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Gorenstein in codimension 4: the general structure theory

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Abstract.

I describe the projective resolution of a codimension 4 Gorenstein ideal, aiming to extend Buchsbaum and Eisenbud's famous result in codimension 3. The main result is a structure theorem stating that the ideal is determined by its $(k+1) \times 2k$ matrix of first syzygies, viewed as a morphism from the ambient regular space to the Spin-Hom variety $\mathrm{SpH}_k \subset \mathrm{Mat}(k+1, 2k)$. This is a general result encapsulating some theoretical aspects of the problem, but, as it stands, is still some way from tractable applications.

This paper introduces the *Spin-Hom varieties* $\mathrm{SpH}_k \subset \mathrm{Mat}(k+1, 2k)$ for $k \geq 3$, that I define as almost homogeneous spaces under the group $\mathrm{GL}(k+1) \times \mathrm{O}(2k)$ (see 2.4). These serve as key varieties for the $(k+1) \times 2k$ first syzygy matrixes of codimension 4 Gorenstein ideals I in a polynomial ring S plus appropriate presentation data; the correspondence takes I to its matrix of first syzygies. Such ideals I are parametrised by an open subscheme of $\mathrm{SpH}_k(S) = \mathrm{Mor}(\mathrm{Spec}\, S, \mathrm{SpH}_k)$. The open condition comes from the Buchsbaum–Eisenbud exactness criterion "What makes a complex exact?" [BE1]: the classifying map $\alpha\colon \mathrm{Spec}\, S \to \mathrm{SpH}_k$ must hit the *degeneracy locus* of SpH_k in codimension ≥ 4.

The map α has *Cramer-spinor coordinates* L_i and σ_J in standard representations $\mathbf{k}^{k+1}$ and $\mathbf{k}^{2^{k-1}}$ of $\mathrm{GL}(k+1)$ and $\mathrm{Pin}(2k)$ (see 3.3), and the $k \times k$ minors of $M_1(I)$ are in the product ideal $I \cdot \mathrm{Sym}^2(\{\sigma_J\})$. The spinors themselves should also be in I, so that the $k \times k$ minors of $M_1(I)$ are in I^3; this goes some way towards explaining the mechanism that makes the syzygy matrix $M_1(I)$ "drop rank by 3 at one go"—it has rank k outside $V(I) = \mathrm{Spec}(S/I)$ and $\leq k-3$ on $V(I)$.

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The results here are not yet applicable in any satisfactory way, and raise almost as many questions as they answer. While Gorenstein codimension 4 ideals are subject to a structure theorem, that I believe to be the correct codimension 4 generalisation of the famous Buchsbaum–Eisenbud theorem in codimension 3 [BE2], I do not say that this makes them tractable.

§1. Introduction

Gorenstein rings are important, appearing throughout algebra, algebraic geometry and singularity theory. A common source is Zariski's standard construction of graded ring over a polarised variety X, L: the graded ring $R(X,L) = \bigoplus_{n\ge0} H^0(X, nL)$ is a Gorenstein ring under natural and fairly mild conditions (cohomology vanishing plus $K_X = k_X L$ for some $k_X \in \mathbb{Z}$, see for example [GW]). Knowing how to construct $R(X,L)$ by generators and relations gives precise answer to questions on embedding $X \hookrightarrow \mathbb{P}^n$ and determining the equations of the image.

1.1. Background and the Buchsbaum–Eisenbud result

I work over a field $\mathbf{k}$ containing $\frac{1}{2}$ (such as $\mathbf{k} = \mathbb{C}$, but see 4.5 for the more general case). Let $S = \mathbf{k}[x_1, \dots, x_n]$ be a positively graded polynomial ring with $\operatorname{wt} x_i = a_i$, and $R = S/I_R$ a quotient of S that is a Gorenstein ring. Equivalently, $\operatorname{Spec} R \subset \operatorname{Spec} S = \mathbb{A}^n_{\mathbf{k}}$ is a Gorenstein graded scheme. By the Auslander–Buchsbaum form of the Hilbert syzygies theorem, R has a minimal free graded resolution $P_\bullet$ of the form

$$\begin{array}{c} 0 \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_c \leftarrow 0 \\ \downarrow \\ R \end{array} \tag{1.1}$$

where $P_0 = S \to R = S/I_R$ is the quotient map, and $P_1 \to S$ gives a minimum set of generators of the ideal I_R. Here the length c of the resolution equals $n - \operatorname{depth} R$, and each P_i is a graded free module of rank b_i. I write $P_i = b_i S$ (as an abbreviation for $S^{\oplus b_i}$), or $P_i = \bigoplus_{j=1}^{b_i} S(-d_{ij})$ if I need to keep track of the gradings. The condition $\operatorname{depth} R = \dim R$ that the depth is maximal characterises the Cohen–Macaulay case, and then $c = \operatorname{codim} R = \operatorname{codim}(\operatorname{Spec} R \subset \operatorname{Spec} S)$. If in addition P_c is a free module of rank 1, so that $P_c \cong S(-\alpha)$ with α the *adjunction number*, then R is a Gorenstein ring of canonical weight $\kappa_R = \alpha - \sum a_i$; for my purposes, one can take this to be the definition of Gorenstein.

Duality makes the resolution (1.1) symmetric: the dual complex $(P_\bullet)^\vee = \mathrm{Hom}_S(P_\bullet, P_c)$ resolves the dualising module $\omega_R = \mathrm{Ext}^c_S(R, \omega_S)$, which is isomorphic to R (or, as a graded module, to $R(\kappa_R)$ with $\kappa_R = \alpha - \sum a_i$), so that $P_\bullet \cong (P_\bullet)^\vee$. In particular the Betti numbers b_i satisfy the symmetry $b_{c-i} = b_i$, or

$$P_{c-i} = \mathrm{Hom}_S(P_i, P_c) \cong \bigoplus_{j=1}^{b_i} S(-\alpha + d_{ij}), \quad \text{where} \quad P_i = \bigoplus_{j=1}^{b_i} S(-d_{ij}).$$

The Buchsbaum–Eisenbud symmetriser trick [BE2] adds precision to this (this is where the assumption $\frac{1}{2} \in S$ comes into play):

There is a symmetric perfect pairing $S^2(P_\bullet) \to P_c$ inducing the duality $P_\bullet \cong (P_\bullet)^\vee$.

The idea is to pass from $P_\bullet$ as a resolution of R to the complex $P_\bullet \otimes P_\bullet$ (the total complex of the double complex) as a resolution of $R \otimes_S R$ (left derived tensor product), then to replace $P_\bullet \otimes P_\bullet$ by its symmetrised version $S^2(P_\bullet)$. In the double complex $P_\bullet \otimes P_\bullet$, one decorates the arrows by signs ± 1 to make each rectangle anticommute (to get $d^2 = 0$). The symmetrised complex $S^2(P_\bullet)$ then involves replacing the arrows by half the sum or differences of symmetrically placed arrows. (This provides lots of opportunities for confusion about signs!)

For details, see [BE2]. The conclusion is that $P_\bullet$ has a $\pm$-symmetric bilinear form that induces perfect pairings $P_i \otimes P_{c-i} \to P_c = S$ for each i, compatible with the differentials.

The Buchsbaum–Eisenbud structure theorem in codimension 3 is a simple consequence of this symmetry, and a model for what I try to do in this paper. Namely, in codimension 3 we have

$$0 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow 0, \tag{1.2}$$

with $P_0 = S$, $P_3 \cong S$, $P_2 = \mathrm{Hom}(P_1, P_3) \cong P_1^\vee$, and the matrix M defining the map $P_1 \leftarrow P_2$ is skew (that is, antisymmetric). If I set $P_1 = nS$ then the respective ranks of the differentials in (1.2) are 1, $n-1$ and 1; since M is skew, his rank must be even, so that $n = 2\nu + 1$. Moreover, the kernel and cokernel are given by the Pfaffians of M, by the skew version of Cramer's rule.

Generalising the Buchsbaum–Eisenbud Theorem to codimension 4 has been a notoriously elusive problem since the 1970s.

1.2. Main aim

This paper starts by describing the shape of the resolution of a codimension 4 Gorenstein ring by analogy with (1.2). The first syzygy matrix $M_1\colon P_1 \leftarrow P_2$ is a $(k+1)\times 2k$ matrix whose $k+1$ rows generically span a maximal isotropic space of the symmetric quadratic form on P_2. The ideal I_R is generated by the entries of the map $L\colon P_0 \leftarrow P_1$, which is determined by the linear algebra of quadratic forms as the linear relation that must hold between the $k+1$ rows of M_1.

This is all uncomplicated stuff, deduced directly from the symmetry trick of [BE2]. It leads to the definition of the Spin-Hom varieties SpH_k in the space of $(k+1)\times 2k$ matrixes (see Section 2.4). The first syzygy matrix M_1 is then an S-valued point of SpH_k, or a morphism $\alpha\colon \operatorname{Spec} S \to \mathrm{SpH}_k$.

The converse is more subtle, and is the main point of the paper. By construction, SpH_k supports a short complex $\mathcal{P}_1 \leftarrow \mathcal{P}_2 \leftarrow \mathcal{P}_3$ of free modules with a certain universal property. If we were allowed to restrict to a smooth open subscheme S^0 of SpH_k meeting the degeneracy locus $\mathrm{SpH}_k^{\mathrm{dgn}}$ in codimension 4, the reflexive hull of the cokernel of M_1 and the kernel of M_2 would provide a complex $\mathcal{P}_\bullet$ that resolves a sheaf of Gorenstein codimension 4 ideals in S^0. (This follows by the main proof below).

