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The mathematical justification of the Bohm criterion in plasma physics

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Abstract.

In plasma physics, the Bohm criterion is known as a necessary condition for the formation of the boundary layer called sheath. Our aim is to validate the Bohm criterion from the mathematical point of view. First, we define the sheath as a stationary solution to the Euler–Poisson equations over a half space since the Bohm criterion is proved to be sufficient for the existence of the stationary solution. Then we study the asymptotic stability of the stationary solution under the degenerate or nondegenerate Bohm criterion.

§1. Introduction

In plasma physics, the Bohm criterion is known as a necessary condition for the formation of a boundary layer called sheath. Here we briefly explain the physical background of our research. Let us consider a situation that a wall is put in a plasma and is negatively charged. Due to this potential gradient, positive ions are attracted to the wall while the electrons are bounced back. Hence the sharp density gradient appears close to the wall. This boundary layer is called a sheath. The research of the sheath dates back to Tonks and Langmuir [8] where the transition between the plasma and the sheath were discussed. Later, Bohm in [2] obtained the necessary condition for the formation of sheath, which is now known as the Bohm criterion. Readers are referred to [1], [6] for reviews and new insights concerning the sheath.

Recently Suzuki in [7] interpreted the sheath to be a monotone stationary solution to the system of Euler–Poisson equations (1) over one-dimensional half space and showed that the Bohm criterion together

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with the physically natural boundary condition on the electric potential is sufficient for the unique existence of a monotone stationary solution. The authors in [5] proved the asymptotic stability of the stationary solution under the degenerate or nondegenerate Bohm criterion and justified the Bohm criterion from the mathematical point of view. The object of the present paper is to summarize these two papers.

§2. Main results

We study the initial boundary value problem to the Euler–Poisson equations (1) over $\mathbb{R}^N_+ := \{(x_1, x') \in \mathbb{R}^N |\ x_1 > 0, x' \in \mathbb{R}^{N-1}\}$ for $N = 1, 2, 3$:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1a}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + K\nabla\rho + \rho\nabla\phi = 0, \tag{1b}$$

$$-\Delta\phi = \rho - e^{\phi}. \tag{1c}$$

This system of equations describes the isothermal flow of positive ions, where unknown functions ρ, u and ϕ stand for the density and the velocity of positive ions and the electrostatic potential, respectively. The positive constant K corresponds to the temperature of ions. The third equations is obtained by assuming the Boltzmann relation: $\rho_e = e^{\phi}$.

We prescribe the initial and the boundary data as

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \quad \inf_{x \in \mathbb{R}^N_+} \rho_0(x) > 0,$$

$$\lim_{x_1 \to \infty} (\rho_0, u_0)(x_1, x') = (\rho_+, u_+, 0, \ldots, 0) \in \mathbb{R}^{N+1}, \tag{2}$$

$$\phi(t, 0, x') = \phi_b \tag{3}$$

for any $x' \in \mathbb{R}^{N-1}$, where $\rho_+ > 0$, u_+ and ϕ_b are constants. The reference point of the value of the potential ϕ is taken as $x_1 = \infty$:

$$\lim_{x_1 \to \infty} \phi(t, x_1, x') = 0 \quad \text{for any } x' \in \mathbb{R}^{N-1}. \tag{4}$$

We easily see that constructing a classical solution to (1c) requires

$$\rho_+ = 1. \tag{5}$$

The planar stationary solution $(\tilde{\rho}, \tilde{u}, 0, \dots, 0, \tilde{\phi})(x_1)$ is a solution to (1) independent of the time variable t or tangential variable x':

$$(\tilde{\rho}\tilde{u})_{x_1} = 0, \tag{6a}$$

$$\left(\tilde{\rho}\tilde{u}^2 + K\tilde{\rho}\right)_{x_1} + \tilde{\rho}\tilde{\phi}_{x_1} = 0, \tag{6b}$$

$$-\tilde{\phi}_{x_1x_1} = \tilde{\rho} - e^{\tilde{\phi}}. \tag{6c}$$

Assumptions corresponding to (2)–(4) are also made, that is,

$$\inf_{x_1\in\mathbb{R}_+} \tilde{\rho}(x_1) > 0, \quad \lim_{x_1\to\infty} (\tilde{\rho}, \tilde{u}, \tilde{\phi})(x_1) = (\rho_+, u_+, 0), \quad \tilde{\phi}(0) = \phi_b. \tag{7}$$

