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# Stability of ground states of NLS with fourth order dispersion

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**Abstract.**

In this paper, we investigate the existence, uniqueness and stability of the ground states of nonlinear Schrödinger type equations with a small fourth order dispersion. Such equations appear in the higher order approximation of the propagation of laser beam in Kerr medium. We show that for the critical case, the ground state, which is unstable in the absence of the fourth order dispersion, becomes stable with small fourth order term.

## §1. Introduction

We consider the following nonlinear Schrödinger type equation with fourth order dispersion term:

$$iu_t = \omega\Delta^2 u - \Delta u - |u|^2 u, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \tag{1.1}$$

where $\omega \in \mathbb{R}$. For the case $\omega = 0$, (1.1) is known as a nonlinear Schrödinger equation and it has received a great deal of attention (see, for example [1]). For the case $\omega \neq 0$, (1.1) has been introduced in the series of work by Karpman and Shagalov (see [7] and references therein) to investigate the effect of small forth order dispersion term in the propagation of laser beam in Kerr medium. From a mathematical point of view, (1.1) without the term $-\Delta u$ has been studied by Fibich, Ilan and Papanicolau [3], who described many properties of the fourth order NLS partially relying on numerical analysis.

In this paper, we consider the existence, uniqueness and the stability of standing wave solutions $u(x,t) = e^{i\mu t}\phi(x)$, where the real valued

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function $\phi$ satisfies

$$0 = \omega\Delta^2\phi - \Delta\phi + \mu\phi - |\phi|^2\phi. \tag{1.2}$$

Our aim is to investigate the difference between the cases $\omega = 0$ and $\omega > 0$ by looking at the stability of standing waves. Here by stability we mean the following.

**Definition 1.1.** *We say that $e^{i\mu t}\phi_\omega$ is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ with the following property: when $||u_0 - \phi_\omega||_{H^2} < \delta$, the solution $u(t)$ of (1.1) with $u(0) = u_0$ exists globally in time and satisfies*

$$\sup_{t>0} \inf_{\theta,y\in\mathbb{R}} ||u(t) - e^{i\theta}\phi_\omega(\cdot - y)||_{H^2} < \varepsilon.$$

*If $e^{i\mu t}\phi_\omega$ is not stable, we say it is unstable.*

The case $\omega = \omega_0$ and $\mu = \mu_0$ can be rescaled to the case $\omega = \mu_0\omega_0$ and $\mu = 1$. Thus, we set $\mu = 1$ without loss of generality.

For the case $\omega = 0$, it is known that all standing waves are unstable [10]. Further, there exists a blow-up solution with a initial data arbitrary near the standing waves. For the case $\omega \neq 0$, it is known that (1.1) is globally well-posed in $H^2(\mathbb{R}^2)$ ([2]). Therefore, for arbitrary small $\omega \neq 0$, the term $\omega\Delta^2 u$ prevents the blowup. In this paper, we show that there exists a positive standing wave which is stable.

We begin with the existence of standing wave for $\omega > 0$. Let

$$\begin{aligned} S_\omega(u) &= 2\omega||\Delta u||_{L^2}^2 + 2||\nabla u||_{L^2}^2 + 2||u||_{L^2}^2 - ||u||_{L^4}^4, \\ N_\omega(u) &= \omega||\Delta u||_{L^2}^2 + ||\nabla u||_{L^2}^2 + ||u||_{L^2}^2 - ||u||_{L^4}^4. \end{aligned}$$

$S_\omega$ is a conservation quantity of (1.1). Then, set

$$G_\omega = \left\{ u \in H^2(\mathbb{R}^2) \setminus \{0\} \mid N_\omega(u) = 0,\ S_\omega(u) = \inf_{N_\omega(v)=0, v\neq 0} S_\omega(v) \right\}.$$

We call $\phi_\omega \in G_\omega$, the ground state.

**Remark 1.** By the Lagrange multiplier method, $\phi_\omega \in G_\omega$ satisfies (1.2) with $\mu = 1$.

By a standard argument using the concentration compactness, we can show the existence of the ground state.

**Proposition 1.** *$G_\omega \neq \emptyset$.*

The main results in this paper are the following two theorems. The first is concerned with the uniqueness of the ground states.