Unfortunately, this is only an adequate description of codimension 4 Gorenstein ideals in the uninteresting case of complete intersection ideals. Any other case necessarily involves smaller strata of SpH_k, where SpH_k is singular. Thus to cover every codimension 4 Gorenstein ring, I am forced into the logically subtle situation of a universal construction whose universal space does not itself support the type of object I am trying to classify, namely a Gorenstein codimension 4 ideal. See 4.3 for further discussion of this point.

Main Theorem 2.5 gives the universal construction. To paraphrase: for a polynomial ring S graded in positive degrees, there is a 1-to-1 correspondence between:

(1) Gorenstein codimension 4 graded ideals $I \subset S$ and
(2) graded morphisms $\alpha\colon \operatorname{Spec} S \to \mathrm{SpH}_k$ for which $\alpha^{-1}(\mathrm{SpH}_k^{\mathrm{dgn}})$ has codimension ≥ 4 in $\operatorname{Spec} S$.

I should say at once that this is intended as a theoretical structure result. It has the glaring weakness that it does not so far make any tractable predictions even in model cases (see 4.7 for a discussion). But it is possibly better than no structure result at all.

1.3. Contents of the paper

Section 2.1 describes the shape of the free resolution and its symmetry, following the above introductory discussion. Section 2.4 defines the Spin-Hom variety $\mathrm{SpH}_k \subset \mathrm{Mat}(k+1, 2k)$, to serve as my universal space. The definition takes the form of a quasihomogeneous space for the complex Lie group $G = \mathrm{GL}(k+1) \times \mathrm{O}(2k)$ or its spin double cover $\mathrm{GL}(k+1) \times \mathrm{Pin}(2k)$. More explicitly, define SpH_k as the closure of the G-orbit $\mathrm{SpH}_k^0 = G \cdot M_0$ of the *typical matrix* $M_0 = \left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right)$ under the given action of $G = \mathrm{GL}(k+1) \times \mathrm{O}(2k)$ on $\mathrm{Mat}(k+1, 2k)$.

The degeneracy locus $\mathrm{SpH}_k^{\mathrm{dgn}}$ is the complement $\mathrm{SpH}_k \setminus \mathrm{SpH}_k^0$. Once these definitions are in place, Section 2.5 states the main theorem, and proves it based on the exactness criterion of [BE1].

The Spin-Hom varieties SpH_k have a rich structure arising from representation theory. A matrix $M_1 \in \mathrm{SpH}_k^0$ can be viewed as an isomorphism between a k-dimensional space in $\mathbf{k}^{k+1}$ and a maximal isotropic space for φ in $\mathbf{k}^{2k}$. This displays SpH_k^0 as a principal $\mathrm{GL}(k)$ bundle over $\mathbb{P}^k \times \mathrm{OGr}(k, 2k)$. Section 3 discusses the properties of the SpH_k in more detail, notably their symmetry under the maximal torus and Weyl group. The spinor and nonspinor sets correspond to the two different spinor components $\mathrm{OGr}(k, 2k)$ and $\mathrm{OGr}'(k, 2k)$ of the maximal isotropic Grassmannian.

I introduce the Cramer-spinor coordinates σ_J in 3.3; the main point is that, for a spinor subset $J \cup J^c$, the $(k+1) \times k$ submatrix of $M_1 \in \mathrm{SpH}_k$ formed by those columns has top wedge factoring as $(L_1, \dots, L_{k+1}) \cdot \sigma_J^2$ where $L\colon P_0 \leftarrow P_1$ is the vector of equations (see Lemma 3.3.2). Ensuring that the appropriate square root σ_J is defined as an element $\sigma_J \in S$ involves the point that, whereas the spinor bundle defines a 2-torsion Weil divisor on the affine orthogonal Grassmannian $a\,\mathrm{OGr}(k, 2k) \subset \bigwedge^k \mathbf{k}^{2k}$ (the affine cone over $\mathrm{OGr}(k, 2k)$ in Plücker space) and on SpH_k, its birational transform under the classifying maps $\alpha\colon \mathrm{Spec}\, S \to \mathrm{SpH}_k$ of Theorem 2.5 is the trivial bundle on $\mathrm{Spec}\, S$.

The spinor coordinates vanish on the degeneracy locus $\mathrm{SpH}_k^{\mathrm{dgn}}$ and define an equivariant morphism $\mathrm{SpH}_k^0 \to \mathbf{k}^{k+1} \otimes \mathbf{k}^{2^{k-1}}$. At the same time, they vanish on the nonspin variety SpH'_k, corresponding to the other component $\mathrm{OGr}'(k, 2k)$ of the Grassmannian of maximal isotropic subspaces; this has nonspinor coordinates, that vanish on SpH_k. Between them, these give set theoretic equations for SpH_k and its degeneracy locus.

The final Section 4 discusses a number of issues with my construction and some open problems and challenges for the future.

§2. The main result

For a codimension 4 Gorenstein ideal I with $k+1$ generators, the module P_2 of first syzygies is a $2k$ dimensional orthogonal space with a nondegenerate (symmetric) quadratic form φ. The $k+1$ rows of the first syzygy matrix $M_1(R)$ span an isotropic subspace in P_2 with respect to φ. Since the maximal isotropic subspaces are k-dimensional, this implies a linear dependence relation $(L_1, \dots, L_{k+1})$ that bases coker M_1 and thus provides the generators of I. A first draft of this idea was sketched in [Ki], 10.2.

2.1. The free resolution

Let $S = \mathbf{k}[x_1, \dots, x_N]$ be the polynomial ring over an algebraically closed field $\mathbf{k}$ of characteristic $\neq 2$, graded in positive degrees. Let I_R be a homogeneous ideal with quotient $R = S/I_R$ that is Gorenstein of codimension 4; equivalently, I_R defines a codimension 4 Gorenstein graded subscheme

$$V(I_R) = \operatorname{Spec} R \subset \mathbb{A}^N_{\mathbf{k}} = \operatorname{Spec} S.$$

Suppose that I_R has $k+1$ generators $L_1, \dots, L_{k+1}$. It follows from the Auslander–Buchsbaum form of the Hilbert syzygies theorem and the symmetriser trick of Buchsbaum–Eisenbud [BE2] that the free resolution of R is

$$0 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow P_4 \leftarrow 0, \tag{2.1}$$

where $P_0 = S$, $P_4 \cong S$, $P_3 = \operatorname{Hom}(P_1, P_4) \cong P_1^\vee$; and moreover, P_2 has a nondegenerate *symmetric* bilinear form $\varphi\colon S^2P_2 \to P_4$ compatible with the complex $P_\bullet$, so that $P_2 \to P_1$ is dual to $P_3 \to P_2$ under φ. The simple cases of 2.3, Examples 2.1–2.3 give a sanity check (just in case you are sceptical about the symmetry of φ).

A choice of basis of P_2 gives φ the standard block form[1] $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Then the first syzygy matrix in (2.1) is $M_1(R) = (A\,B)$, where the two blocks are $(k+1) \times k$ matrixes satisfying

$$(A\,B)\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) {}^t(A\,B) = 0, \tag{2.2}$$

that is, $A\,{}^tB + B\,{}^tA = 0$, or $A\,{}^tB$ is *skew*. I call this a $(k+1)\times 2k$ resolution (meaning that the defining ideal I_R has $k+1$ generators yoked by $2k$ first syzygies).

The number of equations in (2.2) is $\binom{k+2}{2}$. For example, in the typical case $k = 8$, the variety defined by (2.2) involves $\binom{k+2}{2} = 45$

[1] In the graded case this is trivial because φ is homogeneous of degree 0, so is basically a nondegenerate quadratic form on a vector space V_2 with $P_2 = V_2 \otimes S$. See the discussion in 4.5 for the more general case.

quadratic equations in $2k(k+1) = 144$ variables. The scheme V_k defined by (2.2) appears in the literature as the *variety of complexes.* However it is not really the right object – it breaks into 2 irreducible components for spinor reasons, and it is better to study just one, which is my SpH_k.

2.2. The general fibre

Let $\xi \in \operatorname{Spec} S = \mathbb{A}^N$ be a point outside $V(I_R) = \operatorname{Spec} R$ with residue field $K = \mathbf{k}(\xi)$ (for example, a $\mathbf{k}$-valued point, with $K = \mathbf{k}$, or the generic point, with $K = \operatorname{Frac} S$). Evaluating (2.1) at ξ gives the exact sequence of vector spaces

$$0 \leftarrow V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow V_3 \leftarrow V_4 \leftarrow 0 \tag{2.3}$$

over K, where $V_0 = K$, $V_4 \cong K$, $V_1 = (k+1)K$, $V_3 = \operatorname{Hom}(V_1, V_4) \cong V_1^\vee$, and $V_2 = 2kK$ with the nondegenerate quadratic form $\varphi = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Over K, the maps in (2.3) can be written as the matrixes

$$\begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I_k & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2.4}$$

This data determines a fibre bundle over $\mathbb{A}^N \setminus V(I_R)$ with the exact complex (2.3) as fibre, and structure group the orthogonal group of the complex, which I take to be $\mathrm{GL}(k+1) \times \mathrm{O}(2k)$ or its double cover $\mathrm{GL}(k+1) \times \mathrm{Pin}(2k)$.

2.3. Simple examples

Example 2.1. A codimension 4 complete intersection R has $L = (x_1, x_2, x_3, x_4)$ and Koszul syzygy matrix

$$(A\,B) = \begin{pmatrix} -x_4 & \cdot & \cdot & \cdot & x_3 & -x_2 \\ \cdot & -x_4 & \cdot & -x_3 & \cdot & x_1 \\ \cdot & \cdot & -x_4 & x_2 & -x_1 & \cdot \\ x_1 & x_2 & x_3 & \cdot & \cdot & \cdot \end{pmatrix}. \tag{2.5}$$

In this choice, $A = M_{1,2,3}$ has rank 3 and $\bigwedge^3 A = x_4^2 \cdot (x_1, \dots, x_4)$. See 3.3 for spinors. A spinor subset $J \cup J^c$ has an odd number i of columns from A and the complementary $3 - i$ columns from B. For example, columns $1, 5, 6$ give a 4×3 matrix with $\bigwedge^3 M_{1,5,6} = x_1^2 \cdot (x_1, x_2, x_3, x_4)$.

