is rather involved, and will not be presented here. Let us also remark that the Procesi bundle is essential to prove the positivity conjecture for Macdonald polynomials, and we do not have a counter part in our theory.

3(i). Jack symmetric functions

Let us briefly recall the definition of Jack symmetric functions in [19, VI.10]. Our notation follows [19] except that the parameter α is replaced by $\boldsymbol{k}$ here.

Let $\boldsymbol{k}$ be an indeterminate. We define an inner product on the ring $\Lambda_{\mathbb{C}(\boldsymbol{k})}$ of symmetric functions with coefficients in $\mathbb{C}(\boldsymbol{k})$ by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \boldsymbol{k}^{l(\lambda)} z_\lambda, \tag{3.1}$$

where $l(\lambda)$ is the length of a partition λ and $z_\lambda = \prod k^{m_k} m_k!$ for $\lambda = (1^{m_1} 2^{m_2} \cdots)$.

Let $\geq$ denote the *dominance order* on partitions, i.e., $\lambda \geq \mu$ if and only if $|\lambda| = |\mu|$ and

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \quad \text{for all } i. \tag{3.2}$$

If λ', μ' denote conjugate partitions of λ, μ respectively, we have

$$\lambda \geq \mu \Longleftrightarrow \mu' \geq \lambda'.$$

See [19, I.(1.11)] for the proof.

Let m_λ be the monomial symmetric function. A Jack symmetric function $P_\lambda^{(\boldsymbol{k})}$ is characterized by the following two properties

$$P_\lambda^{(\boldsymbol{k})} = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu}^{(\boldsymbol{k})} m_\mu \quad \text{for } u_{\lambda\mu}^{(\boldsymbol{k})} \in \mathbb{C}(\boldsymbol{k}), \tag{3.3}$$

$$\langle P_\lambda^{(\boldsymbol{k})}, P_\mu^{(\boldsymbol{k})} \rangle = 0 \quad \text{if } \lambda \neq \mu. \tag{3.4}$$

Let $\succeq$ be a total order refining the dominance order. Then $\{P_\lambda^{(\boldsymbol{k})}\}$ is obtained from the base $\{m_\lambda\}$ by Gram-Schmidt orthogonalization with respect to $\succeq$. Hence the uniqueness of $P_\lambda^{(\boldsymbol{k})}$ follows. The existence is obvious if we replace $\geq$ by $\succeq$ in (3.3), and this difference of orders is a nontrivial part in the theory of Jack symmetric functions. We briefly recall the proof in [19] in the next subsection.

When $\boldsymbol{k} = 1$, the inner product (3.1) is the standard one. The above two properties are satisfied by Schur functions s_λ.

3(ii). Hamiltonian

For a positive integer N, let us consider the ring $\Lambda_{N,\mathbb{C}(\boldsymbol{k})}$ of symmetric functions in N-variables $x_1, \ldots, x_N$ with $\mathbb{C}(\boldsymbol{k})$-coefficients. (Note that N, $\boldsymbol{k}$ correspond to n, α in [19].)

Let X be an indeterminate and let

$$D_N \overset{\text{def.}}{=} a_\delta(x)^{-1} \sum_{w\in S_N} \varepsilon(w) x^{w\delta} \prod_{i=1}^N \left(X + (w\delta)_i + \boldsymbol{k} x_i \frac{\partial}{\partial x_i} \right),$$

where $\delta = (N-1, N-2, \ldots, 0)$, $a_\delta(x) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant, $\varepsilon(w) = \pm 1$ is the sign of $w \in S_N$, and $(w\delta)_i$ is the i^{th} component of $w\delta$.

For $r = 0, 1, \ldots, N$ let D_N^r denote the coefficient of X^{N-r} in D_N:

$$D_N = \sum_{r=0}^N X^{N-r} D_N^r.$$

By [19, VI, §3. Ex.3(a)] we have

$$D_N m_\lambda(x) = \sum_{w\in S_N} \prod_{i=1}^N (X + N - i + \boldsymbol{k}(w\delta)_i) s_{w\lambda}(x),$$

where

$$s_{w\lambda}(x) = \frac{a_{\delta+w\lambda}(x)}{a_\delta(x)} = \frac{1}{a_\delta(x)} \sum_{w_1\in S_N} \varepsilon(w_1) x^{w_1(w\lambda+\delta)}.$$

(There is a typo in [19, VI, §3, Ex.3(a)]. In the formula (a), λ_i should read β_i or $(w\lambda)_i$ in the current notation.) Since $s_{w\lambda}(x)$ is either zero or is equal to $\pm s_\mu$ for some partition $\mu < \lambda$ (unless $w = 1$), and since the transition matrix between $\{s_\lambda\}$ and $\{m_\lambda\}$ is upper triangular with 1 on diagonal, we get

$$D_N m_\lambda(x) = \sum_{\mu \le \lambda} c_{\lambda\mu}(X; \boldsymbol{k}) m_\mu(x)$$

with $c_{\lambda\mu} \in \mathbb{Z}[X, \boldsymbol{k}]$. See [19, VI, §4, Ex.4]. Here $m_\lambda(x)$ and $s_\lambda(x)$ are zero if $l(\lambda) > N$, and $\{m_\lambda(x)\}_{l(\lambda)\le N}$, $\{s_\lambda(x)\}_{l(\lambda)\le N}$ are bases of Λ_N. The diagonal entries are given by

$$c_{\lambda\lambda}(X; \boldsymbol{k}) = \prod_{i=1}^{N} (X + N - i + \boldsymbol{k}\lambda_i).$$

When we set $\boldsymbol{k}$ a positive real number, $N - i + \boldsymbol{k}\lambda_i$ is strictly decreasing in i. Therefore $c_{\lambda\lambda} \ne c_{\mu\mu}$ for $\lambda \ne \mu$. Therefore there exists $P^{(\boldsymbol{k})}_\lambda$ of the form (3.3) such that

$$D_N P^{(\boldsymbol{k})}_\lambda = c_{\lambda\lambda}(X; \boldsymbol{k}) P^{(\boldsymbol{k})}_\lambda.$$

Moreover D_N is self-adjoint with respect to the finitely many variable version of the inner product (3.1) [19, VI, §3, Ex.3(b)]. Therefore we deduce (3.4). It means that $\{P^{(\boldsymbol{k})}_\lambda\}$ is obtained from the base $\{m_\lambda\}$ by Gram-Schmidt orthogonalization with respect to any total order $\succeq$ compatible with $\ge$.

