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Decay estimates of solutions for nonlinear viscoelastic systems

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Abstract.

In this paper, we consider a class of second order hyperbolic systems of viscoelasticity in the whole space and show the global existence and sharp decay estimates of solutions under the smallness condition on the initial data. The tools that we use here are the time-weighted energy method and the semigroup argument.

§1. Introduction

We study the initial value problem for the nonlinear hyperbolic system of equations which decribes a motion of viscoelastic materials. It is known that the global existence of the solutions is shown by combining the a priori estimate of solutions and the local existence result. Here, the key result, a priori estimate of solutions is obtained with use of the time-weighted energy method introduced by Matsumura [10] and the semigroup argument. As a corollary of the a priori estimates of solutions, we obtain the sharp decay estimates of solutions.

We consider the following nonlinear hyperbolic systems with dissipation:

$$u_{tt} - \sum_j b^j(\partial_x u)_{x_j} + \sum_{j,k} K^{jk} * u_{x_j x_k} + Lu_t = 0, \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \tag{1.2}$$

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Here u is an unknown m-vector function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ Here $b^j(v)$ are smooth m-vector functions of $v = (v_1, \cdots, v_n) \in \mathbb{R}^{mn}$, where $v_j \in \mathbb{R}^m$ corresponds to u_{x_j}; $K^{jk}(t)$ are smooth $m \times m$ real matrix functions of $t \geq 0$ satisfying $K^{jk}(t)^T = K^{kj}(t)$ for each j, k and $t \geq 0$; and L is an $m \times m$ real symmetric constant matrix; the symbol "$*$" denotes the convolution with respect to t, and the superscript "T" denotes the transposed. The elastic term, $b^j(v)$, is given by the following assumption. We assume that there exists a smooth scalar function (the free energy) $\phi(v)$ such that

$$b^j(v) = D_{v_j}\phi(v), \tag{1.3}$$

where $D_{v_j}\phi(v)$ denotes the Fréchet derivative of $\phi(v)$ with respect to $v_j \in \mathbb{R}^m$and put

$$B^{jk}(v) := D_{v_k}b^j(v) = D_{v_k}D_{v_j}\phi(v). \tag{1.4}$$

It then follows that $B^{jk}(v)$ is an $m \times m$ real matrix function satisfying $B^{jk}(v)^T = B^{kj}(v)$ for each j, k and $v \in \mathbb{R}^{mn}$.

We note that the system (1.1) can be rewritten in the following quasi-linear form

$$u_{tt} - \sum_{j,k} B^{jk}(\partial_x u)u_{x_j x_k} + \sum_{j,k} K^{jk} * u_{x_j x_k} + Lu_t = 0. \tag{1.5}$$

We introduce the following symbols of the differential operators associated with (1.5):

$$B_\omega(v) := \sum_{j,k} B^{jk}(v)\omega_j\omega_k, \qquad K_\omega(t) := \sum_{j,k} K^{jk}(t)\omega_j\omega_k \tag{1.6}$$

for $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. We see that $B_\omega(v)$ and $K_\omega(t)$ are real symmetric matrices. As in [1, 2, 3, 4], we impose the following structural conditions on the system (1.1).

[A1] $B_\omega(0)$ is positive definite for each $\omega \in S^{n-1}$, while $K_\omega(t)$ is nonnegative definite for each $\omega \in S^{n-1}$ and $t \geq 0$, and L is real symmetric and nonnegative definite.

[A2] $B_\omega(0) - \mathcal{K}_\omega(t)$ is positive definite for each $\omega \in S^{n-1}$ uniformly in $t \geq 0$, where $\mathcal{K}_\omega(t) := \int_0^t K_\omega(s)\, ds$.

[A3] $K_\omega(0) + L$ is (real symmetric and) positive definite for each $\omega \in S^{n-1}$.

[A4] $K_\omega(t)$ is smooth in $t \geq 0$ and decays exponentially as $t \to \infty$. Precisely, there are positive constants c_0, c_1 and C_0 such that $-c_1 K_\omega(t) \leq \dot{K}_\omega(t) \leq -c_0 K_\omega(t)$ and $-C_0 K_\omega(t) \leq \ddot{K}_\omega(t) \leq C_0 K_\omega(t)$ for $\omega \in S^{n-1}$ and $t \geq 0$, where $\dot{K}_\omega(t) := \partial_t K_\omega(t)$ and $\ddot{K}_\omega(t) := \partial_t^2 K_\omega(t)$.

