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On spherically symmetric gravitational collapse in the Einstein–Gauss–Bonnet theory

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Abstract.

Spherically symmetric gravitational collapse of a scalar field in the Einstein–Gauss–Bonnet theory will be considered. We show some global results for this system, which supports the validity of the cosmic censorship conjecture.

§1. Preliminaries

Let $(M, g_{\mu\nu})$ be a spacetime, where M is an orientable n-dimensional manifold and $g_{\mu\nu}$ is a Lorentzian metric on it. Here, a massless scalar field is assumed as simplest one. The Einstein–Gauss–Bonnet(EGB)-scalar equations [6] are as follows:

$$G_{\mu\nu} + \alpha H_{\mu\nu} = \kappa_n^2 T_{\mu\nu}, \tag{1}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = 0 \tag{2}$$

where α is a positive constant, $T_{\mu\nu}$ is the energy-momentum tensor and

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu},$$

$$H_{\mu\nu} := 2\left[RR_{\mu\nu} - 2R_{\alpha\mu}R^\alpha_\nu - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} + R_\mu^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}\right] - \frac{1}{2}g_{\mu\nu}L_2,$$

where $L_2 := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. Our main theorem is

Theorem 1. *Consider asymptotically flat smooth initial data (S, h, k, ϕ) for the spherically symmetric Einstein–Gauss–Bonnet-scalar equations. Here, S is an $(n-1)$-dimensional spacelike hypersurface, h is an induced metric on S, k is an extrinsic curvature of S. Let (M, g) be*

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the maximal Cauchy development from S and let $\pi : M \to Q^+$ be the projection map to the two dimensional Lorentzian quotient Q^+. Suppose that there exists on asymptotically flat spacelike Cauchy surface $\tilde{S} \subset Q^+$ and a point $p \in \tilde{S}$ such that $\pi^{-1}(p)$ is trapped or marginally trapped and at least one of the connected components $\tilde{S} \setminus \{p\}$ containing an asymptotically flat end is such that $\pi^{-1}(q)$ is not outer anti-trapped or marginally anti-trapped for any q in the component. Then $J^-(\mathcal{I}^+) \cap J^+(\tilde{S}) \subset D^+(\tilde{S}) \subset Q^+$, where $\mathcal{I}^+$ is the future complete null infinity. Moreover, the Penrose-like inequality $r \le r_P(M_f, \alpha, n)$ holds on $\mathcal{H}^+ = \overline{J^-(\mathcal{I}^+) \cap Q^+} \setminus (I^-(\mathcal{I}^+) \cup \overline{\mathcal{I}^+})$, where r_P is the unique positive solution to $M_f = \frac{(n-2)V_{n-2}}{2\kappa_n^2} r^{n-5}(r^2 + \tilde{\alpha})$, r denotes the volume radius function and M_f is the final Bondi mass.

§2. Spherically symmetric spacetimes in n-dimension

Globally hyperbolic spacetimes M with n-dimensional spherical symmetry imply that the group $SO(n-1)$ acts by isometry on M and preserves ϕ. We assume $Q^+ = M/SO(n-1)$, inherits from spacetime metric g the structure of a $1+1$-dimensional Lorentzian manifold with boundary with metric $\tilde{g}$, such that

$$g = \tilde{g} + r^2 d\sigma^2 = -\Omega^2 dudv + r^2\sigma^2,$$

where σ^2 is the standard metric for $(n-2)$-sphere and functions Ω and r depend on only u and v on Q^+. The boundary of Q^+ consists of $\Gamma \cap S$, where Γ is a connected timelike curve and S is a connected spacelike curve (Cauchy surface). $\Gamma \cap S$ is a single point and $r(p) = 0$ if and only if $p \in \Gamma$. Γ is called the centre. In this metric, the EGB-scalar equations (1)-(2) become as follow:

$$(3) \quad D\partial_u\partial_v r = -\frac{\Omega^2}{4r}\left[(n-3)\mu + (n-5)\frac{\tilde{\alpha}}{r^2}\mu^2\right] + \frac{\kappa_n^2}{n-2} r T_{uv},$$

