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Global existence of the spherically symmetric flow of a self-gravitating viscous gas

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Abstract.

In this paper we consider a free-boundary problem for the spherically symmetric flow of a self-gravitating viscous and heat-conductive gas over a rigid sphere. Considering the problem in the Lagrangian-mass coordinate system, we investigate the temporally global existence and behaviour of the solution. Especially, we focus our discussion on the corresponding stationary problem, whose solution gives an asymptotic profile of the solution of the original (time-dependent) problem.

§1. Introduction

We consider a system of equations describing the spherically symmetric motion of a viscous and heat-conductive gas bounded both from above and from below by a free-surface and a central rigid core (sphere), respectively. Both the self-gravitation of the gas and an attractive force due to the core drive the motion of the gas. Such a motion is described by the following equations in the Lagrangian-mass coordinate system

$$
(1.1)\qquad \begin{cases} v_t = (r^2 u)_x, \\[2mm] u_t = r^2 \left(-p + \zeta \dfrac{(r^2 u)_x}{v} \right)_x - G \dfrac{M_c + x}{r^2}, \\[2mm] e_t = \left(-p + \zeta \dfrac{(r^2 u)_x}{v} \right) (r^2 u)_x - 4\mu (r u^2)_x + \left(\dfrac{r^4 \kappa}{v} \theta_x \right)_x \end{cases}
$$

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in $\Omega \times (0,\infty)$, $\Omega := (0,M)$. Here $x \in \overline{\Omega}$ is the mass variable and M represents total mass of the gas. Imposed boundary conditions are

$$
\begin{cases} \left(-p+\zeta\frac{(r^2u)_x}{v}-4\mu\frac{u}{r},\ \theta_x\right)\Big|_{x=M}=(-p_e,\ 0), \\ (u,\ \theta_x)\big|_{x=0}=(0,\ 0). \end{cases} \tag{1.2}
$$

The Eulerian position (radius) of the fluid particle at (x,t) is given by

$$
r=r(x,t)=\left({R_c}^3+3\int_0^x v(s,t)\,\mathrm{d}s\right)^{\frac{1}{3}}
$$

with the radius of the core R_c (a positive constant), which satisfies $r_t = u$ and $r_x = \frac{v}{r^2}$. We seek to find, for any $t > 0$, the specific volume $v = v(x,t)$, the velocity $u = u(x,t)$ and the absolute temperature $\theta = \theta(x,t)$ satisfying (1.1) and (1.2) for given initial conditions

$$
(v,u,\theta)\big|_{t=0}=(v_0,u_0,\theta_0). \tag{1.3}
$$

In this paper we consider only the case that the gas is ideal, namely, the pressure $p = p(v,\theta)$ and the internal energy per unit mass $e = e(v,\theta)$ are given by the equations of state: $p = R\frac{\theta}{v}$, $e = c_{\mathrm{v}}\theta$ with the perfect gas constant R and the specific heat capacity at constant volume c_{v}. Here, μ and ζ are coefficients of the shear and the bulk viscosity, respectively, satisfying $\mu > 0$ and $3\zeta - 4\mu \geq 0$; a constant $p_e \geq 0$ is the external pressure; positive constants G and M_c are the Newtonian gravitational constant and mass of the core, respectively. For simplicity we assume that μ, ζ, c_{v} and κ (heat conductivity) are all positive constants.

Systems of equations related to (1.1), (1.2) have been investigated as some stellar models appearing in the astrophysical arguments (see, for example, [1]). In the paper [2] a large-time behaviour of a stellar model closely-similar to ours was discussed. However, some statements and proofs in [2] seem not to be clear; for example, it looks rather hard to make hold the following estimate (Lemma 9 in [2]):

$$
\int_0^t\int_0^M \theta {v_x}^2\,\mathrm{d}x\,\mathrm{d}\tau \leq C \quad \text{for any } t>0
$$

with some constant $C > 0$ independent of time, which played an important roll for the rest of the estimates in that paper. (The alternative estimate is found below, see Lemma 3.1.) Although, recently, the system (1.1)–(1.3) with $\mu = 0$, $\zeta = \zeta(v)$, $\kappa = \kappa(v,\theta)$ was investigated by Ducomet and Nečasová [3], the asymptotic behaviour of it was not discussed in [3]. Global behaviours of viscous gases are obtained in, for example, [4], [6] with no external force fields; in [13] with an attractive force due to a central core in a fixed annulus domain; in [12] with the self-gravitation of the gas under a one-dimensional motion.