Notations. For a nonnegative integer s, $H^s = H^s(\mathbb{R}^n)$ denotes the Sobolev space of L^2 functions on $\mathbb{R}^n$, equipped with the norm $\|\cdot\|_{H^s}$. For a nonnegative integer l, ∂_x^l denotes the totality of all the l-th order derivatives with respect to $x \in \mathbb{R}^n$. Also, for an interval I and a Banach space X, $C^l(I;X)$ denotes the space of l-times continuously differential functions on I with values in X.

Now we introduce the quantities Q_K, $Q_K^\sharp$ and $Q_K^\flat$ which will be used to describe the dissipation induced by the memory term $\sum_{j,k} K^{jk} * u_{x_j x_k}$ in (1.1):

$$Q_K[\partial_x u] := Q_K^\sharp[\partial_x u] + Q_K^\flat[\partial_x u], \tag{1.7}$$

$$Q_K^\sharp[\partial_x u] := \sum_{j,k} \int_{\mathbb{R}^n} K^{jk}[u_{x_j}, u_{x_k}]\, dx,$$

$$Q_K^\flat[\partial_x u] := \sum_{j,k} \int_{\mathbb{R}^n} \langle K^{jk} u_{x_j}, u_{x_k} \rangle\, dx.$$

Here $A[\psi,\zeta](t) = \displaystyle\int_0^t \langle\, A(t-\tau)(\psi(t)-\psi(\tau)),\, \zeta(t)-\zeta(\tau)\,\rangle\, d\tau$ for $\psi,\ \zeta \in \mathbb{C}^m$, where $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{C}^m$. This was previously introduced by J. E. Muñoz Rivera ([10], [11]) in the study of equations of viscoelasticity with memory (see also [1], [2], [3], [4]).

Throughout the paper, C and c denote various generic positive constants.

§2. Main theorems

This section is devoted to giving the main theorems of our paper. First, we give the main result, the global existence and the sharp decay of solutions to the problem (1.1),(1.2).

To this end, we introduce the time-weighted energy norm $E(t)$ and the corresponding dissipation norm $D(t)$ by

$$E(t)^2 := \sum_{m=0}^{s} E_m(t)^2, \qquad D(t)^2 := \sum_{m=0}^{s} \tilde{D}_m(t)^2 + \sum_{m=0}^{s-1} D_m(t)^2, \tag{2.1}$$

where

$$E_m(t)^2 = \sup_{0\le\tau\le t}(1+\tau)^m\Big(\|(\partial_x^m u_t, \partial_x^{m+1}u)(\tau)\|_{H^{s-m}}^2 + \sum_{l=m}^{s} Q_K[\partial_x^{l+1}u](\tau)\Big),$$

$$\tilde{D}_m(t)^2 = \int_0^t (1+\tau)^m\Big(\|(I-P)\partial_x^m u_t(\tau)\|_{H^{s-m}}^2 + \sum_{l=m}^{s} Q_K[\partial_x^{l+1}u](\tau)\Big)\,d\tau,$$

$$D_{m-1}(t)^2 = \int_0^t (1+\tau)^{m-1}\Big(\|(\partial_x^m u_t, \partial_x^{m+1}u)(\tau)\|_{H^{s-m}}^2 + \sum_{l=m}^{s} Q_K[\partial_x^{l+1}u](\tau)\Big)\,d\tau.$$

Here Q_K is defined in (1.7), I is the identity matrix, and P denotes the orthogonal projection matrix onto $\ker(L)$. We also makes use of the following L^∞ norm:

$$N(t) := \sup_{0\le\tau\le t}\Big\{\|\partial_x u(\tau)\|_{L^\infty} + (1+\tau)\|(\partial_x u_t, \partial_x^2 u)(\tau)\|_{L^\infty}\Big\}. \tag{2.2}$$

Theorem 2.1 (Global existence and sharp decay estimates). *Suppose that all the conditions* [A1]–[A4] *are satisfied. Let $n \ge 1$ and $s \ge s_0 + 2$ with $s_0 = [n/2] + 1$. Suppose that $(u_1, \partial_x u_0) \in H^s \cap L^1$ and put $E_1 := \|(u_1, \partial_x u_0)\|_{H^s} + \|(u_1, \partial_x u_0)\|_{L^1}$. Then there is a positive constant δ_1 such that if $E_1 \le \delta_1$, then the problem* (1.1), (1.2) *has a unique global solution u with $(u_t, \partial_x u) \in C^0([0,\infty); H^s)$. The solution satisfies*

$$E(t) + D(t) + M(t) \le CE_1 \tag{2.3}$$

for $t \ge 0$. In particular, we have the following optimal decay estimate:

$$\|(\partial_x^m u_t, \partial_x^{m+1}u)(t)\|_{H^{s-m-1}} \le CE_1(1+t)^{-n/4-m/2} \tag{2.4}$$

for $t \ge 0$, where $0 \le m \le s-1$.

