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Remark on C_0-semigroups with scaling invariance

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Abstract.

We study C_0-semigroups acting on a Banach space which possess an invariant property with respect to an action of the multiplicative group of positive real numbers (scaling). Some properties on the domains or the spectrum of the associated generators are presented.

§1. Introduction

There are wide classes of evolution equations which possess invariant properties with respect to a scaling and translations. In the abstract settings a scaling and a translation can be considered as an action of the multiplicative group of positive real numbers and of the additive group of real numbers, respectively. Using this, [6] discussed large time behaviors of solutions to nonlinear evolution equations which possess scaling and translation invariance, within the abstract framework based on the semigroup theory; see also [5]. In this paper we present a short remark on properties of generators for C_0-semigroups possessing the invariance with respect to a scaling acting on a Banach space X. Only abstract results will be stated here; for concrete examples, see [5], [6].

§2. Preliminaries

In this section we recall the definition of scaling stated in [6]. Let X be a Banach space and let $\mathcal{L}(X)$ be the Banach space of all bounded linear operators in X. We assume that $(\mathcal{A}, \mathrm{Dom}(\mathcal{A}))$ is a closed linear operator in X which generates a C_0-semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0} \subset \mathcal{L}(X)$.

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Definition 2.1. *We say $\mathcal{R} = \{R_r\}_{r>0} \subset \mathcal{L}(X)$ a scaling acting on X if $\mathcal{R}$ is a strongly continuous action of $\{r \in \mathbb{R} \mid r > 0\}$ on X, i.e.,*

$$R_{r_1 r_2} = R_{r_1} R_{r_2}, \qquad r_1,\ r_2 > 0 \tag{2.1}$$

$$R_1 = I, \tag{2.2}$$

$$\lim_{r' \to r} R_{r'} u = R_r u \quad \text{in } X, \qquad \text{for each } u \in X. \tag{2.3}$$

The generator of $\mathcal{R} = \{R_r\}_{r>0}$ is denoted by $\mathcal{B}$, which is a closed linear operator defined by $\mathcal{B}f = \lim_{h\to 0} h^{-1}(R_{1+h}f - f)$ for $f \in \mathrm{Dom}(\mathcal{B})$, where $\mathrm{Dom}(\mathcal{B}) = \{f \in X \mid \lim_{h\to 0} h^{-1}(R_{1+h}f - f)$ exists $\}$. A scaling $\mathcal{R} = \{R_r\}_{r>0}$ induces an action on $C((0,\infty); X)$, called the scaling induced by $\mathcal{R}$, as follows.

$$\Theta_r(f)(t) = R_r(f(rt)) \qquad r > 0, \quad f \in C((0,\infty); X). \tag{2.4}$$

Definition 2.2. *Let $\mathcal{R} = \{R_r\}_{r>0}$ be a scaling and let $\mathcal{T} = \{\tau_a\}_{a\in\mathbb{R}} \subset \mathcal{L}(X)$ be a C_0-group acting on X.* (i) *We say that $\{e^{t\mathcal{A}}\}_{t\geq 0}$ is invariant with respect to the scaling induced by $\mathcal{R}$ if*

$$R_r e^{rt\mathcal{A}} = e^{t\mathcal{A}} R_r \qquad r > 0, \quad t \geq 0. \tag{2.5}$$

(ii) *We say that $\{e^{t\mathcal{A}}\}_{t\geq 0}$ is invariant with respect to $\mathcal{T}$ if*

$$\tau_a e^{t\mathcal{A}} = e^{t\mathcal{A}} \tau_a \qquad a \in \mathbb{R}, \quad t \geq 0. \tag{2.6}$$

§3. Domains of generators

3.1. Invariance of domains

In this section we investigate invariant properties of the domains of generators when $\{e^{t\mathcal{A}}\}_{t\geq 0}$ is invariant under the scaling induced by $\mathcal{R}$.