**Theorem 1.** *If* $0 < \omega \ll 1$, *then*

$$G_\omega = \left\{ e^{i\theta}\phi_\omega(\cdot - y) \;\middle|\; \theta \in \mathbb{R}, y \in \mathbb{R}^2 \right\}.$$

*In particular, we can take* $\phi_\omega$ *to be positive and radially symmetric.*

The second is concerned with the stability.

**Theorem 2.** *Let* $\phi_\omega \in G_\omega$. *If* $0 < \omega \ll 1$, *then* $e^{it}\phi_\omega$ *is stable.*

There are several related results for the stabilization of ground states of cubic nonlinear Schrödinger equations in $\mathbb{R}^2$. For the linear potential case, see [4] and for the nonlocal nonlinearity case, see [8].

This paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we prove Theorem 2.

## §2. Proof of Theorem 1

In this section we prove Theorem 1. Before starting the proof, we prepare some notations and propositions. Set

$$||u||_\omega^2 := \omega||\Delta u||_{L^2}^2 + ||\nabla u||_{L^2}^2 + ||u||_{L^2}^2.$$

Note that under the condition $N_\omega(u) = 0$, we have $S_\omega(u) = ||u||_\omega^2 = ||u||_{L^4}^4$. For the case $\omega = 0$, it is well known that the minimizer of $S_0$ under the constraint $N_0(u) = 0$, $u \neq 0$ is attained by $\phi_0$ (See, for example, Chapter 8 of [1]), where $\phi_0$ is the unique positive radial solution of (1.2) with $\omega = 0$, $\mu = 1$.

**Proposition 2.** *We have* $G_0 = \left\{ e^{i\theta}\phi_0(\cdot - y) \;\middle|\; \theta \in \mathbb{R}, y \in \mathbb{R}^2 \right\}$. *Further, let* $u_n \in H^1(\mathbb{R}^2) \setminus \{0\}$ *satisfy* $N_0(u_n) = 0$ *and* $S_0(u_n) \to S_0(\phi_0)$ *as* $n \to \infty$. *Then, there exist* $\theta_n \in \mathbb{R}$ *and* $y_n \in \mathbb{R}^2$ *such that* $e^{i\theta_n}u_n(\cdot - y_n) \to \phi_0$ *in* $H^1(\mathbb{R}^2)$ *as* $n \to \infty$.

We first show the convergence in $H^1(\mathbb{R}^2)$ by using the variational characterization of $\phi_0$ (Proposition 2) and $\phi_{\omega_n}$ (Proposition 1).

**Lemma 1.** *Let* $\phi_{\omega_n} \in G_{\omega_n}$, *then there exists* $\{\theta_{\omega_n}\} \subset \mathbb{R}$, $\{y_{\omega_n}\} \subset \mathbb{R}^2$ *such that* $e^{i\theta_{\omega_n}}\phi_{\omega_n}(\cdot - y_{\omega_n}) \to \phi_0$ *as* $\omega_n \to 0$ *in* $H^1(\mathbb{R}^2)$.

*Proof.* Take $a_{\omega_n} > 0$ such that $N_{\omega_n}(a_{\omega_n}\phi_0) = 0$. Then,

$$a_{\omega_n}^2 = \left(\omega_n||\Delta\phi_0||_{L^2}^2 + ||\phi_0||_{H^1}^2\right)/||\phi_0||_{H^1}^2 = 1 + \omega_n||\Delta\phi_0||_{L^2}^2/||\phi_0||_{H^1}^2.$$

Thus, we have $a_{\omega_n} > 1$ and $a_{\omega_n} \to 1$ as $\omega_n \to 0$. Since $\phi_{\omega_n}$ is the minimizer of $S_{\omega_n}$ under the constraint $N_{\omega_n}(\phi_{\omega_n}) = 0$, we have

$$||\phi_{\omega_n}||_{\omega_n}^2 = S_{\omega_n}(\phi_{\omega_n}) \leq S_{\omega_n}(a_{\omega_n}\phi_0) = a_{\omega_n}^2||\phi_0||_{\omega_n}^2.$$

Therefore, we see that $\{\phi_{\omega_n}\}$ is bounded in $H^1(\mathbb{R}^2)$. Next, let $b_{\omega_n} > 0$ such that $N_0(b_{\omega_n}\phi_{\omega_n}) = 0$. Then, since $N_{\omega_n}(\phi_{\omega_n}) = 0$, we have

$$b_{\omega_n}^2 = ||\phi_{\omega_n}||_{H^1}^2/||\phi_{\omega_n}||_{L^4}^4 = 1 - \omega_n||\Delta\phi_{\omega_n}||_{L^2}^2/||\phi_{\omega_n}||_{L^4}^4.$$