$$(4) \; D\partial_u\partial_v \log\Omega = \frac{(n-3)}{r^2}\partial_u r \partial_v r + \frac{k(n-3)}{4r^2}\Omega^2 - \frac{(n-4)r}{2r^2}\partial_u\partial_v r + \frac{\tilde{\alpha}\Omega^2}{2r^4} Z + \frac{\Omega^2}{24r^2}\left(g^{ab}T_{ab} - 4T^u_u\right),$$

$$(5) \qquad D\partial_u(\Omega^{-2}\partial_u r) = -\frac{\kappa_n^2}{n-2} r\Omega^{-2} T_{uu},$$

$$(6) \qquad D\partial_v(\Omega^{-2}\partial_v r) = -\frac{\kappa_n^2}{n-2} r\Omega^{-2}T_{vv},$$

$$(7) \qquad \partial_u\partial_v\phi = -\frac{n-2}{2r}\partial_u\phi\partial_v r - \frac{n-2}{2r}\partial_u r\partial_v\phi.$$

Here, $\tilde{\alpha} = (n-3)(n-4)\alpha$, $D \equiv \left[1 + \frac{2\tilde{\alpha}}{r^2}\mu\right]$, $\mu \equiv 1 + \frac{4\partial_u r\partial_v r}{\Omega^2}$, and

$$\begin{aligned} Z \equiv \quad & -\frac{2(n-8)\mu r\partial_u\partial_v r}{\Omega^2} - \frac{16r^2}{\Omega^4}\left(\partial_u \ln\Omega\partial_u r\partial_v\partial_v r + \partial_v \ln\Omega\partial_v r\partial_u\partial_u r\right) \\ & +(n-5)\mu^2 \\ & +\frac{8r^2}{\Omega^2}\left(\partial_u\partial_u r\partial_v\partial_v r + 4\partial_u \ln\Omega\partial_v \ln\Omega\partial_u r\partial_v r - (\partial_u\partial_v r)^2\right). \end{aligned}$$

From the definition, components of the energy-momentum tensor for the massless scalar field are given by

$$T_{uu} = (\partial_u\phi)^2, \qquad T_{vv} = (\partial_v\phi)^2, \qquad T_{uv} = 0,$$

$$g^{\mu\nu}T_{\mu\nu} = g^{ab}T_{ab} = 2\Omega^{-2}\partial_u\phi\partial_v\phi.$$

Note that $\partial_v r = (1-\mu)\kappa$.

§3. Trapped region

Following the idea of Penrose's closed trapped surfaces, we will divide spacetimes into three regions:

- *Regular region*: $\mathcal{R} = \{q \in Q^+ : \partial_v r > 0, \partial_u r < 0\}$,
- *Trapped region*: $\mathcal{T} = \{q \in Q^+ : \partial_v r < 0, \partial_u r < 0\}$,
- *Marginally trapped region*: $\mathcal{A} = \{q \in Q^+ : \partial_v r = 0, \partial_u r < 0\}$.

In addition, we call $\mathcal{R} \cup \mathcal{A}$ the *non-trapped* region. By the definition and the constraint equations (5) and (6) for $1 + \frac{2\tilde{\alpha}}{r^2}\mu > 0$, we have

Proposition 1. *The followings holds:*

(1) $Q^+ = \mathcal{R} \cup \mathcal{T} \cup \mathcal{A}$.

(2) *If* $(u,v) \in \mathcal{T}$, *then*$(u,v^*) \in \mathcal{T}$ *for* $v^* > v$. *Similarly, if* $(u,v) \in \mathcal{T} \cup \mathcal{A}$, *then*$(u,v^*) \in \mathcal{T} \cup \mathcal{A}$ *for* $v^* > v$.

From the constraint equation (5) under the dominant energy condition, we have

Proposition 2. *Assume* $\partial_u r < 0$ *along* S, *which means that spacetimes do not expand on initial surface. Then,* $\partial_u r < 0$ *in* Q^+.

§4. Gravitational mass

Now, we will define the *generalized Misner–Sharp mass*, which is a useful tool to analyze spherical symmetric gravitational system.

