Advanced Studies in Pure Mathematics 67, 2015
Variational Methods for Evolving Objects
pp. 59–85

Optimal transportation of particles, fluids and currents

Yann Brenier

Abstract.

In these lectures, we review a series of optimal transport (OT) problems of growing complexity. Surprisingly enough, in this seemingly narrow framework, we will encounter nonlinear PDEs of very different type, such as the Monge–Ampère équation, the Euler equations of incompressible fluids, the hydrostatic Boussinesq equations in convection theory, the Born–Infeld equation of electromagnetism, showing the hidden richness of the concept of optimal transportion

§1. Introduction

In these lectures, we discuss the mathematical theory of optimal transport, following Monge (1780) and Kantorovich (1942), and its connection with the theory of inviscid incompressible fluids following Euler (1755) and Arnold (1966). Both theories are currently very active and combine various aspects of geometry, analysis and PDEs. More precisely, we will address the following questions of increasing complexity: 1) Is it possible to transport a fluid, knowing only its initial and final density field, so that the total kinetic energy spent will be as small as possible? 2) Is it possible to transport fluid particles of an incompressible fluid, from given initial positions to given final positions, so that the total kinetic energy spent will be as small as possible? In both case we will establish the existence and uniqueness of a potential which provides, in the first case, the initial velocity of the particles, and, in the second case, the acceleration of the particles during their travel. These results require very few assumptions on the data and open challenging regularity issues

Received August 10,2013.
Revised October 14, 2013.
2010 *Mathematics Subject Classification.* Primary 35Q; Secondary 49, 76.
Key words and phrases. Optimal transport, fluid mechanics, electromagnetism.

on the solutions. They turn out to be useful for many applications, both practical (image processing, inverse problems in cosmology...) and theoretical (geometric and functional inequalities, degenerate parabolic equations...). Finally, we will close the discussion by considerations on
i) optimal transportation of incompressible fluids in an inhomogeneous environment in relation with the theory of convection in geophysics,
ii) optimal transportation of strings in relation with the nonlinear Born–Infeld theory of electromagnetism.
In this review, we will concentrate on the key concepts and the main results, without delivering comprehensive proofs (for which precise references will be provided).

§2. A first, elementary, optimal transport problem

Let us consider (for simplicity) a convex closed bounded domain D in $\mathbb{R}^d$, typically the unit cube. We want to move, inside D, some particles, labelled by a, from their initial position $X_0(a) \in D$ at time $t = 0$ to their final destination $X_T(a)$ at time $t = T$, and we want to spend the smallest possible kinetic energy during the transportation time. In other words, we look for

$$\inf_{(X_t)} \int_0^T \int_{\mathcal{A}} \frac{1}{2} |\frac{d}{dt} X_t(a)|^2 \mu(da) dt \tag{2.1}$$

where $(\mathcal{A}, \mu)$ is an abstract Borel probability space just used to label the particles, $|\cdot|$ the euclidean norm, and $X_t(a)$ the position in D of particle a at time t.

The optimality condition is $\frac{d^2}{dt^2} X_t(a) = 0$ for μ a.e. $a \in \mathcal{A}$. As a matter of fact, this convex minimization problem has an obvious unique optimal solution

$$X_t(a) = (1 - t/T) X_0(a) + t/T\, X_T(a), \tag{2.2}$$

for which particles move along straight lines. (N.B.: if D were a Riemannian manifold without boundary, particles would move along geodesics.) Indeed, the transportation cost of this solution is

$$T^{-1} \int_{\mathcal{A}} \frac{1}{2} |X_T(a) - X_0(a)|^2 \mu(da), \tag{2.3}$$

which is an obvious lower bound for the transportation cost, because of Jensen's inequality.

§3. The quadratic Monge optimal transport problem and the Monge–Ampère equation

Now, let us move to a more difficult problem: X_0, X_T are now unknown, but their "laws" are given. These laws are denoted by ρ_0, ρ_T and defined, as Borel probability measures on D, by

$$\int_D f(x)\rho_0(dx) = \int_{\mathcal{A}} f(X_T(a))\mu(da), \tag{3.1}$$

$$\int_D f(x)\rho_T(dx) = \int_{\mathcal{A}} f(X_T(a))\mu(da), \tag{3.2}$$

for all continuous functions f on $\mathbb{R}^d$. Given ρ_0 and ρ_T, the optimal transportation problem can be rewritten

(3.3)

$$\inf_{X_0,X_T}\{\ T^{-1}\int_{\mathcal{A}}\frac{1}{2}|X_T(a)-X_0(a)|^2\mu(da),\ \ \mathrm{law}(X_0)=\rho_0\ \ \mathrm{law}(X_T)=\rho_T\}$$

and is no longer convex. Using Lagrange multipliers f_0, f_T for constraints (3.1, 3.2), this minimization problem becomes a saddle point problem

(3.4)

$$\begin{cases} \inf_{X_0,X_T}\sup_{f_0,f_T}\quad \int_D f_0(x)\rho_0(dx) + \int_D f_T(x)\rho_T(dx) \\ +\int_{\mathcal{A}}\{T^{-1}\frac{1}{2}|X_T(a)-X_0(a)|^2 - f_0(X_0(a)) - f_T(X_T(a))\}\mu(da). \end{cases}$$

Notice that this Lagrangian does not involve any derivatives in X_0, X_T. It is therefore straightforward to get the *formal* optimality conditions for each fixed a, μ almost surely. The first order conditions are obtained by differentiating in X_0, X_T,

(3.5)

$$X_0(a) = X_T(a) - T(\nabla f_T)(X_T(a)), \quad X_T(a) = X_0(a) - T(\nabla f_0)(X_0(a)).$$

The second order optimality conditions say that the following symmetric matrices must be positive

$$Id - T(D^2 f_T)(X_T(a)) \geq 0, \quad Id - T(D^2 f_0)(X_0(a)) \geq 0. \tag{3.6}$$

Thus, $X_T(a) = (\nabla\Phi)(X_0(a))$, $\mu-$ a.e. where Φ is a convex function, namely

$$\Phi(x) = \frac{1}{2}|x|^2 - T\, f_0(x). \tag{3.7}$$

Using (3.1, 3.2), this implies

$$\int_D f(x)\rho_T(dx) = \int_{\mathcal{A}} f(X_T(a))\mu(da) = \int_{\mathcal{A}} f(\nabla\Phi(X_0(a)))\mu(da)$$

$$= \int_D f(\nabla\Phi(x))\rho_0(dx).$$

In other words, ρ_T is just the image measure of ρ_0 by the map with convex potential $x \to \nabla\Phi(x)$. We may also rewrite the transportation cost in a similar way

$$T^{-1}\int_{\mathcal{A}} \frac{1}{2}|X_T(a) - X_0(a)|^2\mu(da) = T^{-1}\int_{\mathcal{A}} \frac{1}{2}|(\nabla\Phi)(X_0(a)) - X_0(a)|^2\mu(da)$$

$$= T^{-1}\int_D \frac{1}{2}|\nabla\Phi(x) - x|^2\rho_0(dx).$$

and see that $M = \nabla\Phi$ minimizes this cost among all maps M for which ρ_T is the image by M of ρ_0.