Thus, we have $b_{\omega_n} < 1$. Since $\phi_0$ is the minimizer of $S_0$ under the constraint $N_0(\phi_0) = 0$, we have

$$||\phi_0||_{H^1}^2 = S_0(\phi_0) \leq S_0(b_{\omega_n}\phi_{\omega_n}) = ||b_{\omega_n}\phi_{\omega_n}||_{H^1}^2 \leq ||\phi_{\omega_n}||_{H^1}^2.$$

Therefore, we have

$$||\phi_{\omega_n}||_{\omega_n}^2 \leq a_{\omega_n}^2||\phi_{\omega_n}||_{\omega_n}^2 \leq a_{\omega_n}^2\left(\omega_n||\Delta\phi_0||_{L^2}^2 + ||\phi_{\omega_n}||_{H^1}^2\right).$$

This implies,

$$\omega_n||\Delta\phi_{\omega_n}||_{L^2}^2 \leq \omega_n a_{\omega_n}^2||\Delta\phi_0||_{L^2}^2 + (a_{\omega_n}^2 - 1)||\phi_{\omega_n}||_{H^1}^2 \to 0, \ (\omega_n \to 0).$$

On the other hand, by the Sobolev embedding, we have

$$||\phi_{\omega_n}||_{H^1}^2 \leq ||\phi_{\omega_n}||_{\omega_n}^2 = ||\phi_{\omega_n}||_{L^4}^4 \leq C||\phi_{\omega_n}||_{H^1}^4.$$

Thus, $||\phi_{\omega_n}||_{H^1}$ and $||\phi_{\omega_n}||_{L^4}$ are bounded away from 0. Therefore,

$$1 > b_{\omega_n}^2 = 1 - \omega_n||\Delta\phi_{\omega_n}||_{L^2}^2/||\phi_{\omega_n}||_{L^4}^4 \to 1, \ (\omega_n \to 0).$$

Finally, we show $\{b_{\omega_n}\phi_{\omega_n}\}_{n\in\mathbb{N}}$ is a minimizing sequence of $S_0$ under the constraint $N_0(u) = 0$ and $u \neq 0$.

$$||b_{\omega_n}\phi_{\omega_n}||_{H^1}^2 \leq b_{\omega_n}^2||\phi_{\omega_n}||_{\omega_n}^2 \leq b_{\omega_n}^2 a_{\omega_n}^2||\phi_0||_{\omega_n}^2 = ||\phi_0||_{H^1}^2 + o(1),$$

where $o(1) \to 0$ as $\omega_n \to 0$. Proposition 2 and the fact $b_{\omega_n} \to 1$ as $\omega_n \to 0$, gives the conclusion. Q.E.D.

By a standard bootstrap argument, we can prove the following.

**Lemma 2.** *Let $\phi_\omega \in G_\omega$, then there exist $\{\theta_\omega\} \subset \mathbb{R}$, $\{y_\omega\} \subset \mathbb{R}^2$ such that $e^{i\theta_\omega}\phi_\omega(\cdot - y_\omega) \to \phi_0$ as $\omega \to 0$ in $H^m(\mathbb{R}^2)$ for arbitrary $m \in \mathbb{N}$.*

We now show the uniqueness of ground states by using Lemma 2.

*Proof of Theorem* 1. Suppose there exists $\omega_n \to 0$ such that there exist $\phi_{\omega_n}, \psi_{\omega_n} \in G_{\omega_n}$ such that $\phi_{\omega_n}, \psi_{\omega_n} \to \phi_0$ in $H^4(\mathbb{R}^2)$ and there are no $y \in \mathbb{R}^2$, $\theta \in \mathbb{R}$ such that $\phi_{\omega_n} = e^{i\theta}\psi_{\omega_n}(\cdot - y)$. Let

$$\begin{aligned} F_j(\omega_n, y, \theta) &:= \langle\phi_{\omega_n} - e^{i\theta}\psi_{\omega_n}(\cdot - y), \partial_{x_j}\phi_0\rangle, \ j = 1, 2, \\ F_3(\omega_n, y, \theta) &:= \langle\phi_{\omega_n} - e^{i\theta}\psi_{\omega_n}(\cdot - y), i\phi_0\rangle, \end{aligned}$$