Lemma 3.1. *Let* (2.5) *hold. Then $\mathcal{A}R_r f = rR_r\mathcal{A}f$ if $f \in \mathrm{Dom}(\mathcal{A})$.*

Proof. The assertion easily follows from the equality $t^{-1}(e^{t\mathcal{A}}R_r f - R_r f) = R_r t^{-1}(e^{rt\mathcal{A}}f - f)$, which is derived from (2.5). The details are left to the reader. Q.E.D.

Corollary 3.2. *Let* (2.5) *hold. Let $w_{\mathcal{B}}$ be the growth bound of the strongly continuous group $\{R_{e^t}\}_{t\in\mathbb{R}}$. Then for all $f \in \mathrm{Dom}(\mathcal{A})$ and $\mu \in \mathbb{C}$ such that $\mathrm{Re}(\mu) > 1 + w_{\mathcal{B}}$ we have*

$$\mathcal{A}(\mu - \mathcal{B})^{-1} f = (\mu - 1 - \mathcal{B})^{-1}\mathcal{A}f. \tag{3.1}$$

Proof. The assertion follows from the Laplace formula $t^{-1}(e^{t\mathcal{A}} - I)(\mu - \mathcal{B})^{-1}f = \int_0^\infty e^{-\mu s}t^{-1}(e^{t\mathcal{A}} - I)R_{e^s}f ds$. We omitted the details here. Q.E.D.

Lemma 3.3. *Assume that* (2.5) *holds. Then* $\mathcal{B}e^{t\mathcal{A}}f = -t\mathcal{A}e^{t\mathcal{A}}f + e^{t\mathcal{A}}\mathcal{B}f$ *if* $f \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(\mathcal{B})$.

Proof. From (2.5) we have

$$\frac{R_r e^{t\mathcal{A}}f - e^{t\mathcal{A}}f}{r-1} = \frac{R_r(e^{t\mathcal{A}}f - e^{rt\mathcal{A}}f)}{r-1} + \frac{e^{t\mathcal{A}}(R_r f - f)}{r-1} = J_1 + J_2.$$

Then $J_1 \to -t\mathcal{A}e^{t\mathcal{A}}f$ and $J_2 \to e^{t\mathcal{A}}\mathcal{B}f$ in X as $r \to 1$, which yileds $e^{t\mathcal{A}}f \in \mathrm{Dom}(\mathcal{B})$ and $\mathcal{B}e^{t\mathcal{A}}f = -t\mathcal{A}e^{t\mathcal{A}}f + e^{t\mathcal{A}}\mathcal{B}f$. The proof is complete. Q.E.D.

Let $\mathcal{T} = \{\tau_a\}_{a\in\mathbb{R}}$ be a C_0-group acting on X. We denote by D the generator of $\mathcal{T}$, and τ_a will be often written as e^{aD}. Let $\mathcal{R}$ be a scaling acting on X. Then for each $\mu > 0$ a one-parameter family $\mathcal{R}^\mu = \{R_r^{(\mu)}\}_{r>0} \subset \mathcal{L}(X)$, $R_r^{(\mu)} = R_{r^\mu}$, also defines a scaling acting on X whose generator is $\mu\mathcal{B}$. Now we assume that $\mathcal{T}$ is invariant under the scaling induced by $\mathcal{R}^{1/\mu}$ for some $\mu > 0$, i.e.,

$$R_r^{(1/\mu)}e^{raD} = e^{aD}R_r^{(1/\mu)}, \qquad \text{for all } r > 0,\ a \in \mathbb{R}. \tag{3.2}$$

Since the generator of $\mathcal{R}^{1/\mu}$ is given by $\mu^{-1}\mathcal{B}$, Lemma 3.3 implies

Lemma 3.4. *Let* $\mathcal{T} = \{\tau_a\}_{a\in\mathbb{R}}$ *be a* C_0*-group acting on* X. *Assume that* (3.2) *holds for some* $\mu > 0$. *Then*

$$\mathcal{B}\tau_a f = -a\mu D\tau_a f + \tau_a\mathcal{B}f \qquad f \in \mathrm{Dom}(D) \cap \mathrm{Dom}(\mathcal{B}). \tag{3.3}$$

Remark 3.5. For a pair of linear operators L_1, L_2 its commutator is defined by $[L_1, L_2] = L_1L_2 - L_2L_1$. Then (3.3) is formally written as $[\mathcal{B}, \tau_a] = -a\mu D\tau_a$, which is a special case of the condition **(T1)** given in [6, Section 2.2.1]. That is, **(T1)** in [6] represents the symmetry between the scaling $\mathcal{R}$ and the translation $\mathcal{T}$ in the sense of (3.2).