The conditions on the unique existence of the monotone solution to (6) and (7) are obtained in [7]:

Theorem 1. (i) *Let u_+ be a constant satisfying either $u_+^2 \le K$ or $K+1 \le u_+^2$. Then the stationary problem (6) and (7) has a unique monotone solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})(x_1)$ verifying $\tilde{\rho}, \tilde{u}, \tilde{\phi} \in C(\overline{\mathbb{R}_+})$ and $\tilde{\rho}, \tilde{u}, \tilde{\phi}, \tilde{\phi}_{x_1} \in C^1(\mathbb{R}_+)$ if and only if the boundary data ϕ_b satisfies conditions*

$$V(\phi_b) \ge 0 \quad \text{and} \quad \phi_b \le f(|u_+|/\sqrt{K}). \tag{8}$$

where the Sagdeev potential V and the function f is defined by

$$V(\phi) := \int_0^\phi \left[e^\eta - f^{-1}(\eta)\right] d\eta, \quad f(\rho) := -K\log\rho - \frac{u_+^2}{2\rho^2} + \frac{u_+^2}{2}, \tag{9}$$

and the inverse function f^{-1} is defined by adopting the branch which contains the equilibrium point $(\rho, \phi) = (1, 0)$.

(ii) *Let u_+ be a constant satisfying $K < u_+^2 < K+1$. If $\phi_b \ne 0$, then the stationary problem (6) and (7) does not admit any solutions in the function space $C^2(\mathbb{R}_+)$.*

By these results, we see that the condition

$$u_+^2 > K+1, \quad u_+ < 0 \tag{10}$$

together with $|\phi_b| \ll 1$ or

$$u_+^2 = K+1, \quad u_+ < 0 \tag{11}$$

together with $\phi_b \le 0$ and $|\phi_b| \ll 1$ is sufficient for the unique existence and the stability of the monotone stationary solution.

To study the asymptotic stability of the stationary solution, we introduce unknown functions $v := \log\rho$, $\tilde{v} := \log\tilde{\rho}$ and the perturbation

$$(\psi, \eta, \sigma)(t, x_1, x') := (v, u, \phi)(t, x_1, x') - (\tilde{v}, \tilde{U}, \tilde{\phi})(x_1),$$

where $\tilde{U} = (\tilde{u}, 0, \ldots, 0)$. The equations for (ψ, η, σ) are obtained from (1) and (6):

$$\psi_t + u \cdot \nabla\psi + \mathrm{div}\eta + \eta_1 \tilde{v}_{x_1} = 0, \tag{12a}$$

$$\eta_t + u \cdot \nabla\eta + K\nabla\psi + \nabla\sigma + \eta_1 \tilde{U}_{x_1} = 0, \tag{12b}$$

$$-\Delta\sigma = e^{\psi+\tilde{v}} - e^{\tilde{v}} - e^{\sigma+\tilde{\phi}} + e^{\tilde{\phi}}, \tag{12c}$$

and the initial and the boundary data are obtained from (2), (3) and (7):

$$\lim_{x_1\to\infty} (\psi_0, \eta_0)(x_1, x') = (0,0) \quad \text{for any } x' \in \mathbb{R}^{N-1}, \tag{13}$$

$$\text{where} \quad (\psi_0, \eta_0)(x) := (\psi, \eta)(0, x) = (\log\rho_0 - \log\tilde{\rho}, u_0 - \tilde{U}),$$

$$\sigma(t, 0, x') = 0 \quad \text{for any } x' \in \mathbb{R}^{N-1}. \tag{14}$$

Now we linearize (12) around the asymptotic state

$$(\rho, u, \phi) = (\rho_+, u_+, 0, \ldots, 0, 0). \tag{15}$$

By a straightforward computation, we see that the real part of all the spectra are zero. We overcome this difficulty by considering the stability problem in a function space with weight functions $(1 + \beta x_1)^\alpha$ or $e^{\beta x_1}$, following ideas in [4]. In fact, we proved in [5] the next theorem which indicates condition (10) might be sufficient for the stability of the sheath.