Example 2.2. Another easy case is that of a hypersurface section $h = 0$ in a codimension 3 ideal given by the Pfaffians Pf_i of a $(2l+1) \times (2l+1)$ skew matrix M. The syzygy matrix is

$$(A\,B) = \begin{pmatrix} -hI_{2l+1} & M \\ \mathrm{Pf}_1 \dots \mathrm{Pf}_{2l+1} & 0 \dots 0 \end{pmatrix}. \tag{2.6}$$

One sees that a spinor σ_J corresponding to $2l+1-2i$ columns from A and a complementary $2i$ from B is of the form h^{l-i} times a diagonal $2i \times 2i$ Pfaffian of M. Thus the top wedge of the left-hand block A of (2.6) equals $\sigma^2 \cdot (h, \mathrm{Pf}_1, \dots, \mathrm{Pf}_{2l+1})$ where $\sigma = h^l$.

Example 2.3. The extrasymmetric matrix

$$M = \begin{pmatrix} a & b & d & e & f \\ & c & e & g & h \\ & & f & h & i \\ & & & -\lambda a & -\lambda b \\ & & & & -\lambda c \end{pmatrix} \tag{2.7}$$

with a single multiplier λ is the simplest case of a Tom unprojection (see [TJ], Section 9 for details). Let I be the ideal generated by the 4×4 Pfaffians of M. The diagonal entries d, g, i of the 3×3 symmetric top right block are all unprojection variables; thus i appears linearly in 4 equations of the form $i \cdot (a, d, e, g) = \cdots$, and eliminating it projects to the codimension 3 Gorenstein ring defined by the Pfaffians of the top left 5×5 block.

If $\lambda \in S$ is a perfect square, I is the ideal of $\mathrm{Segre}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$ up to a coordinate change, but the Galois symmetry $\sqrt{\lambda} \mapsto -\sqrt{\lambda}$ swaps the two factors. See [TJ], Section 9 for more details, and for several more families of examples; in any of these cases, writing out the resolution matrixes $(A\,B)$ with the stated isotropy property makes a demanding but rewarding exercise for the dedicated student.

By extrasymmetry, out of the 15 entries of M, 9 are independent and 6 repeats. His 4×4 Pfaffians follow a similar pattern. I write the 9 generators of the ideal I of Pfaffians as the vector $L =$

$$\begin{aligned}[\lambda ac + eh - fg,\ -\lambda ab - dh + ef,\ \lambda a^2 + dg - e^2,\\ ah - bg + ce,\ -af + be - cd,\ \lambda b^2 + di - f^2,\\ \lambda bc + ei - fh,\ \lambda c^2 + gi - h^2,\ ai - bh + cf]\end{aligned}$$

Its matrix of first syzygies M_1 is the transpose of

$$
(2.8)\qquad
\begin{array}{ccccccccc}
. & a & b & d & e & . & . & . & . \\
-a & . & c & e & g & . & . & . & . \\
-b & -c & . & f & h & . & . & . & . \\
-d & -e & -f & . & -\lambda a & . & . & . & . \\
-e & -g & -h & \lambda a & . & . & . & . & . \\[6pt]
-h & . & . & \lambda c & . & . & g & -e & . \\
f & -h & . & -\lambda b & \lambda c & -g & . & d & . \\
. & f & . & . & -\lambda b & e & -d & . & . \\[6pt]
i & . & . & . & . & . & -h & f & -\lambda c \\
. & i & . & . & . & h & -f & . & \lambda b \\
. & h & i & . & -\lambda c & . & e & -d & -\lambda a \\
. & . & . & i & . & . & -c & b & -h \\
. & . & . & . & i & c & -b & . & f \\[6pt]
. & -b & . & . & f & -a & . & . & d \\
. & -c & . & . & h & . & -a & . & e \\
c & . & . & -h & . & . & . & -a & g
\end{array}
$$

M_1 is of block form $(A\,B)$ with two 9×8 blocks, and one checks that $LM_1 = 0$, and M_1 is isotropic for the standard quadratic form $J = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, so its kernel is $M_2 = \begin{pmatrix} {}^tB \\ {}^tA \end{pmatrix}$. The focus in (2.8) is on i as an unprojection variable, multiplying d, e, g, a. One recognises its Tom_3 matrix as the top 5×5 block, and the Koszul syzygy matrix of d, e, g, a as Submatrix([6, 7, 8, 14, 15, 16], [6, 7, 8, 9]); compare [KM].

For some of the spinors (see Section 3), consider the 8×9 submatrixes formed by 4 out of the first 5 rows of (2.8), and the complementary 4 rows from the last 8. One calculates their maximal minors with a mild effort:

$$
(2.9)\qquad
\begin{aligned}
\textstyle\bigwedge^8 M_{1,2,3,4,13,14,15,16} &= a^2(af - be + cd)^2 \cdot L, \\
\textstyle\bigwedge^8 M_{1,2,3,5,12,14,15,16} &= a^2(ah - bg + ce)^2 \cdot L, \\
\textstyle\bigwedge^8 M_{1,2,4,5,11,14,15,16} &= a^2(-\lambda a^2 - dg + e^2)^2 \cdot L, \\
\textstyle\bigwedge^8 M_{1,3,4,5,10,14,15,16} &= a^2(-\lambda ab - dh + ef)^2 \cdot L, \\
\textstyle\bigwedge^8 M_{2,3,4,5,9,14,15,16} &= a^2(-\lambda ac - eh + fg)^2 \cdot L.
\end{aligned}
$$

The factor a comes from the 3×3 diagonal block at the bottom right, and the varying factors are the 4×4 Pfaffians of the first 5×5 block. Compare 4.4 for a sample Koszul syzygy.

Exercise 2.4. Apply column and isotropic row operations to put the variable f down a main diagonal of B; check that this puts the complementary A in the form of a skew 8×8 matrix and a row of zeros. Hint: order the rows as $15, 16, 12, 11, 6, 2, 1, 5,\ 7, 8, 4, 3, 14, 10, 9, 13$ and the columns as $1, 2, -3, 4, 5, -7, 8, 9, 6$. (See the website for the easy code.) Do the same for either variable e, h, and the same for any of a, b, c (involving the multiplier λ).

Thus the isotropy condition ${}^tMJM$ can be thought of as *many* skew symmetries.

These examples provide useful sanity checks, with everything given by transparent calculations; it is reassuring to be able to verify the symmetry of the bilinear form on P_2 asserted in Proposition 1, the shape of $A\,{}^tB$ in (2.2), which parity of J gives nonzero spinors σ_J, and other minor issues of this nature.

I have written out the matrixes, spinors, Koszul syzygies etc. in a small number of more complicated explicit examples (see the website). It should be possible to treat fairly general Tom and Jerry constructions in the same style, although so far I do not know how to use this to predict anything useful. The motivation for this paper came in large part from continuing attempts to understand Horikawa surfaces and Duncan Dicks' 1988 thesis [Di], [R1].

2.4. Definition of the Spin-Hom variety SpH_k

Define the *Spin-Hom variety* $\mathrm{SpH}_k \subset \mathrm{Mat}(k+1, 2k)$ as the closure under $G = \mathrm{GL}(k+1) \times \mathrm{O}(2k)$ of the orbit of $M^0 = \left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right)$, the second matrix in (2.4). It consists of isotropic homomorphisms $V_1 \leftarrow V_2$, in other words matrixes M_1 whose $k+1$ rows are isotropic and mutually orthogonal vectors in V_2 w.r.t. the quadratic form φ, and span a subspace that is in the given component of maximal isotropic subspaces if it is k-dimensional.

In more detail, write $\mathrm{SpH}_k^0 = G\cdot M^0 \subset \mathrm{Mat}(k+1, 2k)$ for the orbit, SpH_k for its closure, and $\mathrm{SpH}_k^{\mathrm{dgn}} = \mathrm{SpH}_k \setminus \mathrm{SpH}_k^0$ for the *degeneracy locus*, consisting of matrixes of rank $< k$. Section 3 discusses several further properties of SpH_k and its degeneracy locus $\mathrm{SpH}_k^{\mathrm{dgn}}$.

2.5. The Main Theorem

Assume that S is a polynomial ring graded in positive degrees. Let I be a homogeneous ideal defining a codimension 4 Gorenstein subscheme $X = V(I) \subset \mathrm{Spec}\, S$. Then a choice of minimal generators of I (made up of $k+1$ elements, say) and of the first syzygies between these defines a morphism $\alpha\colon \mathrm{Spec}\, S \to \mathrm{SpH}_k$ such that $\alpha^{-1}(\mathrm{SpH}^{\mathrm{dgn}})$ has the same support as X, and hence codimension 4 in $\mathrm{Spec}\, S$.

Conversely, let $\alpha\colon \operatorname{Spec} S \to \mathrm{SpH}_k \subset \mathrm{Mat}(k+1, 2k)$ *be a morphism for which* $\alpha^{-1}(\mathrm{SpH}^{\mathrm{dgn}})$ *has codimension* ≥ 4 *in* $\operatorname{Spec} S$. *Assume that* α *is graded, that is, equivariant for a positively graded action of* $\mathbb{G}_m$ *on* $\mathrm{SpH}_k \subset \mathrm{Mat}(k+1, 2k)$. *Let* $M_1 = (A\,B)$ *be the matrix image of* α *(the matrix entries of* M_1 *or the coordinates of* α *are elements of* S*). Then by construction* M_1 *and* $J\,{}^tM_1$ *define the two middle morphisms of a complex. I assert that this extends to a complex*

$$0 \leftarrow P_0 \xleftarrow{L} P_1 \xleftarrow{M_1} P_2 \xleftarrow{J\,{}^tM_1} P_3 \xleftarrow{{}^tL} P_4 \leftarrow 0. \tag{2.10}$$

in which $P_0, P_4 \cong S$, *the complex is exact except at* P_0, *and the image of* $L = (L_1, \ldots, L_{k+1})$ *generates the ideal of a Gorenstein codimension* 4 *subscheme* $X \subset \operatorname{Spec} S$.