Definition 1 (Maeda and Nozawa [7]). *The generalized Misner–Sharp mass is*

$$m(u,v) = \frac{(n-2)V_{n-2}r^{n-3}}{2\kappa_n^2}\left(\mu + \frac{\tilde{\alpha}}{r^2}\mu^2\right),$$

where V_{n-2} is the volume of $(n-2)$-sphere.

Using equations (3)–(7), we have evolution of the mass is as follow:

$$\begin{aligned}\partial_u m &= 2r^{n-2}V_{n-2}\Omega^{-2}\left(T_{uv}\partial_u r - T_{uu}\partial_v r\right),\\ \partial_v m &= 2r^{n-2}V_{n-2}\Omega^{-2}\left(T_{uv}\partial_v r - T_{vv}\partial_u r\right).\end{aligned}$$

By these evolution equations for $m(u,v)$, we have the following propositions [7]:

Proposition 3 (Monotonicity). *Monotonicity properties $\partial_u m \leq 0$ and $\partial_v m \geq 0$ hold in $\mathcal{R}$ if the dominant energy condition $T_{uu} \geq 0, T_{vv} \geq 0, T_{uv} \geq 0$ is satisfied.*

Proposition 4 (Positivity). *$m \geq 0$ on any nonspacelike hypersurfaces in $\mathcal{R}$ with regular center if the dominant energy condition holds.*

In the case of massless scalar fields, one can show the positivity of m in whole spacetime.

Corollary 1. *$m \geq 0$ in Q^+ with regular center in the case of massless scalar fields.*

We can also show the following proposition, which is an immediate concequence of the identity $\mu \equiv 1 + \frac{4\partial_u r \partial_v r}{\Omega^2}$, with proposition 2.

Proposition 5. *The relation $1-\mu = 0$ folds on $\mathcal{A}$, $1-\mu < 0$ in $\mathcal{T}$ and $1-\mu > 0$ in $\mathcal{R}$, where*

$$1-\mu = 1 + \frac{r^2}{2\tilde{\alpha}}\left(1 - \sqrt{1 + \frac{8\kappa_n^2\tilde{\alpha}m}{(n-2)V_{n-2}^1 r^{n-1}}}\right).$$

From the above and the positivity of m, we have

Corollary 2. $1 + \frac{2\tilde{\alpha}}{r^2}\mu \geq 1$ *in Q^+ with regular center in the case of massless scalar fields.*

§5. Extension in the non-trapped region

From the local existence theorem and the maximality of the Cauchy development [1], [9], [3], we have the following extension principle:

Proposition 6. *Let* $p \in \overline{Q}^+ \setminus \overline{\Gamma}$, *and* $q \in Q^+ \cap I^-(p)$ *such that* $J^-(p) \cap J^+(q) \setminus \{p\} \subset Q^+$, *and* $N(J^-(p) \cap J^+(q) \setminus \{p\}) < \infty$. *Then* $p \in Q^+$. *Here, given a subset* $Y \subset Q^+ \setminus \Gamma$, *define*

$$N(Y) = \sup\{|\Omega|_1, |\Omega^{-1}|_0, |r|_2, |r^{-1}|_0, |\phi|_1\},$$

where $|f|_k$ *denotes the restriction of the* C^k *norm to* Y.

Now, we can show

Proposition 7. *Let* $p \in \overline{\mathcal{R}} \setminus \overline{\Gamma}$ *and* $q \in \overline{\mathcal{R}} \cap I^-(p)$ *such that* $J^-(p) \cap J^+(q) \setminus \{p\} \subset \mathcal{R} \cap \mathcal{A}$. *Then* $p \in \mathcal{R} \cap \mathcal{A}$.

§6. Infinity

We will define *spatial infinity* and *null infinity* as follows.