We can prove the existence of the global solution by combining the local existence result and the corresponding a priori estimates of solutions. Here, we omit the proof of the local existence result and readers are kindly requested to refer [4] for outline of the proof.

Now we lay a foundation to prove the a priori estimates of solutions. We start it developing the time-weighted energy method for the system

(1.1), which is based on the energy method employed in our previous paper [2]. We list some related works for reference (see [5], [6], [7], [8], [12]).

Our time-weighted energy method, which is based on the previous energy method in [2], can yield the following energy inequality for the problem (1.1), (1.2).

Proposition 2.2 ([3], [4]). *Suppose that all the conditions* [A1]–[A4] *are satisfied. Let $n \geq 1$ and $s \geq [n/2]+2$. Assume that $(u_1, \partial_x u_0) \in H^s$ and put $E_0 = \|(u_1, \partial_x u_0)\|_{H^s}$. Let u be a solution to the problem* (1.1), (1.2) *satisfying $(u_t, \partial_x u) \in C^0([0,T]; H^s)$ for $T > 0$ such that $N_0(T) = \sup_{0\leq\tau\leq T} \|(\partial_x u, \partial_x u_t, \partial_x^2 u)(\tau)\|_{L^\infty}$ is suitably small. Then we have the following time-weighted energy estimate for $t \in [0,T]$:*

$$E(t)^2 + D(t)^2 \leq CE_0^2 + CN(t)D(t)^2. \tag{2.5}$$

Proof. We divide our proof into several steps. First, we apply ∂_x^l to (1.5) to obtain

$$\partial_x^l u_{tt} - \sum_{j,k} B^{jk}(\partial_x u)\partial_x^l u_{x_j x_k} + \sum_{j,k} K^{jk} * \partial_x^l u_{x_j x_k} + L\partial_x^l u_t = f^{(l)}, \tag{2.6}$$

where $f^{(l)} = \sum_{j,k}[\partial_x^l, B^{jk}(\partial_x u)]u_{x_j x_k}$ and $[\,\cdot\,,\,\cdot\,]$ denotes the commutator; notice that $f^{(0)} = 0$. Continuing the computation, we take the inner product of (2.6) with $\partial_x^l u_t$ and integrate in x over $\mathbb{R}^n$. Then we multiply the resulting equation by $(1+t)^m$, integrate with respect to t, and add for l with $m \leq l \leq s$. After tedious computations as in [2], we arrive at the basic energy estimate of the form

$$E_m(t)^2 + \tilde{D}_m(t)^2 \leq CE_0^2 + CN(t)D(t)^2 + mCD_{m-1}(t)^2 \tag{2.7}$$

where $0 \leq m \leq s$. Note that the last term on the right-hand side of (2.7) is absent if $m = 0$. Accordingly, we have the result obtained from the energy method in physical space.(see [2]).

The next step is to obtain the time-weighted dissipation norm for $\partial_x u_t$. For this purpose, we take the inner product of (2.6) with $\sum_{j,k}(K^{jk} * \partial_x^l u_{x_j x_k})_t$ and integrate over $\mathbb{R}^n$. Moreover, we multiply the result by $(1+t)^m$, integrate with respect to t, and add for l with $m \leq l \leq s-1$. Then the technical computations in [2] yield

$$\begin{aligned} &\int_0^t (1+\tau)^m \|\partial_x^{m+1} u_t(\tau)\|_{H^{s-m-1}}^2 \, d\tau \leq CE_0^2 \\ &+ CN(t)D(t)^2 + \lambda \int_0^t (1+\tau)^m \|\partial_x^{m+2} u(\tau)\|_{H^{s-m-1}}^2 \, d\tau \\ &+ C_\lambda\big(E_m(t)^2 + \tilde{D}_m(t)^2\big) + mCD_{m-1}(t)^2 \end{aligned} \tag{2.8}$$

for any $\lambda > 0$, where $0 \le m \le s-1$ and C_λ is a constant depending on λ; the last term on the right-hand side of (2.8) is absent if $m = 0$.

