Advanced Studies in Pure Mathematics 66, 2015
Singularities in Geometry and Topology 2011
pp. 161–172

Some geometric-arithmetic aspects of separated variable curves

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Abstract.

The paper shows certain geometric-arithmetic aspects with some new observations in consideration of variables separated curves, *e.g.* relationships with generalized Chebyshev polynomials, Chebyshev pencils, local variant at infinity of Stothers–Mason *abc*-theorem, Stothers–Langevin pairs, Pell–Abel conics (or polynomial Pell equations), Belyi maps, etc. Several case studies and open problems are discussed.

§1. Introduction

The aim of this paper is to give an account of the generalized Chebyshev polynomial phenomenon somehow mysteriously appeared in several geometric-arithmetic aspects of separated variable curves, *i.e.* curves of the form

$$\Gamma_{f,g} \coloneqq \{f(x) - g(y) = 0\},$$

where f, g are polynomials in one variable. In what follows we shall denote by $C_{f,g}$ the projective closure of $\Gamma_{f,g}$ and by the genus of $C_{f,g}$ we mean the genus of its normalization. Over a number field k the class of separated variable has attracted a great arithmetical interest since [5], in which a little surprising role of Chebyshev polynomials has been already emphasized, and subsequently (*cf.* [7] and references therein). There is also a variety of applications to coding theory and cryptography.

Another interesting topics is an old problem of Severi how to construct explicitly curves with given number of nodes as their only singularities. It turns out that Chebyshev polynomials actually give a very

Received March 22, 2012.
Revised July 30, 2012.
2010 *Mathematics Subject Classification.* Primary 11G30, 14H10.
Key words and phrases. Chebyshev polynomial, Stothers–Mason abc-theorem, Stothers–Langevin pair, Pell–Abel conics.

nice manner for that purpose ([17], [18]). In fact Ritt's second theorem ([19]) describes all rational curves $C_{f,g}$ in the case $(\deg f, \deg g) = 1$ which says roughly that those are the only power functions, or Chebyshev polynomials.

All these together lead to the study of generalized Chebyshev polynomials which were conceived already in [23], *Problème* (C). The generalized Chebyshev curves, *i.e.* if f, g are generalized Chebyshev polynomials, can be studied via data of their plane trees. It gives a combinatorial way in constructing curves with given singularities, say in the simplest case with nodes and cusps.

It should be noted that the Cassels' "monster" ([4]), as mathematicians named it for the time, indicated implicitly a link with Belyi's theorem. This observation was often exploited in the sequence by many authors to the reducibility problem modulo the $FSGC$ (Finite Simple Groups Classification) ([7]). It was exploited also in the author's recent observation from a slightly different angle, namely the pencil point of view ([14]).

The fact that Chebyshev polynomials give simple examples of Stothers–Langevin pairs (the extremal case in the local form of Stothers–Mason *abc*-theorem) can be interpreted for a wider class of generalized Chebyshev polynomials, together with the Pell–Abel conics $x^2 - D(t)\ y^2 = 1$ (or polynomial Pell equations) treated classically in the works of Abel, Chebyshev, Zolotarev, Halphen, Akhiezer et el. It should be remarked that the theory of Pell–Abel equations has recently attracted a new interest. A particular important case of the Stothers–Mason *abc*-theorem is Davenport's bound having a surprisingly natural connection with elliptic surfaces discovered recently by T. Shioda. In this aspect we realize that Stothers–Langevin pairs turn out to be also quite interesting. We shall discuss on all these in the forthcoming paper ([15]).

The first part of this paper is devoted to generalized Chebyshev curves in connection with irreducibility and genus estimation which in view of Siegel's theorem essentially affects on the arithmetic of $\Gamma_{f,g}$. In the second part we show certain arithmetic-geometric aspects of generalized Chebyshev polynomials related to Stothers–Langevin pairs, Pell–Abel conics, etc.

Throughout the paper, unless otherwise stated, we assume the ground field k is the field of complex numbers $\mathbb{C}$.