The ring $\Lambda_{\mathbb{C}(\boldsymbol{k})}$ is defined as the direct sum of projective limit

$$\Lambda_{\mathbb{C}(\boldsymbol{k})} = \bigoplus \varprojlim_N \Lambda^n_{N,\mathbb{C}(\boldsymbol{k})},$$

where $\Lambda^n_{N,\mathbb{C}(\boldsymbol{k})}$ is the degree n part of $\Lambda_{N,\mathbb{C}(\boldsymbol{k})}$, and the inverse system is given by the homomorphism $\Lambda_{M,\mathbb{C}(\boldsymbol{k})} \to \Lambda_{N,\mathbb{C}(\boldsymbol{k})}$ sending $x_{N+1}, \dots, x_M$ to 0. Under the homomorphism $m_\lambda(x_1, \dots, x_M)$ is sent to $m_\lambda(x_1, \dots, x_N)$ if $l(\lambda) \le N$, and to 0 if $l(\lambda) > N$. Therefore Gram–Schmidt orthogonalization is compatible with N, and we get $P^{(\boldsymbol{k})}_\lambda \in \Lambda_{\mathbb{C}(\boldsymbol{k})}$, as the limit of $P^{(\boldsymbol{k})}_\lambda$ above when $N \to \infty$. This finishes the proof of the existence of Jack symmetric functions.

For a later purpose, let us give a formula for D_N^2 ([19, VI, §3, Ex.3(d)]). Suppose that f is a homogeneous polynomial of degree r. Then

$$D_N^2 f = (-\boldsymbol{k}^2 U_N - \boldsymbol{k} V_N + c_N) f,$$

where

$$U_N = \frac{1}{2}\sum_{i=1}^{N} x_i^2 \frac{\partial^2}{\partial x_i^2}, \quad V_N = \sum_{i\neq j} \frac{x_i^2}{x_i - x_j}\frac{\partial}{\partial x_i},$$

and

$$c_N = \frac{1}{2}\boldsymbol{k}^2 r(r-1) + \frac{1}{2}\boldsymbol{k} rN(N-1) + \frac{1}{24}N(N-1)(N-2)(3N-1).$$

We further introduce an operator

$$\square_N^{\boldsymbol{k}} f = (\boldsymbol{k}U_N + V_N - (N-1)r)\, f.$$

This operator has the limit

$$\square^{\boldsymbol{k}} = \varprojlim_N \square_N^{\boldsymbol{k}}. \tag{3.5}$$

See [19, VI, §4, Ex.3(a)] or (3.7) below.

We have

$$\square^{\boldsymbol{k}} P_\lambda^{(\boldsymbol{k})} = e_\lambda(\boldsymbol{k}) P_\lambda^{(\boldsymbol{k})}; \qquad e_\lambda(\boldsymbol{k}) \overset{\text{def.}}{=} n(\lambda')\boldsymbol{k} - n(\lambda), \tag{3.6}$$

where

$$n(\lambda) = \sum (i-1)\lambda_i = \sum \frac{\lambda_i'(\lambda_i'-1)}{2}.$$

Computing how $\square^{\boldsymbol{k}}$ acts on the base $\{p_\lambda\}$ of $\Lambda_{\mathbb{C}(\boldsymbol{k})}$, we obtain the following formula:

$$\begin{aligned}\square^{\boldsymbol{k}} = \frac{\boldsymbol{k}}{2}\sum_{m,n>0} mnp_{m+n}\frac{\partial}{\partial p_m}\frac{\partial}{\partial p_n} + \frac{\boldsymbol{k}-1}{2}\sum_{m>0} m(m-1)p_m \frac{\partial}{\partial p_m} \\ + \frac{1}{2}\sum_{m,n>0}(m+n)p_m p_n \frac{\partial}{\partial p_{m+n}}.\end{aligned} \tag{3.7}$$

Here we regard $\Lambda_{\mathbb{C}(\boldsymbol{k})}$ as a polynomial ring $\mathbb{C}(\boldsymbol{k})[p_1, p_2, \dots]$. This formula will be crucial in §4. It is not present in [19]. The author learned it from [2] when he wrote [22], but it was certainly known much before.

Remark 3.8. The operator D_N^2 is essentially equal to the Calogero-Sutherland hamiltonian, which has been studied intensively in the context of quantum integrable systems. (See [11, §5.5] and the reference therein for example.) It is a trigonometric analog of a quantization of the Calogero-Moser integrable system, appeared in Wilson's work [28], mentioned at [24, a paragraph preceding Theorem 3.46]. At first sight, two appearances of this integrable system have no link: one is a classical system and appears in the deformation of $X^{[n]}$, while the other is quantum and appears in the cohomology of $X^{[n]}$. However, they are connected in a deep way: Bezrukavnikov-Finkelberg-Ginzburg [4] considered the quantized integrable system in *positive characteristic*, and connected it with the derived category of $X^{[n]}$. See also [3] for a nice application of this result.

3(iii). Inner product

After giving the review of the definition and basic properties of Jack symmetric functions, we start study of equivariant homology groups of Hilbert schemes. In this subsection we identify (3.1) with the inner product on $\bigoplus H_T^*(X^{[n]})_{\mathbb{F}}$ induced from (2.4).

Let $\alpha, \beta \in H_T^*(X)_{\mathbb{F}}$. The commutation relation (2.5) implies

$$\langle P_{-n}(\alpha), P_{-n}(\beta)\rangle = n\langle \alpha, \beta\rangle.$$

More generally, we consider an analog of p_λ:

$$P_\lambda(\alpha) = P_{-\lambda_1}(\alpha)P_{-\lambda_2}(\alpha)\cdots$$

for $\lambda = (\lambda_1, \lambda_2, \dots)$. Then we have

$$\langle P_\lambda(\alpha), P_\mu(\beta)\rangle = \delta_{\lambda\mu}\langle \alpha, \beta\rangle^{l(\lambda)} z_\lambda, \tag{3.9}$$

where $z_\lambda = \prod k^{m_k} m_k!$ for $\lambda = (1^{m_1}2^{m_2}\cdots)$.

In particular, we take α and β to be the Poincaré dual of the z-axis, i.e., $\alpha = \beta = \varepsilon_2$. Then

$$\langle \alpha, \beta\rangle = -\frac{\varepsilon_2}{\varepsilon_1}.$$

(See the beginning of §2(i) for the notation ε_1, ε_2.)