2.6. Proof

The first part follows from what I have already said. The converse follows by a straightforward application of the exactness criterion of [BE1].

The complex $P_\bullet$ of (2.10) comes directly from M_1. Namely, define P_0 as the reflexive hull of $\operatorname{coker}\{M_1 \colon P_1 \leftarrow P_2\}$ (that is, double dual); it has rank 1 because M_1 has generic rank k. A graded reflexive module of rank 1 over a graded regular ring is free (this is the same as saying that a Weil divisor on a nonsingular variety is Cartier), so $P_0 \cong S$. Given $P_3 \cong P_1^\vee$, the generically surjective map $S \cong P_0 \leftarrow P_1$ is dual to an inclusion $S \hookrightarrow P_3$ that maps to the kernel of $P_2 \leftarrow P_3$.

The key point is to prove exactness of the complex

$$P_0 \xleftarrow{\varphi_1} P_1 \xleftarrow{\varphi_2} P_2 \xleftarrow{\varphi_3} P_3 \xleftarrow{\varphi_4} P_4 \leftarrow 0,$$

where I write $\varphi_1 = (L_1, \ldots, L_{k+1})$, $\varphi_2 = M_1$, etc. to agree with [BE1]. The modules and homomorphisms $P_0, \varphi_1, P_1, \varphi_2, P_2, \varphi_3, P_3, \varphi_4, P_4$ of this complex have respective ranks $1, 1, k+1, k, 2k, k, k+1, 1, 1$, which accords with an exact sequence of vector spaces, as in (2.3–2.4); this is Part (1) of the criterion of [BE1], Theorem 1.

The second condition Part (2) requires the matrixes of φ_i to have maximal nonzero minors generating an ideal $I(\varphi_i)$ that contains a regular sequence of length i. However, $P_\bullet$ is exact outside the degeneracy locus, that is, at any point $\xi \in \operatorname{Spec} S$ for which $\alpha(\xi) \notin \mathrm{SpH}_k^{\mathrm{dgn}}$, and by assumption, the locus of such points has codimension ≥ 4. Thus the maximal minors of each φ_i generate an ideal defining a subscheme of codimension ≥ 4. In a Cohen–Macaulay ring, an ideal defining a subscheme of codimension $\geq i$ has height $\geq i$. Q.E.D.

§3. Properties of SpH_k and its spinors

This section introduces the *spinors* as sections of the spinor line bundle $\mathcal{S}$ on SpH_k. The *nonspinors* vanish on SpH_k and cut it out in V_k set theoretically. The spinors vanish on the *other component* SpH'_k and cut out set theoretically the degeneracy locus $\mathrm{SpH}_k^{\mathrm{dgn}}$ in SpH_k.

The easy bit is to say that a spinor is the square root of a determinant on $V_k \subset \mathrm{Mat}(k+1, 2k)$ that vanishes to even order on a divisor of SpH_k because it is locally the square of a Pfaffian. The ratio of two spinors is a rational function on SpH_k.

The tricky point is that the spinors are sections of the spinor bundle $\mathcal{S}$ on SpH_k that is defined as a $\mathrm{Pin}(2k)$ equivariant bundle, so not described by any particularly straightforward linear or multilinear algebra. As everyone knows, the spinor bundle $\mathcal{S}$ on $\mathrm{OGr}(k, 2k)$ is the ample generator of $\mathrm{Pic}(\mathrm{OGr}(k, 2k))$, with the property that $\mathcal{S}^{\otimes 2}$ is the restriction of the Plücker bundle $\mathcal{O}(1)$ on $\mathrm{Gr}(k, 2k)$. On the affine orthogonal Grassmannian in Plücker space $a\,\mathrm{Gr}(k, 2k) \subset \bigwedge^k \mathbf{k}^{2k}$, it corresponds to a 2-torsion Weil divisor class. I write out a transparent treatment of the first example in 3.2.

I need to argue that the spinors pulled back to my regular ambient $\mathrm{Spec}\, S$ by the appropriate birational transform are elements of S (that is, polynomials), rather than just sections of a spinor line bundle. The reason that I expect to be able to do this is because I have done many calculations like the Tom unprojection of 2.3, Example 2.3, and it always works. In the final analysis, I win for the banal reason that the ambient space $\mathrm{Spec}\, S$ has no 2-torsion Weil divisors in its class group (because S is factorial), so that the birational transform of the spinor bundle $\mathcal{S}$ to $\mathrm{Spec}\, S = \mathbb{A}^N$ is trivial.

The Cramer-spinor coordinates of the syzygy matrix $M_1 = (A\,B)$ have the potential to clarify many points about Gorenstein codimension 4: the generic rank of M_1 is k, but it drops to $k-3$ on $\mathrm{Spec}\, R$; its $k \times k$ minors are in I_R^3. There also seems to be a possible explanation of the difference seen in examples between k even and odd in terms of the well known differences between the Weyl groups D_k (compare 3.1.3).

3.1. Symmetry

View $\mathrm{GL}(k+1)$ as acting on the first syzygy matrix $M_1(R)$ by row operations, and $\mathrm{O}(2k)$ as column operations preserving the orthogonal structure φ, or the matrix $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. The maximal torus $\mathbb{G}_m^{k+1}$ and Weyl group $A_k = S_{k+1}$ of the first factor $\mathrm{GL}(k+1)$ act in the obvious way by scaling and permuting the rows of M_1.

I need some standard notions for the symmetry of $\mathrm{O}(2k)$ and its spinors. For further details, see Fulton and Harris [FH], esp. Chapter 20 and [CR], Section 4. Write $V_2 = \mathbf{k}^{2k}$ for the $2k$ dimensional vector space with basis $e_1, \dots, e_k$ and dual basis $f_1, \dots, f_k$, making the quadratic form $\varphi = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Write $U = U^k = \langle e_1, \dots, e_k \rangle$, so that $V_2 = U \oplus U^\vee$. The orthogonal Grassmannian $\mathrm{OGr}(k, 2k)$ is defined as the variety of k-dimensional isotropic subspaces that intersect U in even codimension, that is, in a subspace of dimension $\equiv k$ modulo 2.

3.1.1. *The D_k symmetry of* $\mathrm{OGr}(k, 2k)$ *and* SpH_k I describe the D_k Weyl group symmetry of the columns in this notation (compare [CR], Section 4). The maximal torus $\mathbb{G}_m^k$ of $\mathrm{O}(2k)$ multiplies e_i by λ_i and f_i by λ_i^{-1}, and acts likewise on the columns of $M_1 = (A\,B)$. The Weyl group D_k acts on the e_i, f_i and on the columns of $M_1 = (A\,B)$ by permutations, as follows: the subgroup S_k permutes the e_i simultaneously with the f_i; and the rest of D_k swaps *evenly many* of the e_i with their corresponding f_i, thus taking $U = \langle e_1, \dots, e_k \rangle$ to another coordinate k-plane in $\mathrm{OGr}(k, 2k)$. Exercise: The younger reader may enjoy checking that the $k-1$ permutations $s_i = (i, i+1) = (e_i e_{i+1})(f_i f_{i+1})$ together with $s_k = (e_k f_{k+1})(e_{k+1} f_k)$ are involutions satisfying the standard Coxeter relations of type D_k, especially $(s_{k-1} s_k)^2 = 1$ and $(s_{k-2} s_k)^3 = 1$.

3.1.2. *Spinor and nonspinor subsets* The spinor sets $J \cup J^c$ index the spinors σ_J (introduced in 3.3). Let $\{e_i, f_i\}$ be the standard basis of $\mathbf{k}^{2k}$ with form $\varphi = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. There are 2^k choices of maximal isotropic subspaces of $\mathbf{k}^{2k}$ based by a subset of this basis; each is based by a subset J of $\{e_1, \dots, e_k\}$ together with the complementary subset J^c of $\{f_1, \dots, f_k\}$. The spinor subsets are those for which $\#J$ has the same parity as k, or in other words, the complement $\#J^c$ is even; the nonspinor subsets are those for which $\#J$ has the parity of $k-1$. The spinor set indexes a basis σ_J of the spinor space of $\mathrm{OGr}(k, 2k)$, and similarly, the nonspinor set indexes the nonspinors $\sigma'_{J'}$ of his dark twin $\mathrm{OGr}'(k, 2k)$.

The standard affine piece of $\mathrm{OGr}(k, 2k)$ consists of k-dimensional spaces based by k vectors that one writes as a matrix $(I\,A)$ with A a skew $k \times k$ matrix. The spinor coordinates of $(I\,A)$ are the $2i \times 2i$ diagonal Pfaffians of A for $0 \le i \le [\frac{k}{2}]$. They correspond in an obvious way to the spinor sets just defined and *they are the spinors* apart from the quibble about taking an overall square root and what bundle they belong to.

3.1.3. *Even versus odd* The distinction between k even or odd is crucial for anything to do with $\mathrm{O}(2k)$, D_k, spinors, Clifford algebras, etc. The spinor and nonspinor sets correspond to taking a subset J of $\{e_1, \dots, e_k\}$ and the complementary set J^c of $\{f_1, \dots, f_k\}$. The 2^k

choices correspond to the vertices of a k-cube. When k is even this is a bipartite graph; the spinors and nonspinors form the two parts. By contrast, for odd k, both spinors and nonspinors are indexed by the vertices of the k-cube divided by the antipodal involution ([CR], Section 4 writes out the case $k = 5$ in detail).