Theorem 2. *Let us define* $(\Psi, H, \Sigma) := (e^{\beta x_1/2}\psi, e^{\beta x_1/2}\eta, e^{\beta x_1/2}\sigma)$ *and rewrite* (12) *with respect to* (Ψ, H, Σ)*, linearize the result around* (15) *and denote it by* (L)*. Considering this linearized problem over* $\mathbb{R}^N$*, its spectra are given by putting* $\zeta := 1 + |\xi|^2 - \frac{\beta^2}{4} + i\beta\xi_1$ *for* $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ *as*

$$\mu(i\xi) = \frac{\beta u_+}{2} + i\left(-\xi_1 u_+ \pm \sqrt{K\zeta - \frac{1}{\zeta} + 1 - K}\right), \ \frac{\beta u_+}{2} - i\xi_1 u_+$$

where $\beta u_+/2 - i\xi_1 u_+$ *has a multiplicity of* $N - 1$*. Moreover, it holds*

$$\sup_{\xi\in\mathbb{R}^N} Re\,(\mu(i\xi)) = \max\{Re\,(\mu(0))\} = \frac{\beta}{2}\left(u_+ + \sqrt{K + (1 - \beta^2/4)^{-1}}\right).$$

Hence if $u_+ < 0$, $u_+^2 > K + (1 - \beta^2/4)^{-1}$ *are satisfied, the system* (L) *is linearly stable. This condition is fulfilled if and only if* (10) *holds and the positive weight parameter* β *is set suitably small.*

In accordance with the physical observation and the result of the linear stability, the Bohm criterion is validated by showing the unique existence of the global-in-time solution to (1) and its stability under (10). We show similar results under the degenerate condition (11). For the proofs, readers are referred to [5].

Theorem 3. (nondegenerate case)
For $N = 1, 2, 3$, *let* $m = [N/2] + 2$. *Assume* $K > 0$ *and* (10) *hold.*
(i) *Suppose* $(e^{\lambda x_1/2}\psi_0, e^{\lambda x_1/2}\eta_0) \in (H^m(\mathbf{R}^N_+))^{N+1}$ *for some positive constant* λ. *If* $\beta + \left(|\phi_b| + \|(e^{\beta x_1/2}\psi_0, e^{\beta x_1/2}\eta_0)\|_{H^m}\right)/\beta$ *is small enough for* $\beta \in (0, \lambda]$, *the initial boundary value problem* (12)–(14) *has a unique solution as* $(e^{\beta x_1/2}\psi, e^{\beta x_1/2}\eta, e^{\beta x_1/2}\sigma) \in (\mathfrak{X}^0_m([0,\infty)))^{N+1} \times \mathfrak{X}^2_m([0,\infty))$. *Moreover, for some positive constant* α, *the solution verifies*

$$(16)\quad \|(e^{\beta x_1/2}\psi, e^{\beta x_1/2}\eta)(t)\|^2_{H^m} + \|e^{\beta x_1/2}\sigma(t)\|^2_{H^{m+2}} \leq C\|(e^{\beta x_1/2}\psi_0, e^{\beta x_1/2}\eta_0)\|^2_{H^m}\, e^{-\alpha t}.$$

(ii) *Suppose* $\left((1+\beta x_1)^{\lambda/2}\psi_0, (1+\beta x_1)^{\lambda/2}\eta_0\right) \in (H^m(\mathbf{R}^N_+))^{N+1}$ *holds for some* $\lambda \geq 2$ *and* $\beta > 0$. *For any* $\varepsilon \in (0, \lambda]$, *there exists a positive constant* δ *such that if* $(\|((1+\beta x_1)^{\lambda/2}\psi_0, (1+\beta x_1)^{\lambda/2}\eta_0)\|_{H^m} + |\phi_b|)/\beta + \beta \leq \delta$, *the initial boundary value problem* (12)–(14) *has a unique solution as* $((1+\beta x_1)^{\varepsilon/2}\psi, (1+\beta x_1)^{\varepsilon/2}\eta, (1+\beta x_1)^{\varepsilon/2}\sigma) \in (\mathfrak{X}^0_m([0,\infty)))^{N+1} \times \mathfrak{X}^2_m([0,\infty))$. *Moreover, the solution decays algebraically fast:*

