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On the decay property of solutions to the Cauchy problem of the semilinear beam equation with weak damping for large initial data

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Abstract.

This paper is devoted to the study of the initial value problem for some semilinear damped beam equation which has a non-negative definite energy functional. We prove a uniform bound and decay estimates of a solution for the equation with large initial data using the method introduced by Kawashima–Nakao–Ono [3].

§1. Introduction

In this paper we study the initial value problem for the following semilinear damped beam equation

$$\partial_t^2 u + \partial_t u + \partial_x^4 u - \alpha \partial_x^2 u = \partial_x f(\partial_x u), \qquad t > 0, \quad x \in \mathbb{R}, \tag{1}$$

$$u(0,x) = g_0(x), \quad \partial_t u(0,x) = g_1(x), \qquad x \in \mathbb{R}, \tag{2}$$

where $u = u(t,x)$ is an unknown real-valued function, $g_0(x)$ and $g_1(x)$ are given initial data and $\alpha \geq 0$ is a constant. We assume that the nonlinear function $f \in C^2(\mathbb{R})$ satisfies that for some $p > 1$ and any v,

$$f(v)v \geq kF(v) \geq 0, \quad F(v) := \int_0^v f(\eta)d\eta, \tag{3}$$

$$|f'(v)| \leq C|v|^{p-1}, \quad |f(v)| \leq C|v|^p, \tag{4}$$

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where k and C are positive constants. The typical example is given by

$$f(v) = \beta|v|^{p-1}v, \quad p > 1, \tag{5}$$

where $\beta > 0$ is a positive constant.

We refer to the introduction in [6] for the motivations for studying (1)–(2). In [6] and [7], it was proved that the problem (1)–(2) with (5) in the case $\beta < 0$ has small solution with optimal decay, smoothing effect and asymptotic behavior for the solution. The equation (1) with (5) in the case $\beta > 0$ corresponds to the semilinear damped wave equation with absorption

$$\partial_t^2 u + \partial_t u - \Delta u = -\beta|u|^{p-1}u, \quad t > 0, \quad x \in \mathbb{R}^n, \quad \beta > 0. \tag{6}$$

Recently, by choosing a suitable weighted function for the equation (6), Todorova–Yordanov in [8] obtained quite sharp results, and Hayashi–Kaikina–Naumkin in [2] and Nishihara–Zhao in [5] developed their method. Here, we shall use another method to show the decay estimate. Our strategy for the proof of theorems lies in the application of the method introduced by Kawashima–Nakao–Ono [3]. Since unique global existence of solution $u(t) \in C([0,\infty); H^2(\mathbb{R})) \cap C^1([0,\infty); L^2(\mathbb{R}))$ with data $(g_0, g_1) \in H^2 \times L^2$ for (1)–(2) immediately follows from the standard energy method with the help of the contraction mapping principle, we omit the proof of it (see e.g. [1]). Let us state our main results of this paper.

Theorem 1. *Assume that* (3)–(4) *and* $\alpha \geq 0$. *Let* $(g_0, g_1) \in H^2 \times L^2$. *Then the solution* $u(t) \in C([0,\infty); H^2(\mathbb{R})) \cap C^1([0,\infty); L^2(\mathbb{R}))$ *for* (1)–(2) *has the boundedness and decay properties*

$$\|u(t)\|_{L^2} \leq C, \tag{7}$$

$$\|\partial_x^2 u(t)\|_{L^2} + \alpha\|\partial_x u(t)\|_{L^2} + \|\partial_t u(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}. \tag{8}$$

We conclude the introduction by giving notation used in this paper. Let ∂_t and ∂_x be a partial differential operator with respect to a variable t and x, respectively. $L^p(\mathbb{R})$ and $H^k(\mathbb{R})$ are the standard Lebesgue and Sovolev spaces, respectively. We denote the several positive constants by $C > 0$ without confusions. These constants may change from line to line.