For simplicity, I assume that k is even in most of what follows; the common case in applications that I really care about is $k = 8$. Then $J = \emptyset$ and $J^c = \{1, \dots, k\}$ is a spinor set, and the affine pieces represented by $(I\,X)$ and $(Y\,I)$ (with skew X or Y) are in the same component of $\mathrm{OGr}(k, 2k)$. The odd case involves related tricks, but with some notable differences of detail (compare [CR], Section 4).

3.1.4. *The other component* OGr' *and* SpH'_k I write $\mathrm{OGr}'(k, 2k)$ for the *other component* of the maximal isotropic Grassmannian, consisting of subspaces meeting U in odd codimension. Swapping *oddly many* of the e_i and f_i interchanges OGr and OGr′. Likewise, SpH'_k is the closure of the G-orbit of the matrix M'_0 obtained by interchanging one corresponding pair of columns of M_0.

Claim 3.1. *Write* V_k *for the scheme defined by* (2.2) (*that is, the "variety of complexes"*). *It has two irreducible components* $V_k = \mathrm{SpH}_k \cup \mathrm{SpH}'_k$ *containing matrixes of maximal rank* k. *The two components are generically reduced and intersect in the degenerate locus* $\mathrm{SpH}_k^{\mathrm{dgn}}$. (*But one expects* V_k *to have embedded primes at its smaller strata, as in the discussion around* (3.5).)

This follows from the properties of spinor minors Δ_J discussed in Exercise 3.2.1: the Δ_J are $k \times k$ minors defined as polynomials on V_k, and vanish on SpH'_k but are nonzero on a dense open subset of SpH_k.

3.2. A first introduction to $\mathrm{OGr}(k, 2k)$ and its spinors

The lines on the quadric surface provide the simplest calculation, and already have lots to teach us about $\mathrm{OGr}(2, 4)$ and $\mathrm{OGr}(k, 2k)$: the conditions for the 2×4 matrix

$$N = \begin{pmatrix} a & b & x & y \\ c & d & z & t \end{pmatrix} \tag{3.1}$$

to be isotropic for $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ are

$$ax + by = 0, \quad az + bt + cx + dy = 0, \quad cz + dt = 0. \tag{3.2}$$

Three equations (3.2) generate an ideal I_W defining a codimension 3 complete intersection $W \subset \mathbb{A}^8$ that breaks up into two components $\Sigma \sqcup \Sigma'$, corresponding to the two pencils of lines on the quadric surface:

the two affine pieces of $\mathrm{OGr}(2,4)$ that consist of matrixes row equivalent to $(I\,A)$ or $(A\,I)$, with A a skew matrix, have one of the *spinor minors* $\Delta_1 = ad - bc$ or $\Delta_2 = xt - yz$ nonzero, and

$$(3.3) \qquad dx - bz = at - cy = 0 \quad \text{and} \quad dy - bt = -(az - cx)$$

on them. This follows because all the products of Δ_1, Δ_2 with the *non-spinors minors* $dx - bz, at - cy$ are in I_W, as one checks readily. Thus if $\Delta_1 \neq 0$ (say), I can multiply by the adjoint of the first block to get

$$(3.4) \qquad \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b & x & y \\ c & d & z & t \end{pmatrix} = \begin{pmatrix} \Delta_1 & 0 & dx - bz & dy - bt \\ 0 & \Delta_1 & az - cx & at - cy \end{pmatrix}$$

where the second block is skew. Note that

$$(3.5) \qquad \Delta_1 \cdot \left(\Delta_1\Delta_2 - (az - cx)^2\right) \in I_W.$$

If $\Delta_1 \neq 0$, the relations (3.2) imply that we are in Σ. The ideal of Σ is obtained from (3.2) allowing cancellation of Δ_1; in other words $I_\Sigma = [I_W : \Delta_1]$ is the colon ideal with either of the spinor minors Δ_1 or Δ_2.

The second block in (3.4) is only skew mod I_W after cancelling one of $a, b, \dots, t$; similarly $\Delta_1\Delta_2 - (az - cx)^2 \notin I_W$, so that (3.5) involves cancelling Δ_1. Thus a geometric description of $\Sigma, \Sigma' \subset \mathrm{Mat}(k, 2k)$ should usually lead to ideals with embedded primes at their intersection or its smaller strata.

Now by relation (3.5), the Plücker embedding takes $\mathrm{OGr}(2,4)$ to the conic $XZ = Y^2$, with $X = \Delta_1 = ad - bc$, $Y = az - cx$, $Z = \Delta_2 = xt - yz$. This is $(\mathbb{P}^1, \mathcal{O}(2))$ parametrised by u^2, uv, v^2 where u, v base $H^0(\mathbb{P}^1, \mathcal{O}(1))$. Thus $X = u^2$, $Y = uv$ and $Z = v^2$ on $\mathrm{OGr}(2,4)$; the spinors are u and v. The ratio $u : v$ equals $X : Y = Y : Z$. Each of Δ_1 and Δ_2 vanishes on a double divisor, but the quantities $u = \sqrt{\Delta_1}$, $v = \sqrt{\Delta_2}$ are not themselves polynomial.

The conclusion is that the minors Δ_1 and Δ_2 are *spinor squares*, that is, squares of sections u, v of a line bundle $\mathcal{S}$, the spinor bundle on $\mathrm{OGr}(2,4)$. If we view $\mathrm{OGr}(2,4)$ as a subvariety of $\mathrm{Gr}(2,4)$, only $\mathcal{S}^{\otimes 2}$ extends to the Plücker line bundle $\mathcal{O}(1)$. Embedding $\mathrm{OGr}(2,4)$ in the Plücker space $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$ and taking the affine cone gives the affine spinor variety $a\,\mathrm{OGr}(2,4)$ as the cone over the conic, and $\mathcal{S}$ with its sections u, v as the ruling.

In fact $a\,\mathrm{OGr}(2,4)$ and his dark twin $a\,\mathrm{OGr}'$ are two ordinary quadric cones in linearly disjoint vector subspaces of the Plücker space $\bigwedge^2 \mathbb{C}^4$, and the spinor bundle on the union has a divisor class that is a 2-torsion Weil divisor on each component. This picture is of course the orbifold quotient of ± 1 acting on two planes $\mathbb{A}^2$ meeting transversally in $\mathbb{A}^4$.

3.2.1. *Exercise* Generalise the above baby calculation to the subvariety $W_k \subset \mathrm{Mat}(k, 2k)$ of matrixes $(A\,X)$ whose k rows span an isotropic space for $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, or in equations, the $k \times k$ product $A\,{}^tX$ is skew. Assume k is even.

(1) $W_k \subset \mathrm{Mat}(k, 2k)$ is a complete intersection subvariety of codimension $\binom{k+1}{2}$. [Hint: Just a dimension count.]

(2) W_k breaks up into two irreducible components $\Sigma \cup \Sigma'$, where Σ contains the space spanned by $(I\,X)$ with X skew, or more generally, by the span of the columns $J \cup J^c$ for J a spinor set; its nondegenerate points form a principal $\mathrm{GL}(k)$ bundle over the two components $\mathrm{OGr} \sqcup \mathrm{OGr}'$ of the maximal isotropic Grassmannian.

(3) For J a spinor set, the $k \times k$ spinor minor Δ_J of $(A\,X)$ (the determinant of the submatrix formed by the columns $J \cup J^c$) is a polynomial on $\mathrm{Mat}(k \times 2k)$ that vanishes on Σ', and vanishes along a double divisor of Σ, that is, twice a prime Weil divisor D_J.

(4) The Weil divisors D_{J_1} and D_{J_2} corresponding to two spinor sets J_1 and J_2 are linearly equivalent. [Hint: First suppose that J_1 is obtained from J by exactly two transpositions, say $(e_1 f_2)(e_2 f_1)$, and argue as in (3.5) to prove that $\sigma_J \sigma_{J_1}$ restricted to Σ is the square of either minor obtained by just one of the transpositions.]

3.2.2. *Spinors on* $\mathrm{OGr}(k, 2k)$ The orthogonal Grassmann variety $\mathrm{OGr}(k, 2k)$ has a *spinor* embedding into $\mathbb{P}(\mathbf{k}^{2^{k-1}})$, of which the usual Plücker embedding

$$\mathrm{OGr}(k, 2k) \subset \mathrm{Gr}(k, 2k) \hookrightarrow \mathbb{P}\Big(\bigwedge^k \mathbf{k}^{2k}\Big)$$

is the Veronese square. The space of spinors $\mathbf{k}^{2^{k-1}}$ is a representation of the spin double cover $\mathrm{Pin}(2k) \to \mathrm{O}(2k)$.

A point $W \in \mathrm{OGr}(k, 2k)$ is a k-dimensional subspace $W^k \subset \mathbf{k}^{2k}$ isotropic for $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ and intersecting $U = \langle e_1, \dots, e_n \rangle$ in even codimension. I can write a basis as the rows of a $k \times 2k$ matrix N_W. If I view W as a point of $\mathrm{Gr}(k, 2k)$, its Plücker coordinates are all the $k \times k$ minors of N_W. There are $\binom{2k}{k}$ of these (that is, 12870 if $k = 8$), a fraction of which vanish $\mathrm{OGr}(k, 2k)$, as the determinant of a skew matrix of odd size.