§2. Maximally singular curves

Singularities of $\Gamma_{f,g} = \{f(x) - g(y) = 0\}$ are precisely

$$\{(a,b) \in \Gamma_{f,g} \colon f'(a) = g'(b) = 0\}.$$

In other words one has to verify the condition $f(a) = g(b)$ among the net points $\{f'(a) = g'(b) = 0\}$. Hence in order to find curves with possibly maximal number of singularities it is desirable to use polynomials with few critical values.

Example 1.1. The power function z^n has only one critical value $w = 0$. If m, n are coprime and $m < n$, then the quasi-homogeneous curve

$$\mathcal{C}_0^{m,n} \coloneqq \{2^{m-1}x^m - 2^{n-1}y^n = 0\}$$

is an irreducible simply connected curve.

The curve $\mathcal{C}_0^{m,n}$ has a nice symmetric property, as at infinity its projective closure has another quasi-homogeneous singularity at $(1:0:0)$ (so sometimes we say it is simply connected at infinity).

The Zaidenberg–Lin theorem (*cf.* [2]) asserts that it has a unique plane embedding.

Example 1.2. The n-th Chebyshev polynomial of the first kind $T_n(z)$ (of degree n) is defined by the following recurrence relation

$$T_0(z) = 1,\; T_1(z) = z,\; T_n(z) = 2z\, T_{n-1}(z) - T_{n-2}(z),\; \text{for } n \geq 2.$$

Quite often in the computation it is more convenient to use the following property

$$T_n(\cos\varphi) = \cos n\varphi$$

which might be taken also for a definition of $T_n(z)$. It is clear that $T_n(z)$ has only two critical values $w = \pm 1$.

The Chebyshev curve

$$\mathcal{C}_1^{m,n} \coloneqq \{T_m(x) - T_n(y) = 0\}$$

with $(m,n) = 1$ has a plenty, namely $\frac{1}{2}(m-1)(n-1)$, of ordinary double points, all of them are real. Hence $\mathcal{C}_1^{n-1,n}$ is a maximal nodal curve ([9], [18]).

It should be noted that the paper [16] uses a slightly different technique—the existence of perturbed Chebyshev polynomials with three critical values ([23], [3]) to produce maximal nodal curves.

It is known that the family

$$\mathcal{C}_\lambda^{m,n} \coloneqq \{\lambda^{mn}\big(T_m(\lambda^{-n}x) - T_n(\lambda^{-m}y)\big) = 0\},\ \lambda \in [0,1]$$

gives a real morsification for $\mathcal{C}_0^{m,n}$ ([10], [3]).

Problem 1.3. One may ask whether $\mathcal{C}_1^{m,n}$ has a unique plane embedding.

Remarks 1.4. (*i*) It is true when m, n are distinct primes ([2]). During the conference at Fukuoka Prof. A'Campo pointed out that the question is closely related to the existence of special divisors on $\mathcal{C}_1^{m,n}$. For instance, if $m = n = d$, then $\mathcal{C}_0^{m,n}$ is of type (IV_d) (the completely decomposable case), and similarly $\mathcal{C}_1^{d,d}$ being highly reducible (*cf.* below) has many crossings (special position of ordinary double points), hence must be rigid. Presumably one can exploit the technique of [8] in realizing the above idea.

(*ii*) Ritt's second theorem ([19]) essentially asserts that in the notation above if $g(C_{f,g}) = 0$, then it can be parametrized as either (I): $(t^m, t^r P(t)^m), r \in \mathbb{N}$; or (II): $(T_n(t), T_m(t))$. The recent paper [17] exhibits an explicit construction of maximal nodal curves of type (I) using Fermat curves.