Substituting this into (3.9) and comparing the result with (3.1), we find

Proposition 3.10. *Our inner product is equal to* (3.1) *used to define Jack symmetric functions under the identification* $P_\lambda(\varepsilon_2) \leftrightarrow p_\lambda$*, where the parameter* $\boldsymbol{k}$ *is* $-\varepsilon_2/\varepsilon_1$*.*

When $\varepsilon_1 + \varepsilon_2 = 0$, we have $\boldsymbol{k} = -\varepsilon_2/\varepsilon_1 = 1$. It means that our inner product is the standard inner product on symmetric polynomials.

3(iv). Dominance order

Recall that we identify the monomial symmetric function m_λ with the class $[L^\lambda C]$ in Proposition 2.21. In order to understand the characterization of Jack symmetric functions in (3.3, 3.4), our next task is to explain a geometric meaning of the dominance order (3.2). It is given by modifying the stratification introduced in [6, 13].

Let $C = \{\xi = 0\}$ as before. For $i \geq 0$, let (ξ^i) be the ideal of functions vanishing to order $\geq i$ along C. Let $I \in X^{[n]}$ be an ideal of colength n such that the support of $\mathbb{C}[z,\xi]/I$ is contained in C. We consider the sequence $(\lambda'_1, \lambda'_2, \dots)$ of nonnegative integers given by

$$\lambda'_i(I) \stackrel{\text{def.}}{=} \dim \left(\frac{(\xi^{i-1})}{I \cap (\xi^{i-1}) + (\xi^i)} \right). \tag{3.11}$$

The reason why we put the prime become clear later. The sequence in [6, 13] was defined by replacing (ξ^i) by $\mathfrak{m}_x^i$ where $\mathfrak{m}_x$ is the maximal ideal corresponding to a point x. It is clear that $I \supset (\xi^n)$ (see e.g., [13, Lemma 1.1]), hence $\lambda'_i(I) = 0$ for $i \geq n+1$. From the exact sequence

$$0 \to \frac{(\xi^i)}{I \cap (\xi^i)} \to \frac{(\xi^{i-1})}{I \cap (\xi^{i-1})} \to \frac{(\xi^{i-1})}{I \cap (\xi^{i-1}) + (\xi^i)} \to 0,$$

we have

$$\sum_{i=1}^n \lambda'_i(I) = n.$$

If $\xi^i f_1(z), \dots, \xi^i f_d(z)$ form a basis of

$$\frac{(\xi^i)}{I \cap (\xi^i) + (\xi^{i+1})},$$

Then $\xi^{i-1} f_1(z), \dots, \xi^{i-1} f_d(z)$ are linearly independent in

$$\frac{(\xi^{i-1})}{I \cap (\xi^{i-1}) + (\xi^i)}.$$

Hence we have $\lambda'_i(I) \geq \lambda'_{i+1}(I)$. Thus $(\lambda'_1(I), \lambda'_2(I), \dots)$ is a partition of n. Let us denote the partition by $\lambda'(I)$.

For a partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ of n, let $V^{\lambda'}$ be the set of ideals $I \in X^{[n]}$ such that $\mathbb{C}[z,\xi]/I$ is supported on C and $\lambda'(I) = \lambda'$. Since

$$\dim \frac{(\xi^i)}{I \cap (\xi^i)} \leq \sum_{j=i+1}^n \lambda'_j = n - \sum_{j=1}^i \lambda'_j$$

is a closed condition on I, the union

$$\bigcup_{\mu' : \mu' \geq \lambda'} V^{\mu'}$$

is a closed subset of $\{I \in X^{[n]} \mid \operatorname{Supp}(\mathbb{C}[z,\xi]/I) \subset C\}$. Thus we have

$$\text{Closure of } V^{\lambda'} \subset \bigcup_{\mu' \geq \lambda'} V^{\mu'}. \tag{3.12}$$

Suppose that λ' is the conjugate partition of λ. We get the following third description of $L^\lambda C$.

Proposition 3.13. *$L^\lambda C =$ Closure of $V^{\lambda'}$.*

Proof. Let us write $\lambda = (\lambda_1, \dots, \lambda_N)$ with $N = l(\lambda)$. Using the description of $L^\lambda C$ as the closure of W_λ^-, we first check that $W_\lambda^- \subset V^{\lambda'}$. We may take a generic point in $I \in W_\lambda^-$, so $\lim_{t\to\infty} t \cdot I = I_{(\lambda_1),z_1} \cap \cdots \cap I_{(\lambda_N),z_N}$ such that x_i's are distinct points in C and Young diagrams $D_1, \dots$ have only single column, $D_1 = (\lambda_1), \dots$. Since the support of I cannot move as $t \to \infty$, we can decompose $I = I_1 \cap \cdots \cap I_N$ such that $\operatorname{Supp}(\mathbb{C}[z,\xi]/I_k) = \{(z_k, 0)\}$.

Recall that $I_{(\lambda_k),z_k}$ is the ideal $(z - z_k, \xi^{\lambda_k})$. It is contained in $V^{(\lambda_k)'}$, as can be checked directly in the definition. Since $(\lambda_k)' = (1^{\lambda_k})$ is the unique minimum in the dominance order, $V^{(\lambda_k)'}$ is open in L^*C. As $\lim_{t\to\infty} t \cdot I_k = I_{(\lambda_k),x_k}$, we have $t \cdot I_k \in V^{(\lambda_k)'}$ for sufficiently large t. It is clear that $V^{\lambda'}$ is invariant under the $\mathbb{C}^*$-action for any λ. In particular, $I_k \in V^{(\lambda_k)'}$. Since $I = I_1 \cap \cdots \cap I_N$, we have $I \in V^{\lambda'}$. Thus $L^\lambda C \subset$ Closure of $V^{\lambda'}$. As we have $L^*C = \bigsqcup V^{\lambda'}$, we conclude $L^\lambda C =$ Closure of $V^{\lambda'}$. Q.E.D.

3(v). Fixed points and Jack symmetric functions

Recall that the torus fixed points in $X^{[n]}$ are parametrized by partitions λ with $|\lambda| = n$, as we have explained at the beginning of §2(i).