3.2. Similarity transform and the associated generator

As in [6], we define the similarity transform of $\{e^{t\mathcal{A}}\}_{t\geq 0}$ with respect to the scaling induced by $\mathcal{R}$ by

$$S(t) = R_{e^t}e^{(e^t-1)\mathcal{A}}, \qquad t \geq 0. \tag{3.4}$$

This one-parameter family $\{S(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is closely related with the operator $\mathcal{C}$ defined by

$$\mathcal{C} = \mathcal{A} + \mathcal{B}, \quad \mathrm{Dom}(\mathcal{C}) = \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(\mathcal{B}). \tag{3.5}$$

Lemma 3.6. *Let* (2.5) *hold. Then the one parameter family* $\{S(t)\}_{t\geq 0}$ *defined by* (3.4) *is a* C_0*-semigroup acting on* X*, and its generator* A *satisfies* $\mathcal{C} \subset A$*. Moreover,* $w_A \leq w_{\mathcal{B}}$ *holds, where* w_A *and* $w_{\mathcal{B}}$ *are the growth bounds of* $\{S(t)\}_{t\geq 0}$ *and* $\{R_{e^t}\}_{t\in\mathbb{R}}$*, respectively.*

Proof. The assertion that $\{S(t)\}_{t\geq 0}$ defines a C_0-semigroup in X is stated in [6, Lemma 2.1]. The property $w_A \leq w_{\mathcal{B}}$ also follows from $\mathcal{R}_{e^t} e^{(e^t-1)\mathcal{A}} = e^{(1-e^{-t})}\mathcal{R}_{e^t}$, for we have $\|S(t)f\|_X \leq C\|R_{e^t} f\|_X$. From Lemma 3.1 and Lemma 3.3 we see that $Af = \mathcal{A}f + \mathcal{B}f$ for $f \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(\mathcal{B})$. This completes the proof. Q.E.D.

Lemma 3.7. *Let* (2.5) *hold. Then* $\mathcal{B}S(t)f = S(t)\mathcal{C}f - e^t S(t)\mathcal{A}f$ *if* $f \in \mathrm{Dom}(\mathcal{C})$.

Proof. The invariant property $S(t)\mathrm{Dom}(\mathcal{C}) \subset \mathrm{Dom}(\mathcal{C})$ immediately follows from Lemma 3.1 and Lemma 3.3. Furthermore, from Lemma 3.3 we have $\mathcal{B}S(t)f = R_{e^t}\big(-(e^t-1)e^{(e^t-1)\mathcal{A}}\mathcal{A}f + e^{(e^t-1)\mathcal{A}}\mathcal{B}f\big) = S(t)\mathcal{C}f - e^t S(t)\mathcal{A}f$. This completes the proof. Q.E.D.

In the following paragraphs we always assume (2.5). Since A is closed, $\mathcal{C}$ is closable and $\overline{\mathcal{C}} \subset A$ holds by Lemma 3.6.

Theorem 3.8. *Assume that* $\mathrm{Dom}(\mathcal{C})$ *is dense in* X*. Then* $A = \overline{\mathcal{C}}$*.*

Proof. Since $\mathrm{Dom}(\mathcal{C})$ is dense in X and is invariant under the action of $\{S(t)\}_{t\geq 0}$ by Lemma 3.7, $\mathrm{Dom}(\mathcal{C})$ is a core of A; [3, Proposition II-1-7]. This implies $A \subset \overline{\mathcal{C}}$, which completes the proof. Q.E.D.