Definition 2 (Dafermos [4]). *The curve* S *has a unique limit point* $i^0 = (\hat{u}, V)$ *on* $\overline{Q^+} \setminus Q^+$, *which is called spatial infinity. Let* $\mathcal{U}$ *be the set of all* u *defined by*

$$\mathcal{U} := \{u | \sup_{v:(u,v) \in Q^+} r(u,v) = \infty\}.$$

For each $u \in \mathcal{U}$, *there is a unique* $v^*(u)$ *such that* $(u, v^*(u)) \in (\overline{Q^+} \setminus Q^+) \cap J^+(\tilde{S})$. *Define the future null infinity* $\mathcal{I}^+$ *as follows*:

$$\mathcal{I}^+ := \bigcup_{u \in \mathcal{U}: v^*(u) = V} (u, v^*(u)).$$

We will denote $\inf_{\mathcal{I}^+} m$ by M_f, which is called the *final Bondi mass.* One can show that $\mathcal{I}^+$ is non-empty by the standard arguments and is a connected in going null ray with past end point i^0 by adapting [4].

§7. Penrose-like inequality

In this section, we will show Penrose-like inequality for our system. Set the *domain of outer communication* $\mathcal{D} = J^+(S) \cap J^-(\mathcal{I}^+) \cap Q^+$ and one can show $\mathcal{D} \subset \mathcal{R}$. The *event horizon* $\mathcal{H}$ is defined by the future boundary of $\mathcal{D}$ in Q^+.

Lemma 1. *On $\tilde{\mathcal{A}}$,*

$$\frac{(n-2)V_{n-2}}{2\kappa_n^2} r^{n-5}\left(r^2+\tilde{\alpha}\right) \leq M_f.$$

Here, $\tilde{\mathcal{A}}$ is a non-empty achronal curve intersecting all ingoing null curves for $v > v_0$ for sufficiently large v_0, where $\tilde{\mathcal{A}} = \{(u,v) \in \mathcal{A} | (u^*, v) \in \mathcal{R}$ for all $u^* < u$ and $\exists u' : (u', v) \in J^-(\mathcal{I}^+) \cap Q^+\}$ is outermost apparent horizon.

Proof. See [7]. Q.E.D.

Lemma 2. *On $\mathcal{H}$,*

$$\frac{(n-2)V_{n-2}}{2\kappa_n^2} r^{n-5}\left(r^2+\tilde{\alpha}\right) \leq M_f.$$

Proof. Assume the contrary, that is, there is a point $(\tilde{U}, \tilde{V})$ with $r(\tilde{U}, \tilde{V}) = R > r_P$ on the horizon. As shown in [4], there is a neighbourhood of the horizon which is in $\mathcal{R}$:

$$\mathcal{N} = [u_0, u''] \times [\tilde{V}, V] \subset \mathcal{R}, \qquad u_0 < \tilde{U} < u''.$$

In addition, we can chose the region $[\tilde{U}, u''] \times [\tilde{V}, V] \subset \mathcal{R}$ where is inside $\mathcal{H}^+$ with

$$r \geq R'$$

and

$$\begin{aligned} 1-\mu &= 1+\frac{r^2}{2\tilde{\alpha}}\left(1-\sqrt{1+\frac{8\kappa_n^2\tilde{\alpha}m}{(n-2)V_{n-2}r^{n-1}}}\right) \\ &\geq 1+\frac{r^2}{2\tilde{\alpha}}\left(1-\sqrt{1+\frac{8\kappa_n^2\tilde{\alpha}M}{(n-2)V_{n-2}(R')^{n-1}}}\right), \end{aligned}$$

where $R' < R$.

Now, we will show that for any $u^* \in [u_0, u'']$, $\lim_{v^* \to V} r(u^*, v^*) = \infty$, which is a contradiction for the definition of the event horizon. In $\mathcal{R}$, we have

$$M \geq \sup_{\bar{v} \leq \tilde{V}} \int_{u_0}^{u^*} \frac{V_{n-2}}{2} r^{n-2} T_{uu} \frac{1-\mu}{(-\partial_u r)}(\overline{uv}) d\bar{u},$$

by integrating (8). Then, we have the following estimate

$$
\begin{aligned}
&\sup_{\overline{v}\le\tilde{V}}\int_{u_0}^{u^*}\frac{V_{n-2}}{2}\frac{rT_{uu}}{(-\partial_u r)}(\overline{uv})d\overline{u}\\
\le\ & \frac{M}{(R')^{n-3}}\left[1+\frac{(R')^2}{2\tilde{\alpha}}\left(1-\sqrt{1+\frac{8\kappa_n^2\tilde{\alpha}M}{(n-2)V_{n-2}(R')^{n-1}}}\right)\right]^{-1}\\
\equiv\ & C_{M,R',\alpha,n}.
\end{aligned}
$$