Let $\iota\colon (X^{[n]})^T = \bigsqcup_\lambda \{I_\lambda\} \to L^*C \cap X^{[n]}$ be the inclusion of the T-fixed point set. Here note that all fixed points I_λ are contained in L^*C. Let $\zeta\colon L^*C \cap X^{[n]} \to X^{[n]}$ be the inclusion. By the localization theorem, we have isomorphisms

$$H_*^T((X^{[n]})^T)_{\mathbb{F}} \xrightarrow[\cong]{\iota_*} H_*^{T,lf}(L^*C \cap X^{[n]})_{\mathbb{F}} \xrightarrow[\cong]{\zeta_*} H_*^{T,lf}(X^{[n]})_{\mathbb{F}}. \tag{3.14}$$

The leftmost space $H_*^T((X^{[n]})^T)_{\mathbb{F}}$ is the direct sum $\bigoplus_\lambda \mathbb{C}(\varepsilon_1,\varepsilon_2)[I_\lambda]$, in particular, it has a base $[I_\lambda]$. As $L^*C \cap X^{[n]}$ is the union of W_λ^-, which is a vector bundle over an affine space $S^\lambda C$ by §2(iii), a standard argument (as in [24, Ch.5]) show that $[L^\lambda C]$ is a base of the middle space $H_*^{T,lf}(L^*C \cap X^{[n]})_{\mathbb{F}}$. In particular, we have an isomorphism

$$\Lambda \otimes \mathbb{C}(\varepsilon_1,\varepsilon_2) \cong H_*^{T,lf}(L^*C)_{\mathbb{F}}, \tag{3.15}$$

extending the isomorphism $\Lambda \cong H^{lf}_{\mathrm{top}}(L^*C)$.

Let $i_\lambda\colon \{I_\lambda\} \to X^{[n]}$ be the inclusion of the fixed point I_λ to the Hilbert scheme. Therefore $\zeta \circ \iota = \bigsqcup i_\lambda$.

The inverse of the composition $\zeta_* \iota_*$ in (3.14) is given by

$$\sum_\lambda \frac{1}{e(T_{I_\lambda} X^{[n]})} i_\lambda^*(\bullet),$$

where $e(T_{I_\lambda} X^{[n]})$ is the T-equivariant Euler class of the tangent space of $X^{[n]}$ at I_λ. See Theorem 1.13.

Let us consider the $\iota_*^{-1}([L^\lambda C])$. Note that the partition given by (3.11) for $I = I_\mu$ is μ'. Hence $I_\mu \in V^{\mu'}$. Therefore we have $I_\mu \in L^\lambda C$ only if $\lambda \geq \mu$ by (3.12) and Proposition 3.13. This implies that

$$\iota_*^{-1}([L^\lambda C]) \in \bigoplus_{\mu \leq \lambda} \mathbb{C}(\varepsilon_1, \varepsilon_2)[I_\mu]. \tag{3.16}$$

Let us next consider the coefficient of $[I_\lambda]$ in (3.16). Recall that $L^\lambda C$ is defined as the closure of W_λ^- in (2.13). As $I_\lambda \in S^\lambda C$, $L^\lambda C$ is a submanifold in a neighborhood of I_λ. The tangent space $T_{I_\lambda} L^\lambda C$ is the direct sum of weight subspaces of $T_{I_\lambda} X^{[n]}$, whose weights are nonpositive with respect to the $\mathbb{C}^*$-action $t(z, \xi) = (z, t\xi)$. Let us decompose the tangent space $T_{I_\lambda} X^{[n]}$ into $T_{I_\lambda}^{>0} \oplus T_{I_\lambda}^{\leq 0}$, sum of positive and nonpositive weight subspaces. Then the fiber of the normal bundle of $L^\lambda C$ at I_λ is identified with $T_{I_\lambda}^{>0}$. Hence we have

$$i_{\lambda *}^{-1} \zeta_* [L^\lambda C] = \frac{e(T_{I_\lambda}^{>0})}{e(T_{I_\lambda} X^{[n]})} [I_\lambda] = \frac{1}{e(T_{I_\lambda}^{\leq 0})} [I_\lambda]. \tag{3.17}$$

Now we arrive at our main result in this section.

Theorem 3.18. *Under the isomorphism* (3.14) *together with* (3.15), *the class* $\frac{1}{e(T_{I_\lambda}^{\leq 0})}[I_\lambda]$ *corresponds to the Jack symmetric function* $P_\lambda^{(\boldsymbol{k})}$ *with* $\boldsymbol{k} = -\varepsilon_2/\varepsilon_1$.

Proof. Let us check two properties (3.3) and (3.4). The property (3.3) follows from (3.16) and (3.17).

Next note that the composite $\zeta_* \iota_*$ of (3.14) preserves the inner product by the definition of the inner product (2.4), where the inner product on $H_*^T((X^{[n]})^T)_{\mathbb{F}} \cong \bigoplus \mathbb{C}(\varepsilon_1, \varepsilon_2)[I_\lambda]$ is the direct sum of the standard inner product $\langle [I_\lambda], [I_\lambda] \rangle = 1$. Then it is clear that $\langle \frac{1}{e(T_{I_\lambda}^{\leq 0})}[I_\lambda], \frac{1}{e(T_{I_\mu}^{\leq 0})}[I_\mu] \rangle = 0$ if $\lambda \neq \mu$. Q.E.D.

Let us make $e(T_{I_\lambda}^{\leq 0})$ concrete.

Proposition 3.19. *The character of the tangent space of $X^{[n]}$ at the fixed point I_λ is given by the formula*

$$\operatorname{ch} T_{I_\lambda} X^{[n]} = \sum_{s \in \lambda} \left(t_1^{l(s)+1} t_2^{-a(s)} + t_1^{-l(s)} t_2^{a(s)+1} \right).$$

See [24, Prop. 5.8] for the proof. Here $l(s)$, $a(s)$ are the *leg length* and the *arm length* of a square s in the Young diagram corresponding to the partition λ. Our convention is the same as in [24, (5.7)], and also as in [19, (6.14)] except that our Young diagram is rotated by 90° in anti-clockwise. For a later purpose, we also introduce the *leg colength* and the *arm colength* by $l'(s) = i - 1$, $a'(s) = j - 1$:

$$\begin{array}{|c|c|c|c|}
\cline{1-1}
 \\
\cline{1-2}
 & \heartsuit \\
\cline{1-3}
 & \heartsuit & \\
\hline
\clubsuit & s & \spadesuit & \spadesuit \\
\hline
 & \diamondsuit & & \\
\hline
\end{array}
\qquad
\begin{array}{ll}
a(s) & = \text{number of } \heartsuit \\
l(s) & = \text{number of } \spadesuit \\
a'(s) & = \text{number of } \diamondsuit \\
l'(s) & = \text{number of } \clubsuit
\end{array}$$

Corollary 3.20. *The equivariant Euler class of the nonpositive part $T_{I_\lambda}^{\leq 0}$ of the tangent space at the fixed point I_λ is given by*

$$e(T_{I_\lambda}^{\leq 0}) = \prod_{s \in \lambda} ((l(s)+1)\varepsilon_1 - a(s)\varepsilon_2).$$

Comparing this expression with [19, VI, (10.21)], we get

$$[I_\lambda] = \varepsilon_1^{|\lambda|} J_\lambda^{(\boldsymbol{k})},$$

where $J_\lambda^{(\boldsymbol{k})}$ is the integral form of the Jack symmetric function $P_\lambda^{(\boldsymbol{k})}$, defined in [19, VI, (10.22)].