where $\langle u, v\rangle = \mathrm{Re}\int_{\mathbb{R}^2} u\bar{v}\,dx$. Then, $F_j(0,0,0) = 0$ $(j = 1,2,3)$ and $\frac{\partial(F_1,F_2,F_3)}{\partial(y_1,y_2,\theta)}(0,0,0)$ is invertible. Thus, by the inverse function theorem, there exists $y(\omega_n) \in \mathbb{R}^2$ and $\theta(\omega_n) \in \mathbb{R}$ such that $F(\omega_n, y(\omega_n), \theta(\omega_n)) = 0$ and $y(\omega_n) \to 0$, $\theta(\omega_n) \to 0$. Set $\tilde{\psi}_{\omega_n} = e^{i\theta(\omega_n)}\psi_{\omega_n}(\cdot - y(\omega_n))$. Then, we have $\tilde{\psi}_{\omega_n} \to \phi_0$ in $H^m(\mathbb{R}^2)$ for arbitrary $m$.

Now, subtract the two equations which $\phi_{\omega_n}$ and $\tilde{\psi}_{\omega_n}$ satisfies. Then,

$$(\omega_n\Delta^2 - \Delta + 1)w_n = U_{1,n}w_n + U_{2,n}\bar{w}_n,$$

where $w_n = (\phi_{\omega_n} - \tilde{\psi}_{\omega_n})/||\phi_{\omega_n} - \tilde{\psi}_{\omega_n}||_{L^\infty}$ and $U_{1,n} = (\phi_{\omega_n} + \tilde{\psi}_{\omega_n})\bar{\tilde{\psi}}_{\omega_n}$, $U_{2,n} = \phi_{\omega_n}^2$. Note that $(\omega_n\Delta^2 - \Delta + 1)^{-1}$ is a bounded operator from $H^s$ to $H^{s+2}$ with the operator norm bounded by 1. Thus, we have

$$||w_n||_{H^2} \le ||U_{1,n}w_n||_{L^2} + ||U_{2,n}\bar{w}_n||_{L^2} \le ||U_{1,n}||_{L^2} + ||U_{2,n}||_{L^2},$$

where we have used Lemma 2 and $||w_n||_{L^\infty} = 1$. Further, we have

$$\begin{aligned} ||w_n||_{H^4} &\le ||U_{1,n}w_n||_{H^2} + ||U_{2,n}\bar{w}_n||_{H^2} \\ &\le (||U_{1,n}||_{H^2} + ||U_{2,n}||_{H^2})^2, \\ ||w_n||_{H^6} &\le (||U_{1,n}||_{H^4} + ||U_{2,n}||_{H^4})^3. \end{aligned}$$

Since $U_{1,n} \to 2\phi_0$ and $U_{2,n} \to \phi_0^2$ in $H^4(\mathbb{R}^2)$, $||w_n||_{H^6}$ is bounded. Therefore, by the Sobolev embedding $H^6(\mathbb{R}^2) \hookrightarrow W^{4,\infty}(\mathbb{R}^2)$, we see that $||\Delta^2 w_n||_{L^\infty}$ is bounded.

Next, $w_n \to w_0$ in $C^1_{loc}$ for some $w_0$, where $w_0$ satisfies

$$\begin{aligned} Lw &= (-\Delta + 1 - 2\phi_0^2)w_0 - \phi_0^2\bar{w}_0 = 0, \\ \langle w_0, \partial_{x_j}\phi_0\rangle &= \langle w_0, i\phi_0\rangle = 0,\ (j = 1,2). \end{aligned}$$

Since it is known that Ker$L$ is spanned by $\partial_{x_j}\phi_0$, $(j = 1,2)$ and $i\phi_0$ (see [11]), we have $w_0 = 0$.

Now, let $x_n$ such that $|w_n(x_n)| = 1$. Take $s_n$ such that $e^{is_n}w_n(x_n) = 1$. Then, Re $e^{is_n}w_n$ attains its maximum at $x_n$ and satisfies,

$$(2.1)\quad (\omega_n\Delta^2 - \Delta + 1)\mathrm{Re}\ e^{is_n}w_n = \mathrm{Re}\left(e^{is_n}(U_{1,n}w_n + U_{2,n}\bar{w}_n)\right)$$

Then, we have $|x_n| \to \infty$ because $w_n \to 0$ in $C^0_{loc}$. However, we have $|U_{j,n}(x_n)| \to 0$ $(j = 1,2)$, $\omega_n\Delta^2 w_n(x_n) \to 0$ and $-\Delta w_n(x_n) \ge 0$. This contradicts (2.1).