(*iii*) There is another way to derive maximal nodal curves with all real nodes from the classically known theory of algebraic trochoids due to T. Cotterill, S. Roberts, M. A. Wolstenholme, F. Morley et al. since the second half of 19-th century (*cf.* [21] and references therein). Precisely it can be described as follows. By assuming the radii of the fixed and rolling circles to be $1, m$ respectively (m is a rational number < 1) the parametric equations of a hypotrochoid $\mathcal{T}_{m,\varepsilon}$ with parameters m, ε are

$$\begin{cases} x = (1-m)\cos m\varphi + (m+\varepsilon)\cos(1-m)\varphi \\ y = (1-m)\sin m\varphi - (m+\varepsilon)\sin(1-m)\varphi \end{cases}$$

In some cases it is more convenient to consider in the Morley form

$$w = az^p + bz^q,\ |z| = 1,$$

where

$$w \coloneqq x + iy,\ z \coloneqq e^{\frac{i\varphi}{\nu}},\ p \coloneqq \mu,\ q \coloneqq \nu - \mu$$

and $m = \frac{\mu}{\nu}$ in the simplest form.

The hypocycloid $\mathcal{T}_{m,0}$ is of class ν and degree $2\max\{p, q\}$. Clearly $\mathcal{T}_{m,0}$ has ν cusps situated on the fixed circle and a number of nodes. In a sense hypotrochoids are "pertubated" curves giving a (real) morsification for $\mathcal{T}_{m,0}$, as their class now is 2ν. We get therefore maximal nodal curves with all real nodes from (projective closures of) $\mathcal{T}_{m,\varepsilon}$ with $m = \frac{\mu}{2\mu+1}$, $\varepsilon > 0$ (resp. $m = \frac{\mu}{2\mu-1}$, $\varepsilon < 0$).

The exceptional value of ε is $\frac{1}{2\mu+1}$ (resp. $-\frac{1}{2\mu-1}$) where the corresponding curve has a point of multiplicity $2\mu+1$ (resp. $2\mu-1$). It should be noted that the same construction goes through for (affine) epitrochoids, although one can not get maximal nodal curves, because of singularities at infinity.

(*iv*) We make a remark concerning Chebyshev pencils studied in [14]. Assume given a pencil of plane curves of degree d that contains two degenerate members of type IV_d, *i.e.* consisting of d concurrent lines. Then up to coordinate changes it can be given by

$$f(x) - tg(y),\ t \in \mathbb{P}^1.$$

A typical example of this sort is the pencil $(x^d - 1) - t(y^d - 1) = 0$ with $(3IV_d)$ configuration. It is known that up to projective isomorphism pencils with $(3IV_d)$ configuration are unique with the above equation. The other interesting examples of pencils with degenerations at $t = \pm 1$ together with given $(2IV_d)$ come from the so-called Chebyshev pencils $T_m(x) - tT_n(y) = 0$ (see [14] for details).

We proceed now to the notion of *generalized Chebyshev polynomials.* Let $P \in k[z]$ be a polynomial in one variable. Recall that a point $z_0 \in k$ called a critical point of P, if $P'(z_0) = 0$. The value $w_0 = P(z_0)$ of P at z_0 is then called the critical value of P. A generalized Chebyshev polynomial is a polynomial P with at most two critical values, *i.e.* there exist two distinct numbers c_0, c_1 such that if $P'(z) = 0$, then $P(z)$ is either c_0, or c_1.

Following [20], [11] we fix the following data for a generalized Chebyshev polynomial P of degree m:

$< \alpha_1, \dots, \alpha_p; \alpha_{p+1}, \dots, \alpha_{m+1} >$—the valency set (each of the two groups is in decreasing order) respectively at

$\{a_1, \dots, a_p; a_{p+1}, \dots, a_{m+1}\}$—the set of points (or vertices);

$$\sum_{i=1}^{p} \alpha_i = \sum_{j=p+1}^{m+1} \alpha_j = m.$$

So

$$(1.1) \qquad P(z) - c_0 = \prod_{i=1}^{p} (z - a_i)^{\alpha_i} \ ; \ \ P(z) - c_1 = \prod_{i=p}^{m+1} (z - a_j)^{\alpha_j}.$$