As a corollary of the above computation, we give a geometric proof of the norm formula [19, VI, (10.16)].

Proposition 3.21.

$$\langle P_\lambda^{(\boldsymbol{k})}, P_\lambda^{(\boldsymbol{k})} \rangle = \prod_{s \in \lambda} \frac{l(s) + (a(s)+1)\boldsymbol{k}}{l(s) + 1 + a(s)\boldsymbol{k}}.$$

Proof. This is a direct consequence of

$$\langle \frac{1}{e(T_{I_\lambda}^{\leq 0})}[I_\lambda], \frac{1}{e(T_{I_\lambda}^{\leq 0})}[I_\lambda] \rangle = (-1)^n \frac{e(T_{I_\lambda} X^{[n]})}{e(T_{I_\lambda}^{\leq 0})^2}$$

and the expression of $e(T_{I_\lambda}^{\leq 0})$ in Corollary 3.20. Q.E.D.

3(vi). Nested Hilbert scheme and Pieri formula

Let us first explain the compatibility between the convolution product and the fixed point formula.

Suppose M_1, M_2 are smooth T-varieties and $Z \subset M_1 \times M_2$ is a nonsingular T-invariant subvariety. We further assume that the second projection $p_1 \colon Z \to M_1$ is proper, and hence the convolution product

$$H_*^{T,lf}(M_2) \to H_*^{T,lf}(M_1); \bullet \mapsto p_{1*}(p_2^*(\bullet))$$

is well-defined. Let p_1^T, p_2^T denote the restriction of the first and second projections to the fixed point set Z^T. respectively.

Let i_1, i_2, i_Z denote the inclusions of fixed point sets M_1^T, M_2^T, Z^T to M_1, M_2, Z respectively. Let us denote the normal bundles by N_1, N_2, N_Z respectively. We understand that they are union of normal bundles of connected components of M_1^T, M_2^T, Z^T. We do not introduce subscript α unlike in §1(iv). From Theorem 1.13 we obtain

Lemma 3.22. *The following equality holds in* $\operatorname{Hom}(H_*^{T,lf}(M_1^T)_{\mathbb{F}}, H_*^{T,lf}(M_2)_{\mathbb{F}})$.

$$p_{1*}p_2^* i_{2*}\frac{1}{e(N_2)} = i_{1*}p_{1*}^T\frac{1}{e(N_Z)}p_2^{T*}.$$

In fact, we apply Theorem 1.13 to $i_Z^* p_2^* = p_2^{T*} i_2^*$ to invert i_Z^*, i_1^*. Then we use $p_{1*} i_{Z*} = i_{1*} p_{1*}^T$. It is suggestive to note that

$$\frac{e(N_Z)}{p_2^{T*}e(N_2)}$$

is the equivariant Euler class of the virtual normal bundle $N_Z - p_2^{T*}N_2$ of fibers.

We apply this lemma to $P[1] \subset X^{[n]} \times X^{[n-1]} \times X$ in §2(i), which realize the operator $P_{-1}(\alpha)$. This $P[1]$ is known to be nonsingular, while other $P[i]$ with $i > 1$ are singular except for small n. The smoothness was proved in [23] in the context of quiver varieties, and also independently in [7, 26] in the context of Hilbert schemes. It is called the *Hecke correspondence* [21, 23], and the *nested Hilbert scheme* in [7] and also various other literature. It has been used to prove many statements on Hilbert schemes by an induction on n. See [10, 9] for example.

Let us briefly review the proof of smoothness in [23, §5]. The proof works for higher rank case. We represent $X^{[n]}$ and $X^{[n-1]}$ as spaces of quadruples (B_1, B_2, i, j) as in [24, Ch. 2]: Let $W = \mathbb{C}^r$, $V^1 = \mathbb{C}^n$, $V^2 = \mathbb{C}^{n-1}$. Then the framed moduli spaces $M(r,n)$ $(\alpha = 1)$, $M(r, n-1)$ $(\alpha = 2)$ of torsion free sheaves (E, φ) over $\mathbb{P}^2$ of rank r, $c_2 = n$, $n-1$ are respectively spaces of quadruples $(B_1^\alpha, B_2^\alpha, i^\alpha, j^\alpha)$ satisfying

- $B_1^\alpha, B_2^\alpha \in \operatorname{End}(V^\alpha)$, $i^\alpha\colon W \to V^\alpha$, $j^\alpha\colon V^\alpha \to W$,
- $[B_1^\alpha, B_2^\alpha] + i^\alpha j^\alpha = 0$,
- (stability) there is no proper subspace of V^α containing $i^\alpha(W)$ and is invariant under B_1^α, B_2^α

modulo the conjugation under $\mathrm{GL}(V^\alpha)$. We consider V^α as vector bundles over $M(r,n)$, $M(r,n-1)$. We then form a complex of vector bundles over $M(r,n) \times M(r,n-1)$:
(3.23)

$$\operatorname{Hom}(V^1,V^2) \xrightarrow{a} \begin{matrix} \operatorname{Hom}(V^1, Q\otimes V^2) \\ \oplus \\ \operatorname{Hom}(W, V^2) \\ \oplus \\ \operatorname{Hom}(V^1, \bigwedge^2 Q \otimes W) \end{matrix} \xrightarrow{b} \begin{matrix} \bigwedge^2 Q \otimes \operatorname{Hom}(V^1, V^2) \\ \oplus \\ \bigwedge^2 Q \otimes \mathcal{O} \end{matrix},$$

where $Q = \mathbb{C}^2$, and a, b are defined by

$$a(\xi \oplus \lambda) = \left(\xi B_1^1 - B_1^2 \xi\right) \oplus \left(\xi B_2^1 - B_2^2\xi\right) \oplus \xi i^1 \oplus \left(-j^2\xi\right),$$
$$b(C_1 \oplus C_2 \oplus I \oplus J) = \begin{pmatrix} B_1^2 C_2 - C_2 B_1^1 + C_1 B_2^1 - B_2^2 C_1 + i^2 J + I j^1 \\ \operatorname{tr}(i^1 J) + \operatorname{tr}(I j^2) \end{pmatrix}.$$

This is a complex thanks to $[B_1^\alpha, B_2^\alpha] + i^\alpha j^\alpha = 0$ and $\operatorname{tr}(i^1 j^2 \xi) = \operatorname{tr}(\xi i^1 j^2)$.