These results are so far purely formal, but can be made rigorous in the case when ρ_0 is an absolutely continuous measure with respect to the Lebesgue measure. More precisely:

Theorem 1 (Existence and uniqueness of a transport map with convex potential). *Suppose $D \subset \mathbb{R}^d$ is a bounded convex domain. Let ρ_0, ρ_T two Borel probability measures on D. Assume ρ_0 to be absolutely continuous with respect to the Lebesgue measure. Then, among all Borel maps M such that ρ_T is the image of ρ_0 by M, i.e.*

$$\int_D f(x)\rho_T(dx) = \int_D f(M(x))\rho(dx), \quad \forall f \in C(\mathbb{R}^d) \tag{3.8}$$

there exists a unique one with convex potential, i.e. such that $M = \nabla\Phi$, ρ_0 a.e., for some Lipschitz convex function Φ on D. Moreover, among those maps, $M = \nabla\Phi$ is characterized as the unique minimizer of the transportation cost

$$\int_D \frac{1}{2}|M(x) - x|^2\rho_0(dx).$$

Notice that the uniqueness was not obvious from our formal calculation. Also notice that in this theorem, the space $(\mathcal{A}, \mu)$, the time T, the maps X_0, X_T have totally disappeared. Only matter the two probability laws on D. This problem was first considered by Monge in 1780 [Mo], [Ev], with the "linear" transportation cost

$$\int_D \frac{1}{2}|M(x) - x|\rho_0(dx).$$

The modern method of addressing the "quadratic" Monge transportation problem [Br1], [Br3], [KS], [RR] goes back to the seminal work of Kantorovich (one of the few mathematicians who was awarded the Nobel prize in Economics) in 1942 [Ka], [RR].

Sketch of proof. Observe first that the optimal value of the saddle-point problem (3.4) is certainly bounded from below by the "dual" optimization problem, obtained after permuting inf and sup in (3.4), which simply reduces to:

(3.9)
$$\begin{cases} \sup_{f_0, f_T} \quad \int_D f_0(x)\rho_0(dx) + \int_D f_T(x)\rho_T(dx), \quad \text{subject to}: \\ \quad c(X_0(a), X_T(a)) \geq f_0(X_0(a)) + f_T(X_T(a)), \quad \mu \text{ a.e. } a \in \mathcal{A}, \end{cases}$$

where

$$c(x, y) = T^{-1} \int \frac{1}{2}|x - y|^2 \tag{3.10}$$

is called "cost function". Then, the optimal value of this dual problem is itself bounded from below by

(3.11)
$$\begin{cases} \sup_{f_0, f_T} \quad \int_D f_0(x)\rho_0(dx) + \int_D f_T(y)\rho_T(dy), \quad \text{subject to}: \\ \qquad c(x, y) \geq f_0(x) + f_T(y), \quad \forall x, y \in D, \end{cases}$$

which turns out to be a convex maximization problem in $f_0, f_T \in C^0(D)$. Using Rockafellar's duality theorem (see [Brz]), the optimal value of this problem can be computed as a minimum over all nonnegative Borel measures ν over $D \times D$, namely:

(3.12)
$$\begin{cases} \min_{\nu \in (C^0(D \times D))' \geq 0} \quad \int_{D \times D} c(x, y)\nu(dx, dy), \\ \qquad \text{subject to}: \ \forall f_0, f_T \in C^0(D), \\ \int_{D \times D}(f_0(x) + f_T(y))\nu(dx, dy) = \int_D f_0(x)\rho_0(dx) + \int_D f_T(y)\rho_T(dy). \end{cases}$$

This is the celebrated Monge–Kantorovich problem [Ka], for a general cost function c. It turns out that, for our cost function (3.10), both dual problems have at least an optimal solution, namely (f_0, f_T), Lipschitz continuous on D, on one side, and $\nu \geq 0$ in the other side. For all optimal solutions, we necessarily have, at once,

$$\int_{D\times D} (c(x,y) - f_0(x) - f_T(y))\nu(dx,dy) = 0$$

and $c(x,y) - f_0(x) - f_T(y) \geq 0$, $\forall x, y \in D$, which implies

$$c(x,y) - f_0(x) - f_T(y) = 0, \quad \nu - a.e.\ (x,y) \in D \times D.$$

Since f_0 is Lipschitz continuous, by Rademacher's theorem, f_0 is differentiable Lebesgue almost-everywhere on D. Since ρ_0 is assumed to be absolutely continuous with respect to the Lebesgue measure, we deduce that the set of $(x_0, y_0) \in D \times D$, for which $x \in D \to f_0(x)$ is differentiable at x_0 has full ν measure. (Indeed the first projection of ν on D, is nothing but ρ_0). This implies, by differentiation of $x \to c(x, y_0) - f_0(x) - f_T(y_0)$, which reaches its minimum 0 at $x = x_0$, that $\nabla_x c(x_0, y_0) = \nabla f_0(x_0)$. By definition (3.10) of c, we find out

$$y_0 = x_0 - T^{-1}\nabla f_0(x_0), \quad \nu -\ a.e.\ (x_0, y_0) \in D \times D.$$

Thus, ν must be equal to

$$\nu(dx,dy) = \delta(y - x + T^{-1}\nabla f_0(x))\rho_0(dx),$$

which enforces the uniqueness of both ν and f_0 and further leads to all desired results after rather elementary arguments. Notice that a similar proof was obtained by Gangbo, without duality argument [Ga]. Q.E.D.

Regularity results and Monge–Ampère equation

So far, existence and uniqueness results have been obtained just by classical Convex Analysis. Regularity results require more refined analysis, relying on a priori estimates. Let us summarize one of them due to L. Caffarelli [Ca]. (For more recent results and references, we refer to [DF].)

Theorem 2 (Regularity of the optimal transport map with convex potential). *Suppose $D \subset \mathbb{R}^d$ is a bounded convex domain. Assume ρ_0, ρ_T of form*

$$\rho_0(dx) = 1_{\Omega_0}(x)\sigma_0(x)dx, \quad \rho_T(dx) = 1_{\Omega_T}(x)\sigma_T(x)dx,$$

where Ω_0 and Ω_T are two smooth domain included in D, and $\sigma_0, \sigma_T > 0$ are two smooth positive functions on $\mathbb{R}^d$. Assume further that Ω_T is strictly uniformly convex. Then the optimal map $\nabla\Phi$ is a diffeomorphism $\overline{\Omega_0} \to \overline{\Omega_T}$ and solves the Monge–Ampère equation

$$\sigma_T(\nabla\Phi(x))detD^2\Phi(x) = \sigma_0(x), \forall x \in \Omega_0. \tag{3.13}$$

The fact that $\nabla\Phi$ solves the Monge–Ampère equation simply follows from the property that ρ_T is the image of ρ_0 by $\nabla\Phi$. Indeed, we have, for all continuous function f

$$\int_{\Omega_0} f(\nabla\Phi(x))\sigma_0(x)dx = \int_D f(\nabla\Phi(x))\rho_0(dx)$$

$$= \int_D f(x)\rho_T(dx) = \int_{\Omega_T} f(x)\sigma_T(x)dx$$

$$= \int_{\Omega_0} f(\nabla\Phi(x))\sigma_T(\nabla\Phi(x))\mathrm{det}D^2\Phi(x)dx$$

(where for the last equality, we have performed the change of variable $x \to \nabla\Phi(x)$). So, by identification of the first and last integrals for all functions $f \circ \nabla\Phi$, we get the Monge–Ampère equation (3.13).

Applications of transportation method

These type of results have had many applications, in particular for geometric and functional inequalities (such as [Ba]), after the seminal work of R. McCann, published in [Mc]. A striking example is the proof of the isoperimetric inequality and its quantitative version by Figalli, Maggi and Pratelli [FMP]. Theorem 1 has been extended to Riemannian manifolds by R. McCann [Mc1]. Another very successful application is the treatment of many parabolic PDEs by transportation method, following a seminal paper by F. Otto [Ot]. For all these applications, we refer to the books of C. Villani [Vi], [Vi1].