§2. Preliminaries

In this section we prepare several estimates to show the main result. We first define the energy functional of (1)–(2) by

$$
\begin{aligned}
E(t) :=& \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 + \frac{1}{2}\|\partial_x^2 u(t)\|_{L^2}^2 \\
&+ \frac{\alpha}{2}\|\partial_x u(t)\|_{L^2}^2 + \int_{\mathbb{R}} F(\partial_x u(t,x))dx,
\end{aligned} \tag{9}
$$

where F is defined in (3).

Lemma 2. *For any $t_2 \geq t_1 \geq 0$, the solution $u(t) \in C([0,\infty); H^2(\mathbb{R})) \cap C^1([0,\infty); L^2(\mathbb{R}))$ for (1)–(2) satisfies*

$$
E(t_2) + \int_{t_1}^{t_2} \|\partial_t u(s)\|_{L^2}^2 ds = E(t_1), \tag{10}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}} u(t_2,x)\partial_t u(t_2,x)dx - \int_{\mathbb{R}} u(t_1,x)\partial_t u(t_1,x)dx \\
&\quad + \int_{t_1}^{t_2} \left(\|\partial_x^2 u(s)\|_{L^2}^2 + \alpha\|\partial_x u(s)\|_{L^2}^2\right) ds \\
&\quad + \int_{t_1}^{t_2}\int_{\mathbb{R}} u(s,x)\partial_t u(s,x)dxds - \int_{t_1}^{t_2} \|\partial_t u(s)\|_{L^2}^2 ds \\
&= -\int_{t_1}^{t_2}\int_{\mathbb{R}} \partial_x u(s,x) f(\partial_x u(s,x))dxds.
\end{aligned} \tag{11}
$$

Proof. Multiplying (1) by $\partial_t u$ and integrating over $[t_1, t_2] \times \mathbb{R}$, we can easily obtain (10). Multiplying (1) by u and integrating over $\mathbb{R}$ yield

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} u(t,x)\partial_t u(t,x)dx - \|\partial_t u(t)\|_{L^2}^2 + \int_{\mathbb{R}} u(t,x)\partial_t u(t,x)dx \\
&\quad + \|\partial_x^2 u(t)\|_{L^2}^2 + \alpha\|\partial_x u(t)\|_{L^2}^2 = -\int_{\mathbb{R}} \partial_x u(t,x) f(\partial_x u(t,x))dx.
\end{aligned}
$$

Then integrating it over $[t_1, t_2]$, we obtain the equality (11). Q.E.D.

Next lemma is an easy consequence of Lemma 2.

Lemma 3. *For the solution for (1)–(2), we have*

$$
\|u(t)\|_{L^2}^2 \leq C, \quad t \geq 0, \tag{12}
$$

where C is independent of t.

Proof. Note that $vf(v) \geq 0$. Then it follows from (11) that

$$\int_{\mathbb{R}} u(t,x)\partial_t u(t,x)dx - \int_{\mathbb{R}} g_0(x)g_1(x)dx + \|u(t)\|_{L^2}^2 - \|g_0\|_{L^2}^2$$
$$- \int_0^t \|\partial_t u(s)\|_{L^2}^2 ds \leq - \int_0^t \int_{\mathbb{R}} \partial_x u(s,x) f(\partial_x u(s,x)) dx ds \leq 0.$$

Therefore, we have

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\leq \|g_0\|_{L^2}^2 + \|g_0\|_{L^2}\|g_1\|_{L^2} + \frac{1}{2}\|u(t)\|_{L^2}^2 \\ &\quad + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 + E(0) \\ &\leq \|g_0\|_{L^2}^2 + \|g_0\|_{L^2}\|g_1\|_{L^2} + \frac{1}{2}\|u(t)\|_{L^2}^2 + 2E(0),\end{aligned}$$

which implies the desired estimate (12). Q.E.D.