Lemma 3.24. *Consider a and b as linear map between fibers of vector bundles. Then a is injective and b is surjective at any point in $M(r,n) \times M(r,n-1)$.*

Proof. See [23, Lemma 5.2]. Note that V^1, V^2 are swapped and the stability condition is the opposite there. Therefore we need to take transposes of $B_1^\alpha, B_2^\alpha, i^\alpha, j^\alpha$. Q.E.D.

Therefore $\operatorname{Ker} b / \operatorname{Im} a$ forms a vector bundle over $M(r,n) \times M(r,n-1)$ of rank $2rn - r - 1$.

If we omit the factor $\mathcal{O}$ in the third term in (3.23), the complex gives $\operatorname{Ext}^1(E_1, E_2(-\ell_\infty))$, where E_1, E_2 are torsion free sheaves in $M(r,n)$ and $M(r,n-1)$ respectively. This is clear from the proof of the description for $M(r,n)$, $M(r,n-1)$. It can be also checked as follows: we consider $M(r,n) \times M(r,n-1)$ as a component of $\mathbb{C}^*$-fixed point in $M(2r, 2n-1)$, where $\mathbb{C}^*$ acts on $\mathbb{C}^{2r} = \mathbb{C}^r \oplus \mathbb{C}^r$ with weight 0 on the first factor and 1 on the second. The tangent space of $M(2r, 2n-1)$ at $E_1 \oplus E_2$ is $\operatorname{Ext}^1(E_1 \oplus E_2, (E_1 \oplus E_2)(-\ell_\infty))$, and is a $\mathbb{C}^*$-module. The factor $\operatorname{Ext}^1(E_1, E_2(-\ell_\infty))$ is the weight 1 subspace. On the other hand, we compute the tangent space in terms of quadruples, and find that (3.23) without $\mathcal{O}$ is the weight 1 subspace.

We define a section s of $\operatorname{Ker} b/\operatorname{Im} a$ by

$$s = \left(0 \oplus i^2 \oplus (-j^1)\right) \pmod{\operatorname{Im} a}.$$

Then s vanishes if and only if there is ξ such that

$$\xi B_1^1 = B_1^2 \xi, \quad \xi B_2^1 = B_2^2 \xi, \quad \xi i^1 = i^2, \quad j^1 = j^2 \xi.$$

This means we have a homomorphism $E_1 \to E_2$ which is the identity at ℓ_∞. It is injective, as a sheaf homomorphism, since it is so at ℓ_∞. For $r = 1$, this means that $I_1 \subset I_2$. The stability condition implies ξ is surjective, hence $\operatorname{Ker} \xi$ is 1-dimensional. The homomorphisms B_1^1, B_2^2 preserve $\operatorname{Ker} \xi$, hence give an element in $\mathbb{C}^2 = X$. Thus $\operatorname{Zero}(s) = P[1]$ if $r = 1$, and its natural higher rank generalization for $r > 1$.

Now the smoothness of $\operatorname{Zero}(s)$ follows from

Lemma 3.25. *Take a connection ∇ on $\operatorname{Ker} b/\operatorname{Im} a$ and consider the differential ∇s. It is surjective on $\operatorname{Zero}(s)$.*

See [23, Th. 5.7] for the proof. Therefore $\operatorname{Zero}(s)$ is nonsingular of dimension $2rn - r + 1 = \frac{1}{2}\dim(M(r,n) \times M(r,n-1) \times X)$.

Now assume $r = 1$ and consider a T^2-action on $P[1]$ induced by $(t_1, t_2) \cdot (z, \xi) = (t_1 z, t_2 \xi)$. The fixed points are parametrized by pairs of Young diagrams (λ, μ) of size n and $n-1$ respectively such that μ is contained in λ, i.e., $\lambda_i \geq \mu_i$ for all $i \geq 1$. Since the number of boxes differs by 1, the skew diagram $\lambda - \mu$ consists of a single box.

Using the complex (3.23), we can compute the character of the tangent space at (I_λ, I_μ). We consider V^1, V^2 as T-modules according to λ, μ, and Q as the natural T-module. The detail is given in [25, Prop. 5.2]. (Note that $\widehat{M}^0(c)$ has no $\mathbb{C}^2$-factor, so $t_1 + t_2$ below is not present there.)

Proposition 3.26.

$$\operatorname{ch} T_{(I_\mu, I_\lambda)} P[1] = t_1 + t_2 + \sum_{s \in \mu} \left(t_1^{-l_\lambda(s)} t_2^{a_\mu(s)+1} + t_1^{l_\mu(s)+1} t_2^{-a_\lambda(s)} \right).$$

Here the leg and arm lengths are considered with respect to either λ or μ. We use the notation $l_\lambda(s)$, $l_\mu(s)$ to indicate Young diagrams as subscripts.

Let

$$b_\lambda^{(\boldsymbol{k})}(s) \overset{\text{def.}}{=} \frac{\varepsilon_1(l_\lambda(s)+1) - \varepsilon_2 a_\lambda(s)}{\varepsilon_1 l_\lambda(s) - \varepsilon_2(a_\lambda(s)+1)} = \frac{l_\lambda(s) + 1 + \boldsymbol{k} a_\lambda(s)}{l_\lambda(s) + \boldsymbol{k}(a_\lambda(s)+1)}. \tag{3.27}$$

See [19, (10.10)].

Let us denote R the set of boxes in μ which lies in the same row with the box $\lambda - \mu$. For example, when $\lambda - \mu$ is the box marked with $\heartsuit$, R consists of boxes with $\clubsuit$:

$$\begin{array}{|c|c|c|c|c|}
\cline{1-2}
 & \\
\cline{1-3}
 & & \\
\cline{1-4}
\clubsuit & \clubsuit & \clubsuit & \heartsuit \\
\hline
 & & & & \\
\hline
\end{array}$$

In the notation [19, VI, §6], it is $C_{\lambda/\mu} - R_{\lambda/\mu}$. (Note that our Young diagram is rotated by $90°$.)