§4. The optimal incompressible transport problem and the Euler equations

Let us now consider a third optimal transportation problem, where the maps X_0, X_T are entirely known, but the motion of particles is supposed to be *incompressible* in the sense that at *any* time $t \in [0, T]$, the law of X_t is supposed to be the (normalized) Lebesgue measure on D, which can be translated by

$$\int_{\mathcal{A}} f(X_t(a))\mu(da) = \int_D f(x)dx \tag{4.1}$$

for all continuous functions f on $\mathbb{R}^d$ and all $t \in [0,T]$, or, alternately,

$$\int_0^T \int_{\mathcal{A}} p_t(X_t(a))\mu(da)dt = \int_0^T \int_D p_t(x)dx\, dt, \tag{4.2}$$

for all continuous functions $(t,x) \to p_t(x)$ on $\mathbb{R} \times \mathbb{R}^d$. We still want to find the minimal transportation cost

$$\inf_{(X_t)} T \int_0^T \int_{\mathcal{A}} \frac{1}{2}|\frac{d}{dt}X_t(a)|^2 \mu(da)dt. \tag{4.3}$$

(Notice that a simple scaling argument shows that this optimal cost does not depend on T.) This optimal incompressible transport can be written as a saddle point problem

$$\left\{ \begin{array}{l} \frac{1}{2}d^2(X_0, X_T) \\ \quad = \inf_{(X_t)} \sup_{(p_t)} \ \int_0^T \int_{\mathcal{A}} \{\frac{1}{2}|\frac{d}{dt}X_t(a)|^2 - p_t(X_t(a))\}\mu(da)dt \\ \quad\quad + \int_0^T \int_D p_t(x)dx\, dt. \end{array} \right. \tag{4.4}$$

The formal first order optimality conditions read

$$\frac{d^2}{dt^2}X_t(a) + (\nabla p_t)(X_t(a)) = 0, \quad \mu \text{ a.e.} \tag{4.5}$$

Meanwhile, the second order optimality condition clearly involves p_t, but in a non obvious way, which might be crucial to understand the regularity of p. (We conjecture that $p_t(x)$ should be a semi-concave function of x.) Equations (4.5) are nothing but the famous Euler equations, introduced in 1755 to describe the motion of "ideal" incompressible fluids [Eu], [Ar], [Ar1], [AK], [Li]. These equations are still of paramount importance in natural science, in particular for atmosphere and ocean sciences. Usually, it is assumed that the trajectories $t \to X_t(a)$ can be recovered by integrating a "velocity field" $v_t(x) \in \mathbb{R}^d$ so that $\frac{d}{dt}X_t(a) = v_t(X_t(a))$. Thus, at least formally, (4.5), through the chain rule, can be translated by a partial differential equation for v and p

$$\partial_t v_t + (v_t \cdot \nabla)v_t + \nabla p_t = 0. \tag{4.6}$$

In addition, the incompressibility condition (4.1) leads to

$$0 = \frac{d}{dt}\int_{\mathcal{A}} f(X_t(a))\mu(da) = \int_{\mathcal{A}} (\nabla f)(X_t(a)) \cdot v_t(X_t(a))\mu(da)$$

$$= \int_D \nabla f(x) \cdot v_t(x)dx$$

for all C^1 function f on $\mathbb{R}^d$, which exactly means that, for all t, v_t is a divergence free vector field on D and parallel to the boundary ∂D. Finally, observe the simple expression of the transportation cost in terms of v:

$$T\int_0^T \int_{\mathcal{A}} \frac{1}{2}|\frac{d}{dt}X_t(a)|^2\mu(da)dt = T\int_0^T \int_{\mathcal{A}} \frac{1}{2}|v_t(X_t(a))|^2\mu(da)dt$$

$$= T\int_0^T \int_D \frac{1}{2}|v_t(x)|^2 dx\, dt.$$

This suggest the following definition of approximate solutions to the optimal incompressible transport (OIT) problem:

Definition 3. [Approximate solutions to the OIT problem] Let $D \subset \mathbb{R}^d$ be a bounded convex domain with $d \geq 2$. For $\varepsilon > 0$, we first say that a smooth time dependent divergence-free vector field v_t, compactly supported in the interior of $[0,T] \times D$, is an ε admissible solution of the OIT problem with data T, X_0, X_T if v_t almost carries X_0 to X_t, in the sense that

$$\int |g_t(X_0(a)) - X_T(a)|^2\mu(da) \leq \varepsilon^2,$$

where $g_t(x)$ denotes the unique solution of $\frac{d}{dt}g_t(x) = v_t(g_t(x)),\ \ g_0(x) = x$.
Next, we define

$$\Delta(X_0, X_T) = \liminf_{\varepsilon \to 0} \sqrt{T\int_0^T \int_D |v_t(x)|^2 dx\, dt}$$

over all $\varepsilon-$ admissible solutions. Notice that, by a rather obvious scaling argument, $\Delta(X_0, X_T)$ does not depend on T.
Then we say that an $\varepsilon-$ admissible solutions is an $\varepsilon-$ approximate solutions whenever

$$\int_0^T \int_D \frac{1}{2}|v_t(x)|^2 dx\, dt \leq \frac{1}{2T}\Delta(X_0, X_T)^2 + \varepsilon^2.$$

Then, the following can be proven:

Theorem 4 (Existence and uniqueness of a pressure gradient). *Let $D \subset \mathbb{R}^d$ be a bounded convex domain with $d \geq 2$. Let X_0, X_T be two incompressible maps (in the sense of (4.1)). Then there is a unique pressure gradient $\nabla p_t(x)$ such that every ε solution $v_t(x)$ of the OIT problem with data X_0, X_T, in the sense of definition (3), is almost solution of the Euler equations, in the sense that*

$$\partial_t v_t + (v_t \cdot \nabla) v_t + \nabla p_t$$

converges to zero in the distributional sense in the interior of $[0,T] \times D$, as ε goes to zero.

Theorem 5 (Regularity of the pressure gradient). *The pressure gradient is locally in $]0,T[$ a square integrable function of t valued in the space of locally bounded measures in the interior of D.*
In addition, there is an example of data for which the hessian $D_x^2 p(t,x)$ is a bounded measure with a singular part and, more precisely $p(t,x)$ is just semi-concave in x with kinks.

Strategy of proof. The method of proof is rather similar to the Kantorovich method we used for the quadratic optimal transportation problem. We start with the saddle-point problem (4.4). Its optimal value is certainly bounded from below by the dual optimization problem obtained by permuting *inf* and *sup*, which reduces to:
(4.7)

$$\begin{cases} \sup_{(p_t)} \int_0^T \int_D p_t(x) dx\, dt + \int_{\mathcal{A}} K_p[X_0(a), X_T(a)] \mu(da), \\ K_P[x,y] = \inf_{\xi_0 = x,\, \xi_T = y} \int_0^T \{\frac{1}{2} |\frac{d}{dt}\xi_t|^2 - p_t(\xi_t)\} dt, \quad \forall x, y \in D, \end{cases}$$

(where $t \to \xi_t$ is a typical C^1 curve in D). Next, using classical tools of control theory (or alternately "weak-KAM" theory), we express the functional K_p in terms of Hamilton–Jacobi equations, benefiting from the simplifying assumption that D is convex. More precisely, we can express it as a supremum over Lipschitz functions $\phi_t(x)$ over $Q = [0,T] \times D$, namely:
(4.8)

$$\begin{cases} K_P[x,y] = \sup_\phi \ \phi_T(y) - \phi_0(x), \quad \text{subject to :} \\ \partial_t \phi_t(x) + \frac{1}{2} |\nabla_x \phi_t(x)|^2 + p_t(x) \leq 0, \ \ a.e.\ (t,x) \in Q = [0,T] \times D. \end{cases}$$