Next we give a sufficient condition to lead $A = \mathcal{C}$ when X is a Hilbert space. For a linear operator L in a Hilbert space X we denote by L^* the adjoint operator of L in X.

Theorem 3.9. *Let* X *be a Hilbert space. Let* $\mathrm{Dom}(\mathcal{C})$ *be dense in* X *and* $\mathrm{Dom}(\mathcal{C}) \subset \mathrm{Dom}(\mathcal{A}^*) \cap \mathrm{Dom}(\mathcal{B}^*)$*. Assume that there are positive constants* $a_1,\ a_2,\ b_1,\ b_2,\ C'$*, such that* $a_1 + b_1 < 1,\ a_2 + b_2 < 1$*, and*

$$\|(\mathcal{A} - \mathcal{A}^*)f\|_X^2 \leq a_1\|\mathcal{A}f\|_X^2 + a_2\|\mathcal{B}f\|_X^2 + C'\|f\|_X^2, \tag{3.6}$$

$$\|(\mathcal{B} + \mathcal{B}^*)f\|_X^2 \leq b_1\|\mathcal{A}f\|_X^2 + b_2\|\mathcal{B}f\|_X^2 + C'\|f\|_X^2, \tag{3.7}$$

for all $f \in \mathrm{Dom}(\mathcal{C})$*. Then* $A = \mathcal{C}$*.*

Proof. By Theorem 3.8 it suffices to show that $\mathcal{C}$ is closed. Let $f \in \mathrm{Dom}(\mathcal{C})$. Then

$$\text{(3.8)} \quad \|(\mathcal{A}+\mathcal{B})f\|_X^2 = \|\mathcal{A}f\|_X^2 + \|\mathcal{B}f\|_X^2 + \langle \mathcal{B}f, \mathcal{A}^*f\rangle_X - \langle \mathcal{A}f, \mathcal{B}^*f\rangle_X + \langle \mathcal{A}f, (\mathcal{B}+\mathcal{B}^*)f\rangle_X + \langle \mathcal{B}f, (\mathcal{A}-\mathcal{A}^*)f\rangle_X.$$

From (3.6) and (3.7) we have

$$\text{(3.9)} \quad \begin{aligned} &\langle \mathcal{A}f, (\mathcal{B}+\mathcal{B}^*)f\rangle_X + \langle \mathcal{B}f, (\mathcal{A}-\mathcal{A}^*)f\rangle_X \\ \geq\ & -\frac{1+a_1+b_1}{2}\|\mathcal{A}f\|_X^2 - \frac{1+a_2+b_2}{2}\|\mathcal{B}f\|_X^2 - C\|f\|_X^2, \end{aligned}$$

for some $C > 0$. Next we observe from Lemma 3.3 that $\langle \mathcal{B}(e^{t\mathcal{A}}f - f), f\rangle_X = -t\langle \mathcal{A}e^{t\mathcal{A}}f, f\rangle_X + \langle (e^{t\mathcal{A}} - I)\mathcal{B}f, f\rangle_X$. Since X is reflexive, the adjoint operator of $e^{t\mathcal{A}}$ is given by $e^{t\mathcal{A}^*}$ ([3, Propositions I-5-14, II-2-6], and thus we have $\langle t^{-1}(e^{t\mathcal{A}}f - f), \mathcal{B}^*f\rangle_X = -\langle \mathcal{A}e^{t\mathcal{A}}f, f\rangle_X + \langle \mathcal{B}f, t^{-1}(e^{t\mathcal{A}^*}f - f)\rangle_X$. Taking the limit $t \to 0$ leads to the equality

$$\text{(3.10)} \quad \langle \mathcal{B}f, \mathcal{A}^*f\rangle_X - \langle \mathcal{A}f, \mathcal{B}^*f\rangle_X = \langle \mathcal{A}f, f\rangle_X.$$

Collecting (3.8)–(3.10), we finally get the inequality

$$\text{(3.11)} \quad \|(\mathcal{A}+\mathcal{B})f\|_X^2 \geq \frac{1-a_1-b_1}{4}\|\mathcal{A}f\|_X^2 + \frac{1-a_2-b_2}{2}\|\mathcal{B}f\|_X^2 - C\|f\|_X^2,$$

for some $C > 0$ if $f \in \mathrm{Dom}(\mathcal{C})$. The estimate (3.11) is enough to conclude that $\mathcal{C}$ is closed. This completes the proof. Q.E.D.