From equation (5), we have

$$\partial_u A=\frac{\kappa_n^2 rT_{uu}}{(n-2)\partial_u r}\left(1+\frac{2\tilde{\alpha}}{r^2}\mu\right)^{-1}A,$$

where $A=\dfrac{\partial_v r}{1-\mu}$. Thus, we have

$$\int_{u_0}^{u^*}\frac{\partial_u A}{A}d\overline{u}\ge C_{M,R',\alpha,n}.$$

From this inequality, the following estimate is obtained

$$\frac{\partial_v r}{1-\mu}(u^*,v^*)\ge Ce^{C_{M,R',\alpha,n}}\frac{\partial_v r}{1-\mu}(u_0,v^+),$$

then $\partial_v r(u^*,v^*)\ge Ce^{C_{M,R',\alpha,n}}\partial_v r(u_0,v^*)$. This means that $r(u^*,v^*)\to\infty$ as $v^*\to V$ since $r(u_0,v^*)\to\infty$ as $v^*\to V$. Then, $(u^*,V)\in\mathcal{I}^+$. This is a contradiction. Q.E.D.

§8. Completeness of future null infinity

The purpose of this section is to prove completeness of future null infinity and then to finish the proof of Theorem 1.

Lemma 3. *If $\mathcal{A}\cup\mathcal{T}$ is non-empty, then $\mathcal{I}^+$ is future complete.*

Proof. Define the vector field

$$X(u,v)=\frac{\Omega^2(u_0,v)}{\Omega^2(u,v)}\frac{\partial}{\partial u}$$

on $J^-(\mathcal{I}^+)\cap Q^+$. This is parallel along all ingoing null rays and along the curve $u=u_0$. We need to show

$$I\equiv\int_{u_0}^{\tilde{U}}\frac{du}{X(u,v)\cdot u}\to\infty\quad\text{as}\quad v\to V.$$

Now, we have

$$
\begin{aligned}
I &\geq \int_{u_0}^{u^*(v)} \frac{\Omega^2(u,v)}{\Omega^2(u_0,v)} du \\
&= \int_{u_0}^{u^*(v)} \left[\frac{-\partial_u r(u,v)}{-\partial_u r(u_0,v)} \exp\left\{ \frac{\kappa_n^2}{n-2} \int_{u_0}^{u} \frac{r T_{uu}}{\partial_u r} \left(1 + \frac{2\tilde{\alpha}}{r^2}\mu\right)^{-1} d\bar{u} \right\} \right] du \\
&\geq e^{-C_{M,R',\alpha,n}} \frac{1}{-\partial_u r(u_0,v)} \int_{u_0}^{u^*(v)} -\partial_u r(u,v) du \\
&= e^{-C_{M,R',\alpha,n}} \frac{r(u_0,v) - R}{-\partial_u r(u_0,v)},
\end{aligned}
$$

from (5). Define $B = \dfrac{\partial_u r}{1-\mu} < 0$. From (6), we have

$$
\partial_v B = \frac{\kappa_n^2 r T_{vv}}{(n-2)\partial_v r} \left(1 + \frac{2\tilde{\alpha}}{r^2}\mu\right)^{-1} B.
$$

Integrating this equation, we have the following estimate

$$
-\partial_u r(u_0, v) \leq C e^{C_{M,R',\alpha,n}} \quad \text{for} \quad \forall v \geq v_0.
$$

Then $I \to \infty$ as $v \to V$. Q.E.D.

Thus, Propositions 6 and 7 and Lemmas 2 and 3 are give the proof of Theorem 1.

In summary, the similar results are already given [2], [3], [4], [5], [8] in the Einstein theory. Our result is a generalization to the EGB theory, which is a more general one of gravitational theory.

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