Thus, the optimal value of the dual problem (4.7) can be bounded from below by:
(4.9)

$$\begin{cases} \sup_{(p_t,\phi_t)} \int_0^T \int_D p_t(x)dx\, dt + \int_{\mathcal{A}}(\phi_T(X_T(a),a) - \phi_0(X_0(a),a))\mu(da), \\ \quad \text{subject to}: \ \partial_t\phi_t(x,a) + \frac{1}{2}|\nabla_x\phi_t(x,a)|^2 + p_t(x) \le 0, \ \ \forall\ (t,x,y,a), \end{cases}$$

where $\phi_t(x,a)$ is assumed to be continuous in a and C^1 in (t,x). At this level, we have obtained a concave optimization problem in (p,ϕ) and, again, we can rely on Rockafellar's duality theorem to compute its optimal value and find out:
(4.10)

$$\begin{cases} \inf_{(c,m)} \int_{[0,T]\times D\times\mathcal{A}} \frac{1}{2}|\frac{dm(t,x,a)}{dc(t,x,a)}|^2 dc(t,x,a), \quad \text{subject to}: \\ \int_{[0,T]\times D\times\mathcal{A}}[\nabla_x\phi_t(x,a)\cdot dm(t,x,a) + (\partial_t\phi_t(x,a) + p_t(x))dc(t,x,a)] \\ \quad = \int_{\mathcal{A}}[\phi_T(X_T(a),a) - \phi_0(X_0(a),a)]d\mu(a) + \int_{[0,T]\times D} p_t(x)dtdx, \\ \qquad\qquad \forall(p,\phi), \end{cases}$$

where c, m should be understood as Borel measures on $Q' = [0,T]\times D\times\mathcal{A}$, respectively valued in $\mathbb{R}_+$ and $\mathbb{R}^d$, $v(t,x,a) = \frac{dm}{dc}(t,x,a)$ denoting the Radon–Nikodym derivative of m with respect to c. This is only the first step of the proof which is much more involved than the one needed for the quadratic transportation problem. In particular, the existence of an optimal solution (ϕ,p) is far from being obvious. (However, the existence of an optimal pair (c,m) is easy to prove.) Anyway, we get approximate solutions $(\phi_\epsilon,p_\epsilon)$ with approximate optimality conditions:

Proposition 6. *For every small $\epsilon > 0$, there are continuous functions $\phi_\epsilon(t,x,a)$ on Q' and $p_\epsilon(t,x)$ on Q, with $\partial_t\phi_\epsilon$, $\nabla_x\phi_\epsilon$ continuous on Q' and $\int_D p_\epsilon(t,x)dx = 0$, such that, for every optimal solution (c,m),*

$$\partial_t\phi_\epsilon + \frac{1}{2}|\nabla_x\phi_\epsilon|^2 + p_\epsilon \le 0 \tag{4.11}$$

and

$$\int_{Q'}(|\partial_t\phi_\epsilon + \frac{1}{2}|\nabla_x\phi_\epsilon|^2 + p_\epsilon| + |v - \nabla_x\phi_\epsilon|^2)dc \le \epsilon^2. \tag{4.12}$$

Next, a key approximate regularity result is obtained in [Br5]:

Proposition 7. *Let $Q'_\tau = [\tau, T-\tau] \times D \times \mathcal{A}$, for $\tau > 0$. Let $x \in D \to w(x) \in \mathbf{R}^d$ be a smooth divergence-free vector field compactly supported in the interior of D and let $s \in \mathbf{R} \to \xi_s^w(x) \in D$ be the integral curve of w passing through x at $s = 0$. Then,*

$$\int_{Q'_\tau} |\nabla_x \phi_\epsilon(t,x,a) - v(t,x,a)|^2 dc(t,x,a) \le C\epsilon^2, \tag{4.13}$$

$$\int_{Q'_\tau} |\nabla_x \phi_\epsilon(t,x,a)|^2 dc(t,x,a) \le C, \tag{4.14}$$

$$\int_{Q'_\tau} |\nabla_x \phi_\epsilon(t+\eta, \xi_\delta^w(x), a) - \nabla_x \phi_\epsilon(t,x,a)|^2 dc(t,x,a) \le (\epsilon^2 + \eta^2 + \delta^2)C, \tag{4.15}$$

for all optimal solution (c, m) and all η, δ and $\epsilon > 0$ small enough, where C depends only on D, T, τ and w.

If c were bounded away from zero (which cannot be expected), this would imply that

$$\int_{Q'_\tau} (|\partial_t v|^2 + |\nabla_x v|^2) dc \le C, \tag{4.16}$$

by letting first $\epsilon \to 0$ (to get v instead of $\nabla_x \phi_\epsilon$), then $\delta, \eta \to 0$. Unfortunately, $c(t,x,a)$ is expected to be a singular measure, possibly highly concentrated, and this bound cannot be proven. However a uniform bound on $\int |\nabla p_\epsilon|$ can be obtained. A non rigorous argument is as follows. Starting from (4.12), letting $\epsilon \to 0$, we formally get

$$\partial_t \phi + \frac{1}{2}|\nabla_x \phi|^2 + p = 0, \quad c - a.e.$$

Differentiating in x, we (very!) formally get

$$\partial_t v + (v.\nabla_x) v + \nabla_x p = 0, \quad c - a.e.$$

Then, integrating in $a \in \mathcal{A}$ with respect to c,

$$\int_{\mathcal{A}} (\partial_t v + (v.\nabla_x) v) c(t,x,da) = -\nabla_x p,$$

and, by Schwarz inequality,

$$(\int |\nabla_x p|)^2 \leq \int |\partial_t v|^2 dc \int dc + \int |\nabla_x v|^2 dc \int |v|^2 dc.$$

All these calculations are incorrect. However, the formal idea is made rigorous in [Br5], by working only on the ϕ_ϵ and using finite differences instead of derivatives, leading to:

Proposition 8. *The family* (∇p_ϵ) *converges in the sense of distributions toward a unique limit* ∇p*, depending only on* (X_0, X_T)*, which is a locally bounded measure in the interior of* Q *and is uniquely defined by*

(4.17)

$$\nabla p(t,x) = -\partial_t \int v(t,x,a)c(t,x,da) - \nabla_x . \int (v \otimes v)(t,x,a)c(t,x,da),$$

for ALL optimal solution $(c, m = cv)$.

Then, the rest of the proof relies on approximation theorems for incompressible motions, similar to those obtained by Shnirelman in [Sh1] (see also [Sh], [Ne], [BG]). For more details, we refer to [Sh], [Br2], [Br4], [Sh1], [Br5], [Br8], [AF], [AF1], [BFS]. Q.E.D.

Regularity: results and conjectures

The partial regularity obtained for the pressure field is an output of the strategy developped for the existence and uniqueness of the pressure field in [Br5] and was improved by Ambrosio–Figalli in [AF], [AF1]. At the present level of knowledge, the main issue, in terms of regularity theory, is now to fill the gap between the *obtained* regularity ($\nabla_x p$ is a locally bounded measure), and the *conjectured* regularity: $D_x^2 p$ is a locally bounded measure, or, better, $p(t,x)$ is semi-concave in x. Indeed, the existence of explicit examples of optimal solutions (related to an earlier work of Duchon–Robert [DR]), for which the pressure field has some kink singularities and is not better than semi-concave in x has been established in [Br10].

§5. Optimal transport and convection theory

Adding a potential to the transportation cost does not essentially modify the OIT. We get

$$\inf_{(X_t)} T \int_0^T \int_{\mathcal{A}} \{\frac{1}{2}|\frac{d}{dt}X_t(a)|^2 - \Phi(t, X_t(a), a)\}\mu(da)dt, \tag{5.1}$$

where $\Phi(t,x,a)$ denotes the added potential. From the saddle-point formulation

$$(5.2) \qquad \begin{cases} \inf_{(X_t)} \sup_{(p_t)} \int_0^T \int_{\mathcal{A}} \{\frac{1}{2}|\frac{d}{dt}X_t(a)|^2 - p_t(X_t(a)) \\ \\ -\Phi(t, X_t(a), a)\}]\mu(da)dt + \int_0^T \int_D p_t(x)dx\, dt, \end{cases}$$

we see that the added potential does not play any role if it does not involve the label variable a. (Then, it can be entirely absorbed in the pressure p, which is a well known phenomenon in fluid mechanics.) Let us concentrate on the simplest (and still relevent) situation when $\Phi(t,x,a) = -F(t,a)\cdot x$, where $F \in \mathbb{R}^d$ is a given function of t and a. (As we will discuss later, this takes into account non-trivial effects in fluid mechanics, for instance the buoyancy force, which plays a crucial role in ocean-atmosphere dynamics.) Thus, we are going to consider the corresponding saddle-point formulation, namely

$$(5.3) \qquad \begin{cases} \inf_{(X_t)} \sup_{(p_t)} \int_0^T \int_{\mathcal{A}} \{\frac{1}{2}|\frac{d}{dt}X_t(a)|^2 - p_t(X_t(a)) \\ \\ +F(t,a)\cdot X_t(a)\}\mu(da)dt + \int_0^T \int_D p_t(x)dx\, dt. \end{cases}$$