Next we look for another sufficient condition to ensure $A = \mathcal{C}$, which can be applied also for the case when X is not a Hilbert space. For this purpose we follow the arguments by Metafune et al [7], where the domains of the Ornstein–Uhlenbeck operators are discussed.

Theorem 3.10. *Let X be a Banach space of class $\mathcal{HT}$. Assume that $\lambda_0 - \mathcal{A}$, $\mu_0 - \mathcal{B} \in BIP(X)$ for some $\lambda_0 > w_{\mathcal{A}}$, $\mu_0 > 1 + w_{\mathcal{B}}$, and that the strong parabolicity condition $\theta_{\lambda_0-\mathcal{A}} + \theta_{\mu_0-\mathcal{B}} < \pi$ holds. Here θ_L is the power angle of $L \in BIP(X)$. Then $A = \mathcal{C}$.*

Proof. For the definitions of $\mathcal{HT}$, $BIP(X)$, and the power angle, we refer to [4]. As in [7], the proof is based on the closedness result for noncommuting operators by Monniaux and Prüss [8], which is a significant extension of the classical Dore–Venni theorem [2]. We write $\mathcal{A}_0 = \mathcal{A} - \lambda_0$ and $\mathcal{B}_0 = \mathcal{B} - \mu_0$ for simplicity of notations. Then from Corollary 3.2 we have $\mathcal{A}(\mu - \mathcal{B}_0)^{-1}(-\mathcal{A}_0)^{-1} = (\mu - 1 - \mathcal{B}_0)^{-1}\mathcal{A}(-\mathcal{A}_0)^{-1}$ for all $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) \geq 0$. Hence from the definition of $\mathcal{A}_0$ it is not difficult to derive $\mathcal{A}_0(\mu-\mathcal{B}_0)^{-1}(-\mathcal{A}_0)^{-1} = -(\mu-1-\mathcal{B}_0)^{-1} - \lambda_0\{(\mu-\mathcal{B}_0)^{-1} - (\mu-1-\mathcal{B}_0)^{-1}\}(-\mathcal{A}_0)^{-1}$, which yield by the resolvent equation,

(3.12)

$$\begin{aligned}
&[(-\mathcal{A}_0)^{-1}, (\mu-\mathcal{B}_0)^{-1}] \\
&\quad = (-\mathcal{A}_0)^{-1}\{(\mu-\mathcal{B}_0)^{-1} - (\mu-1-\mathcal{B}_0)^{-1}\} \\
&\quad\quad - \lambda_0(-\mathcal{A}_0)^{-1}\{(\mu-\mathcal{B}_0)^{-1} - (\mu-1-\mathcal{B}_0)^{-1}\}(-\mathcal{A}_0)^{-1} \\
&\quad = (-\mathcal{A}_0)^{-1}(\mu-\mathcal{B}_0)^{-1}(\mu-1-\mathcal{B}_0)^{-1}\{\lambda_0(-\mathcal{A}_0)^{-1} - 1\}.
\end{aligned}$$