From physical point of view, it is expected that the solution of our problem (1.1)–(1.3) converges to the corresponding stationary solution as $t \to +\infty$ in some sense. For barotropic viscous fluids, in [14] a stationary problem was discussed, and unique existence of the stationary solution was obtained under a certain restricted condition. In the present paper we obtain the condition similar to the one in [14] guaranteeing unique existence of the stationary solution of our problem (see Proposition 1).

§2. Statements of theorems

Let $Q_T := \Omega \times (0, T)$. Firstly, note that we already have the following result concerning unique global solvability of our problem up to any fixed time T. (Some notations are found in, for example, [5].)

Theorem 1. *Let $\alpha \in (0,1)$. Assume that $p_e > 0$ and the initial data in* (1.3)

$$(v_0,\, u_0,\, \theta_0) \in C^{1+\alpha}(\overline{\Omega}) \times (C^{2+\alpha}(\overline{\Omega}))^2$$

satisfy the corresponding compatibility conditions and $v_0,\, \theta_0 > 0$ on $\overline{\Omega}$. Then there exists a unique solution (v, u, θ) of the initial-boundary value problem (1.1)–(1.3) *such that*

$$(v,\, v_x,\, v_t,\, u,\, \theta) \in (C_{x,\,t}^{\alpha,\,\frac{\alpha}{2}}(\overline{Q_T}))^3 \times (C_{x,\,t}^{2+\alpha,\,1+\frac{\alpha}{2}}(\overline{Q_T}))^2$$

for any positive number T and the following inequalities hold:

$$|v,\, v_x,\, v_t|_{Q_T}^{(\alpha)} + |u,\, \theta|_{Q_T}^{(2+\alpha)} \le C, \qquad v,\, \theta \ge C^{-1} \text{ in } \overline{Q_T}, \tag{2.1}$$

where C is a positive constant dependent on the initial data and T.

For the proof of Theorem 1, see [11], in which the case of not only polytropic and ideal gas, but also radiative and reactive gas was discussed.

In order to investigate asymptotic behaviour of the solution, we need to derive some uniform-in-time estimates of the solution. Let us consider a function $\tilde{v} = \tilde{v}(x)$ and a positive constant $\bar{\theta}$ satisfying

$$\left\{ \begin{aligned} &\frac{R\bar{\theta}}{\tilde{v}} = p_e + \int_x^M G\,\frac{M_c + s}{\bar{r}[\tilde{v}]^4}\,\mathrm{d}s \quad \text{for } x \in [0, M], \\ &\int_0^M \left(c_v\bar{\theta} + p_e\tilde{v} - G\,\frac{M_c + x}{\bar{r}[\tilde{v}]} \right) \mathrm{d}x = E_0. \end{aligned} \right. \tag{2.2}$$

Here

$$E_0 := \int_0^M \left(\frac{1}{2} {u_0}^2 + c_{\mathrm{v}}\theta_0 + p_e v_0 - G \frac{M_c + x}{\bar{r}[v_0]} \right) \mathrm{d}x,$$

$$\bar{r}[V](x) := \left({R_c}^3 + 3 \int_0^x V(s)\, \mathrm{d}s \right)^{\frac{1}{3}}$$

for an arbitrary function $V = V(x)$. Our main result is

Theorem 2. *Let T be an arbitrary positive number. Assume that α, μ, ζ and the initial data satisfy the hypotheses of Theorem 1 and that there exist a function $\tilde{v} \in C^1(\overline{\Omega})$ and a positive constant $\bar{\theta}$ satisfying (2.2). Moreover the condition*

$$p_e - \frac{R}{c_{\mathrm{v}}} \cdot \frac{GM(M_c + M/2)}{{R_c}^4} \left(1 + \frac{4C_0}{{R_c}^3} \right) > 0 \tag{2.3}$$

is assumed to be satisfied with

$$C_0 := \frac{1}{p_e} \left[E_0 + \frac{GM(M_c + M/2)}{R_c} \right].$$

Then there exists a positive constant C independent of T such that the inequality (2.1) holds for the solution (v, u, θ) of the initial-boundary value problem (1.1)–(1.3). Moreover, the solution (v, u, θ) converges to the state $(\tilde{v}, 0, \bar{\theta})$ as $t \to +\infty$ in the sense of $H^1(\Omega) \cap C(\overline{\Omega})$.