Finally we show the positivity. Set, $a_\omega = (1 + \sqrt{1-4\omega})/2$ and $b_\omega = (1 - \sqrt{1-4\omega})/2\omega$. Note that we have

$$\omega\Delta^2 - \Delta + 1 = (-\omega\Delta + a_\omega)(-\Delta + b_\omega).$$

Let $K_\omega$ be the integral kernel of $(-\omega\Delta + a_\omega)^{-1}$. That is

$$(2.2) \qquad (-\omega\Delta + a_\omega)^{-1}u = K_\omega * u.$$

It is known that $K_\omega$ is a positive function. Now, minimizing $S_\omega$ under the constraint $N_\omega(u) = 0$, $u \neq 0$ is equivalent to minimizing $\tilde{S}_\omega$ under the constraint $\tilde{N}(v) = 0$, $v \neq 0$ where

$$\begin{aligned} \tilde{S}_\omega(v) &= 2||\nabla v||_{L^2}^2 + 2b_\omega||v||_{L^2}^2 - ||(-\omega\Delta + a_\omega)^{-1/2}v||_{L^4}^4, \\ \tilde{N}_\omega(v) &= ||\nabla v||_{L^2}^2 + b_\omega||v||_{L^2}^2 - ||(-\omega\Delta + a_\omega)^{-1/2}v||_{L^4}^4. \end{aligned}$$

Taking $u = (-\omega\Delta + a_\omega)^{-1/2}v$, one can show that these two problems are equivalent. Now, we show that if $v_\omega$ is a minimizer, then also $|v_\omega|$ is a minimizer. Indeed, we have

$$||\nabla v||_{L^2}^2 \geq ||\nabla |v|||_{L^2}^2, \quad ||(-\omega\Delta + a_\omega)^{-1/2}v||_{L^4}^4 \leq ||(-\omega\Delta + a_\omega)^{-1/2}|v|||_{L^4}^4,$$

by the positivity of the integral kernel. Thus, we have

$$\tilde{S}_\omega(v_\omega) \geq \tilde{S}_\omega(|v_\omega|), \quad \tilde{N}_\omega(v_\omega) \geq \tilde{N}_\omega(|v_\omega|).$$

Now, suppose $0 = \tilde{N}_\omega(v_\omega) > \tilde{N}_\omega(|v_\omega|)$. Then, there exists some $t \in (0,1)$ such that $\tilde{N}_\omega(t|v_\omega|) = 0$. However, since $\tilde{S}_\omega(v_\omega) \geq \tilde{S}_\omega(|v_\omega|) > \tilde{S}_\omega(t|v_\omega|)$, we have a contradiction. Thus, we have $\tilde{N}_\omega(v_\omega) = \tilde{N}_\omega(|v_\omega|)$. This shows that also $|v_\omega|$ is a minimizer. Now, $u_\omega := (-\omega\Delta + a_\omega)^{-1/2}|v_\omega|$ is a minimizer of the original minimizing problem. Since $K_\omega$ is positive, we see that $u_\omega$ is positive. Therefore, from the uniqueness result, we have that $\phi_\omega$ is positive. Q.E.D.

## §3. Proof of Theorem 2

We show the stability of the ground state by using the stability criterion of Grillakis, Shatah and Strauss [5]. To use the criterion, we need to check the spectral property of the linearized operator and smoothness of the map $\omega \mapsto \phi_\omega$.

**Lemma 3.** *Let $0 < \omega \ll 1$ and let $\phi_\omega \in G_\omega$ be given as a positive radially symmetric function. Then, the following holds.*

(i) $\mathrm{Ker}(\omega\Delta^2 - \Delta + 1 - 3\phi_\omega^2) = \mathrm{span}\langle \partial_{x_1}\phi_\omega, \partial_{x_2}\phi_\omega \rangle$.