For example in the case of Chebyshev polynomials one has $c_0 = -1,\ c_1 = 1$ and:

$< 2, \dots, 2, 1; 2, \dots, 2, 1 >$,
$< \cos\frac{\pi}{2m+1}, \dots, \cos\frac{(2m-1)\pi}{2m+1}, -1; \cos\frac{2m\pi}{2m+1}, \dots, \cos\frac{2\pi}{2m+1}, 1 >$
(for $T_{2m+1}(z)$);

$< 2, \dots, 2; 2, \dots, 2, 1, 1 >$,
$< \cos\frac{\pi}{2m}, \dots, \cos\frac{(2m-1)\pi}{2m}; \cos\frac{\pi}{m}, \dots, \cos\frac{(m-1)\pi}{m}, -1, 1 >$
(for $T_{2m}(z)$).

Let Q be another generalized Chebyshev polynomial of degree n with same critical values c_0, c_1 and let $< \beta_1, \dots, \beta_q; \beta_{q+1}, \dots, \beta_{n+1} >$, $\{b_1, \dots, b_q; b_{q+1}, \dots, b_{n+1}\}$ be as above with $\sum\limits_{i=1}^{q} \beta_i = \sum\limits_{j=q+1}^{n+1} \beta_j = n$,

$$(1.2) \qquad Q(z) - c_0 = \prod_{i=1}^{q} (z - b_i)^{\beta_i} \ ; \ \ Q(z) - c_1 = \prod_{j=q+1}^{n+1} (z - b_j)^{\beta_j}.$$

Let us consider the generalized Chebyshev curve

$$\Gamma_{P,Q} = \{P(x) - Q(y) = 0\}$$

under the assumption $(m, n) = 1$ as above. An estimation for the genus formula of $C_{f,g}$ was given in [5] for an arithmetical application. Later along this line of ideas a general formula was obtained by [6]. In the case of $\Gamma_{P,Q}$ it can be read from the combinatorial data of P and Q as shown in the following theorem.

Theorem 1.5. *With notation we have the following formula for the genus of $C_{P,Q}$*

$$2g(C_{P,Q}) = (m-1)(n-1) - \sum_{\substack{1 \le i \le p \\ 1 \le j \le q}} \big[(\alpha_i - 1)(\beta_j - 1) - d_{ij}\big] - \sum_{\substack{p+1 \le i \le m+1 \\ q+1 \le j \le n+1}} \big[(\alpha_i - 1)(\beta_j - 1) - d_{ij}\big]$$

where $d_{ij} = \gcd(\alpha_i, \beta_j)$.

For a sketch of a possibly simple proof one may argue as follows. We consider $C_{P,Q}$ as a fiber product of two coverings defined by P and Q. From (1.1) and (1,2) one sees that at (a_i, b_j) of $\Gamma_{P,Q}$ we have a singularity of type $x^{\alpha_i} = y^{\beta_j}$. With singularities at infinity it is desirable to deal a little more technically. It remains to use Plücker's formula for plane curves.

Thus in this approach we get a nice combinatorial way of constructing certain classes of curves with singularities.

§3. On Pell–Abel conics and related questions

First we formulate the local form at infinity of Stothers–Mason (*abc*-)theorem. For a polynomial $P \in k[z]$ we denote by $d(P) := \deg(P)$, $r(P) :=$ the number of distinct roots of P.

Theorem 2.1. (*cf.* [22], [12]). *For two distinct polynomials $R, S \in k[z]$ not all constant we have*

$$r(RS) + d(R - S) \geq \max\{d(R), d(S)\} + 1. \tag{2.1}$$

If the equality holds in (2.1), we call (R, S) a Stothers–Langevin pair.