The finer embedding of $\mathrm{OGr}(k, 2k)$ is by spinors. The spinors σ_J are sections of the spinor line bundle $\mathcal{S}$, 2^{k-1} of them (which is 128 if $k = 8$, about 1/100 of the number of Plücker minors). Each comes by taking a $k \times k$ submatrix formed by a spinor subset of columns of N_W

(in other words, restricting to an isotropic coordinate subspace of $\mathbf{k}^{2k}$ in the specified component $\operatorname{OGr}(k,2k)$), taking its $2\kappa\times 2\kappa$ minor (where $\kappa=\left[\frac{k}{2}\right]$) and factoring it as the perfect square of a section of $\mathcal{S}$. The only general reason for a $2\kappa\times 2\kappa$ minor to be a perfect square is that the submatrix is skew in some basis; in fact, as in (3.4), after taking one fixed square root of a determinant, and making a change of basis, the maximal isotropic space can be written as $(I\,X)$ with X skew, and the spinors are all the Pfaffians of X.

3.3. Cramer-spinor coordinates on SpH_k

3.3.1. *Geometric interpretation* A point of the open orbit $\mathrm{SpH}_k^0\subset\mathrm{SpH}_k$ is a matrix M of rank k; it defines an isomorphism from a k-dimensional subspace of V_1 (the column span of M) to its row span, a maximal isotropic subspace of V_2 in the specified component $\operatorname{OGr}(k,2k)$. Therefore the nondegenerate orbit $\mathrm{SpH}_k^0\subset\mathrm{SpH}_k$ has a morphism to $\mathbb{P}(V_1^\vee)\times\operatorname{OGr}(k,2k)$ that makes it a principal $\mathrm{GL}(k)$ bundle. The product $\mathbb{P}(V_1^\vee)\times\operatorname{OGr}(k,2k)$ is a projective homogeneous space under $G=\mathrm{GL}(k+1)\times\mathrm{Pin}(2k)$

It embeds naturally in the projectivisation of $\mathbf{k}^{k+1}\otimes\mathbf{k}^{2^{k-1}}$, with the second factor the space of spinors. This is the representation of G with highest weight vector $v=(0,\dots,0,1)\otimes(1,0,\dots,0)$. The composite

$$\mathrm{SpH}_k^0\to\mathbb{P}(V_1^\vee)\times\operatorname{OGr}(k,2k)\hookrightarrow\mathbb{P}(\mathbf{k}^{k+1}\otimes\mathbf{k}^{2^{k-1}})\tag{3.6}$$

takes the typical matrix M_0 (or equivalently, the complex (2.4)) to v.

The Cramer-spinor coordinates of $\alpha\in\mathrm{SpH}_k(S)$ are the bihomogeneous coordinates under the composite map (3.6).

3.3.2. *Spinors as polynomials* The spinors σ_J occur naturally as sections of the spinor line bundle $\mathcal{S}$ on $\operatorname{OGr}(k,2k)$, and so have well defined pullbacks to SpH_k^0 or to any scheme T with a morphism $\alpha\colon T\to\mathrm{SpH}_k^0$. For σ_J to be well defined in $H^0(\mathcal{O}_T)$, the pullback of the spinor line bundle to T must be trivial.

Lemma 3.2. *Let $\alpha\in\operatorname{Mor}(\operatorname{Spec}S,\mathrm{SpH}_k)=\mathrm{SpH}_k(S)$ be a classifying map as in Theorem 2.5 and write $M_1\in\operatorname{Mat}(S,k+1,2k)$ for its matrix (with entries in S). Then for a spinor set $J\cup J^c$ (as in 3.1.2), the $(k+1)\times k$ submatrix N_J of M_1 with columns $J\cup J^c$ has*

$$\bigwedge^k N_J=L\cdot\sigma_J^2,\tag{3.7}$$

where $L=(L_1,\dots,L_{k+1})$ generates the cokernel of M_1, and $\sigma_J\in S$.

3.4. Proof

A classifying map $\alpha \in \mathrm{SpH}_k(S)$ as in Theorem 2.5 restricts to a morphism α from the nondegenerate locus $\operatorname{Spec} S \setminus V(I_R)$ to SpH_k^0; on the complement of $V(I_R)$, the matrix M_1 has rank k, and its kth wedge defines the composite morphism to the product $\mathbb{P}^k \times \mathrm{Gr}(k, 2k)$ in its Segre embedding:

$$\text{(3.8)} \quad \operatorname{Spec} S \setminus V(I_R) \to \mathrm{SpH}_k^0 \to \mathbb{P}^k \times \mathrm{OGr}(k, 2k) \hookrightarrow \mathbb{P}^k \times \mathrm{Gr}(k, 2k) \subset \mathbb{P}\left(\mathbf{k}^{k+1} \otimes \bigwedge^k V^{2k}\right).$$

The entries of $\bigwedge^k N_J$ are $k+1$ coordinates of this morphism, and are of the form $L_i \cdot \sigma_J^2$ already on the level of $\mathbb{P}^k \times \mathrm{OGr}(k, 2k)$.

Note that $\operatorname{Spec} S \setminus V(I_R)$ is the complement in $\operatorname{Spec} S = \mathbb{A}^N$ of a subset of codimension ≥ 4 so has trivial Pic. Each maximal minor of N_J splits as L_i times a polynomial that vanishes on a divisor that is a double (because it is the pullback of the square of a spinor); therefore the polynomial is a perfect square in S. Q.E.D.

The following statement is the remaining basic issue that I am currently unable to settle in general.

Conjecture 3.3. *Under the assumptions of Lemma* 3.3.2, $\sigma_J \in I_R$.

This is clear when R is reduced, that is, I_R is a radical ideal. Indeed if σ_J is a unit at some generic point $\xi \in V(I_R) = \operatorname{Spec} R$, then (3.7) implies that I_R is generated at ξ by the $k \times k$ minors of the $(k+1) \times k$ matrix N_J; these equations define a codimension 2 subscheme of $\operatorname{Spec} S$, which is a contradiction. This case is sufficient for applications to construction of ordinary varieties, but not of course to Artinian subschemes of $\mathbb{A}^4$.

The conjecture also holds under the assumption that I_R is generically a codimension 4 complete intersection. Indeed, the resolution of I_R near any generic point $\xi \in V(I_R)$ is then the 4×6 Koszul resolution of the complete intersection direct sum some nonminimal stuff that just add invertible square matrix blocks. Then both the L_i and the σ_J are locally given by Example 2.1.

At present, the thing that seems to make the conjecture hard is that the definition of the σ_J and the methods currently available for getting formulas for them consists of working on the nondegenerate locus of SpH_k: choose a block diagonal form and take the Pfaffian of a skew complement, ... This is just not applicable at points $\sigma \in V(I_R)$.

The conjecture could possibly be treated by a more direct understanding of the spin morphism $\operatorname{Spec} S \to \mathbf{k}^{2k}$ defined by spinors and nonspinors, not passing via the square root of the Plücker morphism as I do implicitly in Lemma 1 by taking $\bigwedge^k$.

§4. Final remarks, open problems

4.1. Birational structure and dimension of SpH_k

A general $M = (A\,B) \in \mathrm{SpH}_k$ has $k+1$ rows that span a maximal isotropic space $U \in \mathrm{OGr}(k,2k)$ and $2k$ columns that span a k-dimensional vector subspace of $\mathbf{k}^{k+1}$, that I can view as a point of $\mathbb{P}^k$; thus SpH_k^0 is a principal $\mathrm{GL}(k)$ bundle over $\mathbb{P}^k \times \mathrm{OGr}(k,2k)$. In particular, $\dim \mathrm{SpH}_k = k^2 + k + \binom{k}{2} = \frac{3k^2+k}{2}$.

The tangent space to SpH_k at the general point $M_0 = \left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is calculated by writing an infinitely near matrix as $M_0 + \begin{pmatrix} A'_k & B'_k \\ a_{k+1} & b_{k+1} \end{pmatrix}$; here the blocks A'_k and B'_k are $k \times k$ matrixes, and a_{k+1} and b_{k+1} are $1 \times k$ rows. Then the tangent space to V_k defined by $A\,{}^tB = 0$ is the affine subspace obtained by setting B'_k to be skew and $b_{k+1} = 0$. Therefore SpH_k has codimension $\binom{k+1}{2} + k$ and dimension $2k(k+1) - \binom{k+1}{2} - k = \frac{3k^2+k}{2}$.

It is interesting to observe that the set of equations (2.2) express $\mathrm{SpH}_k \cup \mathrm{SpH}'_k$ as an almost complete intersection. Namely, (2.2) is a set of $\binom{k+1}{2}$ equations in $\mathbb{A}^{2k(k+1)}$ vanishing on a variety of dimension $\frac{3k^2+k}{2}$, that is, of codimension $\binom{k+1}{2} - 1$.

4.2. Intermediate rank

The Spin-Hom variety SpH_k certainly contains degenerate matrixes M_1 of rank $k-1$ or $k-2$, but any morphism $\mathrm{Spec}\, S \to \mathrm{SpH}_k$ that hits one of these must hit the degeneracy locus in codimension ≤ 3, so does not correspond to anything I need here. The following claim must be true, but I am not sure where it fits in the logical development.

Claim 4.1. *Every point $P \in \mathrm{SpH}_k$ corresponds to a matrix $M_1 = (A\,B)$ of rank $\le k$. If a morphism $\alpha\colon \mathrm{Spec}\, S \to \mathrm{SpH}_k$ takes ξ to a matrix M_1 of rank $k+1-i$ for $i = 1,2,3,4$ then $\alpha^{-1}(\mathrm{SpH}_k^{\mathrm{dgn}})$ has codimension $\le i$ in a neighbourhood of ξ. In other words, a morphism α that is regular in the sense of my requirement never hits matrixes M_1 of rank intermediate between k and $k-3$; and if α is regular then $\alpha^{-1}(\mathrm{SpH}_k^{\mathrm{dgn}})$ has codimension exactly* 4.