The optimality equations read

$$(5.4) \qquad \frac{d^2}{dt^2}X_t(a) + (\nabla p_t)(X_t(a)) = F(t,a), \quad \mu \text{ a.e.}$$

and do not look very different from the Euler equations, at first glance. We are now interested in the case when the additional force $F(t,a)$ is modified by the environment and more specifically when

$$(5.5) \qquad \partial_t F(t,a) = G_t(X(t,a)),$$

where G is a given, sufficiently smooth, function describing the local change brought to F by the environment. We call function G the "change" function, (This is relevent, for instance, in the case of convective motions when the salinity —or the temperature— of fluid parcels is modified by the environment. Of course, this is very important in the context of climate change studies: the change of salinity in the sea due to the melting of the continental ice is just an obvious example.) As we did for the Euler equations, assume, for a moment, their exists a sufficiently smooth velocity field $v_t(x) \in \mathbb{R}^d$ so that $\frac{d}{dt}X_t(a) = v_t(X_t(a))$. Let us also choose the space of labels A so that $a \in \mathcal{A} \to X_t(a) \in D$

is a smooth diffeomorphism at each time t. For that purpose we may set $(\mathcal{A}, \mu) = (D, dx)$ and set $X_0(a) = a$, so that fluid particles are just labelled by their initial position in D, a very common choice in fluid mechanics. Then, it makes sense to introduce the "Eulerian" field $f_t(x) = F(t, a)$ whenever $x = X_t(a)$. We can now express the modified Euler equations (5.4) together with (5.5) just in terms of (v_t, p_t, f_t, G_t):

(5.6)

$$D_t v_t + \nabla p_t = f_t, \quad D_t f_t = G_t, \quad D_t = \partial_t + v_t \cdot \nabla, \quad \nabla \cdot v_t = 0, \quad v_t \parallel \partial D.$$

The resulting evolution equations are substantially more difficult than the Euler equations themselves and very little is known about their mathematical analysis: essentially only the existence of smooth solutions for short times is known. Mathematics get easier when a dissipation term is added to the Euler equations and we, then, get the Navier–Stokes equation:

(5.7)

$$D_t v_t + \nabla p_t - \nu \Delta v_t = f_t, \quad D_t f_t = G_t, \quad D_t = \partial_t + v_t \cdot \nabla, \quad \nabla \cdot v_t = 0, \quad v_t \parallel \partial D,$$

where $\nu > 0$ is the viscosity of the fluid that we may assume as small as desired but positive. Provided additional boundary conditions are added (typically $v_t = 0$ along ∂D), the existence (but not the uniqueness) of global weak solutions is guaranteed for any reasonable initial condition, say $v_0, f_0 \in L^2(D, \mathbb{R}^d)$, by combining the Leray theory of Navier–Stokes equations and the DiPerna–Lions theory of advection equations [Li], [DL], [NP]. (Global smooth solutions can be obtained for two-space dimensions, as in [Ch], [HL], [DP].) We call these equations, with some abuse, the Navier–Stokes–Boussinesq convection (NSBC) equations, because they describe convective motions under the Boussinesq approximation. An interesting rescaling of the equations is obtained when we assume the "change" function G to be of small amplitude of order $\varepsilon \ll 1$, slowly varying in time: $G \to \varepsilon G_{\varepsilon t}(x)$. Then we may consider the evolution on large time scales of order ε^{-1}. Accordingly, we perform the following rescaling of the equations:

$$t = t/\varepsilon, \quad v = \varepsilon v, \tag{5.8}$$

and get the following rescaled equations

$$\begin{gathered} \varepsilon^2 D_t v_t + \nabla p_t + \varepsilon \nu \Delta v_t = f_t, \quad D_t f_t = G_t, \\ D_t = \partial_t + v_t \cdot \nabla, \quad \nabla \cdot v_t = 0, \quad v_t \parallel \partial D. \end{gathered} \tag{5.9}$$

Notice the interesting property enjoyed, at least formally, for all solutions of this equations:

$$\frac{d}{dt}\int_D \zeta(f_t(x))dx = \int_D (\nabla\zeta)(f_t(x))\cdot G_t(x)dx. \ \forall f \in C^\infty(\mathbb{T}^d). \tag{5.10}$$

Surprisingly enough, in this relation, ε, ν and even $v_t(x)$ are completely absent. It is now natural to consider the formal limit obtained as $\varepsilon \to 0$, namely:

$$\nabla p_t = f_t, \quad D_t f_t = G_t, \quad D_t = \partial_t + v_t\cdot\nabla, \quad \nabla\cdot v_t = 0, \quad v_t \parallel \partial D. \tag{5.11}$$

These equations, that we call Hydrostatic Boussinesq convection (HBC) equations, still abusively, have an intriguing structure. There is no more evolution equation for v and, in some sense, v becomes a multiplier for the constraint that, at any time t, f_t exactly balances the pressure force (in a hydrostatical way) and, therefore, stays always a gradient. As a matter of fact, v_t can be solved, at each t, after "curling" these equations. For example, in the 3D case, using notation $\nabla\times$ for the curl operator, we find

$$\nabla\times D_x^2 p_t(x)\cdot v_t = \nabla\times G_t, \quad \nabla\cdot v_t = 0. \tag{5.12}$$

Assuming t to be fixed and p_t to be known, this "static" linear PDE in v_t gets elliptic, with appropriate boundary conditions along ∂D, provided that the Hessian matrix $D_x^2 p_t$ is bounded away from zero and infinity, in the sense of symmetric matrices, uniformly in x. This a striking occurrence of convexity in this fluid mechanics framework. It turns out that this convexity condition (known as the Cullen–Purser stability condition in the framework of convection theory [CP], [Ho]) plays a crucial role in the mathematical analysis of the HBC equations and their rigorous derivation from the NSBC equations. However, simple examples show that strong uniform convexity cannot be sustained for large time, in general. Therefore, it is quite appealing to relax the strong convexity condition and consider generalized solutions to the HBC equations, for which the force field f_t is *always* a field with *convex* potential

$$f_t = \nabla p_t, \quad D_x^2 p_t \geq 0.$$

Then, according to Theorem 1, f_t can be entirely recovered from relation (5.10)! This leads to a concept of solutions, very similar to the concept of "entropy solutions" for hyperbolic conservation laws (for which we

refer to Dafermos' book [Da]. Let us now quote two results taken from [BC], [Br11].
From a technical viewpoint, it is easier (and still relevent) to substitute for D the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.We then define the class $\mathcal{C}$ of all maps $x \in \mathbb{R}^d \to M(x) = x + \nabla\phi(x)$ such that ϕ is Lipschitz continuous and $\mathbb{Z}^d-$ periodic and $I + D^2\phi \geq 0$. We call them "periodic maps with convex potential".

Theorem 9 (Existence of global solutions for the HBC equations). *Let $D = \mathbb{T}^d$ and $\mathcal{C}$ the class of all periodic maps with convex potential:*

$$\mathcal{C} = \{x \to x + \nabla\phi(x), \quad \phi \in Lip(\mathbb{T}^d), \quad I + D^2\phi \geq 0\}.$$

Then, for every $f_0 \in \mathcal{C}$, there is a global solution of the HBC equations (5.11)

$$t \to f_t \in C^0(\mathbb{R}_+; L^2(\mathbb{T}^d))$$

in the sense that, for all t, f_t belongs to $\mathcal{C}$ and satisfies (5.10), namely

$$\frac{d}{dt}\int_D \zeta(f_t(x))dx = \int_D (\nabla\zeta)(f_t(x)) \cdot G_t(x)dx, \;\; \forall f \in C^\infty(\mathbb{T}^d).$$

Theorem 10 (Derivation of the HBC equations from the NSBC equations). *Let $D = \mathbb{T}^d$ and $(f_t(x) = x + \nabla\phi_t(x), v_t(x))$ be a smooth solution of the HBC equations* (5.11) *on some time interval* $[0,T]$ *such that*

$$\alpha \leq I + D^2\phi_t(x) \leq \alpha^{-1}, \tag{5.13}$$

for some constant $\alpha > 0$. Then, this solution can be obtained as the limit of a solution to the NSBC equations (5.9) as $\varepsilon \to 0$.