If $s \in \mu \backslash R$, we have $l_\lambda(s) = l_\mu(s)$. If $s \in R$, we have $l_\lambda(s) = l_\mu(s)+1$ and $a_\lambda(s) = a_\mu(s)$. Therefore

$$\begin{aligned}
&e(T_{(I_\lambda, I_\mu)} P[1]) \\
&= \varepsilon_2 e(T_{I_\mu}^{>0}) e(T_{I_\lambda}^{\le 0}) \prod_{s\in R} \frac{-\varepsilon_1 l_\lambda(s) + \varepsilon_2(a_\lambda(s)+1)}{-\varepsilon_1 l_\mu(s) + \varepsilon_2(a_\mu(s)+1)} \frac{\varepsilon_1(l_\mu(s)+1) - \varepsilon_2 a_\mu(s)}{\varepsilon_1(l_\lambda(s)+1) - \varepsilon_2 a_\lambda(s)} \\
&= \varepsilon_2 e(T_{I_\mu}^{>0}) e(T_{I_\lambda}^{\le 0}) \prod_{s\in R} \frac{b_\mu^{(\boldsymbol{k})}(s)}{b_\lambda^{(\boldsymbol{k})}(s)},
\end{aligned}$$

where $T_{I_\mu}^{>0}$, $T_{I_\lambda}^{\le 0}$ are sum of positive and negative weight spaces as before.

Since $P_\mu^{(\boldsymbol{k})} = \frac{1}{e(T_{I_\mu}^{\le 0})}[I_\mu]$ and $p_1 = P_{-1}(\varepsilon_2)$, we get

Theorem 3.28. *We have*

$$p_1 P_\mu^{(\boldsymbol{k})} = \sum_\lambda \prod_{s\in R} \frac{b_\lambda^{(\boldsymbol{k})}(s)}{b_\mu^{(\boldsymbol{k})}(s)} P_\lambda^{(\boldsymbol{k})},$$

where the summation runs over λ with $|\lambda| = |\mu| + 1$, containing μ.

This is a special case of Pieri formulas for Jack symmetric functions [19, VI, (6.24)]. When $\boldsymbol{k} = 1$, we have $b_\lambda^{(\boldsymbol{k})}(s) = 1$. Then the above is specialized to a classical Pieri formula for Schur functions.

§4. Virasoro algebra

The goal of this section is to construct a representation of the Virasoro algebra on $\bigoplus H_*^T(X^{[n]})$ for $X = \mathbb{C}^2$. Lehn [14] constructed a representation for an arbitrary quasiprojective surface X. Our proof is completely different from Lehn's, and based on the geometric construction of Jack symmetric functions in the previous section. The key is that the Hamiltonian $\square^{\boldsymbol{k}}$ gives the Virasoro algebra. On the other hand, Chern classes of the tautological bundle $\mathcal{V}$ on $X^{[n]}$ is diagonalized by the fixed points base $\{[I_\lambda]\}$. Therefore it is, more or less, obvious that $\square^{\boldsymbol{k}}$ is related to $\mathcal{V}$. As far as the author knows, this observation was not mentioned explicitly in the literature before, but it has been well-known among experts.

4(i). Insertions and coproducts

Note that our Heisenberg generators $P_i(\alpha)$ are 'colored' by (co)homology classes α of the base space X. When we consider a (normal ordered) product of Heisenberg operators, such as a vertex operator, it is sometime natural to color it a single cohomology class instead of multiples of them. Such a coloring is given naturally by considering a coproduct on $H_T^*(X)$. This was first noticed by Lehn [14] when he considered Virasoro generators, and subsequently used by other people. Here we use a slightly modified version in [20]. We only consider the case $X = \mathbb{C}^2$, but note that the framework makes sense for any X with or without the T-action.

Let $\Delta\colon H_T^*(X) \to H_T^*(X)\otimes H_T^*(X)$ be the adjoint of the cup product $\cup\colon H_T^*(X)\otimes H_T^*(X) \to H_T^*(X)$ with respect to $\langle\ ,\ \rangle$ (2.6), the negative of the intersection pairing. More concretely, it is $H_T^*(\mathrm{pt})$-linear, and hence is given by

$$\Delta(1) = -1 \otimes \mathrm{PD}[0] = -1 \otimes \varepsilon_1\varepsilon_2,$$

where PD[0] is the Poincaré dual of the class [0]. We also consider its iteration

$$\Delta^n(1) = (-1)^n \otimes \underbrace{\varepsilon_1\varepsilon_2 \otimes \cdots \otimes \varepsilon_1\varepsilon_2}_{n \text{ times}}.$$

Note that we can iterate Δ in various ways, say $(\Delta\otimes 1)\Delta$, $(1\otimes\Delta)\Delta$, but the result is the same thanks to the coassociativity. Thus we have $\Delta^n(1) = (-\varepsilon_1\varepsilon_2)^n \cdot 1 \otimes \cdots \otimes 1$.

We define

$$(P_mP_n)(\alpha) \overset{\text{def.}}{=} \sum_i P_m(\alpha_i')P_n(\alpha_i'')$$

if $\Delta(\alpha) = \sum \alpha_i' \otimes \alpha_i''$. Similarly we define

$$:P_mP_n:(\alpha) \overset{\text{def.}}{=} \sum_i :P_m(\alpha_i')P_n(\alpha_i''):.$$

Products of more than two operators are defined in the same way, using $\Delta^n(\alpha)$.

Remark 4.1. Our Δ is the negative of $\delta\colon H_T^*(X) \to H_T^*(X)\otimes H_T^*(X)$, the pushforward homomorphism associated with the diagonal embedding, used in [14, §3.1].

4(ii). The first Chern class of the tautological bundle

Let $\mathcal{Z}$ be the universal family on $X^{[n]}$, which is a subvariety of $X \times X^{[n]}$. Let p denote the projection to $X^{[n]}$. Then $p_*\mathcal{O}_{\mathcal{Z}}$ is a vector bundle of rank n over $X^{[n]}$. In the description [24, Th. 1.9], it is the vector bundle associated with the principal $GL_n(\mathbb{C})$-bundle $\widetilde{H} \to X^{[n]} = \widetilde{H}/GL_n(\mathbb{C})$ with respect to the vector representation of $GL_n(\mathbb{C})$. Let us denote it by $\mathcal{V}$. It is denoted by $\mathcal{O}^{[n]}$ in [14]. It is called the *tautological bundle.*

We consider the multiplication of the first Chern class $c_1(\mathcal{V})$ of $\mathcal{V}$ as an operator on $H_T^*(X^{[n]})$.