§3. Proof of Theorem 1

In this section we shall prove Theorem 1. The latter part of the proof is known as the procedure of Nakao [4].

Proof of Theorem 1. We first show that for any $t \geq 0$

$$\sup_{t \leq s \leq t+1} E(s) \leq C(E(t) - E(t+1)) + C\sqrt{E(t) - E(t+1)}. \tag{13}$$

Since from (10)

$$E(t) - E(t+1) = \int_t^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds, \tag{14}$$

then we have

$$E(t) - E(t+1) \geq \int_t^{t+\frac{1}{4}} \|\partial_t u(s)\|_{L^2}^2 ds, \tag{15}$$

$$E(t) - E(t+1) \geq \int_{t+\frac{3}{4}}^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds. \tag{16}$$

From the mean value theorem there exists some $\tau_1 \in [t, t+1/4]$ and $\tau_2 \in [t+3/4, t+1]$ such that

$$\frac{1}{4}\|\partial_t u(\tau_1)\|_{L^2}^2 = \int_t^{t+\frac{1}{4}} \|\partial_t u(s)\|_{L^2}^2 ds, \tag{17}$$

$$\frac{1}{4}\|\partial_t u(\tau_2)\|_{L^2}^2 = \int_{t+\frac{3}{4}}^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds. \tag{18}$$

From (11) and (12) we have

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2} \left\{ \|\partial_x^2 u(s)\|_{L^2}^2 + \alpha \|\partial_x u(s)\|_{L^2}^2 + \int_{\mathbb{R}} \partial_x u(s,x) f(\partial_x u(s,x)) dx \right\} ds \\
&\quad \le \|u(\tau_1)\|_{L^2} \|\partial_t u(\tau_1)\|_{L^2} + \|u(\tau_2)\|_{L^2} \|\partial_t u(\tau_2)\|_{L^2} \\
&\qquad + \int_t^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds + \int_t^{t+1} \|u(s)\|_{L^2} \|\partial_t u(s)\|_{L^2} ds \\
&\quad \le C \|\partial_t u(\tau_1)\|_{L^2} + C \|\partial_t u(\tau_2)\|_{L^2} \\
&\qquad + \int_t^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds + C \int_t^{t+1} \|\partial_t u(s)\|_{L^2} ds \qquad (19) \\
&\quad \le C \left(\int_t^{t+\frac{1}{4}} \|\partial_t u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} + C \left(\int_{t+\frac{3}{4}}^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\
&\qquad + \int_t^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds + C \left(\int_t^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\
&\quad \le E(t) - E(t+1) + C\sqrt{E(t) - E(t+1)},
\end{aligned}
$$

by virtue of (14)–(18). On the other hand, we see that

$$
\int_{\tau_1}^{\tau_2} \|\partial_t u(s)\|_{L^2}^2 ds \le \int_t^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds = E(t) - E(t+1). \tag{20}
$$

Since $vf(v) \ge kF(v)$, combining (19) and (20) yields

$$
\int_{\tau_1}^{\tau_2} E(s) ds \le C(E(t) - E(t+1)) + C\sqrt{E(t) - E(t+1)}. \tag{21}
$$

From the mean value theorem, there exists $\tau_0 \in [\tau_1, \tau_2]$ such that

$$
(\tau_2 - \tau_1) E(\tau_0) = \int_{\tau_1}^{\tau_2} E(s) ds.
$$

Since $\tau_2 - \tau_1 \ge 1/2$, we have for some $\tau_0 \in [\tau_1, \tau_2]$

$$
E(\tau_0) \le C(E(t) - E(t+1)) + C\sqrt{E(t) - E(t+1)}. \tag{22}
$$

For any $\tau \in [\tau_0, t+1]$ it follows from (10) that

$$
E(\tau_0) = E(\tau) + \int_{\tau_0}^{\tau} \|\partial_t u(s)\|_{L^2}^2 ds \ge E(\tau).
$$