Sketch of proof. The existence of global solutions is a rather straightforard consequence of the quadratic optimal transportation theory. The local existence of smooths solution can be established adaptating ideas of G. Loeper in [Lo] (which rely on a careful study of the underlying Monge–Ampère equation and use the Dini regularity of its solution). Finally, the rigorous derivation relies on an interesting (and very unusual) application of the so-called "relative entropy method", for which we refer to [Da], [LV] in the framework of hyperbolic conservation laws, to [Sa] for the hydrodynamic limit of the Boltzmann equations, and to [Br6] —in close connection with [Gr]— for the hydrostatic limit of the Euler equations. Let us sketch a proof. Since we are working on the periodic box $\mathbb{R}^d/\mathbb{Z}^d$, it is convenient to use only periodic unknowns, namely

$$z_t(x) = f_t(x) - x = \nabla\phi_t(x), \quad z_t^\epsilon(x) = f_t^\epsilon(x) - x, \tag{5.14}$$

where (f_t, v_t) and $(f_t^\epsilon, v_t^\epsilon)$ are respectively solutions to the limit equations (5.11) and to the full NSBC equations (5.9). We also introduce the "periodic Legendre–Fenchel" transform of ϕ:

$$\psi_t(y) = -\inf_{x\in\mathbb{R}^d} \frac{1}{2}|x-y|^2 + \phi_t(x), \tag{5.15}$$

and notice that

$$\alpha \le I + D^2\psi_t(y) \le \alpha^{-1} \tag{5.16}$$

follows from (5.13) by Legendre duality. Then, we introduce the "relative entropy" (or Bregman) function

$$\eta^{\Psi_t}(y, y'') = \Psi_t(y') - \Psi_t(y) - \Psi_t(y)\cdot(y'-y) \sim |y'-y|^2, \tag{5.17}$$

where $\Psi_t(y) = \frac{1}{2}|y|^2 + \psi_t(y)$, which controls $|y'-y|^2$ because of (5.16). Accordingly, we introduce the "relative-entropy" functional

$$e_t = \int \eta^{\Psi_t}(x + z_t(x), x + z_t^\varepsilon(x))\, dx, \tag{5.18}$$

where the integral in x is performed over $\mathbb{R}^d/\mathbb{Z}^d$. This functional controls the squared L^2 distance between z_t and z_t^ε. We also introduce the augmented functional

$$\tilde{e}_t = e_t + \frac{\varepsilon}{2}\int |v^\varepsilon - v|^2\, dx \tag{5.19}$$

and, after lengthy but simple calculations, get

$$\frac{d}{dt}\tilde{e}_t \le (\tilde{e}_t + \epsilon)c, \tag{5.20}$$

where c depends only on the limit solution (f, v) on a fixed finite time interval $[0, T]$ on which (f, v) is smooth. From this estimate (5.20), we immediately deduce that $z - z^\varepsilon$ is of order $O(\sqrt{\epsilon})$ in $L^\infty([0,T], L^2(\mathbb{R}^d/\mathbb{Z}^d))$. Notice that (5.20) could not be obtained by substituting the crude squared L^2 norm for the more sophisticated relative entropy functional e_t. (Then, c would be substituted for by c/ϵ and no convergence could be established!). Details are provided in [Br11]. Q.E.D.

§6. Optimal transport of currents

So far, we have only considered motions of particles. It is also interesting to consider strings moving in the Minkowski space, i.e. $\mathbb{R}^4$ with metric $\mathrm{diag}(-1,1,1,1)$. For simplicity we only consider loops. They are denoted

$$(t,s) \in \mathbb{R} \times \mathbb{T} \to (t, X(t,s,a)) \in \mathbb{R}^4$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and a denotes the label $a \in \mathcal{A}$ of each string. A classical action for a string (that substitutes for the kinetic energy for particles) is the Nambu–Goto action [Na], [Po], which is nothing but its area computed according to the Minkowski metric $\mathrm{diag}(-1,1,1,1)$, namely:

$$-\int_{t,s} \sqrt{(1-|\partial_t X|^2)|\partial_s X|^2 + (\partial_t X \cdot \partial_s X)^2}. \tag{6.1}$$

To the collection of all strings labelled by $a \in \mathcal{A}$, we may associate their "current", namely the time-dependent divergence-free vector field $(t,x) \in \mathbb{R}^4 \to B(t,x) \in \mathbb{R}^3$ defined as:

$$B(t,x) = \int_{\mathcal{A}} \int_{\mathbb{T}} \partial_s X(t,s,a) \delta(x - X(t,s,a)) ds\, \mu(da), \tag{6.2}$$

or, equivalently

$$\int_x A(x) \cdot B(t,x) = \int_{\mathcal{A}} \int_{\mathbb{T}} \partial_s X(t,s,a) A(X(t,s,a)) ds\, \mu(da), \tag{6.3}$$

for all t and all compactly supported smooth function A. [In this framework, the "current" B plays the role that the "law" was playing for particles in the second problem of optimal transportation we have covered so far.] Let us now make a rather strong assumption that, at each time t, $\mathbb{R}^3$ is foliated by the strings in the sense that $(s,a) \to X(t,s,a)$ is one-to-one (just like spaghetti filling the space, or, if one prefers, like lasagne filling the space-time). Then, we can define a velocity field $v(t,x)$ such that, for all strings:

$$\partial_t X(t,s,a) = v(t, X(t,s,a)) \tag{6.4}$$

and we can write the "transportation cost"

$$-\int_{t,x} \sqrt{(1-|v(t,x)|^2)|B(t,x)|^2 + (v(t,x) \cdot B(t,x))^2}\ . \tag{6.5}$$

The fields B and v are related by the compatibility condition

$$\partial_t(\partial_s X) = \partial_s(\partial_t X) = 0,$$

that can be translated by differentiating (6.3) with respect to t and, then, integrating by part in $s \in \mathbb{T}$, which leads to:

$$\partial_t B_i + \sum_{j=1,d} \partial_j(v_j B_i - v_i B_j) = 0, \tag{6.6}$$

which can be written, using vectorial notations,

$$\partial_t B + \nabla \times (B \times v) = 0. \tag{6.7}$$

Thus, introducing $(t,x) \to A(t,x) \in \mathbb{R}^3$ as a Lagrange multiplier for constraint (6.7), the optimal transportation problem for strings can be expressed as a saddle-point problem problem

$$\left\{ \begin{array}{l} \inf_B \sup_{A,v} \int_{t,x} \sqrt{(1-|v|^2)|B|^2 + (v \cdot B)^2} \\ \qquad -B \cdot \partial_t A + (B \times v) \cdot \nabla \times A \end{array} \right. \tag{6.8}$$

which, of course, must be supplemented by appropriate boundary conditions. Introducing

$$(t,x) \in \mathbb{R}^4 \to E(t,x) = B(t,x) \times v(t,x) \in \mathbb{R}^3, \tag{6.9}$$

equation (6.7) just means

$$\partial_t B + \nabla \times E = 0, \tag{6.10}$$

and the problem (6.8) can be written very simply in terms of E and B

$$\left\{ \begin{array}{l} \inf_B \sup_{A,E} \int_{t,x} \sqrt{|B|^2 - |E|^2} \\ -B \cdot \partial_t A + E \cdot \nabla \times A, \quad \text{subject to } E \cdot B = 0 \text{ , pointwise.} \end{array} \right. \tag{6.11}$$

Notice the additional *algebraic* constraint

$$E(t,x) \cdot B(t,x) = 0 \text{ , pointwise,} \tag{6.12}$$

which is just enough to enforce the existence of v such that $E = B \times v$ and allows us to completely disregard the field v.