(ii) $\omega\Delta^2 - \Delta + 1 - 3\phi_\omega^2$ *has only one negative eigenvalue and it is uniformly bounded away from zero with respect to $\omega$.*

(ii) $\sigma(\omega\Delta^2 - \Delta + 1 - 3\phi_\omega^2) \cap (0,\infty)$ *is bounded away from zero.*

**Lemma 4.** *There exists $\varepsilon_0 > 0$ such that $\omega \mapsto \phi_\omega$ is $C^1((0,\varepsilon_0); H^2(\mathbb{R}^2))$. Further, $\partial_\omega\phi_\omega$ is bounded in $L^2(\mathbb{R}^2)$ for $\omega \in (0,\varepsilon_0)$.*

Since the proofs of Lemmas 3, 4 are standard, we omit the proof.

Now, let $\psi_\omega(x) = \omega^{1/2}\phi_\omega(\omega^{1/2}x)$. Then, $e^{i\omega t}\psi_\omega$ is a solution of

$$iu_t = \Delta^2 u - \Delta u - |u|^2 u, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{3.1}$$

Therefore, to prove Theorem 2, it suffices to show that $e^{i\omega t}\psi_\omega$ is stable under the flow of (1.1). Let

$$\begin{aligned} S_{1,\omega}(u) &= 2(||\Delta u||_{L^2}^2 + ||\nabla u||_{L^2}^2 + \omega||u||_{L^2}^2) - ||u||_{L^4}^4, \\ d(\omega) &= S_{1,\omega}(\psi_\omega). \end{aligned}$$

By Lemmas 3, 4, we can apply the stability criterion of Grillakis, Shatah and Strauss, which is the following.

**Theorem 3.** *Let $0 < \omega \ll 1$. Then $e^{i\omega t}\psi_\omega$ is stable if $d''(\omega) > 0$.*

**Remark 2.** $d'(\omega) = \langle S'_{1,\omega}(\psi_\omega), \partial_\omega \psi_\omega\rangle + 2||\psi_\omega||_{L^2}^2 = 2||\psi_\omega||_{L^2}^2$. Therefore, by Lemma 4, $d$ is twice differentiable if $0 < \omega \ll 1$.

**Lemma 5.** *For sufficiently small $\omega > 0$, $d''(\omega) > 0$.*

*Proof.* Recall the equation $\phi_\omega$ satisfies,

$$0 = \omega\Delta^2\phi_\omega - \Delta\phi_\omega + \phi_\omega - \phi_\omega^3. \tag{3.2}$$

Differentiating (3.2) by $\omega$, we obtain

$$-\Delta^2\phi_\omega = (\omega\Delta^2 - \Delta + 1 - 3\phi_\omega^2)\partial_\omega\phi_\omega. \tag{3.3}$$

By computing $\int_{\mathbb{R}^2}(3.2) \times \partial_\omega\phi_\omega\, dx - \int_{\mathbb{R}^2}(3.3) \times \phi_\omega\, dx$, we have

$$\int_{\mathbb{R}^2} |\Delta\phi|^2\, dx = 2\int_{\mathbb{R}^2} \phi_\omega^3 \partial_\omega\phi_\omega\, dx. \tag{3.4}$$

Now, we compute $d''(\omega)$. Since $d(\omega) = ||\psi_\omega||_{L^4}^4 = \omega||\phi_\omega||_{L^4}^4$, we have

$$\begin{aligned} d'(\omega) &= ||\phi_\omega||_{L^4}^4 + 4\omega\int_{\mathbb{R}^2} \phi_\omega^3\partial_\omega\phi_\omega\, dx = ||\phi_\omega||_{L^4}^4 + 2\omega||\Delta\phi_\omega||_{L^2}^2, \\ d''(\omega) &= \int_{\mathbb{R}^2} \left(4\phi_\omega^3\partial_\omega\phi_\omega + 2|\Delta\phi_\omega|^2 + 4\omega\Delta\phi_\omega\Delta\partial_\omega\phi_\omega\right) dx. \end{aligned}$$

Thus, we have

$$d''(\omega) = 4||\Delta\phi_\omega||_{L^2}^2 + 4\omega\int_{\mathbb{R}^2} \Delta\phi_\omega\Delta\partial_\omega\phi_\omega\, dx.$$

By Lemmas 2 and 4, we have

$$\left|\omega\int_{\mathbb{R}^2} \Delta\phi_\omega\Delta\partial_\omega\phi_\omega\, dx\right| \le \omega||\Delta^2\phi_\omega||_{L^2}||\partial_\omega\phi_\omega||_{L^2} \to 0, \ (\omega \to 0).$$

Therefore, we have the conclusion. Q.E.D.

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