Lemma 4.2. *In the fixed points base $\{[I_\lambda]\}$, the operator $c_1(\mathcal{V}) \cup \bullet$ is diagonalized:*

$$c_1(\mathcal{V}) \cup [I_\lambda] = -\left(n(\lambda)\varepsilon_1 + n(\lambda')\varepsilon_2\right)[I_\lambda]. \tag{4.3}$$

Proof. Since $[I_\lambda]$ is the class of a fixed point, $c_1(\mathcal{V}) \cup [I_\lambda]$ is given by $c_1(\mathcal{V}|_{I_\lambda})[I_\lambda]$. The formula of the character of $\mathcal{V}|_{I_\lambda}$ is given in the proof of [24, Prop. 5.8]. Then the equivariant first Chern class is

$$c_1(\mathcal{V}|_{I_\lambda}) = -\sum_{s\in\lambda}(l'(s)\varepsilon_1 + a'(s)\varepsilon_2) = -n(\lambda)\varepsilon_1 - n(\lambda')\varepsilon_2.$$

Q.E.D.

The above lemma is easy to prove, but has the following remarkable consequence.

Corollary 4.4. $c_1(\mathcal{V}) \cup \bullet$ *is equal to $\varepsilon_1\Box^{\boldsymbol{k}}$ under the isomorphisms* (3.14, 3.15), *where $\Box^{\boldsymbol{k}}$ is given by* (3.5).

Proof. Recall that $\Box^{\boldsymbol{k}}$ is diagonalized in Jack symmetric functions $P_\lambda^{(\boldsymbol{k})}$ with eigenvalues $e_\lambda(\boldsymbol{k}) = n(\lambda')\boldsymbol{k} - n(\boldsymbol{k})$, see (3.6). Since $P_\lambda^{(\boldsymbol{k})}$ corresponds to the (normalized) fixed point class in Theorem 3.18, the assertion follows as $\varepsilon_1 e_\lambda(\boldsymbol{k}) = c_1(\mathcal{V}|_{I_\lambda})$ in (4.3). Q.E.D.

Let us write $\Box^{\boldsymbol{k}}$ in terms of Heisenberg generators $P_m(\alpha)$. Recall our identification Proposition 3.10 and the commutation relation $[P_i(\varepsilon_2), P_j(\varepsilon_2)] = i\delta_{i+j,0}\boldsymbol{k}\,\mathrm{id}$, we rewrite (3.7) by

$$p_m \leftrightarrow P_{-m}(\varepsilon_2), \qquad m\frac{\partial}{\partial p_m} \leftrightarrow \frac{1}{\boldsymbol{k}}P_m(\varepsilon_2) = -P_m(\varepsilon_1).$$

We get

$$
\begin{aligned}
&\varepsilon_1\square^{\boldsymbol{k}} \\
&= -\sum_{m,n>0}\Big(\frac{\varepsilon_2}{2}P_{-m-n}(\varepsilon_2)P_m(\varepsilon_1)P_n(\varepsilon_1)+\frac{\varepsilon_1}{2}P_{-m}(\varepsilon_2)P_{-n}(\varepsilon_2)P_{m+n}(\varepsilon_1)\Big) \\
&\quad +\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{m>0}(m-1)P_{-m}(\varepsilon_2)P_m(\varepsilon_1).
\end{aligned}
$$

Noticing $\Delta(1)=\varepsilon_1\varepsilon_2$, $\Delta^2(1)=(\varepsilon_1\varepsilon_2)^2$, and $K_X=-\varepsilon_1-\varepsilon_2$, we rewrite this expression as follows. ($P_0(\alpha)$ is understood as 0.) It is a special case of Lehn's result which holds for any X.

Theorem 4.5 (Lehn [14])**.**

$$
\begin{aligned}
c_1(\mathcal{V})\cup\bullet = -\frac{1}{3!}\sum_{m_1+m_2+m_3=0} &:P_{m_1}P_{m_2}P_{m_3}:(1) \\
&+\frac{1}{4}\sum_m(|m|-1):P_{-m}P_m:(K_X).
\end{aligned}
$$

The commutator with $P_n(\alpha)$ is given by

$$
[c_1(\mathcal{V})\cup\bullet, P_n(\alpha)] = \frac{n}{2}\sum_{l+m=n}:P_lP_m:(\alpha)-\frac{n(|n|-1)}{2}P_n(K_X\cup\alpha).
$$

The formula is presented in this form in [14, Th. 3.10] up to the sign convention. Since $\bigoplus_n H_*^T(X^{[n]})$ is an irreducible representation of the Heisenberg algebra, two formulas are equivalent.

Remark 4.6. Note that $\mathfrak{q}_n(\alpha)$ in [14] is our $P_{-n}(\alpha)$.

The terms

$$
L_n(\alpha)\overset{\text{def.}}{=}\frac{1}{2}\sum_{l+m=n}:P_lP_m:(\alpha)
$$

satisfy the Virasoro relation

(4.7)
$$
[L_n(\alpha),L_m(\beta)] = (n-m)L_{n+m}(\alpha\beta)-\frac{n^3-n}{12}\delta_{n+m,0}\langle c_2(X)\alpha,\beta\rangle\,\mathrm{id},
$$

where $c_2(X)=\varepsilon_1\varepsilon_2$. It appears as the composite

$$
H_T^*(X)\xrightarrow{\Delta}H_T^*(X)\otimes H_T^*(X)\xrightarrow{\cup}H_T^*(X)
$$

is multiplication with $-c_2(X)$. We leave a check of (4.7) to the reader.

Remark 4.8. It would be nice if one could deduce Lehn's result for general X from Theorem 4.5. In fact, Li-Qin-Wang [16] (see also [15]) proved that operators given by cup products of classes

$$p_{1*}(\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n})p_2^*(\mathrm{td}_X \cup \alpha)),$$

are expressed by universal polynomials in Heisenberg operators coupled with $\alpha \cup \{1_X, K_X, K_X^2, c_2(X)\}$, independent of X. Here $\mathcal{Z}_n$ is the universal subscheme in $X^{[n]} \times X$, p_1, p_2 are projection from $X^{[n]} \times X$ to $X^{[n]}$ and X respectively, and td_X is the Todd class of X. By Riemann-Roch, $c_1(\mathcal{V})$ is the degree 2 part of the above for $\alpha = 1_X$. Thanks to the universality, it is enough to determine the polynomials for $X = \mathbb{C}^2$. However the universality in [16] was derived from Lehn's result, and hence this argument does not work.

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