4.3. The degeneracy locus as universal subscheme

The proof in 2.6 doesn't work for SpH_k itself in a neighbourhood of a point of $\mathrm{SpH}_k^{\mathrm{dgn}}$, because taking the reflexive hull, and asserting that P_0 is locally free works only over a regular scheme. Moreover, it is not just the proof that goes wrong. I don't know what happens over the strata of $\mathrm{SpH}_k^{\mathrm{dgn}}$ where M_1 drops rank by only 1 or 2.

We discuss the speculative hope that $\mathrm{SpH}_k^{\mathrm{dgn}} \subset \mathrm{SpH}_k$ has a description as a kind of universal codimension 4 subscheme, with the inclusions enjoying some kind of Gorenstein adjunction properties. But if this is to be possible at all, we must first discard uninteresting components of $\mathrm{SpH}_k^{\mathrm{dgn}}$ corresponding to matrixes of intermediate rank $k-1$ or $k-2$.

It is possible that there is some universal blowup of some big open in SpH_k that supports a Gorenstein codimension 4 subscheme and would be a universal space in a more conventional sense. Or, as the referee suggests, there might be a more basic sense in which appropriate codimension 4 components Γ of the degeneracy locus are *universal Gorenstein embeddings*, meaning that the adjunction calculation $\omega_\Gamma = \mathrm{Ext}^4_{\mathcal{O}_{\mathrm{SpH}}}(\mathcal{O}_\Gamma, \omega_{\mathrm{SpH}})$ for the dualising sheaf is locally free and commutes with regular pullbacks.

4.4. Koszul syzygies

Expressing the generators of I as a function of the entries of the syzygy matrix is essentially given by the map $\bigwedge^2 P_1 \to P_2$ that writes the Koszul syzygies as linear combinations of the minimal syzygies.

The L_i are certainly linear combinations of the entries of M_1. More precisely, since the $2k$ columns of M_1 provide a minimal basis for the syzygies, they cover in particular the Koszul syzygies $L_i \cdot L_j - L_j \cdot L_i \equiv 0$. This means that for every $i \neq j$ there is column vector v_{ij} with entries in S such that $M_1 v_{ij} = (\dots, L_j, \dots, L_i, \dots)$ is the column vector with L_j in the ith place and L_i in the jth and 0 elsewhere. For example, referring to Example 2.3, you might enjoy the little exercise in linear algebra of finding the vector

$$v = (-\lambda c, \lambda b, 0, 0, 0, d, e, g, 0, 0, 0, 0, 0, 0, 0, 0) \quad \text{for which}$$
$$v\,{}^tM_1 = (-\lambda ab - dh + ef, -\lambda ac - eh + fg, 0, 0, 0, 0, 0, 0, 0, 0),$$

where ${}^tM_1$ is the matrix of (2.8), and similarly for 35 other values of i, j.

4.5. More general ambient ring S

I restrict to the case of ideals in a graded polynomial ring over a field of characteristic $\neq 2$ in the belief that progress in this case will surely be followed by the more general case of a regular local ring. Then P_2 is still a free module, with a perfect symmetric bilinear form $S^2(P_2) \to P_4$, with respect to which $P_1 \leftarrow P_2$ is the dual of $P_2 \leftarrow P_3$. This can be put in the form $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ over the residue field $\mathbf{k}_0 = S/m_S$ of S if we assume that $k(S)$ is algebraically closed and contains $\frac{1}{2}$; we can do the same over S

itself if we assume that S is complete (to use Hensel's Lemma). At some point if we feel the need for general regular rings, we can probably live with a perfect quadratic form φ and the dualities it provides, without the need for the normal form $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$.

4.6. More general rings and modules

Beyond the narrow question of Gorenstein codimension 4, one could ask for the structure of any free resolution of an S-module M or S-algebra R. As in 2.2, one can say exactly what the general fibre is, and think of the complex $P_\bullet$ as a fibre bundle over $S \setminus \operatorname{Supp} M$ with some product of linear groups as structure group. If we are doing R-algebras, the complex $P_\bullet$ also has a symmetric bilinear structure, that reduces the structure group. My point is that if we eventually succeed in making some progress with Gorenstein codimension 4 rings, we might hope to also get some ideas about Cohen–Macaulay codimension 3 and Gorenstein codimension 5.

For example, in vague terms, there is a fairly clear strategy how to find a key variety for the resolution complexes of Gorenstein codimension 5 ideals, by analogy with my Main Theorem 2.5. In this case, the resolution has the shape

$$0 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow P_4 \leftarrow P_5 \leftarrow 0, \tag{4.1}$$

with $P_0 = S$, $P_1 = (a+1)S$, $P_2 = (a+b)S$ and $P_3, \dots, P_5$ their duals. The complex is determined by two syzygy matrixes $M_1 \in \operatorname{Mat}(a+1, a+b)$ of generic rank a defining $P_1 \leftarrow P_2$ and a symmetric $(a+b) \times (a+b)$ matrix M_2 of generic rank b defining $P_2 \leftarrow P_3 = P_2^\vee$, constrained by the complex condition $M_1 M_2 = 0$. The "general fibre" is given by the pair $M_1 = \left(\begin{smallmatrix} I_a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $M_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_b \end{smallmatrix}\right)$, the appropriate key variety is its closed orbit under $\mathrm{GL}(a+1) \times \mathrm{GL}(a+b)$. The maximal nonzero minors of M_1 and M_2 define a map to a highest weight orbit in

$$\operatorname{Hom}\Bigl(\bigwedge^a P_2, \bigwedge^a P_1\Bigr) \times \operatorname{Sym}^2\Bigl(\bigwedge^b P_2\Bigr).$$

4.7. Difficulties with applications

I expand what the introduction said about the theory currently not being applicable. We now possess hundreds of constructions of codimension 4 Gorenstein varieties, for example, the Fano 3-folds of [TJ], but their treatment (for example, as Kustin–Miller unprojections) has almost nothing to do with the structure theory developed here. My Main Theorem 2.5 does not as it stands construct anything, because it does not say how to produce morphisms $\alpha\colon \operatorname{Spec} S \to \mathrm{SpH}_k$, or predict their properties. The point that must be understood is not the key variety SpH_k itself, but rather the space of morphisms $\mathrm{Mor}(\operatorname{Spec} S, \mathrm{SpH}_k)$, which may be intractable or infinitely complicated (in the sense of Vakil's Murphy's law [Va]); there are a number of basic questions here that I do not yet understand.

Even given α, we do not really know how to write out the equations $(L_1, \dots, L_{k+1})$, other than by the implicit procedure of taking hcfs of $k \times k$ minors. One hopes for a simple formula for the defining relations L_i as a function of the first syzygy matrix $M_1 = (A\,B)$. Instead, one gets the vector $(L_1, \dots, L_{k+1})$ by taking out the highest common factor from $\bigwedge^k M_I$ for any spinor subset I, asserting that it is a perfect square σ_J^2. The disadvantage is that as it stands this is only implicitly a formula for the L_i.

4.8. Obstructed constructions

One reason that $\mathrm{Mor}(S, \mathrm{SpH}_k)$ is complicated is that the target is big and singular and needs many equations. However, there are also contexts in which S-valued points of much simpler varieties already give families of Gorenstein codimension 4 ideals that are obstructed in interesting ways.

Given a 2×4 matrix $A = \left(\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}\right)$ with entries in a regular ring S, the 6 equations $\bigwedge^2 A = 0$ define a Cohen–Macaulay codimension 3 subvariety $V \subset \operatorname{Spec} S$. An elephant $X \in |-K_V|$ is then a Gorenstein subvariety of codimension 4 with a 9×16 resolution. If we are in the "generic" case with 8 independent indeterminate entries, V is the affine cone over $\mathrm{Segre}(\mathbb{P}^1 \times \mathbb{P}^3)$, and X is a cone over a divisor of bidegree $(k, k+2)$ in $\mathrm{Segre}(\mathbb{P}^1 \times \mathbb{P}^3)$.

Although $X \subset V$ is a divisor, if we are obliged to treat it by equations in the ambient space $\operatorname{Spec} S$, it needs 3 equations in "rolling factors format". The general case of this is contained in Dicks' thesis [Di], [R1]: choose two vectors m_1, m_2, m_3, m_4 and n_1, n_2, n_3, n_4, and assume that the identity

$$\sum a_i n_i \equiv \sum b_i m_i \tag{4.2}$$

holds as an equality in the ambient ring S. Then the 3 equations

$$\sum a_i m_i = \sum b_i m_i \equiv \sum a_i n_i = \sum b_i n_i = 0 \tag{4.3}$$

define a hypersurface $X \subset V$ that is an elephant $X \in |-K_V|$ and thus a Gorenstein subvariety with 9×16 resolution.

The problem in setting up the data defining X is then to find solutions in S of (4.2). In other words, these are S-valued points of the affine quadric cone Q_{16}, or morphisms $\operatorname{Spec} S \to Q_{16}$. How to map a regular ambient space to the quadratic cone Q_{16} is a small foretaste of the more general problem of the classifying map $\operatorname{Spec} S \to \mathrm{SpH}_k$. This case is discussed further in [Ki], Example 10.8, which in particular writes out explicitly the relation between (4.3) and the classifying map $\operatorname{Spec} S \to \mathrm{SpH}_k$ of Theorem 2.5.

There are many quite different families of solutions to this problem, depending on what assumptions we make about the graded ring S, and how general we take the matrix A to be; different solutions have a number of important applications to construction and moduli of algebraic varieties, including my treatment of the Horikawa quintic n-folds.

Another illustration of the phenomenon arises in a recent preprint of Catanese, Liu and Pignatelli [CLP]. Take the 5×5 skew matrix

$$(4.4) \qquad M = \begin{pmatrix} v & u & z_2 & D \\ & z_1 & y & m_{25} \\ & & l & m_{35} \\ & & & m_{45} \end{pmatrix}$$

with entries in a regular ring S_0, and suppose that v, u, z_2, D forms a regular sequence in S. Assume that the identity

$$(4.5) \qquad z_1 m_{45} - y m_{35} + l m_{25} \equiv av + bu + cz_2 + dD$$

holds as an equality in S_0. The identity (4.5) puts the Pfaffian $\mathrm{Pf}_{23.45}$ in the ideal (v, u, z_2, D); the other 4 Pfaffians are in the same ideal for the trivial reason that every term involves one entry from the top row of M.