Let ϕ_0, ψ_0 be positive numbers such that $\phi_0 > \theta_{-\mathcal{A}_0}$, $\psi_0 > \theta_{-\mathcal{B}_0}$, and $\phi_0+\psi_0 < \pi$. Since each of $(-\mathcal{A}_0)^{-1}(\mu-\mathcal{B}_0)^{-1}$, $(\mu-\mathcal{B}_0)^{-1}(-\mathcal{A}_0)^{-1}$, and $(-\mathcal{A}_0)^{-1}(\mu-\mathcal{B}_0)^{-1}(\mu-1-\mathcal{B}_0)^{-1}\{1-\lambda_0(-\mathcal{A}_0)^{-1}\}$ is holomorphic with respect to μ in the sector $\sum_{\pi-\psi_0} := \{z \in \mathbb{C} \mid z \neq 0,\ |\arg z| < \pi-\psi_0\}$, the equality (3.12) holds for all $\mu \in \sum_{\pi-\psi_0}$. Then from (3.12) we have

$$\|\mathcal{A}_0(\lambda-\mathcal{A}_0)^{-1}[(-\mathcal{A}_0)^{-1}, (\mu-\mathcal{B}_0)^{-1}]\|_{\mathcal{L}(X)} \leq \frac{C}{(1+|\lambda|)(1+|\mu|)^2}$$

for all $\lambda \in \sum_{\pi-\phi_0}$, $\mu \in \sum_{\pi-\psi_0}$. By [8, Corollary 2] the operator $\mathcal{C}_0 = \mathcal{A}_0+\mathcal{B}_0$ with $\mathrm{Dom}(\mathcal{C}_0) = \mathrm{Dom}(\mathcal{C})$ is closed and $\nu_0-\mathcal{C}_0$ is sectorial for some $\nu_0 \geq 0$. Since X is of class $\mathcal{HT}$, it is reflexive. Thus $\mathrm{Dom}(\mathcal{C}) = \mathrm{Dom}(\mathcal{C}_0)$ is dense in X by [4, Proposition 2.1.1]. Hence from Theorem 3.8 we have $A = \overline{\mathcal{C}} = \mathcal{C}$. This completes the proof.

Q.E.D.

§4. Spectral property of $\mathcal{A}+\mathcal{B}$

Let $\sigma_p(L)$ be the set of point spectrum of a linear operator L in X. With the definition of $\mathcal{C}$ in (3.5) we show that if $\mathcal{C}$ is closed and if $\mathcal{A}$ is injective in addition, then one eigenvalue of $\mathcal{C}$ produces infinitely many eigenvalues of $\mathcal{C}$ which reflect the symmetry of the scaling invariance. The argument used here is almost same as [6, Lemma 6.2].

Theorem 4.1. *Set* $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. *Let* $\mathcal{C}$ *be the linear operator defined by* (3.5). (i) *Let* $0 \in \sigma_p(\mathcal{C})$ *and let* $U \in X$ *be an associated eigenfunction. If either* $\mathcal{A}$ *or* $\mathcal{B}$ *is injective, then* -1 *is an eigenvalue of* $\overline{\mathcal{C}}$ *and* $\mathcal{B}U$ *is an eigenfunction to the eigenvalue* -1 *of* $\overline{\mathcal{C}}$. (ii) *Assume that* $\mathcal{C}$ *is closed. Let* $\mu \in \sigma_p(\mathcal{C})$ *and let* $U \in X$ *be an associated eigenfunction. If either* $\mathcal{A}$ *is injective or* $\sigma_p(\mathcal{B})\cap\{\mu-k \mid k \in \mathbb{N}_0\} = \emptyset$, *then* $\{\mu-k \mid k \in \mathbb{N}_0\} \subset \sigma_p(\mathcal{C})$. *Moreover,* $\mathcal{A}^kU$ *is an eigenfunction to the eigenvalue* $\mu-k$ *of* $\mathcal{C}$.

Proof. The assertion (i) is essentially proved in [6, Lemma 6.2] and here we show (ii) only. Under the assumptions of (ii) we will prove by the induction of k that $\mu - k$ is an eigenvalue of $\mathcal{C}$ and $\mathcal{A}^k U (\neq 0)$ is an associated eigenfunction. The case $k = 0$ follows from the assumptions. Suppose that the assertion is true for k. Then we have $\mathcal{A}^k U \in \mathrm{Dom}(\mathcal{C})\setminus\{0\}$ and

$$\mathcal{A}^{k+1}U + \mathcal{B}\mathcal{A}^k U = (\mu - k)\mathcal{A}^k U. \tag{4.1}$$