$$(17)\quad \|((1+\beta x_1)^{\varepsilon/2}\psi, (1+\beta x_1)^{\varepsilon/2}\eta)(t)\|^2_{H^m} + \|(1+\beta x_1)^{\varepsilon/2}\sigma(t)\|^2_{H^{m+2}} \leq C\|((1+\beta x_1)^{\lambda/2}\psi_0, (1+\beta x_1)^{\lambda/2}\eta_0)\|^2_{H^m}(1+\beta t)^{-(\lambda-\varepsilon)}.$$

Theorem 4. (degenerate case)
For $N = 1, 2, 3$, *let* $m = [N/2] + 2$. *Assume* $K > 0$ *and* (11) *hold. Let* $\lambda_0 = 5.5693\ldots$ *be the unique real solution to the equation* $\lambda_0(\lambda_0-1)(\lambda_0-2) - 12(\lambda_0+2) = 0$ *and* $\lambda \in [4, \lambda_0)$ *is satisfied. For any* $\varepsilon \in (0, \lambda]$ *and* $\theta \in (0, 1]$, *there exists a positive constant* δ *such that if* $\phi_b \in [-\delta, 0)$, $\beta/((K+1)|\phi_b|/6)^{1/2} \in [\theta, 1]$, $\left((1+\beta x_1)^{\lambda/2}\psi_0, (1+\beta x_1)^{\lambda/2}\eta_0\right) \in (H^m(\mathbf{R}^N_+))^{N+1}$ *and* $\|((1+\beta x_1)^{\lambda/2}\psi_0, (1+\beta x_1)^{\lambda/2}\eta_0)\|_{H^m}/\beta^3 \leq \delta$ *are satisfied, the initial boundary value problem* (12)–(14) *has a unique solution as* $\left((1+\beta x_1)^{\varepsilon/2}\psi, (1+\beta x_1)^{\varepsilon/2}\eta, (1+\beta x_1)^{\varepsilon/2}\sigma\right) \in (\mathfrak{X}^0_m([0,\infty)))^{N+1} \times \mathfrak{X}^2_m([0,\infty))$. *Moreover the solution verifies the decay estimate*

$$(18)\quad \|((1+\beta x_1)^{\varepsilon/2}\psi, (1+\beta x_1)^{\varepsilon/2}\eta)(t)\|^2_{H^m} + \|(1+\beta x_1)^{\varepsilon/2}\sigma(t)\|^2_{H^{m+2}} \leq C\|((1+\beta x_1)^{\lambda/2}\psi_0, (1+\beta x_1)^{\lambda/2}\eta_0)\|^2_{H^m}(1+\beta t)^{-(\lambda-\varepsilon)/3}.$$

Notation. For a real number x, $[x]$ denotes a maximum integer which does not exceed x. For a nonnegative integer $l \geq 0$, $H^l(\mathbb{R}_+^N)$ denotes the l-th order Sobolev space in the L^2 sense, equipped with the norm $\|\cdot\|_{H^l}$. We denote by $C^k([0,T];H^l(\mathbb{R}_+^N))$ the space of k-times continuously differentiable functions on the interval $[0,T]$ with values in $H^l(\mathbb{R}_+^N)$. The function space $\mathfrak{X}_i^j$ $(i=0,1,2,3,\ j=0,1,2)$ is defined by

$$\mathfrak{X}_i^j([0,T]) := \bigcap_{k=0}^{i} C^k([0,T];H^{j+i-k}(\mathbb{R}_+^N)),\ \ \mathfrak{X}_i([0,T]) := \mathfrak{X}_i^0([0,T]).$$

Lastly, C denotes a generic positive constant.

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