In the third step, we create the time-weighted dissipation norm which coressponds to $\partial_x^2 u$. For this purpose, first we rewrite the equation (1.1) as follows:

$$(2.9)\quad u_{tt} - \sum_{j,k} B^{jk}(0) u_{x_j x_k} + \sum_{j,k} K^{jk} * u_{x_j x_k} + L u_t = \sum_j g^j(\partial_x u)_{x_j},$$

where $g^j(\partial_x u) := b^j(\partial_x u) - \sum_k B^{jk}(0) u_{x_k} = O(|\partial_x u|^2)$. Then we apply ∂_x^{l+1} to (2.9), take the inner product with $\partial_x^{l+1} u$, and integrate over $\mathbb{R}^n$. Moreover, we multiply the result by $(1+t)^m$, integrate with respect to t, and add for l with $m \le l \le s-1$. Then the technical computations as in [2] give

$$(2.10)\quad \begin{aligned} &\int_0^t (1+\tau)^m \|\partial_x^{m+2} u(\tau)\|_{H^{s-m-1}}^2 \, d\tau \le C E_0^2 \\ &+ C N(t) D(t)^2 + C \int_0^t (1+\tau)^m \|\partial_x^{m+1} u_t(\tau)\|_{H^{s-m-1}}^2 \, d\tau \\ &+ C\big(E_m(t)^2 + \tilde{D}_m(t)^2\big) + mC D_{m-1}(t)^2, \end{aligned}$$

where $0 \le m \le s-1$; the last term on the right-hand side of (2.10) is absent if $m = 0$. Now we combine (2.8) and (2.10). Taking $\alpha > 0$ suitably small and using the definition of $D_m(t)$, we have

$$(2.11)\quad \begin{aligned} &D_m(t)^2 \le C E_0^2 + C N(t) D(t)^2 \\ &+ C\big(E_m(t)^2 + \tilde{D}_m(t)^2\big) + mC D_{m-1}(t)^2, \end{aligned}$$

where $0 \le m \le s-1$. Moreover, substituting (2.7) into (2.11), we obtain

$$(2.12)\quad D_m(t)^2 \le C E_0^2 + C N(t) D(t)^2 + mC D_{m-1}(t)^2$$

for $0 \le m \le s-1$, where the last term on the right-hand side of (2.12) is absent if $m = 0$.

Finally, we apply to (2.12) the induction argument with respect to m and conclude that

$$(2.13)\quad D_m(t)^2 \le C E_0^2 + C N(t) D(t)^2$$

for $0 \le m \le s-1$. Moreover, substituting (2.13) into (2.7), we have

$$(2.14)\quad E_m(t)^2 + \tilde{D}_m(t)^2 \le C E_0^2 + C N(t) D(t)^2$$

for $0 \le m \le s$. Now, summing up (2.14) and (2.13) for m with $0 \le m \le s$ and $0 \le m \le s-1$, respectively, and then adding the resultant two inequalities, we arrive at the desired estimate (2.5). This completes the proof of Proposition 2.2 Q.E.D.

Now we are at the last stage of obtaining the a priori estimates of solutions. Here, we close the time-weighted L^∞ norm, $N(t)$, defined in (2.5) suitably. In this case, we require a sharp decay estimates of solutions to the problem (1.1), (1.2). To measure the sharp decay of solutions, we introduce the time-weighted norm, $M(t)$, by

$$(2.15)\quad M(t) := \sum_{m=0}^{s-1} \sup_{0\le\tau\le t} (1+\tau)^{n/4+m/2} \|(\partial_x^m u_t, \partial_x^{m+1} u)(\tau)\|_{H^{s-m-1}}.$$

By applying the Gagliardo–Nirenberg inequality, we can control this $M(t)$ by $N(t)$ as follows:

$$(2.16)\qquad N(t) \le CM(t)$$

for $n \ge 1$ and $s \ge s_0 + 2$, where $s_0 = [n/2]+1$. In fact, we have

$$(2.17)\qquad \|\partial_x u(t)\|_{L^\infty} \le CM(t)(1+t)^{-n/2},$$

$$(2.18)\qquad \|(\partial_x u_t, \partial_x^2 u)(\tau)\|_{L^\infty} \le CM(t)(1+t)^{-n/2-1/2},$$

where $s \ge s_0+1$ in (2.17) and $s \ge s_0 + 2$ in (2.18).