Recall the well-known identity that could be derived immediately from the defining $T_n(z) = \cos(n \arccos z)$

$$T_n^2(z) - (z^2 - 1)\, U_n^2(z) = 1, \tag{2.2}$$

where $U_n(z) := \frac{1}{n}\, T_n'(z)$ thus giving an example of a Stothers–Langevin pair $(T_n^2, (z^2 - 1)\, U_n^2)$.

On the other hand (2.2) is one of the simplest cases of the Pell-Abel equation $x^2 - Dy^2 = 1$ with $D(z) = z^2 - 1$. In general every monic polynomial $P \in k[z]$ of degree $d(P) = n$ gives rise to a solution of a Pell–Abel equation as follows. Let

$$P(z)^2 - 1 = \prod_{i=1}^{r} (z - a_i)^{\alpha_i}, \ \sum_{i=1}^{r} \alpha_i = 2n$$

be the complete factorization. Then by putting

$$D_P(z) := \prod_{\alpha_i \text{ is odd}} (z - a_i)$$

we get

$$P^2(z) - D_P(z)\, Q^2(z) = 1, \tag{2.3}$$

with

$$Q(z) := \prod_{i=1}^{r} (z - a_i)^{[\frac{\alpha_i}{2}]},$$

where $[\,\cdot\,]$ denotes the integer part.

We shall need two technical lemmas (*cf.* [20], [12]).

Lemma 2.2. *In the notation above $r \geq n+1$ with equality if and only if P is a generalized Chebyshev polynomial with critical values ± 1.*

Lemma 2.3. *If P is a non-constant polynomial, then $r(P) = d(P) - d(P,P')$, where $(P,Q) := \gcd(P,Q)$.*

Clearly if P is a generalized Chebyshev polynomial with critical values ± 1, then (2.3) gives us a Stothers–Langevin pair, or more generally

Proposition 2.4. *Let P be a generalized Chebyshev polynomial with two distinct critical values c_0, c_1. Then $(P_1^2, P_1^2 - 1)$ is a Stothers–Langevin pair, where $P_1(z) := \frac{2}{c_1 - c_0}\big(P(z) - \frac{c_1 - c_0}{2}\big)$.*

The proof follows from (1.1) and by the above remark.

For the converse we have

Theorem 2.5. *For a monic polynomial P the pair $(P^2, P^2 - 1)$ is a Stothers–Langevin pair if and only if P is a polynomial with at most three critical values at $\pm 1, 0$. In particular if P is square-free (i.e. with simple roots), then it is a generalized Chebyshev polynomial with critical values ± 1.*

Proof. In the notation of (2.3) from the condition of Theorem and (2.1)

$$r(P) + r = 2n + 1, \tag{2.4}$$

or by using Lemma 2.2

$$n - d(P,P') + 2n - d(P.P', P^2 - 1) = 2n + 1$$

which is equivalent to

$$n - 1 = d(P', P(P^2 - 1)). \tag{2.5}$$

Clearly (2.5) means that $P'(z)|P(P^2-1)$ which implies the first statement of the theorem.

A more direct argument could be proceeded as follows. Writing

$$P(z)=\prod_{j=1}^{r(P)}(z-b_j)^{\beta_j},\ \sum_{j=1}^{r(P)}\beta_j=n$$

one has

$$\prod_{j=1}^{r(P)}(z-b_j)^{\beta_j-1}\prod_{i=1}^{r}(z-a_i)^{\alpha_i-1}|P'(z). \tag{2.6}$$

Hence

$$n-r(P)+2n-r\leq \deg(P')=n-1,$$

which is nothing but (2.1) for this case. In view of (2.4) we have an identity in (2.6), and thus the first statement of the theorem follows.

A particular case of the theorem can be seen easily from Lemma 2.3. The proof is completed.

It should be noted that either (2.5), or (2.6) is equivalent to the assertion—"P^2 is a generalized Chebyshev polynomial with critical values $0,1$"—the fact that is true generally would be deduced by using a Belyi-type argument.