From Lemma 3.3 and (4.1) we get $\mathcal{C}t^{-1}(e^{t\mathcal{A}}\mathcal{A}^k U - \mathcal{A}^k U) = t^{-1}\{e^{t\mathcal{A}}\mathcal{A}^{k+1}U + \mathcal{B}e^{t\mathcal{A}}\mathcal{A}^k U - \mathcal{A}^{k+1}U - \mathcal{B}\mathcal{A}^k U\} = t^{-1}\{e^{t\mathcal{A}}\mathcal{A}^{k+1}U - te^{t\mathcal{A}}\mathcal{A}^{k+1}U + e^{t\mathcal{A}}\mathcal{B}\mathcal{A}^k U + (k-\mu)\mathcal{A}^k U\} = -e^{t\mathcal{A}}\mathcal{A}^{k+1}U + (\mu - k)t^{-1}(e^{t\mathcal{A}}\mathcal{A}^k U - \mathcal{A}^k U)$. Hence $\mathcal{C}(e^{t\mathcal{A}}\mathcal{A}^k U - \mathcal{A}^k U)/t$ converges to $(\mu - k - 1)\mathcal{A}^{k+1}U$ in X as $t \to 0$, while $(e^{t\mathcal{A}}\mathcal{A}^k U - \mathcal{A}^k U)/t$ converges to $\mathcal{A}^{k+1}U$ in X as $t \to 0$. Since $\mathcal{C}$ is closed, this implies that $\mathcal{A}^{k+1}U \in \mathrm{Dom}(C)$ and $\mathcal{C}\mathcal{A}^{k+1}U = (\mu - k - 1)\mathcal{A}^{k+1}U$. Suppose that $\mathcal{A}^{k+1}U = 0$. Then $\mathcal{B}\mathcal{A}^k U = (\mu - k)\mathcal{A}^k U$ by (4.1). Thus $\mathcal{A}^k U$ must be 0 since $\mathcal{A}$ is injective or $\sigma_p(\mathcal{B}) \cap \{\mu - k \mid k \in \mathbb{N}_0\} = \emptyset$, which is a contradiction. So $\mathcal{A}^{k+1}U$ is an eigenfunction to the eigenvalue $\mu - k - 1$ of $\mathcal{C}$. This completes the proof. Q.E.D.

Next we consider the case $\{e^{t\mathcal{A}}\}_{t\geq 0}$ is also invariant with respect to some strongly continuous groups. We will assume that:
(s1) $\{e^{t\mathcal{A}}\}_{t\geq 0}$ is invariant under the scaling induced by $\mathcal{R}$; see (2.5).
(s2) There are n strongly continuous groups $\mathcal{T}^{(j)} = \{\tau_a^{(j)}\}_{a\in\mathbb{R}}$, $1 \leq j \leq n$, acting on X such that they commute with each other, i.e., $\tau_a^{(i)}\tau_{a'}^{(j)} = \tau_{a'}^{(j)}\tau_a^{(i)}$, $a, a' \in \mathbb{R}$, $1 \leq i, j \leq n$, and that $\{e^{t\mathcal{A}}\}_{t\geq 0}$ is invariant with respect to each $\mathcal{T}^{(j)}$; see (2.6).
(s3) For each j there is $\mu_j > 0$ such that $\mathcal{T}^{(j)}$ is invariant with respect to $\mathcal{R}^{1/\mu_j}$; see (3.2).

These assumptions imply three symmetries; **(s1)** between semigroup and scaling, **(s2)** between semigroup and translations, **(s3)** between translations and scaling. As in Theorem 4.1, when $\sigma_p(\mathcal{C}) \neq \emptyset$ there are infinitely many eigenvalues of $\mathcal{C}$ in X under some conditions on the domains of generators. We denote by D_j the generator of $\mathcal{T}^{(j)}$ in X.