Applying the semigroup argument, we can derive the following inequality for $M(t)$. Our semigroup argument is based on the decay property for the linearized system studied in [1]. Readers are kindly requested to refer our paper [4] for more details.

Proposition 2.3. *Suppose that all the conditions* [A1]–[A4] *are satisfied. Let* $n \ge 1$ *and* $s \ge s_0+1$ *with* $s_0 = [n/2]+1$. *Suppose that* $(u_1, \partial_x u_0) \in H^s \cap L^1$ *and put* $E_1 := \|(u_1, \partial_x u_0)\|_{H^s} + \|(u_1, \partial_x u_0)\|_{L^1}$. *Let* u *be a solution to the problem* (1.1), (1.2) *satisfying* $(u_t, \partial_x u) \in C^0([0,T]; H^s)$ *for some* $T>0$ *such that* $N_0(T) = \sup_{0\le\tau\le T} \|(\partial_x u, \partial_x u_t, \partial_x^2 u)(\tau)\|_{L^\infty}$ *is suitably small. Then we have the following inequality for* $t \in [0,T]$:

$$(2.19)\qquad M(t) \le CE_1 + CE(t)M(t) + CM(t)^2.$$

Proof of Theorem 2.1. Substituting (2.16) in (2.5) and noting that $E_0 \le E_1$, we get

$$(2.20)\qquad E(t)^2 + D(t)^2 \le CE_1^2 + CM(t)D(t)^2.$$

On the other hand, we multiply (2.19) by $M(t)$. After a simple manipulation, we have

$$M(t)^2 \le CE_1^2 + CE(t)M(t)^2 + CM(t)^3. \tag{2.21}$$

Let $X(t) := E(t) + D(t) + M(t)$. Then (2.20) and (2.21) give $X(t)^2 \le CE_1^2 + CX(t)^3$. This inequality can be solved as $X(t) \le CE_1$, provided that E_1 is sufficiently small. This give the a priori estimates of solutions. In particular, we have $M(t) \le CE_1$, which implies the sharp decay estimates of solutions (2.4). This completes the proof of Theorem 2.1. Q.E.D.

References

[1] P. M. N. Dharmawardane, J. E. Muñoz Rivera and S. Kawashima, Decay property for second order hyperbolic systems of viscoelastic materials, J. Math. Anal. Appl., **366** (2010), 621–635.

[2] P. M. N. Dharmawardane, T. Nakamura and S. Kawashima, Global solutions to quasi-linear hyperbolic systems of viscoelasticity, Kyoto J. Math., **51** (2011), 467–483.

[3] P. M. N. Dharmawardane, T. Nakamura and S. Kawashima, Time-weighted energy method for quasi-linear hyperbolic systems of viscoelasticity, Proc. Japan Acad. Ser. A Math. Sci., **87** (2011), 99–102.

[4] P. M. N. Dharmawardane, T. Nakamura and S. Kawashima, Decay estimates of solutions for quasi-linear hyperbolic systems of viscoelasticity, SIAM J. Math. Anal., **44** (2012), 1976–2001.

[5] K. Ide and S. Kawashima, Decay property of regularity-loss type and nonlinear effects for dissipative Timoshenko system, Math. Models Methods Appl. Sci., **18** (2008), 1001–1025.

[6] S. Kawashima and W.-A. Yong, Decay estimates for hyperbolic balance laws, Z. Anal. Anwend., **28** (2009), 1–33.

[7] Y. Liu and S. Kawashima, Global existence and asymptotic behavior of solutions for quasi-linear dissipative plate equation, Discrete Contin. Dyn. Syst., **29** (2011), 1113–1139.

[8] Y. Liu and S. Kawashima, Global existence and decay of solutions for a quasi-linear dissipative plate equation, J. Hyperbolic Differ. Equ., **8** (2011), 591–614.

[9] A. Matsumura, An energy method for the equations of motion of compressible viscous and heat-conductive fluids, MRC Technical Summary Report, Univ. of Wisconsin-Madison, #**2194** (1981).

[10] J. E. Muñoz Rivera, Asymptotic behaviour in linear viscoelasticity, Quart. Appl. Math., **52** (1994), 628–648.

[11] J. E. Muñoz Rivera, M. G. Naso and F. M. Vegni, Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory, J. Math. Anal. Appl., **286** (2003), 692–704.

[12] Y. Ueda and S. Kawashima, Decay property of regularity-loss type for the Euler–Maxwell system, Methods Appl. Anal., **18** (2011), 245–267.

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