Then we have

$$
\sup_{\tau_0 \le \tau \le t+1} E(\tau) \le C(E(t) - E(t+1)) + C\sqrt{E(t) - E(t+1)}. \tag{23}
$$

Similarly, from (10), for any $\tau \in [t, \tau_0]$, we see that

$$\begin{aligned} E(\tau) &= E(\tau_0) + \int_\tau^{\tau_0} \|\partial_t u(s)\|_{L^2}^2 ds \\ &\le E(\tau_0) + \int_t^{t+1} \|\partial_t u(s)\|_{L^2}^2 ds \\ &= E(\tau_0) + E(t) - E(t+1). \end{aligned}$$

Then we have

$$\sup_{t \le \tau \le \tau_0} E(\tau) \le C(E(t) - E(t+1)) + C\sqrt{E(t) - E(t+1)}. \tag{24}$$

From (23) and (24) we arrive at (13).

Next, we shall show

$$E(t) \le C(1+t)^{-1}. \tag{25}$$

From (13) we have

$$\begin{aligned} \sup_{t \le s \le t+1} E(s)^2 &\le C\left\{(E(t) - E(t+1)) + \sqrt{E(t) - E(t+1)}\right\}^2 \\ &\le C\left(E(t) - E(t+1)\right)^2 + C\left(E(t) - E(t+1)\right) \\ &\le C\left(E(t) - E(t+1)\right)\left(E(t) - E(t+1) + 1\right) \\ &\le C\left(E(t) - E(t+1)\right)\left(E(0) + 1\right) \\ &\le C\left(E(t) - E(t+1)\right). \end{aligned}$$

Here, we assume that there exists some $T \ge 0$ such that $E(T) = 0$. Then, from (10) we see that for any $t \ge T$

$$E(t) + \int_T^t \|\partial_t u(s)\|_{L^2}^2 ds = E(T) = 0,$$

which implies $E(t) = 0$ for any $t \ge T$. Thus the desired estimate (25) obviously holds true. On the other hand, we assume that $E(t) > 0$ for any $t \ge 0$. Setting $y(t) = E(t)^{-1}$, then we have

$$\begin{aligned} y(t+1) - y(t) &= \int_0^1 \frac{d}{d\theta}\left\{\theta E(t+1) + (1-\theta)E(t)\right\}^{-1} d\theta \\ &= (E(t) - E(t+1)) \int_0^1 \left\{\theta E(t+1) + (1-\theta)E(t)\right\}^{-2} d\theta \\ &\ge C \sup_{t \le s \le t+1} E(s)^2 \int_0^1 \left\{\theta E(t+1) + (1-\theta)E(t)\right\}^{-2} d\theta. \end{aligned}$$

Since $0 \leq \theta \leq 1$, it follows that

$$\frac{1}{\{\theta E(t+1)+(1-\theta)E(t)\}^2} \geq \frac{1}{\{E(t+1)+E(t)\}^2} \geq \frac{1}{4 \sup_{t \leq s \leq t+1} E(s)^2}.$$

Therefore, there exists some positive constant C_0 independent of t such that

$$y(t+1)-y(t) \geq C_0 > 0.$$

For every $t \geq 2$ there exists a unique number $N \in \mathbb{N}$ such that $N \leq t < N+1$. Then we obtain

$$\begin{aligned} y(t) &\geq y(t-1)+C_0 \\ &\geq y(t-2)+2C_0 \\ &\geq \cdots \\ &\geq y(t-N)+NC_0 \\ &> NC_0 \geq (t-1)C_0. \end{aligned}$$

Hence, we conclude that

$$E(t) \leq C(t-1)^{-1} \leq C(1+t)^{-1}$$

for $t \geq 2$. Since $E(t)$ is bounded in $0 \leq t \leq 2$, we obtain (25). This completes the proof. Q.E.D.

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