We consider the stationary problem corresponding to the problem (1.1), (1.2). Namely, find a solution $V = V(x; \bar{\Theta})$ of the following boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{R\bar{\Theta}}{V} \right) = -G \frac{M_c + x}{\bar{r}[V]^4} \quad \text{in } \Omega, \qquad V\big|_{x=M} = \frac{R\bar{\Theta}}{p_e} \tag{2.4}$$

for some positive number $\bar{\Theta}$. We note that a function $\tilde{v}$ mentioned in Theorem 2 is a solution of the problem (2.4). It is easy to confirm that the problem (2.4) has solutions for any number $\bar{\Theta} > 0$, by using Schauder's fixed point theorem (see [14]). Concerning the uniqueness of the solution, we have a sufficient condition for it. Namely,

Proposition 1. *Assume that the condition*

$$p_e - \frac{GM(M_c + M/2)}{{R_c}^7} \cdot \frac{4RM\bar{\Theta}}{p_e} > 0 \tag{2.5}$$

is satisfied. Then there exists a unique solution $V \in C^1(\overline{\Omega})$ of the boundary value problem (2.4).

In the subsection 3.1 we give sketch of the proof of Proposition 1. From physical point of view, it is natural that a number $\bar{\Theta}$ is determined so that it satisfies the equality $g(\bar{\Theta}) = 0$, where

$$g(\bar{\Theta}) := \int_0^M \left(c_v \bar{\Theta} + p_e V - G \frac{M_c + x}{\bar{r}[V]} \right) \mathrm{d}x - E_0$$

with the solution $V = V(x; \bar{\Theta})$ of the problem (2.4) (see (2.2)). Since a number $\bar{\Theta}$ satisfying $g(\bar{\Theta}) = 0$ (if it exists) has an estimate

$$M\bar{\Theta} \leq \frac{1}{c_v} \left[E_0 + \frac{GM(M_c + M/2)}{R_c} \right],$$

the uniqueness condition (2.5) becomes satisfied as far as we are in the situation of (2.3). We can say that if R_c is sufficiently large, then the equation $g(\bar{\Theta}) = 0$ has a solution near

$$\bar{\Theta}_0 := \frac{1}{M} \frac{E_0}{c_v + R}.$$

Although the present author expects that the equation $g(\bar{\Theta}) = 0$ is also solvable under the condition (2.3), a more relaxed one than "R_c is sufficiently large", that is not obvious in this paper.

§3. Outline of proofs of theorems

3.1. Proof of Proposition 1

Here we give a proof only for the uniqueness. We mainly follow the manner found in [14]. Let

$$p(V) := \frac{R\bar{\Theta}}{V}, \qquad f(h) := -G \frac{M_c + x}{(3h)^{\frac{4}{3}}}, \qquad h := \frac{\bar{r}[V]^3}{3}.$$

Firstly, from (2.4) we easily have

$$\int_0^M [(p_e - p(V))\varphi' - f(h)\varphi] \, \mathrm{d}x = 0 \tag{3.1}$$

for any $\varphi \in H^1(\Omega)$ with $\varphi(0) = 0$. Let V_1 and V_2 be two solutions of (2.4), and take $\varphi = h_1 - h_2$ with $h_i := \bar{r}[V_i]^3/3$ $(i = 1, 2)$. Then, from (3.1) we obtain

$$\mathcal{B}(V_1, V_2) := \int_0^M \big[(p(V_1) - p(V_2))(V_1 - V_2) + (f(h_2) - f(h_1))(h_1 - h_2) \big] \, \mathrm{d}x = 0$$

We see that $\mathcal{B}(V_1, V_2)$ is estimated from below as

$$\mathcal{B}(V_1, V_2) \geq \frac{{p_e}^2}{R\bar{\Theta}} \|V_1 - V_2\|^2 - \max_{x \in \overline{\Omega}} (h_1 - h_2)^2 \cdot \int_0^M \frac{4G(M_c + x)}{{R_c}^7} \, \mathrm{d}x.$$

Since Cauchy–Schwarz' inequality gives

$$\max_{x \in \overline{\Omega}} (h_1 - h_2)^2 \leq M \|V_1 - V_2\|^2,$$

we obtain

$$\mathcal{B}(V_1, V_2) \geq \left(\frac{{p_e}^2}{R\bar{\Theta}} - \frac{4GM^2(M_c + M/2)}{{R_c}^7} \right) \|V_1 - V_2\|^2. \tag{3.2}$$

If $V_1 \not\equiv V_2$, then the inequality (3.2) under the hypothesis (2.5) contradicts the fact $\mathcal{B}(V_1, V_2) = 0$. Q.E.D.

3.2. Outline of proofs of Theorems 1 and 2

Proof of Theorem 1 is achieved by continuation of the temporally local solution with the help of suitable *a priori* estimates of the solution. The fundamental theorem about local solvability of the problem had been already established in [7], [8], [9], which are applicable to our problem without essential modifications. In [11] we also have had *a priori* estimates of the solution.