Theorem 4.2. *Assume that* **(s1)**, **(s2)**, **(s3)** *hold and that* $\mathcal{C}$ *is closed. Assume that each* D_j *is injective and* $\mathrm{Dom}(\mathcal{C}) \subset \cap_{j=1}^n \mathrm{Dom}(D_j)$. *Let* $\mu \in \sigma_p(\mathcal{C})$ *and let* $U \in X$ *be an associated eigenfunction. Then* $\{\mu - \sum_{j=1}^n k_j\mu_j \mid k_j \in \mathbb{N}_0,\ j = 1, \cdots, n\} \subset \sigma_p(\mathcal{C})$. *Moreover,* $D_1^{k_1}\cdots D_n^{k_n}U$ *is an eigenfunction to the eigenvalue* $\mu - \sum_{j=1}^n k_j\mu_j$ *of* $\mathcal{C}$ *in* X.

Proof. We follows the arguments in [6, Lemma 6.2]. From **(s2)** it is easy to see that $D_iD_jf = D_jD_if$ if $f \in \mathrm{Dom}(D_j) \cap \mathrm{Dom}(D_jD_i)$, where $\mathrm{Dom}(D_jD_i) = \{f \in \mathrm{Dom}(D_i) \mid D_if \in \mathrm{Dom}(D_j)\}$. Hence by taking the assumption $\mathrm{Dom}(\mathcal{C}) \subset \cap_{j=1}^{n} \mathrm{Dom}(D_j)$ into account, it suffices to show that $D_j U$ is an eigenfunction to the eigenvalue $\mu - \mu_j$ of $\mathcal{C}$ in X, for the general cases then follow by the induction on k_i. Since $U \in \mathrm{Dom}(\mathcal{C})$ satisfies $\mathcal{A}U + \mathcal{B}U = \mu U$, we have $\mathcal{A}\tau_a^{(j)} U + \tau_a^{(j)} \mathcal{B}U = \mu \tau_a^{(j)} U$. Here we used the property $\tau_a^{(j)} \mathcal{A}U = \mathcal{A}\tau_a^{(j)} U$ which follows from (2.6). By (3.2) and $\mathrm{Dom}(\mathcal{C}) \subset \cap_{j=1}^{n} \mathrm{Dom}(D_j)$ we have from Lemma 3.4 that $\tau_a^{(j)} \mathcal{B}U = \mathcal{B}\tau_a^{(j)} U - [\mathcal{B}, \tau_a^{(j)}]U = \mathcal{B}\tau_a^{(j)} U + a\mu_j \tau_a^{(j)} D_j U$. This yields $\mathcal{C}\tau_a^{(j)} U = \mu \tau_a^{(j)} U - a\mu_j \tau_a^{(j)} D_j U$, that is, $\mathcal{C}a^{-1}(\tau_a^{(j)} U - U) = \mu a^{-1}(\tau_a^{(j)} U - U) - \mu_j \tau_a^{(j)} D_j U$. Since $(\tau_a^{(j)} U - U)/a$ converges to $D_j U$ in X as $a \to 0$, by the closedness of $\mathcal{C}$ we have $D_j U \in \mathrm{Dom}(\mathcal{C})$ and $\mathcal{C}D_j U = (\mu - \mu_j) D_j U$. Since D_j is injective and U is not trivial, $\mu - \mu_j$ is an eigenvalue of $\mathcal{C}$ and $D_j U$ is an associated eigenfunction. This completes the proof. Q.E.D.

Combining Theorem 4.1 and Theorem 4.2, we have

Corollary 4.3. *Assume that the assumptions in Theorem* 4.2 *hold. Let* $\mathcal{A}$ *be injective. Then* $\{\mu - \sum_{j=1} k_j \mu_j - k_0 \mid k_j \in \mathbb{N}_0,\ j = 0, 1, \cdots, n\} \subset \sigma_p(\mathcal{C})$. *Moreover,* $\mathcal{A}^{k_0} D_1^{k_1} \cdots D_n^{k_n} U$ *is an eigenfunction to the eigenvalue* $\mu - \sum_{j=1} k_j \mu_j - k_0$ *of* $\mathcal{C}$ *in* X.

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