This strange looking variational problem has an interesting physical interpretation. Indeed it can be seen as the formal limit of

$$(6.13)\qquad \begin{cases} \inf_B \sup_{A,E} \ \int_{t,x} \sqrt{\lambda^2 + |B|^2 - |E|^2 - \lambda^{-2}(B\cdot E)^2} \\ \qquad\qquad -B\cdot \partial_t A + E\cdot \nabla\times A, \end{cases}$$

as the parameter $\lambda \to 0$. For $\lambda > 0$, this model has been introduced in 1934 by M. Born and L. Infeld [BI], [Bo], [Po], [GH] as a non-linear correction to Maxwell's equations of Electromagnetism in vaccum:

$$(6.14)\qquad \begin{cases} \partial_t E - \nabla\times B = 0, \\ \partial_t B + \nabla\times E = 0. \end{cases}$$

Indeed, the Maxwell limit can be obtained from the Born–Infeld model, when λ tends to infinity, through the saddle-point formulation:

$$(6.15)\qquad \begin{cases} \inf_B \sup_{A,E} \ \int_{t,x} \frac{1}{2}\{|B|^2 - |E|^2\} \\ \qquad -B\cdot \partial_t A + E\cdot \nabla\times A, \end{cases}$$

while the optimal transportation problem for currents corresponds to the limit $\lambda = 0$. Surprisingly enough, some convexity properties are hidden in the optimal transportation of currents and, more generally, in the richer Born–Infeld model. To simplify the discussion, we fix $\lambda = 1$ in the Born–Infeld model. The optimality conditions for the BI saddle-point formulation (6.13) can be obtained in "Hamiltonian form" in the following way. Let us introduce the (partial) Legendre transform

$$(6.16)\qquad h(D,B) = \sup_{E\in\mathbb{R}^3} E\cdot D + \sqrt{1+|B|^2-|E|^2-(E\cdot B)^2}, \quad \forall D,B\in\mathbb{R}^3,$$

namely,

$$(6.17)\qquad h(D,B) = \sqrt{1+|B|^2+|D|^2+|D\times B|^2}.$$

Thanks to h, we can rewrite the saddle-point problem (6.13) simply as

$$(6.18)\qquad \inf_B \sup_A \int_{t,x} h(\nabla\times A, B) - B\cdot\partial_t A$$

Then, optimality equations for B and $D = \nabla \times A$ can be easily obtained:

$$(6.19)\qquad \begin{cases} \partial_t B + \nabla \times (\frac{\partial h}{\partial D}(D,B)) = 0, \\ \partial_t D - \nabla \times (\frac{\partial h}{\partial B}(D,B)) = 0. \end{cases}$$

These equations are just the Born–Infeld equations (for $\lambda = 1$) written in "Hamiltonian form". They enjoy an additional conservation law for h:

$$(6.20)\qquad \partial_t(h(D,B)) + \nabla \cdot (D \times B) = 0.$$

These equations have the very nice structure of a first order system of conservation laws. For given smooth intial conditions, first order system of conservation laws are known to be well-posed provided (more or less) there is an additional conservation law for a strictly convex functions of the unknowns [Da]. This could be happily the case for the Born–Infeld equations if $h(D,B)$ were a strongly convex function of D and B. Unfortunately, h is not convex at all in the large and is strictly convex only in a neighborhood of $(D,B) = (0,0)$. [Of course, there is no problem for the Maxwell limit, for which h just becomes $\frac{1}{2}(|D|^2 + |B|^2)$. On the contrary, for the limit $\lambda = 0$, h becomes widely non-convex: $\sqrt{|B|^2 + |D \times B|^2}$.] However, convexity can be fully restaured by *augmenting* the Born–Infeld equations, as done in [Br7]:

Theorem 11 (Hidden convexity in the Born–Infeld equations). *As parameter λ is fixed to 1, the Born–Infeld equations* (6.19) *can be seen as the restriction on the 6 dimensional invariant manifold*

$$(6.21)\qquad \{(B,D,P,h) \in (\mathbb{R}^3)^3 \times]0,+\infty[\,;\ \ P = D \times B,\ \ 1+|B|^2+|D|^2+|P|^2 = h^2\}$$

of the following augmented 10×10 *system of first-order conservation laws:*

$$(6.22)\qquad \begin{cases} \partial_t B + \nabla \times \frac{B \times P}{h} + \nabla \times \frac{D}{h} = 0, \\ \partial_t D + \nabla \times \frac{D \times P}{h} - \nabla \times \frac{B}{h} = 0, \\ \partial_t h + \nabla \cdot P = 0, \\ \partial_t P + \nabla \cdot \frac{P \otimes P}{h} = \nabla \cdot \frac{B \otimes B}{h} + \nabla \cdot \frac{D \otimes D}{h} + \nabla(\frac{1}{h})\,. \end{cases}$$

The augmented Born–Infeld equations (6.22) *enjoy an additional conservation law for the strictly convex function*:

$$(6.23) \qquad (D,B,P,h) \in (\mathbb{R}^3)^3 \times]0,+\infty[\to \frac{1+|B|^2+|D|^2+|P|^2}{h}\,.$$

From the physical point of view, we observe a striking property of the augmented system: it has classical Gallilean invariance under the transforms

$$(6.24) \qquad (t,x) \to (t, x+Vt), \quad (D,B,P/h,h) \to (D,B,P/h-V,h),$$

for all constant velocity $V \in \mathbb{R}^3$. In addition it has the structure of a coupled matter-field structure, where the electromagnetic field is described by (D,B) while h can be considered has the density of matter and P its momentum. In sharp contrast, the BI equations are Lorentzian invariant and are just a field equation without (apparent) interaction with matter.

Going back to our original transportation problem, we can assert

Theorem 12 (Hidden convexity in the optimal transportation of currents)**.** *The optimal transportation of currents can be described as the restriction to the* 5 *dimensional invariant manifold*

$$(6.25) \quad \{(B,P,h) \in (\mathbb{R}^3)^2 \times]0,+\infty[\,; \quad P\cdot B = 0, \quad |B|^2+|P|^2 = h^2\}$$

of the augmented 7×7 *system of first-order conservation laws*:

$$(6.26) \qquad \begin{cases} \partial_t B + \nabla \times \frac{B\times P}{h} = 0, \\ \partial_t h + \nabla \cdot P = 0\,, \\ \partial_t P + \nabla \cdot \frac{P\otimes P}{h} = \nabla \cdot \frac{B\otimes B}{h}. \end{cases}$$

This augmented system enjoys an additional conservation law for the strictly convex "entropy" function:

$$(6.27) \qquad (B,P,h) \in (\mathbb{R}^3)^2 \times]0,+\infty[\to \frac{|B|^2+|P|^2}{h}.$$

This convexification property is vaguely reminiscent of the reduction of minimal surface equations to the Dirichlet equation by using "isothermal" coordinates, but we do not know if a precise connection can be established.

References

[AF] L. Ambrosio and A. Figalli, Geodesics in the space of measure-preserving maps and plans, Arch. Ration. Mech. Anal., **194** (2009), 421–462.

[AF1] L. Ambrosio and A. Figalli, On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations, Calc. Var. Partial Differential Equations, **31** (2008), 497–509.

[AK] V. I. Arnold and B. Khesin, Topological Methods in Hydrodynamics, Springer-Verlag, 1998.

[Ar] V. I. Arnold, Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier (Grenoble), **16** (1966), 319–361.

[Ar1] V. I. Arnold, Mathematical Methods of Classical Mechanics, Grad. Texts in Math., **60**, Springer-Verlag, 1989.

[Ba] F. Barthe, On a reverse form of the Brascamp–Lieb inequality, Invent. Math., **134** (1998), 335–361.