To prove Theorem 2, first we get the following estimates in suitable Sobolev spaces: for some generalized derivatives of the solution

$$(v_{xt},\ v_{tt},\ u_{xxx},\ u_{xt},\ \theta_{xxx},\ \theta_{xt}) \in (L^2(0, T; L^2(\Omega)))^6$$

we find a constant C independent of T such that

$$\begin{cases} \sup\limits_{t \in [0,T]} \|(v,\ v_x,\ v_t,\ v_{xt})(t)\| \leq C, \\ \|u,\ \theta - \bar{\theta}\|_{E(Q_T)} + \displaystyle\int_0^T \|(v - \tilde{v},\ v_x - \tilde{v}_x,\ v_{tt})(t)\|^2 \, \mathrm{d}t \leq C, \\ C^{-1} \leq v,\ \theta \leq C, \quad |u| \leq C \quad \text{in } \overline{Q_T} \end{cases}$$

with the norm

$$\|u\|_{E(Q_T)} := \|u\|_{E_1(Q_T)} + \|u_x\|_{E_1(Q_T)} + \|u_{xx}\|_{E_1(Q_T)} + \|u_t\|_{E_1(Q_T)},$$

$$\|u\|_{E_1(Q_T)} := \left(\sup_{t \in [0,T]} \|u(t)\|^2 + \int_0^T \|u_x(t)\|^2 \, \mathrm{d}t \right)^{\frac{1}{2}}.$$

In addition to getting these bounds, we also obtain that $\|v-\tilde{v}\|_{H^1(\Omega)\cap C(\overline{\Omega})}$, $\|u,\ \theta-\bar{\theta}\|_{H^2(\Omega)\cap C(\overline{\Omega})}$, $\|u_t, \theta_t\|$ and $\sup_{x\in\overline{\Omega}} |u_x, \theta_x|$ all converge to zero as time goes to infinity. The next lemma plays an essential roll in deriving the above estimates and the convergence results.

Lemma 3.1. *If the condition* (2.3) *is satisfied, then it holds*

$$\int_0^t \|(v-\tilde{v})(\tau)\|^2 \,\mathrm{d}\tau \le C$$

for any $t \in [0, T]$, *where* C *is a positive constant independent of* T.

To obtain this lemma, we use the following energy equality:

$$\begin{aligned}
(3.3)\quad & \frac{\mathrm{d}}{\mathrm{d}t}\int_0^M \left[\frac{u}{r^2}\int_0^x v(v-\tilde{v})\,\mathrm{d}s + \frac{\zeta}{2}(v-\tilde{v})^2\right]\mathrm{d}x \\
&\qquad + \int_0^M \left(p_e + \int_x^M G\,\frac{M_c+s}{r^4}\,\mathrm{d}s\right)(v-\tilde{v})^2\,\mathrm{d}x \\
= & \int_0^M \frac{u}{r^2}\left\{\left[(2v-\tilde{v})r^2u\right]_0^x - \int_0^x (2v_x-\tilde{v}_x)r^2u\,\mathrm{d}s\right\}\mathrm{d}x \\
&\qquad + \int_0^M \left(4\mu\frac{u}{r}\Big|_{x=M} - \int_x^M \frac{2u^2}{r^3}\,\mathrm{d}s\right)v(v-\tilde{v})\,\mathrm{d}x \\
&\qquad + \int_0^M R(\theta-\bar{\theta})(v-\tilde{v})\,\mathrm{d}x \\
&\qquad + \int_0^M \left[\int_x^M G(M_c+s)\left(\frac{1}{\bar{r}[\tilde{v}]^4} - \frac{1}{r^4}\right)\mathrm{d}s\right]\tilde{v}(v-\tilde{v})\,\mathrm{d}x.
\end{aligned}$$

This equality is derived by multiplying the momentum equation

$$\left(\frac{u}{r^2}\right)_t = \left(\frac{\zeta}{v}(r^2u)_x - \hat{p}\right)_x, \qquad \hat{p} := p - f(x,t),$$

$$f(x,t) := p_e - 4\mu\frac{u}{r}\Big|_{x=M} + \int_x^M \left(\frac{2u^2}{r^3} + G\,\frac{M_c+s}{r^4}\right)\mathrm{d}s$$

by $\int_0^x v(v-\tilde{v})\,\mathrm{d}s$, and integrating it over $[0, M]$. When we are in the situation of (2.3), the last two terms of the right-hand side of (3.3) can be controlled.

Complete proof of Theorem 2 will be found in the forthcoming (full) paper based on [10].

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