This is a new way of setting up the data for a Kustin–Miller unprojection: write $Y \subset \operatorname{Spec} S_0$ for the codimension 3 Gorenstein subscheme defined by the Pfaffians of M. It contains the codimension 4 complete intersection $V(v, u, z_2, D)$ as a codimension 1 subscheme, and unprojecting V in Y adjoins an unprojection variable x_2 having 4 linear equations $x_2 \cdot (v, u, z_2, D) = \cdots$, giving a codimension 4 Gorenstein ring with 9×16 resolution.

The problem of how to fix (4.5) as an identity in S_0 is again a question of the S_0-valued points of a quadric cone, this time a quadric Q_{14} of rank 14. [CLP], Proposition 5.13 find two different families of solutions, and exploit this to give a local description of the moduli of their surfaces.

At first sight this looks a bit like a Jerry_{15} unprojection. In fact one of the families of [CLP] (the one with $c_0 = B_x = 0$) can easily be massaged to a conventional Jerry_{15} having a double Jerry structure (compare [TJ], 9.2), but this does not seem possible for the more interesting family in [CLP] with $D_x = (l/c_0)B_x$.

Question Do these theoretical calculations contain the results of [Di], [CLP] and the like?

Answer Absolutely not. They may provide a framework that can produce examples, or simplify and organise the construction of examples. To get complete moduli spaces, it is almost always essential to use other methods, notably infinitesimal deformation calculations or geometric constructions.

Question The fact that S can have various gradings seems to add to the complexity of the space $\mathrm{Mor}(S, \mathrm{SpH}_k)$, doesn't it?

Answer That may not be the right interpretation—we could perhaps think that $\mathrm{Mor}(S, \mathrm{SpH}_k)$ (or even the same just for $\mathrm{Mor}(S, Q_{2k})$ into a quadric of rank $2k \geq 4$) is infinite dimensional and infinitely complicated, so subject to Murphy's law [Va], but that when we cut it down to graded in given degrees, it becomes finitely determined, breaking up into a number of finite dimensional families that may be a bit singular, but can be studied with success in favourable cases.

4.9. Problem session

4.9.1. *Computing project* It is a little project in computer algebra to write an algorithm to put the projective resolution (2.1) in symmetric form. This might just be a straightforward implementation of the Buchsbaum–Eisenbud symmetrised complex $S^2P_\bullet$ outlined in Section 1. Any old computer algebra package can do syzygies, but as far as I know, none knows about the symmetry in the Gorenstein case.

We now have very many substantial working constructions of codimension 4 Gorenstein varieties. We know in principle that the matrix of first syzygies can be written out in the $(A\,B)$ form of (2.8), but as things stand, it takes a few hours or days of pleasurable puzzling to do any particular case.

4.9.2. *Linear subvarieties* What are the linear subvarieties of SpH_k? The linear question may be tractable, and may provide a partial answer to the quest for an explicit structure result.

The Spin-Hom variety SpH_k is defined near a general point by quadratic equations, so its linear subspaces can be studied by the tangent-cone construction by analogy with the linear subspaces of quadrics, Segre products or Grassmannians: the tangent plane T_P at $P \in V$ intersects V in a cone, so that linear subspaces of V through P correspond to linear subspaces in the base of the cone. Now choose a point of the projected variety and continue.

Presumably at each stage there are a finite number of strata of the variety in which to choose our point P, giving a finite number of types of Π up to symmetry. I believe that the two famous cases of the Segre models of $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are maximal linear space of SpH_8.

It is possible that this method can be used to understand more general morphisms $\operatorname{Spec} S \to \mathrm{SpH}_k$ from the regular space $\operatorname{Spec} S$. In this context, it is very suggestive that Tom and Jerry [TJ] are given in terms of linear subspaces of $\mathrm{Gr}(2,5)$. In this case, the intersection with a tangent space is a cone over $\mathbb{P}^1 \times \mathbb{P}^2$, so it is clear how to construct all linear subspaces of $\mathrm{Gr}(2,5)$, and equally clear that there are two different families, and how they differ.

4.9.3. *Breaking the A_k and D_k symmetry* Experience shows that the bulk constructions of Gorenstein codimension 4 ideals do not have the symmetry of the Buchsbaum–Eisenbud Pfaffians in codimension 3. The equations and syzygies invariably divide up into subsets that one is supposed to treat inhomogeneously. For example, in the 9×16 unprojection cases, the defining equations split into two sets, the 5 Pfaffian equations of the variety in codimension 3 not involving the unprojection variable s, and the 4 unprojection equations that are linear in s.

The columns of the syzygy matrix $(A\,B)$ are governed by the algebraic group $\mathrm{Spin}(2k)$ of type D_k, whereas its rows are governed by $\mathrm{GL}(k+1)$ of type A_k. The common bulk constructions of Gorenstein codimension 4 ideals seem to to accommodate the A_k symmetry of the rows of M_1 and the D_k symmetry of its columns by somehow breaking both to make them compatible. This arises if you try to write the 128 spinor coordinates σ_J as linear combinations of the 9 relations $(L_1, \dots, L_{k+1})$, so relating something to do with the columns of M_1 to its rows. This symmetry breaking and its effect is fairly transparent in 2.3, Example 2.2, (2.6).

Example 2.3 is more typical. (This case comes with three different Tom projections, so may be more amenable.) Of the 128 spinors σ_J, it turns out that 14 are zero, 62 are of the form a monomial times one of the relations L_i (as in (2.9)), and the remainder are more complicated (probably always a sum of two such products). Mapping this out creates a correspondence from spinor sets to relations, so from the rows of M_1 to its columns; there is obviously a systematic structure going on here, and nailing it down is an intriguing puzzle. How this plays out more generally for Kustin–Miller unprojection [KM], [PR] and its special cases Tom and Jerry [TJ] is an interesting challenge.

4.9.4. *Open problems* To be useful, a structure theory should make some predictions. I hope that the methods of this paper will eventually be applicable to start dealing with issues such as the following:

- $k = 3$. A 4×6 resolution is a Koszul complex.
- $k = 4$. There are no almost complete intersection Gorenstein ideals. Equivalently, a 5×8 resolution is nonminimal: if X is Gorenstein codimension 4 and $(L_1, \dots, L_5)$ generate I_X then the first syzygy matrix M_1 has a unit entry, making one of the L_i redundant. This is a well known theorem of Kunz [K], but I want to deduce it by my methods.
- $k = 5$. Is it true that a 6×10 resolution is a hypersurface in a 5×5 Pfaffian as in 2.3, Example 2.2?

  The same question for more general odd k: are hypersurfaces in a codimension 3 Gorenstein varieties the only cases? Is this even true for all the known examples in the literature? This might relate to my even versus odd remark in 3.1.3.
- $k = 6$. I would like to know whether every case of 7×12 resolution is the known Kustin–Miller unprojection from a codimension 4 complete intersection divisor in a codimension 3 complete intersection.
- $k = 8$. As everyone knows, the main case is 9×16. How do we apply the theory to add anything useful to the huge number of known examples?

There are hints that something along these lines may eventually be possible, but it is not in place yet.

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References

[TJ] G. Brown, M. Kerber and M. Reid, Fano 3-folds in codimension 4, Tom and Jerry. Part I, Compos. Math., **148** (2012), 1171–1194.

[BE1] D. Buchsbaum and D. Eisenbud, What makes a complex exact?, J. Algebra, **25** (1973), 259–268.

[BE2] D. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math., **99** (1977), 447–485.

[CLP] F. Catanese, W. Liu and R. Pignatelli, The moduli space of even surfaces of general type with $K^2 = 8$, $p_g = 4$ and $q = 0$, J. Math. Pures Appl. (9), **101** (2014), 925–948.

[CR] A. Corti and M. Reid, Weighted Grassmannians, In: Algebraic Geometry, de Gruyter, Berlin, 2002, pp. 141–163.

[Di] D. Dicks, Surfaces with $p_g = 3, K^2 = 4$ and extension-deformation theory, 1988, Warwick Ph.D. thesis, 125 + vi pp. (available from my website).

[FH] W. Fulton and J. Harris, Representation Theory. A First Course, Grad. Texts in Math., **129**, Springer-Verlag, 1991.

[GW] S. Goto and K. Watanabe, On graded rings. I, J. Math. Soc. Japan, **30** (1978), 179–213.

[K] E. Kunz, Almost complete intersections are not Gorenstein rings, J. Algebra, **28** (1974), 111–115.

[KM] A. Kustin and M. Miller, Constructing big Gorenstein ideals from small ones, J. Algebra, **85** (1983), 303–322.

[PR] S. A. Papadakis and M. Reid, Kustin–Miller unprojection without complexes, J. Algebraic Geom., **13** (2004), 563–577; preprint, arXiv:math/0011094.

[Ki] M. Reid, Graded rings and birational geometry, In: Proc. of Algebraic Geometry Symposium, Kinosaki, October, 2000, (ed. K. Ohno), pp. 1–72, get from www.warwick.ac.uk/staff/Miles.Reid/3folds.

[R1] M. Reid, Surfaces with $p_g = 3$, $K^2 = 4$ according to E. Horikawa and D. Dicks, In: Proceedings of Algebraic Geometry Mini-Symposium, Tokyo, December, 1989, pp. 1–22.

[Va] R. Vakil, Murphy's law in algebraic geometry: badly-behaved deformation spaces, Invent. Math., **164** (2006), 569–590.

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