[BFS] M. Bernot, A. Figalli and F. Santambrogio, Generalized solutions for the Euler equations in one and two dimensions, J. Math. Pures Appl. (9), **91** (2009), 137–155.

[Bo] M. Born, Ann. Inst. H. Poincaré, 1937.

[BI] M. Born and L. Infeld, Foundations of the new field theory, Proc. Roy. Soc. London A, **144** (1934), 425–451.

[Br1] Y. Brenier, Décomposition polaire et réarrangement monotone des champs de vecteurs, C. R. Acad. Sci. Paris Sér. I Math., **305** (1987), 805–808.

[Br2] Y. Brenier, The least action principle and the related concept of generalized flows for incompressible perfect fluids, J. Amer. Math. Soc., **2** (1989), 225–255.

[Br3] Y. Brenier, Polar factorisation and monotone rearrangement of vector-valued functions, Comm. Pure Appl. Math., **44** (1991), 375–417.

[Br4] Y. Brenier, The dual least action principle for an ideal, incompressible fluid, Arch. Rational Mech. Anal., **122** (1993), 323–351.

[Br5] Y. Brenier, Minimal geodesics on groups of volume-preserving maps, Comm. Pure Appl. Math., **52** (1999), 411–452.

[Br6] Y. Brenier, Remarks on the derivation of the hydrostatic limit of the Euler equations, Bull. Sci. Math., **127** (2003), 585–595.

[Br7] Y. Brenier, Hydrodynamic structure of the augmented Born-Infeld equations, Arch. Ration. Mech. Anal., **172** (2004), 65–91.

[Br8] Y. Brenier, Generalized solutions and hydrostatic approximation of the Euler equations, Physica D, **237** (2008), 1982–1988.

[Br9] Y. Brenier, Optimal Transport, Convection, Magnetic Relaxation and Generalised Boussinesq equations, J. Nonlinear Sci., **19** (2009), 547–570.

[Br10] Y. Brenier, Remarks on the minimizing geodesic problem in inviscid incompressible fluid mechanics, Calc. Var. Partial Differential Equations, **47** (2013), 55–64.

[Br11] Y. Brenier, Rearrangement, convection, convexity and entropy, to appear in Philosophical Transactions of the Royal Society A.

[BC] Y. Brenier and M. Cullen, Rigorous derivation of the x-z semigeostrophic equations, Commun. Math. Sci., **7** (2009), 779–784.

[BG] Y. Brenier and W. Gangbo, L^p approximation of maps by diffeomorphisms, Calc. Var., **16** (2003), 147–164.

[Brz] H. Brezis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Math. Stud., **5**, 1973.

[Ca] L. Caffarelli, The regularity of mappings with a convex potential, J. Amer. Math. Soc., **5** (1992), 99–104.

[Ch] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, Adv. Math., **203** (2006), 497–513.

[CP] M. Cullen and R. Purser, An extended Lagrangian theory of semi-geostrophic frontogenesis, J. Atmospheric Sci., **41** (1984), 1477–1497

[Da] C. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Springer-Verlag, 2000.

[DP] R. Danchin and M. Paicu, Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data, Comm. Math. Phys., **290** (2009), 1–14.

[DF] G. De Philippis and A. Figalli, $W^{2,1}$ regularity for solutions of the Monge-Ampère equation, Invent. Math., **192** (2013), 55–69.

[DL] R. DiPerna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math., **98** (1989), 511–547.

[DR] J. Duchon and R. Robert, Relaxation of the Euler equations and hydrodynamic instabilities, Quarterly Appl. Math., **50** (1992), 235–255.

[EM] D. Ebin and J. Marsden, Groups of diffeomorphisms and the notion of an incompressible fluid, Ann. of Math. (2), **92** (1970), 102–163.

[Eu] L. Euler, Opera Omnia, Series Secunda, **12**, 274–361.

[Ev] L. C. Evans, Partial differential equations and Monge–Kantorovich mass transfer, In: Current Developments in Mathematics, International Press, Cambridge, 1997, pp. 26–78.

[FMP] A. Figalli, F. Maggi and A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities, Invent. Math., **182** (2010), 167–211.

[Ga] W. Gangbo, An elementary proof of the polar factorization of vector-valued functions, Arch. Rational Mech. Anal., **128** (1994), 381–399.

[Gr] E. Grenier, On the derivation of homogeneous hydrostatic equations, M2AN Math. Model. Numer. Anal., **33** (1999), 965–970.

[GH] G. W. Gibbons and C. Herdeiro, Born–Infeld theory and stringy causality, Phys. Rev. D (3), **63** (2001), 064006.

[Ho] B. Hoskins, The Geostrophic Momentum Approximation and the Semi-Geostrophic Equations, J. Atmospheric Sci., **32** (1975), 233–242.

[HL] T. Hou and C. Li, Global well-posedness of the viscous Boussinesq equations, Discrete Contin. Dyn. Syst., **12** (2005), 1–12.

[Ka] L. Kantorovich, On the translocation of masses, Doklady Acad. Sci. URSS (N.S.), **37** (1942), 199–201.

[KS] M. Knott and C. S. Smith, On the optimal mapping of distributions, J. Optim. Theory Appl., **43** (1984), 39–49.

[LV] N. Leger and A. Vasseur, Relative entropy and the stability of shocks and contact discontinuities for systems of conservation laws with non-BV perturbations, Arch. Ration. Mech. Anal., **201** (2011), 271–302.

[Li] P.-L. Lions, Mathematical Topics in Fluid Mechanics. Vol. 1. Incompressible Models, Oxford Lecture Ser. Math. Appl., Oxford Univ. Press, 1996.

[Lo] G. Loeper, A fully nonlinear version of the incompressible Euler equations: the semigeostrophic system, SIAM J. Math. Anal., **38** (2006), 795–823.

[Mc] R. McCann, A convexity principle for interacting gases, Adv. Math., **128** (1997), 153–179.

[Mc1] R. McCann, Polar factorization of maps on Riemannian manifolds, Geom. Funct. Anal., **11** (2001), 589–608.

[Na] Y. Nambu, Quark model and the factorization of the Veneziano amplitude, In: Proceedings of the International Conference on Symmetries and Quark Models, Wayne State Univ., 1969, Gordon and Breach, pp. 269–277.

[Ne] Y. Neretin, Categories of bistochastic measures and representations of some infinite-dimensional groups, Russian Acad. Sci. Sb. Math., **75** (1993), 197–219.

[Mo] G. Monge, Mémoire sur la théorie des déblais et des remblais, In: Académie Royale des Sciences de Paris, Mémoires de Mathématique et de Physique, 1781, pp. 666–704.

[NP] A. Nouri and F. Poupaud, An existence theorem for the multifluid Navier-Stokes problem, J. Differential Equations, **122** (1995), 71–88.

[Ot] F. Otto, Dynamics of labyrinthine pattern formation in magnetic fluids: a mean-field theory, Arch. Rational Mech. Anal., **141** (1998), 63–103.

[Po] J. Polchinski, String theory. Vol. I., Cambridge Univ. Press, 1998.

[RR] S. Rachev and L. Rueschendorf, Mass transportation problems, Springer-Verlag, 1998.

[Sa] L. Saint-Raymond, Convergence of solutions to the Boltzmann equation in the incompressible Euler limit, Arch. Ration. Mech. Anal., **166** (2003), 47–80.

[Sh] A. Shnirelman, On the geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid, Math. Sbornik USSR, **56** (1987), 79–105.

[Sh1] A. I. Shnirelman, Generalized fluid flows, their approximation and applications, Geom. Funct. Anal., **4** (1994), 586–620.

[Vi] C. Villani, Topics in Optimal Transportation, Grad. Stud. Math., **58**, Amer. Math. Soc., Providence, RI, 2003.

[Vi1] C. Villani, Optimal Transport. Old and New, Grundlehren Math. Wiss., **338**, Springer-Verlag, 2009.

CNRS, CMLS, Ecole Polytechnique
FR-91128 Palaiseau
France
E-mail address: brenier@math.unice.fr