Remarks 2.6. (*i*) The converse problem to (2.3) turned out to be rather subtle. The reason should be clear by looking at the Dirichlet's principle applied at a crucial step to the equation $x^2-Dy^2=1$. Here we are asked to find polynomial solutions of the equation

$$x^2-D(t)\ y^2=1 \tag{2.7}$$

for a given square-free polynomial D of even degree $d(D)=2m$.

The theory of polynomial Pell equations was begun in the work of Abel ([1]). Abel's investigation emphasized on the study of hyperelliptic integrals of the form

$$\int\frac{\rho(t)}{\sqrt{D(t)}}\ dt \tag{2.8}$$

for some polynomial $\rho(t)$ of degree $d(\rho)\leq m-1$, $m\geq 2$. Abel's theorem states:

(†) for $d(\rho) \leq m-2$ (2.8) is not integrable in elementary functions;

(††) if $d(\rho) = m-1$, then the existence of a ρ such that (2.8) is integrable in elementary functions is equivalent to the existence of a solution $(x, y) = (P(t), Q(t))$ to (2.7), more precisely $\rho = \frac{P'}{Q}$ for such a solution, and clearly then

$$\int \frac{\rho(t)}{\sqrt{D(t)}}\, dt = \log(P(t) + Q(t)\sqrt{D(t)})$$

Abel has also proved that this is equivalent to the fact that $\sqrt{D}$ can be represented as a periodic continued fraction.

(*ii*) The most interesting condition equivalent to the above in Abel's theorem is

$$(Tors) \qquad P_\infty^+ - P_\infty^- \text{ is a torsion point on } Jac(C_D)$$

where C_D denotes the hyperelliptic curve given in the affine form

$$(C_D) \qquad u^2 = D(t).$$

We briefly summarize several appoaches leading to this condition. The curve C_D has two points at infinity, say P_∞^+, P_∞^-. By using a direct substitution method and Laurent expansion of square root $\sqrt{D}$ at P_∞^- one finds a condition for solubility of (2.7) in terms of certain Hankel determinants which implies the condition $(Tors)$. This can be seen also working directly with units in the function field $k(C_D)$.

In fact the Pell–Abel equation could be considered from the pencil viewpoint, *i.e.* as a pencil of conics

$$\mathcal{C} \to \mathbb{P}^1$$

given in the equation by (2.7) (a similar idea was exploited in [13]). According to the general theory it is the same as considering $\mathcal{C}_\eta$ (the generic fibre) over the function field $K := k(\mathbb{P}^1)$.

The pencil becomes constant after the base change

$$C_D \to \mathbb{P}^1$$

(*i.e.* working over the field extension $k(C_D) = K(\sqrt{D})$) whose fibres are

$$(C_0) \qquad y^2 = x^2 - 1$$

Therefore we get the isomorphism

$$\mathcal{C}_\eta(K) \cong Hom_k(Jac(C_D), C_0)$$

for the set of K-rational points on $\mathcal{C}_\eta$.

Putting $\mathfrak{m} := P_\infty^+ + P_\infty^-$ one sees that solutions of (2.7) are in $1-1$ correspondence with $\mathfrak{m}$-integral points of $\mathcal{C}_\eta$. The latter set is nothing but $Hom_k(J_\mathfrak{m}, \mathbb{G}_m)$, where $J_\mathfrak{m}$ denotes the generalized Jacobian *w.r.t.* the module $\mathfrak{m}$. It is not difficult to see the non-emptiness of $Hom_k(J_\mathfrak{m}, \mathbb{G}_m)$ means again the condition $(Tors)$.

There is also a nice relation to Jacobi operators, nonlinear equations, etc. (*cf.* [15] for details and references).

Acknowledgments. The research was partially supported by JSPS–VAST Bilateral Joint Projects. I would like to thank the organizers of the sixth Franco–Japan Symposium on Singularities held at Fukuoka in September, 5–10, 2011, for the kind invitation and hospitality. The author's thanks go also to Professors N. A'Campo, A. Ashikaga, K. Konno and M. Oka for encouragement and helpful discussions.

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